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# Maximal immediate extensions of valued differential fields

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Abstract

We show that every valued differential field has an immediate strict extension that is spherically complete. We also discuss the issue of uniqueness up to isomorphism of such an extension.

## *Introduction*

In this paper a *valued differential field* is a valued field $K$ of equicharacteristic zero, equipped with a derivation $\partial\colon K \to K$ that is continuous with respect to the valuation topology on the field. (The difference with [**1**, **2**] is that there the definition did not include the continuity requirement.)

Let $K$ be a valued differential field. Unless specified otherwise, $\partial$ is the derivation of $K$, and we let $v\colon K^\times = K \setminus \{0\} \to \Gamma = v(K^\times)$ be the valuation, with valuation ring $\mathcal{O} = \mathcal{O}_v$ and maximal ideal $\smallo = \smallo_v$ of $\mathcal{O}$; we use the subscript $K$, as in $\partial_K$, $v_K$, $\Gamma_K$, $\mathcal{O}_K$, $\smallo_K$, if we wish to indicate the dependence of $\partial$, $v$, $\Gamma$, $\mathcal{O}$, $\smallo$ on $K$. We denote the residue field $\mathcal{O}/\smallo$ of $K$ by $\operatorname{res}(K)$. When the ambient $K$ is clear from the context we often write $a'$ instead of $\partial(a)$ for $a \in K$, and set $a^\dagger := a'/a$ for $a \in K^\times$.

By [**2**, Section 4.4], the continuity requirement on $\partial$ amounts to the existence of a $\phi \in K^\times$ such that $\partial\smallo \subseteq \phi\smallo$; the derivation of $K$ is said to be *small* if this holds for $\phi = 1$, that is, $\partial\smallo \subseteq \smallo$. By an *extension of* $K$ we mean a valued differential field extension of $K$. Let $L$ be an extension of $K$. We identify $\Gamma$ in the usual way with an ordered subgroup of $\Gamma_L$ and $\operatorname{res}(K)$ with a subfield of $\operatorname{res}(L)$, and we say that $L$ is an *immediate* extension of $K$ if $\Gamma = \Gamma_L$ and $\operatorname{res}(K) = \operatorname{res}(L)$. We call the extension $L$ of $K$ *strict* if for every $\phi \in K^\times$,

$$\partial\smallo \subseteq \phi\smallo \Rightarrow \partial_L\smallo_L \subseteq \phi\smallo_L, \quad \partial\mathcal{O} \subseteq \phi\smallo \Rightarrow \partial_L\mathcal{O}_L \subseteq \phi\smallo_L.$$

With these conventions in place, our goal is to establish the following:

Theorem. *Every valued differential field has an immediate strict extension that is spherically complete.*

We consider this as a differential analogue of Krull's well-known theorem in [**5**, § 13] that every valued field has a spherically complete immediate valued field extension. (Recall that for a valued field the geometric condition of spherical completeness is equivalent to the algebraic condition of being maximal in the sense of not having a proper immediate valued field extension.) In our situation, strictness is analogous to the extended derivation 'preserving the norm'. Weakening the theorem by dropping 'strict' would still require strictness at various places in the proof, for example when using Lemma 4.5 and in coarsening arguments at the end of Section 6.

Throughout this paper $K$ is a valued differential field. For the sake of brevity we say that $K$ has the *Krull property* if $K$ has a spherically complete immediate strict extension. Let us first consider two trivial cases:

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*Case* $\Gamma = \{0\}$. Then $K$ itself is a spherically complete immediate strict extension of $K$, and thus $K$ has the Krull property.

*Case* $\partial = 0$. Take a spherically complete immediate valued field extension $L$ of the valued field $K$. Then $L$ with the trivial derivation is a spherically complete immediate strict extension of $K$, so $K$ has the Krull property.

Thus towards proving our main theorem we can assume $\Gamma \neq \{0\}$ and $\partial \neq 0$ when convenient. We shall freely use facts (with detailed references) from Sections 3.4, 4.1, 4.2, 4.3, 4.4, 4.5, 5.7, 6.1, 6.2, 6.3, 6.5, 6.6, 6.9, 9.1, 9.2, 10.5, and 11.1 in [**2**].

