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On Spin Distributions for Generic *p*-Spin Models

Antonio Auffinger[1] · **Aukosh Jagannath**[2]

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Abstract
We provide an alternative formula for spin distributions of generic p-spin glass models. As a main application of this expression, we write spin statistics as solutions of partial differential equations and we show that the generic p-spin models satisfy multiscale Thouless–Anderson–Palmer equations as originally predicted in the work of Mézard–Virasoro (J Phys 46(8):1293–1307, 1985).

Keywords Spin glasses · Thouless–Anderson–Palmer equations · Cavity equations

1 Introduction

Let H_N be the Hamiltonian for the mixed p-spin model on the discrete hypercube $\{+1, -1\}^N$,

$$H_N(\sigma) = \sum_{p\geq 2} \frac{\beta_p}{N^{(p-1)/2}} \sum_{1\leq i_1,\dots,i_p\leq N} g_{i_1,\dots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{1.1}$$

where $\{g_{i_1,\dots,i_p}\}$ are i.i.d. standard Gaussian random variables. Observe that if we let

$$\xi(x) = \sum_{p\in\mathbb{N}} \beta_p^2 x^p,$$

then the covariance of H_N satisfies

$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = N\xi(R_{1,2}),$$

where $R_{\ell,\ell'} := \frac{1}{N}\sum_{i=1}^N \sigma_i^\ell\sigma_i^{\ell'}$ is the normalized inner-product between σ^ℓ and $\sigma^{\ell'}$, $\ell, \ell' \geq 1$. We let G_N to be the Gibbs measure associated to H_N. In this note, we will be concerned with generic p-spin models, that is, those models for which the linear span of the set $\{1\} \cup \{x^p : p \geq 2,\ \beta_p \neq 0\}$ is dense in $(C([-1, 1]), ||\cdot||_\infty)$.

✉ Aukosh Jagannath
aukosh@math.harvard.edu

Antonio Auffinger
tuca@northwestern.edu

1 Department of Mathematics, Northwestern University, Evanston, USA

2 Department of Mathematics, Harvard University, Cambridge, USA

Generic p-spin models are central objects in the study of mean field spin glasses. They satisfy the Ghirlanda–Guerra identities [16]. As a consequence, if we let $(\sigma^\ell)_{\ell\geq 1}$ be i.i.d. draws from G_N, and consider the array of overlaps $(R_{\ell\ell'})_{\ell,\ell'\geq 1}$, then it is known [17] that this array satisfies the ultrametric structure proposed in the physics literature [13]. Moreover, it can be shown (see, e.g., [18]) that the limiting law of R_{12} is given by the Parisi measure, ζ, the unique minimizer of the Parisi formula [5,21].

In [19], a family of invariance principles, called the cavity equations, were introduced for mixed p-spin models. It was shown there that if the spin array

$$(\sigma_i^\ell)_{1\leq i\leq N, 1\leq \ell}. \tag{1.2}$$

satisfies the cavity equations then they can be uniquely characterized by their overlap distributions. It was also shown that mixed p-spin models satisfy these cavity equations modulo a regularizing perturbation that does not affect the free energy. In fact, it can be shown (see Proposition 1.1 below) by a standard argument that generic models satisfy these equations without perturbations. Consequently, the spin distributions are characterized by ζ as well by the results of [19].

Panchenko also showed that the Bolthausen–Sznitman invariance [8] can be utilized to provide a formula for the distribution of spins [18,19]. The main goal of this note is to present an alternative expression for spin distributions of generic models in terms of a family of branching diffusions. This new way of describing the spin distributions provides expressions for moments of spin statistics as solutions of certain partial differential equations. We show a few examples and applications in Sect. 5. One of our main applications is that these spin distributions satisfy a multi-scale generalization of the Thouless–Anderson–Palmer (TAP) equations similar to that suggested in [14,15]. This complements the authors previous work on the Thouless–Anderson–Palmer equations for generic p-spin models at finite particle number [6].

1.1 Main Results

In this paper we assume that the reader is familiar with the theory of spin distributions. For a textbook introduction, see [18, Chapter 4]. We include the relevant definitions and constructions in the Appendix for the reader's convenience. The starting point of our analysis is the following observation, which says that the generic p-spin models satisfy the cavity equations. These equations are stated in (A.3).

Proposition 1.1 *Let ν be a limit of the spin array (1.2) for a generic p-spin model. Then ν satisfies the cavity equations (A.3) for $r = 0$. In particular, ν is unique.*

Let $q_* > 0$ and U be a positive, ultrametric subset of the sphere of radius $\sqrt{q_*}$ in $L^2([0,1])$ in the sense that for any $x, y, z \in U$, we have $(x, y) \geq 0$ and $\|x - z\| \leq \max\{\|x - y\|, \|y - z\|\}$. Define the driving process on U to be the Gaussian process, $B_t(\sigma)$, indexed by $(t, \sigma) \in [0, q_*] \times U$, which is centered, a.s. continuous in time and measurable in space, with covariance

$$\mathrm{Cov}_B((t_1, \sigma^1), (t_2, \sigma^2)) = (t_1 \wedge t_2) \wedge (\sigma^1, \sigma^2). \tag{1.3}$$

Put concretely, for each fixed σ, $B_t(\sigma)$ is a Brownian motion and for finitely many (σ^i), $(B_t(\sigma^i))$ is a family of branching Brownian motions whose branching times are given by the inner products between these σ^i.

We then define the cavity field process on U as the solution, $Y_t(\sigma)$, of the SDE

$$\begin{cases} dY_t(\sigma) = \sqrt{\xi''(t)} dB_t(\sigma) \\ Y_0(\sigma) = h. \end{cases} \tag{1.4}$$

Let ζ be the Parisi measure for the generic p-spin model. Let u be the unique weak solution to the Parisi initial value problem on $(0, 1) \times \mathbb{R}$,

$$\begin{cases} u_t + \frac{\xi''(t)}{2} \left(u_{xx} + \zeta([0, t]) u_x^2 \right) = 0, \\ u(1, x) = \log \cosh(x). \end{cases} \tag{1.5}$$

For the definition of weak solution in this setting and basic properties of u see [11]. We now define the local field process, $X_t(\sigma)$, to be the solution to the SDE

$$\begin{cases} dX_t(\sigma) = \xi''(t)\zeta([0, t])u_x(t, X_t(\sigma))dt + dY_t(\sigma) \\ X_0(\sigma) = h. \end{cases} \tag{1.6}$$

Finally, let the magnetization process be $M_t(\sigma) = u_x(t, X_t(\sigma))$. We will show that the process $X_{q_*}(\sigma)$ is related to a re-arrangement of $Y_{q_*}(\sigma)$. If we view σ as a state, then $M_{q_*}(\sigma)$ will be the magnetization of this state. The basic properties of these processes, e.g., existence, measurability, continuity, etc, are studied briefly in Appendix A.1. We invite the reader to compare their definitions to [14, Eq. IV.51] and [8, Eq. 0.20] (see also [3]). We remind the reader here that the support of the asymptotic Gibbs measure for a generic p-spin model is positive and ultrametric by Panchenko's ultrametricity theorem and Talagrand's positivity principle [18], provided we take $q_* = \sup \operatorname{supp}(\zeta)$.

Now, for a fixed measurable function f on $L^2([0, 1])$, write the measure μ_σ^f, on $\{-1, 1\} \times \mathbb{R}$ as the measure with density $p(s, y; f)$ given by

$$p(s, y; f) \propto e^{sy} e^{-\frac{(y-f)^2}{2(\xi'(1)-\xi'(q_*))}}.$$

Observe that by an application of Girsanov's theorem (see specifically [12, Lemma 8.3.1]), the measure above is equivalently described as the measure on $\{-1, 1\} \times \mathbb{R}$ such that for any bounded measurable ϕ,

$$\int \phi \, d\mu_\sigma^f := \mathbb{E} \left(\frac{\sum_{s \in \{\pm 1\}} \phi(s, X_1) e_1^{X_1 s}}{2 \cosh(X_1)} \middle| X_{q_*}(\sigma) = f(\sigma) \right). \tag{1.7}$$

For any bounded measurable ϕ, we let $\langle \phi \rangle_\sigma^f$, denote its expected value with respect to μ_σ^f. When it is unambiguous we omit the superscript for the boundary data. For multiple copies, $(s_i, y_i)_{i=1}^\infty$, drawn from the product $\mu_\sigma^{\otimes \infty}$, we also denote the average by $\langle \cdot \rangle_\sigma$.

Let μ be a random measure on $L^2([0, 1])$ such that the corresponding overlap array satisfies the Ghirlanda–Guerra identities (see Appendix A.2 for the definition of these identities). Consider the law of the random variables (S, Y) defined through the relation:

$$\mathbb{E} \langle \phi(S, Y) \rangle = \mathbb{E} \int \langle \phi \rangle_\sigma^X \, d\mu(\sigma) \tag{1.8}$$

and the random variables (S', Y') defined through the relation

$$\mathbb{E} \left\langle \phi(S', Y') \right\rangle = \mathbb{E} \int \langle \phi \rangle_\sigma^Y \frac{\cosh(Y_{q_*}(\sigma))}{\int \cosh(Y_{q_*}(\sigma)) d\mu(\sigma)} d\mu(\sigma).$$

Let $(S_i, Y_i)_{i\geq 1}$ be drawn from $\left(\mu_\sigma^X\right)^{\otimes\infty}$ and (S_i', Y_i') be drawn from $\left(\mu_\sigma^Y\right)^{\otimes\infty}$ where σ is drawn from μ. For i.i.d. draws $(\sigma^\ell)_{\ell\geq 1}$ from $\mu^{\otimes\infty}$, we define (S_i^ℓ, Y_i^ℓ) and $(S_i'^\ell, Y_i'^\ell)$ analogously.

The main result of this note is the following alternative representation for spins from cavity invariant measures. We let $\mathscr{M}_{inv}^\xi$ denote the space of law of exchangeable arrays with entries in $\{\pm 1\}$ that satisfy the cavity equations and the Ghirlanda–Guerra identities.

Theorem 1.2 *We have the following.*

(1) *For any generic model ξ and any asymptotic Gibbs measure μ, let $(\sigma_\ell)_{\ell\geq 1}$ be i.i.d. draws from μ, let (S_i^ℓ, Y_i^ℓ) and $(S_i'^\ell, Y_i'^\ell)$ be defined as above with $\sigma = \sigma^\ell$. Then these random variables are equal in distribution.*

(2) *For any measure ν in $\mathscr{M}_{inv}^\xi$, let (s_i^ℓ) denote the array of spins and μ denote its corresponding asymptotic Gibbs measure. Let (S_i^ℓ) be defined as above with $\zeta = \mathbb{E}\mu^{\otimes^2}((\sigma^1, \sigma^2) \in \cdot)$. Then we have*

$$(s_i^\ell) \overset{(d)}{=} (S_i^\ell).$$

Remark 1.3 In [18,19], Panchenko obtained first a description of the laws of (s_i^ℓ) in a finite replica symmetry breaking regime (i.e., when ζ consists of finitely many atoms) using Ruelle probability cascades (see (3.1)). By sending the number of levels of replica symmetry breaking to infinity, he obtains a formula that is valid for any generic p-spin [18, Theorem 4.2]. This is a key step in our proof of Theorem 1.2. At finite replica symmetry breaking, the connection to the process X_t can already be seen in [8, pp. 249–250] as a consequence of the Bolthausen–Sznitman invariance principle.

Let us now briefly present an application of this result. Let ν be the spin distribution for a generic model and let μ be the corresponding asymptotic Gibbs measure. Let $\sigma \in \text{supp}(\mu)$ and fix $q \in [0, q_*]$, where $q_* = \sup \text{supp}(\zeta)$. Let

$$B(\sigma, q) = \left\{\sigma' \in \text{supp}(\mu) : (\sigma, \sigma') \geq q\right\}$$

be the set of points in the support of μ that are of overlap at most q with σ. Recall that by Panchenko's ultrametricity theorem [17], we may decompose

$$\text{supp}\ \mu = \cup_\alpha B(\sigma^\alpha, q)$$

where this union is disjoint. If we call $W_\alpha = B(\sigma^\alpha, q)$, we can then consider the law of (s, y), the spin and the cavity field, but now conditionally on W_α. That is, let $\langle\cdot\rangle_\alpha$ denote the conditional law $\mu(\cdot|W_\alpha)$. We then have the following result.

Theorem 1.4 (*Mezard–Virasoro multiscale Thouless–Anderson–Palmer equations*) *We have that*

$$\langle s\rangle_\alpha = u_x\left(q, \langle y\rangle_\alpha - \int_q^1 \xi''(t)\zeta([0,t])dt \cdot \langle s\rangle_\alpha\right)$$

where again u_x is the first spatial derivative of the Parisi PDE corresponding to ζ.

2 Cavity Equations for Generic Models

2.1 Decomposition and Regularity of Mixed p-Spin Hamiltonians

In this section, we present some basic properties of mixed p-spin Hamiltonians. Let $1 \leq n < N$. For $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N$, $\rho(\sigma) = (\sigma_{n+1}, \ldots, \sigma_N) \in \Sigma_{N-n}$, we can write the

Hamiltonian H_N as

$$H_N(\sigma) = \tilde{H}_N(\sigma) + \sum_{i=1}^{n} \sigma_i y_{N,i}(\rho) + r_N(\sigma). \tag{2.1}$$

where the processes $\tilde{H}_N$, $y_{N,i}$ and r_N satisfy the following lemma.