Special cases of the main theorem are in [**2**]: By [**2**, Corollary 6.9.5], if $K$ has small derivation and $\partial\mathcal{O} \not\subseteq \smallo$, then $K$ has a spherically complete immediate extension with small derivation; in [**2**, Corollary 11.4.10] we obtained spherically complete immediate extensions of certain asymptotic fields. What is new compared to the proofs of these special cases? Mainly the notion of *strict extension*, the invariant convex subgroup $S(\partial)$ of $\Gamma$, the *flexibility* condition on $K$, and the lemmas about these (related) concepts; see Sections 1, 3, 4, 6. We also generalize in Section 2 the notion of *Newton degree* from [**2**, 11.1, 11.2] to our setting. This gives us the tools to adapt in Section 5 the proofs of these special cases to deriving our main theorem for $K$ such that $\Gamma^{>}$ has no least element and $S(\partial) = \{0\}$. Section 6 shows how that case extends to arbitrary $K$ using coarsening by $S(\partial)$.

We give special attention to asymptotic fields, a special kind of valued differential field introduced in [**2**, Section 9.1]: $K$ is *asymptotic* if for all nonzero $f, g \in \smallo$,

$$f \in g\smallo \iff f' \in g'\smallo.$$

For us, $H$-fields are asymptotic fields of particular interest, see [**2**, Section 10.5]: An *$H$-field* is an *ordered* valued differential field $K$ whose valuation ring $\mathcal{O}$ is convex and such that, with $C = \{f \in K : f' = 0\}$ denoting the constant field of $K$, we have $\mathcal{O} = C + \smallo$, and for all $f \in K$, $f > C \Rightarrow f' > 0$. Hardy fields extending $\mathbb{R}$ are $H$-fields. Our theorem answers some questions about Hardy fields and $H$-fields that have been around for some time. For example, it gives the following positive answer to Question 2 in Matusinski [**6**]. (However, in [**6**] the notion of $H$-field is construed too narrowly.) See also the remarks at the end of Section 3.

COROLLARY. *Each $H$-field has an immediate spherically complete $H$-field extension.*

(Here strictness of the extension is automatic by Lemma 1.11 below.) This corollary follows from our main theorem in conjunction with the following: Any immediate strict extension of an asymptotic field is again asymptotic by Lemma 1.12 below; and any immediate asymptotic extension $L$ of an $H$-field $K$ has a unique field ordering extending that of $K$ in which $\mathcal{O}_L$ is convex; equipped with this ordering, $L$ is an $H$-field by [**2**, Lemma 10.5.8].

*Uniqueness*

By Kaplansky [**4**], a valued field $F$ of equicharacteristic zero has up to isomorphism over $F$ a unique spherically complete immediate valued field extension. In Section 7 we prove such uniqueness in the setting of valued differential fields, but only when the valuation is discrete. We also discuss there a conjecture from [**2**] about this, and recent progress on it.

In Section 8 we give an example of an $H$-field where such uniqueness fails. Here we use some basic facts related to transseries from Sections 10.4, 10.5, 13.9, and Appendix A in [**2**].

*Notations and conventions*

We borrow these notational conventions from [**2**]. For the reader's convenience we repeat what is most needed in this paper. We set $\mathbb{N} := \{0, 1, 2, \dots\}$ and let $m$, $n$ range over $\mathbb{N}$.

A valuation (tacitly, on a field) takes values in an ordered (additively written) abelian group $\Gamma$, where 'ordered' here means 'totally ordered', and for such $\Gamma$,

$$\Gamma^{<} := \{\gamma \in \Gamma : \ \gamma < 0\}, \quad \Gamma^{\leqslant} := \{\gamma \in \Gamma : \ \gamma \leqslant 0\},$$

and likewise we define the subsets $\Gamma^{>}$, $\Gamma^{\geqslant}$, and $\Gamma^{\neq} := \Gamma \setminus \{0\}$ of $\Gamma$. For $\alpha, \beta \in \Gamma$, $\alpha = o(\beta)$ means that $n|\alpha| < |\beta|$ for all $n \geqslant 1$.