Lemma 2.1 *There exist centered Gaussian processes $\tilde{H}_N, y_N, r_N$ such that (2.1) holds and*

$$\begin{aligned}
\mathbb{E}\tilde{H}_N(\sigma^1)\tilde{H}_N(\sigma^2) &= N\xi\left(\frac{N-1}{N}R_{12}\right),\\
\mathbb{E}y_N^i(\sigma^1)y_N^j(\sigma^2) &= \delta_{ij}(\xi'(R_{12}) + o_N(1)),\\
\mathbb{E}r_N(\sigma^1)r_N(\sigma^2) &= O(N^{-1}).
\end{aligned}$$

Furthermore, there exist positive constant C_1 and C_2 so that with probability at least $1 - e^{-C_1 N}$,

$$\max_{\sigma\in\Sigma_{N-1}} |r_N(1,\sigma) - r_N(-1,\sigma)| \le \frac{C_2}{\sqrt{N}},$$

and a positive constant C_3 so that

$$\mathbb{E}\exp\left(2\max_{\sigma\in\Sigma_{N-1}} |r_N(1,\sigma) - r_N(-1,\sigma)|\right) \le C_3. \tag{2.2}$$

Proof The lemma is a standard computation on Gaussian processes. Let us focus on the case $n = 2$. The general case is analagous. Furthermore, to simplify the exposition we will consider the pure p-spin model. The mixed case follows by linearity. Here, we set

$$\begin{aligned}
\tilde{H}_N(\rho(\sigma)) &= N^{-\frac{p-1}{2}} \sum_{2\le i_1,\dots,i_p\le N} g_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p},\\
y_N(\rho(\sigma)) &= N^{-\frac{p-1}{2}} \sum_{k=1}^{p} \sum_{\substack{2\le i_1,\dots,i_p\le N\\ i_k=1}} g_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p}, \quad \text{and,}\\
r_N(\sigma_1,\rho(\sigma)) &= N^{-\frac{p-1}{2}} \sum_{l=2}^{p} \sigma_1^{\ell} \sum_{2\le i_1,\dots,i_{p-\ell}\le N} J_{i_1\dots i_{p-\ell}}\sigma_{i_1}\dots\sigma_{i_{p-\ell}},
\end{aligned}$$

where $g_{i_1,\dots,i_p}$ are as above and $J_{i_1\dots i_{p-\ell}}$ are centered Gaussian random variables with variance equal to $\binom{p}{\ell}$: $J_{i_1\dots i_{p-\ell}}$ is the sum of the $g_{i_1\dots i_p}$ where the index 1 appears exactly ℓ times. Computing the variance of these three Gaussian processes give us the the first three statements of the Lemma. For the second to last and last statement, note that for any $\sigma \in \Sigma_{N-1}$, $r(1,\sigma) - r(-1,\sigma)$ is a centered Gaussian process with variance equal to

$$\frac{4}{N^{p-1}} \sum_{\ell=3,\ \ell \text{ odd}}^{p} \binom{p}{\ell}(N-1)^{p-\ell} \le \frac{C_p}{N^2},$$

for some constant C_p. A standard application of Borell's inequality and the Sudakov–Fernique's inequality [1] gives us the desired result. □

We now turn to the proof that generic models satisfy the cavity equations. The argument is fairly standard—see for example [18, Chapter 3, Theorem 3.6].

Proof of Proposition 1.1 Fix n sites and a C_l as in (A.3). By site symmetry, we may assume that these are the last n sites. Our goal is then to show that

$$\mathbb{E}\prod_{l\le q}\left\langle\prod_{i\in C_l}\sigma_i\right\rangle=\mathbb{E}\prod_{i\le q}\frac{\left\langle\prod_{i\in C_l}\tanh(g_{\xi',i}(\sigma))\mathcal{E}_n\right\rangle}{\langle\mathcal{E}_n\rangle}+o_N(1). \tag{2.3}$$

With this observation in hand, note that by Lemma 2.1, the left side of (2.3) is equivalent to

$$\mathbb{E}\prod_{l\le q}\frac{\left\langle\prod_{i\in C_l}\tanh(y_{N,i}(\sigma))\mathcal{E}_{n,0}\right\rangle_{G'}}{\langle\mathcal{E}_{n,0}\rangle_{G'}^q},$$

where G' is the Gibbs measure for $\tilde{H}_N$ on Σ_{N-n}.

By a localization and Stone–Weierstrass argument, we see that it suffices to show that

$$\mathbb{E}\prod_{l\le q}\left\langle\prod_{i\in C_l}\tanh(y_i(\sigma))\mathcal{E}_n\right\rangle_{G'}\langle\mathcal{E}_n\rangle_{G'}^k=\mathbb{E}\prod_{l\le q}\left\langle\prod_{i\in C_l}\tanh(g_{\xi' i}(\sigma))\mathcal{E}_n\right\rangle_{G}\langle\mathcal{E}_n\rangle_{G}^k+o_N(1).$$

Evidently, this will follow provided the limiting overlap distribution for $\mathbb{E}G'^{\otimes\infty}$ and $\mathbb{E}G^{\otimes\infty}$ are the same. As generic models are known to have a unique limiting overlap distribution (by Lemma 3.6 of [18]), it suffices to show that in fact the overlap distribution of law of H'_N and H_{N-n} are the same. Observe that

$$\left|\mathrm{Cov}_{H'}(\sigma^1,\sigma^2)-\mathrm{Cov}_H(\sigma^1,\sigma^2)\right|=N\left|\xi\left(\frac{N}{N+n}R_{12}\right)-\xi(R_{12})\right|\le C(\xi,n),$$

uniformly for $\sigma^1,\sigma^2\in\Sigma_{N-n}$, so that by a standard interpolation argument (see, e.g., [18, Theorem 3.6]) we have that the free energy of these two systems is the same in the limit $N\to\infty$. An explicit differentiation argument (see [18, Theorem 3.7]) combined with [18, Theorem 2.13] shows that the overlap distributions are the same. □

3 Proofs of Representation Formulas

We now turn to the proofs of the results at infinite particle number. Before we can state these results we need to recall certain basic results of Panchenko from the theory of spin distributions [18,19]. The notation here follows [18, Chapter 4] (alternatively, see the Appendix below).

3.1 Preliminaries

We begin with the observation that if we apply the cavity equations, (A.3) with $n=m$ and $r=0$, we get that,

$$\mathbb{E}\prod_{l\le q}\prod_{i\in C_l}s_i^l=\mathbb{E}\frac{\prod_{l\le q}\mathbb{E}'\prod_{i\in C_l}\tanh(G_{\xi',i}(\bar{\sigma}))\prod_{i\le n}\cosh(G_{\xi',i}(\bar{\sigma}))}{\left(\mathbb{E}'\prod_{i\le n}\cosh(G_{\xi',i})\right)^q}.$$

Note that the righthand side is a function of only the overlap distribution of $\bar{\sigma}$ corresponding to ν. Let the law of R_{12} be denoted by ζ. Suppose that ζ consists of $r+1$ atoms. Then,

since μ satisfies the Ghirlanda–Guerra identities by assumption, we know that this can also be written as

$$\mathbb{E}\prod_{l\le q}\prod_{i\in C_l} s_i^l = \mathbb{E}\frac{\prod_{l\le q}\sum_\alpha w_\alpha \prod_{i\in C_l}\tanh(g_{\xi',i}(h_\alpha))\prod_{i\le n}\cosh(g_{\xi',i}(\bar\sigma))}{\left(\sum w_\alpha\prod_{i\le n}\cosh(g_{\xi',i}(\bar\sigma))\right)^q}. \tag{3.1}$$

Here, $(w_\alpha)_{\alpha\in\partial\mathcal{A}_r}$ are the weights corresponding to a $RPC(\zeta)$ and $\{h_\alpha\}_{\alpha\in\mathcal{A}_r}$ are the corresponding vectors, with $\mathcal{A}_r=\mathbb{N}^r$ viewed as a tree with r levels.

For a vertex α of a tree, we denote by $|\alpha|$ the depth of α, that is its (edge or vertex) distance from the root. We denote by $p(\alpha)$ to be the set of vertices in the path from the root to α. For two vertices α,β, we let $\alpha\wedge\beta$ denote their least common ancestor, and we say that $\alpha\precsim\beta$ if $\alpha\in p(\beta)$. In particular $\alpha\precsim\alpha$. We say that $\alpha\nsim\beta$ if neither $\alpha\precsim\beta$ nor $\beta\precsim\alpha$.

In this setting, it is well known that $g_{\xi'}(h_\alpha)$ has the following explicit version. Let $(\eta_\alpha)_{\alpha\in\mathcal{A}_r}$ be i.i.d. gaussians, then

$$g_{\xi'}(h_\alpha)=\sum_{\beta\precsim\alpha}\eta_\beta\left(\xi'(q_{|\beta|})-\xi'(q_{|\beta|-1})\right)^{1/2}.$$

It was then showed by Panchenko that the above also has the following representation in terms of "tilted" variables η' as follows.

We define the following family of functions $Z_p:\mathbb{R}^p\to\mathbb{R}$ with $0\le p\le r$ recursively as follows. Let

$$Z_r(x)=\log\cosh\left(\sum_{i=1}^r x_i(\xi'(q_i)-\xi'(q_{i-1}))^{1/2}\right)$$

and let

$$Z_p(x)=\frac{1}{\zeta([0,q_p])}\log\int\exp\left(\zeta([0,q_p])\cdot Z_{p+1}(x,z)\right)d\gamma(z) \tag{3.2}$$

where $d\gamma$ is the standard gaussian measure on $\mathbb{R}$. We then define the transition kernels

$$K_p(x,dx_{p+1})=\exp\left(\zeta([0,q_p])\left(Z_{p+1}(x,x_{p+1})-Z_p(x,x_{p+1})\right)\right)d\gamma(x_{p+1}).$$

Also, we define η'_α as as the random variable with law $K_{|\alpha|}((\eta_\beta)_{\beta\precsim\alpha},\cdot)$. Finally, define

$$g'_{\xi'}(h_\alpha)=\sum_{\beta\precsim\alpha}\eta'_\beta\left(\xi'(q_{|\beta|})-\xi'(q_{|\beta-1|})\right)^{1/2}.$$

Define $g'_{\xi',i}$ analogously. We then have the following proposition.

Proposition 3.1 (Panchenko [19]) *Let w_α be as above and let*

$$w'_\alpha=\frac{w_\alpha\prod_{i\le n}\cosh(g_{\xi',i}(h_\alpha))}{\sum w_\alpha\prod_{i\le n}\cosh(g_{\xi',i}(h_\alpha))}$$

Then we have

$$\left((w'_\alpha,g_{\xi',i}(h_\alpha))\right)_\alpha\overset{(d)}{=}\left((w_\alpha,g'_{\xi',i}(h_\alpha))\right)_\alpha.$$

If we apply this proposition to (3.1), we have that

$$\mathbb{E}\prod_{l\le q}\prod_{i\in C_l}s_i^l=\mathbb{E}\prod_{l\le q}\sum_\alpha w_\alpha\prod_{i\in C_l}\tanh(g'_{\xi',i}(h_\alpha)).$$

3.2 Proof of Theorem 1.2.

We now turn to proving Theorem 1.2. We begin with the following two lemmas.

Lemma 3.2 *Let h_α, η_α, g_ξ, $g_\xi^{'}$ be as above. We then have the following equalities in distribution*

$$\begin{aligned}(g(h_\alpha))_\alpha &\stackrel{(d)}{=} \left(B_{q_*}(h_\alpha)\right)_\alpha \\ \left(g_{\xi'}(h_\alpha)\right)_\alpha &\stackrel{(d)}{=} \left(Y_{q_*}(h_\alpha)\right)_\alpha \\ \left(g'_{\xi'}(h_\alpha)\right)_\alpha &\stackrel{(d)}{=} \left(X_{q_*}(h_\alpha)\right)_\alpha .\end{aligned}$$

Proof Observe by the independent increments property of Brownian motion, we have that

$$(\eta_\alpha) \stackrel{(d)}{=} (B(q_{|\alpha|}, h_\alpha) - B(q_{|\alpha|-1}, h_\alpha)).$$

This yields the first two equalities. It remains to see the last equality.

To this end, fix h_α, and consider the process X_t thats solves the SDE (1.6). Then if Y_t is distributed like Y as above with respect to some measure Q , then by Girsanov's theorem [12, Lemma 8.3.1], we have that with respect to the measure P with Radon–Nikodym derivative

$$\frac{dP}{dQ}(t) = e^{\int_0^t \zeta([0,s])du_s},$$

the process Y_t has the same law as X_t. In particular, for the finite collection of times q_i we have that

$$\begin{aligned}\mathbb{E}_P F(X_{q_0} \ldots, X_{q_r}) &= \int F(Y_{q_0}, \ldots, Y_{q_r}) e^{\int_0^t \zeta du(s,Y_s)} dQ(Y) \\ &= \int F(Y_{t_1}, \ldots, Y_{t_k}) \prod_{i=0}^{k} e^{\zeta([0,q_k])\left(u(q_k,Y_{q_k}) - u(q_{k-1},Y_{q_{k-1}})\right)} dQ.\end{aligned}$$

By recognizing the law of (B_{q_k}) and (Y_{q_k}) as Gaussian random variables, and (3.2) as the Cole-Hopf solution of the Parisi IVP (1.5), $u(q_k, x) = Z_k(x)$, the result follows. □

We now need the following continuity theorem. This is intimately related to continuity results commonly used in the literature, though the method of proof is different.