For a field $E$ we set $E^{\times} := E \setminus \{0\}$. Let $E$ be a valued field with valuation $v\colon E^{\times} \to \Gamma_E = v(E^{\times})$, valuation ring $\mathcal{O}_E$ and maximal ideal $\mathcal{o}_E$ of $\mathcal{O}_E$. When the ambient valued field $E$ is clear from the context, then for $a, b \in E$ we set

$$\begin{aligned} &a \asymp b :\Leftrightarrow va = vb, \quad && a \preccurlyeq b :\Leftrightarrow va \geqslant vb, \quad && a \prec b :\Leftrightarrow va > vb, \\ &a \succcurlyeq b :\Leftrightarrow b \preccurlyeq a, && a \succ b :\Leftrightarrow b \prec a, && a \sim b :\Leftrightarrow a - b \prec a. \end{aligned}$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$, and that $\sim$ is an equivalence relation on $E^{\times}$; let $a^{\sim}$ be the equivalence class of an element $a \in E^{\times}$ with respect to $\sim$. We use *pc-sequence* to abbreviate *pseudocauchy sequence;* see [**2**, Sections 2.2, 3.2]. Let also a valued field extension $F$ of $E$ be given. Then we identify in the usual way $\operatorname{res}(E)$ with a subfield of $\operatorname{res}(F)$, and $\Gamma_E$ with an ordered subgroup of $\Gamma_F$.

Next, let $E$ be a differential field of characteristic zero (so the field $E$ is equipped with a single derivation $\partial\colon E \to E$, as in [**2**]). Then we have the differential ring $E\{Y\} = E[Y, Y', Y'', \dots]$ of differential polynomials in an indeterminate $Y$, and we set $E\{Y\}^{\neq} := E\{Y\} \setminus \{0\}$. Let $P = P(Y) \in E\{Y\}$ have order at most $r \in \mathbb{N}$, that is, $P \in E[Y, Y', \dots, Y^{(r)}]$. Then $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$, as in [**2**, Section 4.2], with $\boldsymbol{i}$ ranging over tuples $(i_0, \dots, i_r) \in \mathbb{N}^{1+r}$, $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(r)})^{i_r}$, and the coefficients $P_{\boldsymbol{i}}$ are in $E$, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. For such $\boldsymbol{i}$ we set

$$|\boldsymbol{i}| := i_0 + i_1 + \cdots + i_r, \quad \|\boldsymbol{i}\| := i_1 + 2i_2 + \cdots + r i_r.$$

The *degree* and the *weight* of $P \neq 0$ are, respectively,

$$\deg P := \max\big\{|\boldsymbol{i}| : \ P_{\boldsymbol{i}} \neq 0\big\} \in \mathbb{N}, \quad \operatorname{wt} P := \max\big\{\|\boldsymbol{i}\| : \ P_{\boldsymbol{i}} \neq 0\big\} \in \mathbb{N}.$$

For $d \in \mathbb{N}$, we let $P_d := \sum_{|\boldsymbol{i}|=d} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ be the *homogeneous part of degree $d$* of $P$, so $P = \sum_{d \in \mathbb{N}} P_d$ where $P_d = 0$ for all but finitely many $d \in \mathbb{N}$. We also use the decomposition $P = \sum_{\boldsymbol{\sigma}} P_{[\boldsymbol{\sigma}]} Y^{[\boldsymbol{\sigma}]}$; here $\boldsymbol{\sigma}$ ranges over words $\boldsymbol{\sigma} = \sigma_1 \cdots \sigma_d \in \{0, \dots, r\}^*$, $Y^{[\boldsymbol{\sigma}]} := Y^{(\sigma_1)} \cdots Y^{(\sigma_d)}$, all $P_{[\boldsymbol{\sigma}]} \in E$ and $P_{[\boldsymbol{\sigma}]} \neq 0$ for only finitely many $\boldsymbol{\sigma}$, and $P_{[\boldsymbol{\sigma}]} = P_{[\pi(\boldsymbol{\sigma})]}$ for all $\boldsymbol{\sigma} = \sigma_1 \cdots \sigma_d$ and permutations $\pi$ of $\{1, \dots, d\}$, with $\pi(\boldsymbol{\sigma}) = \sigma_{\pi(1)} \cdots \sigma_{\pi(d)}$. We set $\|\boldsymbol{\sigma}\| := \sigma_1 + \cdots + \sigma_d$ for $\boldsymbol{\sigma} = \sigma_1 \cdots \sigma_d$, so $\|\boldsymbol{i}\| = \|\boldsymbol{\sigma}\|$ whenever $Y^{\boldsymbol{i}} = Y^{[\boldsymbol{\sigma}]}$. We also use for $a \in E$ the *additive conjugate* $P_{+a} := P(a + Y) \in E\{Y\}$ and the *multiplicative conjugate* $P_{\times a} := P(aY) \in E\{Y\}$. If $P \notin E$, the *complexity* of $P$ is the triple $(r, s, t) \in \mathbb{N}^3$ where $r$ is the order of $P$, $s$ is the degree of $P$ in $Y^{(r)}$, and $t$ is the total degree of $P$ (so $s, t \geqslant 1$). For the purpose of comparing complexities of differential polynomials we order $\mathbb{N}^3$ lexicographically. Thus for $P, Q \in E\{Y\} \setminus E$, the complexity of $P$ and the complexity of $Q$ are less than the complexity of $PQ$.