Let $\mathcal{Q}_d$ denote the set of $d \times d$ matrices of the form

$$\mathcal{Q}_d = \{(q_{ij})_{i,j\in[d]} : q_{ij} \in [0,1],\ q_{ij} = q_{jk},\ q_{ij} \geq q_{ik} \wedge q_{kj}\ \forall i, j, k\}.$$

Note that this set is a compact subset of $\mathbb{R}^{d^2}$. Consider the space $\Pr([0,1])$ equipped with the weak-* topology. Then the product space $\Pr([0,1]) \times \mathcal{Q}_d$ is compact Polish. For any $Q \in \mathcal{Q}_d$, let $(\sigma^i(Q))_{i=1}^d \subset \mathcal{H}$ be a collection of vectors whose gram-matrix is Q. We can then define the functional

$$\mathcal{R}(\zeta, Q) = \mathbb{E}\prod_{i=1}^{d} u_x(q_*, X_{q_*}(\sigma^i)).$$

Lemma 3.3 *We have that $\mathcal{R}$ is well-defined and is jointly continuous.*

Proof Let $(\sigma^i)_{i=1}^d$ be any collection with overlap matrix Q. Recall the infinitesimal generator, L^{lf}, of the collection $\left(X_t(\sigma^i)\right)$ from (A.1). Observe that L^{lf} depends on (σ^i) only through their overlap matrix, which is Q. Thus the law is determined by this matrix and $\mathcal{R}$ is well-defined.

We now turn to proving continuity. As $\Pr([0,1]) \times \mathcal{Q}$ is compact Polish, it suffices to show that for $\zeta_r \to \zeta$ and $Q^r = (q_{ij}^r)$ with $q_{ij}^r \to q_{ij}$, $1 \le i, j \le l$,

$$\mathcal{R}(\zeta_r, Q^r) \to \mathcal{R}(\zeta, Q),$$

as $r \to \infty$.

Let a_{ij}^r and b_i^r be the coefficients of the diffusion associated to the local field process X^{ζ_r, Q^r}. By (A.1), we have

$$a_{ij}^r(t) = \mathbf{1}_{\{t \le q_{ij}^r\}}, \quad b_i^r(t,\cdot) = \xi''\zeta_r u_x^r(t,\cdot),$$

where u^r is the solution to the Parisi initial value problem corresponding to ζ_r. These coefficients are all uniformly bounded, measurable in time and smooth in space. Furthermore, ξ is continuous, so that

$$\begin{aligned}\int_0^t \left(|a_{ij}^r(s) - a_{ij}(s)| + \sup_x |b^r(s,x) - b(s,x)|\right) ds \\ \le |q_{ij}^r - q_{ij}| + \int_0^t \sup_x |\zeta_r([0,s])\, u_x^r(t,x) - \zeta([0,s])\, u_x(t,x)| ds \to 0 \end{aligned} \tag{3.3}$$

as $r \to \infty$ since u_x^r converges uniformly to u_x by [4, Prop. 1] as $\zeta_r \to \zeta$.

By Stroock–Varadhan's theorem [20, Theorem 11.1.4], the convergence from (3.3) implies that the laws of the solutions to the corresponding martingale problems converge. As $(x_1, \ldots, x_d) \mapsto \prod_{i=1}^d \tanh(x_i)$ is a continuous bounded function we obtain the continuity of F. □

We may now turn to the proof of the main theorem of this section.

Proof of Theorem 1.2 Suppose first that ζ consists of $r+1$ atoms. In this setting the result has already been proved by the aforementioned results of Panchenko combined with Lemma 3.2. The main task is to prove these results for general ζ. To this end, let $\zeta_r \to \zeta$ be atomic. Denote the spins corresponding to these measures by $s_{i,r}^l$.

Correspondingly, for any collection of moments we have

$$\mathbb{E} \prod_{l \le q} \prod_{i \in C_l} s_{i,r}^l = \mathbb{E} \langle \mathcal{R}(\zeta_r, Q) \rangle .$$

Recall that the overlap distribution converges in law when $\zeta_r \to \zeta$, thus by Lemma 3.3 and a standard argument,

$$\mathbb{E} \langle \mathcal{R}(\zeta_r, Q) \rangle_r \to \mathbb{E} \langle \mathcal{R}(\zeta, Q) \rangle = \mathbb{E} \left\langle \prod_{i \le q} \prod_{i \in C_l} \tanh(X_{q_*}^i(\sigma^l)) \right\rangle .$$

However, as the overlap distribution determines the spin distribution, we see that

$$\mathbb{E} \prod_{l \le q} \prod_{i \in C_l} s_{i,r}^l \to \mathbb{E} \prod_{l \le q} \prod_{i \in C_l} s_i^l = \mathbb{E} \frac{\left\langle \prod_{l \le q} \prod_{i \in C_l} \tanh(Y_{q_*}^i(\sigma^l)) \prod_{i \le n} \cosh(Y_{q_*}^i(\sigma^l)) \right\rangle}{\left\langle \prod_{i \le n} \cosh(Y_{q_*}^i(\sigma^l)) \right\rangle^q}.$$

This yields both results. □

4 Proof of Theorem 1.4

We now prove that the TAP equation holds at infinite particle number. Before stating this proof we point out two well-known [5,12] but useful facts: the magnetization process $u_x(s, X_s(\sigma))$ is a martingale for fixed σ and $u_x(t, x) = \tanh(x)$ for $t \geq q_* = sup\ \text{supp}\ \zeta$.

Proof of Theorem 1.4 Consider $\langle s \rangle_\alpha$, if we compute the joint moments of this expectation

$$\langle s \rangle_\alpha^k = u_x(q, X_q^\sigma)^k$$

for any $\sigma \in W_\alpha$. In fact, jointly,

$$\prod_{\alpha \in A} \langle s \rangle_\alpha^{k_\alpha} = \prod_{\alpha \in A} u_x(q, X_q^{\sigma^\alpha})^{k_\alpha}$$

for $|A| < \infty$. Thus in law,

$$\langle s \rangle_\alpha = u_x(q, X_q^{\sigma^\alpha}). \tag{4.1}$$

By a similar argument

$$\langle y \rangle_\alpha = \mathbb{E}\left(X_1^{\sigma^\alpha} | \mathcal{F}_q\right),$$

where $\mathcal{F}_q$ is the sigma algebra of $\sigma((B_q^\alpha(\sigma))_{\sigma \in \text{supp}\mu})$. However,

$$\begin{aligned}
\mathbb{E}\left(X_1^{\sigma^\alpha} | \mathcal{F}_q\right) &= X_q^{\sigma^\alpha} + \int_q^1 \xi''(s)\zeta([0,s])\mathbb{E}\left(u_x(s, X_s^{\sigma^\alpha}) | \mathcal{F}_q\right) ds \\
&= X_q^{\sigma^\alpha} + \int_q^1 \xi''(s)\zeta([0,s])ds \cdot u_x(q, X_q^{\sigma^\alpha}) \\
&= X_q^{\sigma^\alpha} + \int_q^1 \xi''(s)\zeta([0,s])ds \cdot \langle s \rangle_\alpha
\end{aligned}$$

where the first line is by definition, (1.6), of X^σ, and the second line follows from the martingale property of the magnetization process. Solving this for $X_q^{\sigma^\alpha}$ yields,

$$X_q^{\sigma^\alpha} = \langle y \rangle_\alpha - \int_q^1 \xi''(s)\zeta([0,s])ds \cdot \langle s \rangle_\alpha .$$

Combining this with (4.1), yields the result. □

5 Evaluation of Spin Statistics

Using spin distributions, one can obtain formulae for expectations of products of spins, either through the directing function σ or by taking limits of expressions using Ruelle cascades. The goal of this section is to explain how one can obtain expressions for such statistics as the solutions of certain partial differential equations. The input required will be the overlap distribution $\zeta(t)$. In particular, one can in principle evaluate these expression using standard methods from PDEs or numerically. Rather than developing a complete calculus of spin statistics, we aim to give a few illustrative examples.