For a valued differential field $K$ we construe the differential fraction field $K\langle Y \rangle$ of $K\{Y\}$ as a valued differential field extension of $K$ by extending $v\colon K^{\times} \to \Gamma$ to the valuation $K\langle Y \rangle^{\times} \to \Gamma$ by requiring $vP = \min vP_{\boldsymbol{i}}$ for $P \in K\{Y\}^{\neq}$.

## 1. *Preliminaries*

We recall some basics about valued differential fields, mainly from Section 4.4 and Chapter 6 of [**2**], and add further material on compositional conjugation, strict extensions, the set $\Gamma(\partial) \subseteq \Gamma$, the convex subgroup $S(\partial)$ of $\Gamma$, and coarsening. We finish this preliminary section with facts about the dominant degree of a differential polynomial as needed in the next section. *In this section $\phi$ ranges over $K^\times$.*

*Compositional conjugation*

The compositional conjugate $K^\phi$ of $K$ is the valued differential field that has the same underlying valued field as $K$, but with derivation $\phi^{-1}\partial$. Let $L$ be an extension of $K$. Then $L^\phi$ extends $K^\phi$, and

$$L \text{ strictly extends } K \Longleftrightarrow L^\phi \text{ strictly extends } K^\phi.$$

Therefore, $K$ has the Krull property if and only if $K^\phi$ has the Krull property: $L$ is a spherically complete immediate strict extension of $K$ if and only if $L^\phi$ is a spherically complete immediate strict extension of $K^\phi$. Moreover,

$$\partial\smallo \subseteq \phi\smallo \Longleftrightarrow \text{the derivation } \phi^{-1}\partial \text{ of } K^\phi \text{ is small.}$$

Thus for the purpose of showing that $K$ has the Krull property it suffices to deal with the case that its derivation $\partial$ is small.

*Strict extensions*

Suppose that $\partial$ is small. Then $\partial\mathcal{O} \subseteq \mathcal{O}$ by [2, Lemma 4.4.2], so $\partial$ induces a derivation

$$a + \smallo \mapsto (a + \smallo)' := a' + \smallo$$

on the residue field $\operatorname{res}(K)$; the residue field of $K$ with this derivation is called the *differential residue field of* $K$ and is denoted by $\operatorname{res}(K)$ as well. Note that the derivation of $\operatorname{res}(K)$ is trivial if and only if $\partial\mathcal{O} \subseteq \smallo$.

The field $\mathbb{C}((t))$ of Laurent series with derivation $\partial = d/dt$ and the usual valuation, where $\mathcal{O} = \mathbb{C}[[t]]$ and $\smallo = t\mathbb{C}[[t]]$, is a valued differential field, since $\partial\smallo = \mathcal{O} = t^{-1}\smallo$. It is an example of a valued differential field with $\partial\mathcal{O} \subseteq \mathcal{O}$, but $\partial\smallo \nsubseteq \smallo$. On the other hand, under a mild assumption on $\Gamma$ we do have $\partial\mathcal{O} \subseteq \mathcal{O} \Rightarrow \partial\smallo \subseteq \smallo$:

LEMMA 1.1. *Suppose $\partial\smallo \subseteq \mathcal{O}$ and $\Gamma^>$ has no least element. Then $\partial\smallo \subseteq \smallo$.*

*Proof.* For $f \in \smallo$ we have $f = gh$ with $g, h \in \smallo$, so $f' = g'h + gh' \in \smallo$. □

LEMMA 1.2. *Suppose $\partial\smallo \subseteq \smallo$ and $\partial\mathcal{O} \nsubseteq \smallo$. Then for all $\phi$: $\partial\smallo \subseteq \phi\smallo \Leftrightarrow \phi \succcurlyeq 1$.*

*Proof.* From $\partial\smallo \subseteq \phi\smallo$, we get $\phi^{-1}\partial\smallo \subseteq \smallo$, so the derivation $\phi^{-1}\partial$ is small, and thus $\phi^{-1}\partial\mathcal{O} \subseteq \mathcal{O}$, hence $\partial\mathcal{O} \subseteq \phi\mathcal{O}$, which in view of $\partial\mathcal{O} \nsubseteq \smallo$ gives $\phi \succcurlyeq 1$. For the converse, note that if $\phi \succcurlyeq 1$, then $\smallo \subseteq \phi\smallo$. □

This leads easily to:

LEMMA 1.3. *Suppose that $\partial$ is small and the extension $L$ of $K$ has small derivation. Then the differential residue field $\operatorname{res}(L)$ of $L$ is an extension of the differential residue field $\operatorname{res}(K)$ of $K$. If in addition $\partial\mathcal{O} \nsubseteq \smallo$, then $L$ is a strict extension of $K$.*

LEMMA 1.4. *Let $L$ be an algebraic extension of $K$. Then $L$ strictly extends $K$.*

*Proof.* [2, Proposition 6.2.1] says that if the derivation of $K$ is small, then so is the derivation of $L$. Now, if $\partial\smallo \subseteq \phi\smallo$, then $\phi^{-1}\partial$ is small, hence $\phi^{-1}\partial_L$ is small, and thus $\partial_L\smallo_L \subseteq \phi\smallo_L$. Next, assume $\partial\mathcal{O} \subseteq \phi\smallo$. Then $\phi^{-1}\partial$ is small and induces the trivial derivation on $\operatorname{res}(K)$. Hence $\phi^{-1}\partial_L$

is small, and the derivation it induces on $\operatorname{res}(L)$ extends the trivial derivation on $\operatorname{res}(K)$, so is itself trivial, as $\operatorname{res}(L)$ is algebraic over $\operatorname{res}(K)$. Thus $\phi^{-1}\partial_L\mathcal{O}_L \subseteq \smallo_L$, that is, $\partial_L\mathcal{O}_L \subseteq \phi\smallo_L$. □

In this lemma the derivation of $L$ is assumed to be continuous for the valuation topology, because of the meaning we assigned to *extension of* $K$ and to *valued differential field*. In the proof of the lemma we used [**2**, Proposition 6.2.1], but that proposition does not assume this continuity. Thus if we drop the implicit assumption that the derivation of $L$ is continuous, then Lemma 1.4 goes through, with the continuity of this derivation as a consequence.

For immediate extensions, strictness reduces to a simpler condition:

LEMMA 1.5. *Let $L$ be an immediate extension of $K$ such that for all $\phi$, if $\partial\smallo \subseteq \phi\smallo$, then $\partial_L\smallo_L \subseteq \phi\smallo_L$. Then $L$ is a strict extension of $K$.*

*Proof.* Suppose $\partial\mathcal{O} \subseteq \phi\smallo$. Given $f \in \mathcal{O}_L$ we have $f = g(1+\varepsilon)$ with $g \in \mathcal{O}$ and $\varepsilon \in \smallo_L$, hence $f' = g'(1+\varepsilon) + g\varepsilon' \in \phi\smallo_L$. □

The following related fact will also be useful:

LEMMA 1.6. *Suppose that $\partial$ is small and $L$ is an immediate extension of $K$ such that $\partial_L\smallo_L \subseteq \mathcal{O}_L$. Then $\partial_L$ is small.*

*Proof.* If $a \in \smallo_L$, then $a = b(1+\varepsilon)$ with $b \in \smallo$, $\varepsilon \in \smallo_L$, so $a' = b'(1+\varepsilon) + b\varepsilon' \in \smallo_L$. □

Let us record the following observations on extensions $M \supseteq L \supseteq K$.

(1) If $M \supseteq K$ is strict, then $L \supseteq K$ is strict.
(2) If $M \supseteq L$ and $L \supseteq K$ are strict, then so is $M \supseteq K$.
(3) If $L$ is an elementary extension of $K$, then $L \supseteq K$ is strict.
(4) Any divergent pc-sequence in $K$ pseudoconverges in some strict extension of $K$; this is an easy consequence of (3), (cf. [**2**, Remark after Lemma 2.2.5]).