At the heart of these calculations is the following key observation: the magnetization process for any finite collection $(\sigma^i)_{i=1}^n$ is a family of branching martingales whose independence properties mimics that of the tree encoding of their overlap arrays. (This can be

formalized using the language of Branchingales. See [7] for more on this.) In this section we focus on two examples: two spin statistics, i.e., the overlap, and three spin statistics. One can of course write out general formulas, however, we believe that these two cases highlight the key ideas. In particular, the second case is the main example in [15], where this is calculated using replica theory. The reader is encouraged to compare the PDE and martingale based discussion here with the notion of "tree operators" in that paper. For the remainder of this subsection, all state measures should be taken with boundary data $f(\sigma) = X_{q_*}(\sigma)$.

5.1 Two Spin Statistics

We first aim to study two spin statistics. As the spins take values ± 1, there is only one nontrivial two spin statistic, namely $\mathbb{E}s_1^1 s_1^2$ where the subscript denotes the site index and the superscript denotes the replica index. Observe that by (1.8), we have that

$$\mathbb{E}s_1^1 s_1^2 = \mathbb{E}\int \langle s\rangle_{\sigma^1}\cdot\langle s\rangle_{\sigma^2}\, d\mu^{\otimes 2} = \mathbb{E}\int \mathbb{E}\prod_{i=1}^{2} u_x(q_*, X_{q_*}^{\sigma^i}) d\mu^{\otimes 2}.$$

Observe that it suffices to compute $\mathbb{E}u_x(q_*, X_{q_*}^{\sigma^1})u_x(q_*, X_{q_*}^{\sigma^2})$. There are a few natural ways to compute this. Let $q_{12} = (\sigma^1, \sigma^2)$. One method is to observe that if $\Phi = \Phi_{q_{12}}$ solves

$$\begin{cases} (\partial_t + L_t^{lf})\Phi = 0 & [0,1]\times\mathbb{R}^2 \\ \Phi(1,x,y) = \tanh(x)\tanh(y) \end{cases},$$

where L^{lf} is the infinitesimal generator for the local field process (see (A.1)), then

$$\mathbb{E}u_x(q_*, X_{q_*}^{\sigma^1})u_x(q_*, X_{q_*}^{\sigma^2}) = \Phi_{q_{12}}(0,h).$$

One can study this problem using PDE methods or Ito's lemma. This yields the expression

$$\mathbb{E}s_1^1 s_1^2 = \int \Phi_s(0,h)d\zeta(s).$$

Alternatively, note that, by the branching martingale property of the magnetization process, we have that

$$\mathbb{E}u_x(q_*, X_{q_*}^{\sigma^1})u_x(q_*, X_{q_*}^{\sigma^2}) = \mathbb{E}u_x(q_{12}, X_{q_{12}})^2,$$

yielding the alternative expression

$$\mathbb{E}s_1^1 s_1^2 = \int \mathbb{E}u_x^2(s, X_s)d\zeta(s).$$

In the case that ζ is the Parisi measure for ξ (for this notation see [4,12]), it is well-known that on the support of ζ,

$$\mathbb{E}u_x^2(s, X_s) = s$$

so that

$$\mathbb{E}s_1^1 s_1^2 = \int s d\zeta(s).$$

This resolves a question from [8, Remark 5.5].

5.2 Three Spin Statistics

We now turn to computing more complicated statistics. We focus on the case of the three spin statistic, $\mathbb{E}s_1^1 s_1^2 s_1^3$, as we believe this to be illustrative of the essential ideas and it is the main example give in the paper of Mézard–Virasoro [15].

We say a function $f : [0, 1]^k \to \mathbb{R}$ is *symmetric* if for every $\pi \in S_k$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(k)}) = f(x_1, \ldots, x_k)$$

In the following, we denote by $dQ(R^n)$ the law of the overlap array $R^n = (R_{ij})_{ij\in[n]}$. We say that such a function has *vanishing diagonal* if $f(x, \ldots, x) = 0$. We will always assume that Q satisfies the Ghirlanda–Guerra identities. Our goal is to prove the following:

Theorem 5.1 *We have that*

$$\mathbb{E}s_1^1 s_1^2 s_1^3 = \frac{3}{4} \int \int \mathbb{E}u_x(b \vee a, X_{b\vee a})^2 u_x(a \wedge a, X_{a\wedge a}) d\zeta(a) d\zeta(b).$$

As a starting point, again observe that from the properties of state measures, (1.7),

$$\mathbb{E}s_1^1 s_1^2 s_1^3 = \mathbb{E} \int \langle s \rangle_{\sigma^1} \cdot \langle s \rangle_{\sigma^2} \cdot \langle s \rangle_{\sigma^3} \, d\mu^{\otimes 3}.$$

Denote the integrand by

$$\mathcal{R}(\sigma^1, \sigma^2, \sigma^3) = \langle s \rangle_{\sigma^1} \cdot \langle s \rangle_{\sigma^2} \cdot \langle s \rangle_{\sigma^3}.$$

The proof of this result will follow from the following two lemmas.

Lemma 5.2 *We have the following.*

(1) *Suppose that $g(x, y)$ is a continuous, symmetric function. Then*

$$\int g(R_{12}, R_{13}) dQ = \frac{1}{2} \int \int g(x, y) d\zeta(x) d\zeta(y) + \int g(x, x) d\zeta(x).$$

(2) *Suppose that $f(x, y, z)$ is a continuous symmetric function with vanishing diagonal. Then*

$$\int f(R_{12}, R_{13}, R_{23}) dQ = \frac{3}{2} \int f(R_{12} \vee R_{13}, R_{12} \wedge R_{13}, R_{12} \wedge R_{13}) dQ.$$

(3) *Suppose that $f(x, y, z)$ is as above and such that $h(x, y) = f(x \vee y, x \wedge y, x \wedge y)$ is continuous. Then*

$$\int f(R_{12}, R_{13}, R_{23}) dQ = \frac{3}{4} \int \int h(x, y) d\zeta(x) d\zeta(y).$$