*The set* $\Gamma(\partial)$

Note that if $a, b \in K^\times$, $a \preccurlyeq b$, and $\smallo \subseteq a\smallo$, then $\smallo \subseteq b\smallo$. The set $\Gamma(\partial) \subseteq \Gamma$, denoted also by $\Gamma_K(\partial)$ if we need to specify $K$, is defined as follows:

$$\Gamma(\partial) := \{v\phi : \ \partial\smallo \subseteq \phi\smallo\}.$$

This is a nonempty downward closed subset of $\Gamma$, with an upper bound in $\Gamma$ if $\partial \neq 0$. Moreover, $\Gamma(\partial) < v(\partial\smallo)$. Lemma 1.2 has a reformulation:

COROLLARY 1.7. *If $\partial\smallo \subseteq \smallo$ and $\partial\mathcal{O} \nsubseteq \smallo$, then $\Gamma(\partial) = \Gamma^{\leqslant}$.*

LEMMA 1.8. *If $v\phi \in \Gamma(\partial)$ is not maximal in $\Gamma(\partial)$, then $\partial\mathcal{O} \subseteq \phi\smallo$.*

*Proof.* Let $a \in K^\times$ be such that $v\phi < va \in \Gamma(\partial)$. Then $a^{-1}\partial$ is small, so $a^{-1}\partial\mathcal{O} \subseteq \mathcal{O}$, and thus $\partial\mathcal{O} \subseteq a\mathcal{O} \subseteq \phi\smallo$. □

COROLLARY 1.9. *Suppose that $\Gamma_K(\partial)$ has no largest element, $L$ extends $K$, and for all $\phi$, if $\partial\smallo \subseteq \phi\smallo$, then $\partial_L\smallo_L \subseteq \phi\smallo_L$. Then $L$ strictly extends $K$.*

*Proof.* If $v\phi \in \Gamma_K(\partial)$, then $v\phi \in \Gamma_L(\partial)$, but $v\phi$ is not maximal in $\Gamma_L(\partial)$, and thus $\partial\mathcal{O}_L \subseteq \phi\smallo_L$ by Lemma 1.8. □

LEMMA 1.10. *If $L$ strictly extends $K$ with $\Gamma_L = \Gamma$, then $\Gamma_L(\partial) = \Gamma_K(\partial)$.*

*The case of asymptotic fields*

In this subsection we assume familiarity with Sections 6.5, 9.1, and the early parts of Section 9.2 in [2]. Recall that $K$ is said to be asymptotic if for all $f, g \in K$ with $0 \neq f, g \prec 1$ we have $f \prec g \Longleftrightarrow f' \prec g'$. In that case we put $\Psi := \{v(f^\dagger) : f \in K^\times, f \not\asymp 1\} \subseteq \Gamma$; if we need to make the dependence on $K$ explicit we denote $\Psi$ by $\Psi_K$. We recall from [2, Section 9.1] that then $\Psi < v(f')$ for all $f \in \smallo$. An asymptotic field $K$ is said to be *grounded* if $\Psi$ has a largest element, and *ungrounded* otherwise.

LEMMA 1.11. *Suppose that $K$ and $L$ are asymptotic fields, and $L$ is an immediate extension of $K$. Then $L$ is a strict extension of $K$.*

*Proof.* Assume $\partial\smallo \subseteq \smallo$; we show that $\partial\smallo_L \subseteq \smallo_L$. (Using Lemma 1.5, apply this to $\phi^{-1}\partial$ in the role of $\partial$, for $v\phi \in \Gamma(\partial)$.) Now $\partial\smallo \subseteq \smallo$ means that there is no $\gamma \in \Gamma^<$ such that $\Psi \leqslant \gamma$, by [2, Lemma 9.2.9]. Since $\Gamma_L = \Gamma$, we have $\Psi_L = \Psi$, and so there is no $\gamma \in \Gamma_L^<$ such that $\Psi_L \leqslant \gamma$, which gives $\partial\smallo_L \subseteq \smallo_L$. □

Here is a partial converse. (The proof assumes familiarity with asymptotic couples.)

LEMMA 1.12. *Suppose that $K$ is asymptotic and $L$ strictly extends $K$ with $\Gamma_L = \Gamma$. If $K$ is ungrounded or $\operatorname{res}(K) = \operatorname{res}(L)$, then $L$ is asymptotic.*

*Proof.* Let $a \in L^\times$, $a \not\asymp 1$. Then $a = bu$ with $b \in K^\times$ and $a \asymp b$, so $u \asymp 1$ and $a^\dagger = b^\dagger + u^\dagger$. Using $\Gamma = \Gamma_L$ and the equivalence of (i) and (ii) in [2, Proposition 9.1.3] applied to $L$, we see that for $L$ to be asymptotic it is enough to show that $a^\dagger \asymp b^\dagger$, which in turn will follow from $u' \prec b^\dagger$ in view of $u' \asymp u^\dagger$.