Proof The first claim follows immediately from the Ghirlanda–Guerra identities. The last item is implied by the first two. It remains to prove the second claim. By symmetry of f and ultrametricity we have that

$$\begin{aligned} \int f(R_{12}, R_{13}, R_{23}) dQ = 3 &\int_{R_{12} > R_{13}} f(R_{12}, R_{13}, R_{13}) dQ \\ &+ \int_{R_{12} = R_{13} = R_{23}} f(R_{12}, R_{12}, R_{12}) dQ \end{aligned}$$

The second term is zero by the vanishing diagonal property of f, so that,

$$RHS = 3\int_{R_{12}\geq R_{13}} f(R_{12}, R_{13}, R_{13})dQ = \frac{3}{2}\int h(R_{12}, R_{13})dQ,$$

using again the vanishing diagonal property and the definition of h. □

Lemma 5.3 *There is a continuous, symmetric function of three variables defined on the set of ultrametric* $[0, 1]^3$ *such that the function* $\mathcal{R}(\sigma^1, \sigma^2, \sigma^3) = f(R_{12}, R_{13}, R_{23})$. *This function has vanishing diagonal, and satisfies*

$$f(a, b, b) = \mathbb{E}u_x(b, X_b)^2 u_x(a, X_a) \tag{5.1}$$

for $a \leq b$.

Remark 5.4 This is to be compared with [15, Eq. 34].

Proof That it is a continuous, symmetric function of the overlaps is obvious. It suffices to show (5.1). To this end, observe that without loss of generality $R_{12} \geq R_{13} = R_{23}$. In this case, denoting $R_{12} = b$ and $R_{23} = R_{13} = a$, we have that

$$\begin{aligned}
\mathcal{R}(\sigma^1, \sigma^2, \sigma^3) &= \mathbb{E}u_x(1, X^{\sigma^1})u_x(1, X^{\sigma^2})u_x(1, X^{\sigma^3}) \\
&= \mathbb{E}u_x(b, X_b^1)u_x(b, X_b^2)u_x(b, X_b^3) \\
&= \mathbb{E}u_x(b, X_b^1)^2 u_x(b, X_b^3) \\
&= \mathbb{E}u_x(b, X_b^1)^2 u_x(a, X_a^3) \\
&= \mathbb{E}u_x(b, X_b)^2 u_x(a, X_a).
\end{aligned}$$

In the second line, we used independence and the martingale property. In the third line we used that the driving processes are identical in distribution until that time. In the fourth line we use the martingale property and independence of local fields again. The final result comes from the fact that the driving process for the three spins is equivalent until a. □

We can now prove the main result of this subsection:

Proof of Theorem 5.1 Recall that

$$\mathbb{E}s_1^1 s_1^2 s_3^3 = \mathbb{E}\left\langle \mathbb{E}\mathcal{R}(\sigma^1, \sigma^2, \sigma^3)\right\rangle.$$

The result then follow by combining Lemma 5.3 and part 3. of Lemma 5.2. □

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A Appendix

A.1 On the Driving Process and Its Descendants

We record here the following basic properties of the driving process, cavity field process, local field process, and magnetization process.

Lemma A.1 *Let U be a positive ultrametric subset of a separable Hilbert space that is weakly closed and norm bounded equipped with the restriction of the Borel sigma algebra. Let $B_t(\sigma)$ be the process defined in (1.3). We have the following:*

(1) *The covariance structure is positive semi-definite.*
(2) *There is a version of this process that is jointly measurable and continuous in time.*
(3) *For each σ, $B_t(\sigma)$ has the law of a brownian motion so that stochastic integration with respect to $B_t(\sigma)$ is well-defined.*

Proof We begin with the first. To see this, simply observe that if $\alpha_i \in \mathbb{R}$, (t_i, σ_i) are finitely many points in $[0, q_*] \times U$ and $\sigma_* \in U$, then

$$\begin{aligned}\sum \alpha_i \alpha_j \left(t_i \wedge t_j \wedge (\sigma_i, \sigma_j)\right) &= \sum \alpha_i \alpha_j \int \mathbb{1}\{s \le t_i\} \mathbb{1}\left\{s \le t_j\right\} \mathbb{1}\left\{s \le (\sigma_i, \sigma_j)\right\} ds \\ &\ge \sum \alpha_i \alpha_j \int \mathbb{1}\{s \le t_i\} \\ &\quad \mathbb{1}\left\{s \le t_j\right\} \mathbb{1}\{s \le (\sigma_i, \sigma_*)\} \mathbb{1}\left\{s \le (\sigma_j, \sigma_*)\right\} ds \\ &= ||\sum \alpha_i \mathbb{1}\{s \le t_i \wedge (\sigma_i, \sigma_*)\}||_{L^2} \ge 0.\end{aligned}$$

We now turn to the second. Observe first that, since $[0, q_*] \times U$ is separable and $\mathbb{R}$ is locally compact, $B_t(\sigma)$ has a separable version. Furthermore, observe that $B_t(\sigma)$ is stochastically continuous in norm, that is as $(t, \sigma) \to (t_0, \sigma_0)$ in the norm topology, $P(\left|B_t(\sigma) - B_{t_0}(\sigma_0)\right| > \epsilon) \to 0$. Thus since U is weakly-closed and norm bounded it is compact in the weak topology. Thus it has a version that is jointly measurable by [9, Theorem IV.4.1]. Note then, since the covariance of $B_t(\sigma)$ for fixed σ is that of Brownian motion and $B_t(\sigma)$ is separable, it is in fact continuous by [9, Theorem IV.5.2].

The third property was implicit in the proof of the second. □

We now observe the following consequence of the above proposition:

Corollary A.2 *Let U be a positive ultrametric subset of a separable Hilbert space that is weakly closed and norm bounded. Then the cavity field process, $Y_t(\sigma)$, the local field process, $X_t(\sigma)$, and the magnetization process, $M_t(\sigma)$, exist, are continuous in time and Borel measurable in σ.*

In the above, the following observation regarding the infinitesimal generator of the above processes will be of interest.

Lemma A.3 *Let $(\sigma^i)_{i=1}^n \subset U$ where U is as above. Then we have the following.*

(1) *The driving process satisfies the bracket relation*

$$\left\langle B(\sigma^1), B(\sigma^2)\right\rangle_t = \begin{cases} t & t \le (\sigma, \sigma') \\ 0 & t > (\sigma, \sigma') \end{cases}.$$

(2) *The cavity field process satisfies the bracket relation*

$$\left\langle Y(\sigma^1), Y(\sigma^2)\right\rangle_t = \begin{cases} \xi'(t) & t \le (\sigma^1, \sigma^2) \\ 0 & else \end{cases}.$$

(3) *The local fields process satisfies the bracket relation*

$$\left\langle X(\sigma^1), X(\sigma^2)\right\rangle_t = \begin{cases} \xi'(t) & t \le (\sigma^1, \sigma^2) \\ 0 & else \end{cases}$$

and has infinitesimal generator

$$L_t^{lf} = \frac{\xi''(t)}{2}\left(\sum a_{ij}(t)\partial_i\partial_j + 2\sum b_i(t,x)\partial_i\right) \tag{A.1}$$

where $a_{ij}(t) = \mathbb{1}\left\{t \le (\sigma^i, \sigma^j)\right\}$ *and* $b_i(t,x) = \zeta([0,t]) \cdot u_x(t,x)$.

Proof We begin with the first claim. To see this, observe that by construction,

$$B_t(\sigma^1) = B_t(\sigma^2)$$

for $t \le (\sigma^1, \sigma^2)$, thus the bracket above is just the bracket for Brownian motion. If $t > (\sigma^1, \sigma^2) := q$, then the increments $B_t(\sigma^1) - B_q(\sigma^1)$ and $B_t(\sigma^2) - B_q(\sigma^2)$ are independent Brownian motions. This yields the second regime. By elementary properties of Itô processes, we obtain the brackets for Y_t and X_t from this argument. It remains to obtain the infinitesimal generator for the local fields process.

To this end, observe that if $f = f(t, x_1, \ldots, x_k)$ is a test function, then Itô's lemma applied to the process $(X_t(\sigma^i))_{i=1}^n$ yields

$$\begin{aligned} df &= \partial_t f \cdot dt + \sum_i \partial_{x_i} f \cdot dX_t(\sigma^i) + \frac{1}{2}\cdot\sum \partial_{x_i}\partial_{x_j} f \cdot d\left\langle X_t(\sigma^i), X_t(\sigma^j)\right\rangle \\ &= \Bigg(\partial_t f + \sum_i \partial_{x_i} f \cdot \left(\xi''(t)\zeta(t)u_x(t, X_t(\sigma^i)\right) \\ &\quad + \frac{\xi''}{2}\sum \mathbb{1}\left\{t \le (\sigma^i, \sigma^j)\right\}\partial_{x_i}\partial_{x_j} f\Bigg) dt + dMart \end{aligned}$$

where $dMart$ is the increment for some martingale. Taking expectations and limits in the usual fashion then yields the result. □

A.2 The Cavity Equations and Ghirlanda–Guerra Identities

In this section, we recall some definitions for completeness. For a textbook presentation, see [18, Chapters 2 and 4]. Let $\mathscr{M}$ be the set of all measures on the set $\{-1, 1\}^{\mathbb{N}\times\mathbb{N}}$ that are exchangeable, that is, if (s_i^ℓ) has law $\nu \in \mathscr{M}$, then

$$(s_{\pi(i)}^{\rho(\ell)}) \overset{(d)}{=} (s_i^\ell)$$

for any permutations π, ρ of the natural numbers. The Aldous–Hoover theorem [2,10], states that if (s_i^ℓ) is the random variable induced by some measure $\nu \in \mathscr{M}$, then there is a measurable function of four variables, $\sigma(w, u, v, x)$, such that

$$(s_i^\ell) \overset{(d)}{=} (\sigma(w, u_\ell, v_i, x_{\ell i}))$$

where $w, u_\ell, v_i, x_{\ell i}$ are i.i.d. uniform $[0, 1]$ random variables. We call this function a *directing function* for ν. The variables s_i^ℓ are called the spins sampled from ν.

For any ν in $\mathscr{M}$ with directing function σ, let $\bar{\sigma}(w, u, v) = \int \sigma(w, u, v, x)dx$. Note that since σ is $\{\pm 1\}$-valued, this encodes all of the information of $\sigma(w, u, v, \cdot)$. Define the

measure μ on the Hilbert space, $\mathcal{H} = L^2([0,1], dv)$, by the push-forward of du through the map $u \mapsto \bar{\sigma}(w, u, \cdot)$,

$$\mu = (u \mapsto \bar{\sigma}(w, u, \cdot))_* du.$$

The measure μ is called the asymptotic Gibbs measure corresponding to ν.

A measure ν in $\mathscr{M}$ is said to satisfy the Ghirlanda–Guerra identities if the law of the overlap array satisfies the following property: for every $f \in C([-1,1]^n)$ and $g \in C([-1,1])$, we have

$$\mathbb{E}\left\langle f(R^n) \cdot g(R_{1,n+1})\right\rangle = \frac{1}{n}\left[\mathbb{E}\left\langle f(R^n)\right\rangle \cdot \mathbb{E}\left\langle g(R_{12})\right\rangle + \sum_{k=2}^{n} \mathbb{E}\left\langle f(R^n) \cdot g(R_{1k})\right\rangle\right], \quad \text{(A.2)}$$

where by the bracket, $\langle\cdot\rangle$, we mean integration against the relevant products of μ with itself.

A measure ν is said to satisfy the cavity equations if the following is true. Fix the directing function σ and $\bar{\sigma}$ as above. Let $g_{\xi'}(\bar{\sigma})$ denote the centered Gaussian process indexed by $L^2([0,1], dv)$ with covariance

$$\mathbb{E}\left[g_{\xi'}\big(\bar{\sigma}(w,u,\cdot)\big) g_{\xi'}\big(\bar{\sigma}(w,u',\cdot)\big)\right] = \xi'\left(\int \bar{\sigma}(w,u,v)\bar{\sigma}(w,u',v)dv\right)$$

and let $G'_{\xi}(\bar{\sigma}) = g_{\xi'}(\bar{\sigma}) + z(\xi'(1) - \xi'(||\bar{\sigma}(w,u,\cdot)||^2_{L^2(dv)}))^{1/2}$. Let $g_{\xi',i}$ and $G_{\xi',i}$ be independent copies of these processes. Let $n, m, q, r, l \geq 1$ be such that $n \leq m$ and $l \leq q$. Let $C_l \subset [m]$ and let $C_l^1 = C_l \cap [n]$ and $C_l^2 = C_l \cap (n + [m])$. Let

$$U_l = \int \mathbb{E}' \prod_{i \in C_l^1} \tanh G_{\xi',i}(\bar{\sigma}(w,u,\cdot) \prod_{i \in C_l^2} \bar{\sigma}_i \mathcal{E}_{n,r} du$$

where $\mathbb{E}'$ is expectation in z, $\bar{\sigma}_i = \bar{\sigma}(w,u,v_i)$, $\theta(t) = \xi'(t)t - \xi(t)$, and where

$$\mathcal{E}_{n,r} = \exp\left(\sum_{i \leq n} \log\cosh(G_{\xi',i}(\bar{\sigma}(w,u,\cdot)) + \sum_{k \leq r} G_{\theta,k}(\bar{\sigma}(w,u,\cdot))\right).$$

Let $V = \mathbb{E}'\mathcal{E}_{n,r}$. The *cavity equations* for $n, m, q, r \geq 1$ are then given by

$$\mathbb{E}\prod_{l \leq q} \mathbb{E}' \prod_{i \in C_l} \bar{\sigma}_i = \mathbb{E}\frac{\prod_{l \leq q} U_l}{V^q}. \quad \text{(A.3)}$$

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