Suppose that $\Psi$ has no largest element. Take $\phi$ with $v(b^\dagger) < v(\phi) \in \Psi$. Then $v\phi < v(\partial\smallo)$, so $\partial\smallo \subseteq \phi\smallo$, hence $\partial\smallo_L \subseteq \phi\smallo_L$, which by [2, Lemma 4.4.2] gives $\partial\mathcal{O}_L \subseteq \phi\mathcal{O}_L$, and thus $u' \preccurlyeq \phi \prec b^\dagger$.

Next, suppose $\operatorname{res}(K) = \operatorname{res}(L)$. Then in the above we could have taken $u = 1 + \varepsilon$ with $\varepsilon \prec 1$. Then $\partial\smallo \subseteq b^\dagger\smallo$, so $\partial\smallo_L \subseteq b^\dagger\smallo_L$, hence $u' = \varepsilon' \prec b^\dagger$. □

For an example of a nonstrict extension $L \supseteq K$ of asymptotic fields, take $K = \mathbb{R}$ with the trivial valuation and trivial derivation, and $L = \mathbb{R}((t))$ with the natural valuation $f \mapsto \operatorname{order}(f) \colon L^\times \to \mathbb{Z}$ given by $\operatorname{order}(f) = k$ for

$$f = f_k t^k + f_{k+1}t^{k+1} + \cdots$$

with $f_k \neq 0$ and all coefficients $f_{k+n} \in \mathbb{R}$, and derivation $\partial = d/dt$ given by $\partial(f) = \sum_{k \neq 0} k f_k t^{k-1}$ for $f = \sum_k f_k t^k$. Note that $\Gamma_L(\partial) = \mathbb{Z}^<$, so $0 \notin \Gamma_L(\partial)$.

The following is not needed, but is somehow missing in [2].

LEMMA 1.13. *If $K$ is asymptotic and $\Gamma^>$ has a least element, then $K$ is grounded.*

*Proof.* Suppose $f \in \smallo$, $f \neq 0$, and $v(f) = \min(\Gamma^>)$. Replacing $K$ by $K^\phi$ where $\phi = f^\dagger$ we arrange $v(f^\dagger) = 0$. There is no $\gamma \in \Gamma$ with $0 < \gamma < v(f)$, which in view of $v(f) = v(f')$ and $\Psi < v(\partial\smallo)$ gives $0 = \max\Psi$. □

Another class of valued differential fields considered more closely in [2] is the class of *monotone* fields: By definition, $K$ is monotone if and only if $f^\dagger \preccurlyeq 1$ for all $f \in K^\times$. If $K$ is monotone, then so is any strict extension $L$ of $K$ with $\Gamma_L = \Gamma$, by [2, Corollary 6.3.6].

Note that $K$ has a monotone compositional conjugate if and only if for some $\phi \in K^\times$ we have $f^\dagger \preccurlyeq \phi$ for all $f \in K^\times$. If $K$ has a monotone compositional conjugate, then clearly any strict extension $L$ of $K$ with $\Gamma_L = \Gamma$ has as well.

*The stabilizer of* $\Gamma(\partial)$

By this we mean the convex subgroup

$$S_K(\partial) := \big\{\gamma \in \Gamma : \ \Gamma(\partial) + \gamma = \Gamma(\partial)\big\}$$

of $\Gamma$, also denoted by $S(\partial)$ if $K$ is clear from the context. Thus

$$S(\partial)^{\geqslant} = \big\{\gamma \in \Gamma^{\geqslant} : \ \Gamma(\partial) + \gamma \subseteq \Gamma(\partial)\big\}.$$

Note that $\Gamma(\partial)$ is a union of cosets of $S(\partial)$. We have $\Gamma(a\partial) = \Gamma(\partial) + va$ and $S(\partial) = S(a\partial)$ for all $a \in K^\times$. So $S(\partial)$ is invariant under compositional conjugation. Normalizing $\partial$ so that $0 \in \Gamma(\partial)$ has the effect that $S(\partial) \subseteq \Gamma(\partial)$.

Lemma 1.14. *Suppose that $K$ is asymptotic. Then $S(\partial) = \{0\}$.*

*Proof.* We have $\Psi < v(\partial\mathcal{O})$, so $\Psi \subseteq \Gamma(\partial)$. Therefore, if $\gamma \in \Gamma^>$, say $\gamma = vg$, $g \in K^\times$, then $\beta := v(g^\dagger) \in \Psi \subseteq \Gamma(\partial)$, yet $\beta + \gamma = v(g') \notin \Gamma(\partial)$, hence $\gamma \notin S(\partial)$. $\square$

The following is also easy to verify.

Lemma 1.15. *If $\Gamma(\partial)$ has a supremum in $\Gamma$, then $S(\partial) = \{0\}$.*

In particular, if $\Gamma(\partial)$ has a maximum, then $S(\partial) = \{0\}$. If $\partial \neq 0$ and $\Gamma$ is archimedean, then clearly also $S(\partial) = \{0\}$. In Section 3 we need the following:

Lemma 1.16. *Suppose $S(\partial) = \{0\}$. Then for any $\varepsilon \in \Gamma^>$ there are $\gamma \in \Gamma(\partial)$ and $\delta \in \Gamma \setminus \Gamma(\partial)$ such that $\delta - \gamma \leqslant \varepsilon$.*

*Proof.* Let $\varepsilon \in \Gamma^>$. Then $\varepsilon \notin S(\partial)$, so we get $\gamma \in \Gamma(\partial)$ with $\delta := \gamma + \varepsilon \notin \Gamma(\partial)$. $\square$

For cases where $S(\partial) \neq \{0\}$, let $\boldsymbol{k}$ be a field of characteristic zero with a valuation $w\colon \boldsymbol{k} \to \Delta = w(\boldsymbol{k}^\times)$, and let $K = \boldsymbol{k}((t))$ be the field of Laurent series over $\boldsymbol{k}$. Then we have the valuation $f \mapsto \operatorname{order}(f)\colon K^\times \to \mathbb{Z}$, where $\operatorname{order}(f) = k$ means that $f = f_k t^k + f_{k+1}t^{k+1} + \cdots$ with $f_k \neq 0$ and all coefficients $f_{k+n} \in \boldsymbol{k}$. We combine these two valuations into a single valuation $v\colon K^\times \to \Gamma$ extending the valuation $w$ on $\boldsymbol{k}$, with $\Gamma$ having $\Delta$ and $\mathbb{Z}$ as ordered subgroups, $\Delta$ convex in $\Gamma$, and $\Gamma = \Delta + \mathbb{Z}$; it is given by $v(f) = w(f_k) + k$, with $k = \operatorname{order}(f)$. Next, we equip $K$ with the derivation $\partial = t \cdot d/dt$ given by $\partial(f) = \sum_k k f_k t^k$ for $f = \sum_k f_k t^k$. Then $K$ with the valuation $v$ and the derivation $\partial$ is a (monotone) valued differential field, with $\partial\mathcal{O} = \big\{f \in K : \ \operatorname{order}(f) \geqslant 1\big\}$. It follows easily that

$$\Gamma(\partial) = \{\gamma \in \Gamma : \ \gamma \leqslant \delta \text{ for some } \delta \in \Delta\},$$

and thus $S(\partial) = \Delta$.

*Coarsening*

We begin with reminders about coarsening from [2, Sections 3.4, 4.4]. Let $\Delta$ be a convex subgroup of $\Gamma$. This yields the ordered abelian quotient group $\dot{\Gamma} = \Gamma/\Delta$ of $\Gamma$, with the coarsened valuation

$$\dot{v} = v_\Delta \colon K^\times \to \dot{\Gamma}, \quad \dot{v}(a) := v(a) + \Delta$$

on the underlying field of $K$. The $\Delta$-coarsening $K_\Delta$ of $K$ is the valued differential field with the same underlying differential field as $K$, but with valuation $\dot{v}$. Its valuation ring is

$$\dot{\mathcal{O}} = \{a \in K : \ va \geqslant \delta \text{ for some } \delta \in \Delta\} \supseteq \mathcal{O},$$

with maximal ideal

$$\dot{\smallo} = \{a \in K : \ va > \Delta\} \subseteq \smallo.$$

The residue field

$$\dot{K} := \operatorname{res}(K_\Delta) = \dot{\mathcal{O}}/\dot{\smallo}$$

is itself a valued field with valuation $v\colon \dot{K}^\times \to \Delta$ given by

$$v(a + \dot{\smallo}) := v(a) \quad \text{ for } a \in \dot{\mathcal{O}} \setminus \dot{\smallo},$$

and with valuation ring $\{a + \dot{\smallo} : a \in \mathcal{O}\}$. We identify $\operatorname{res}(K)$ with $\operatorname{res}(\dot{K})$ via $\operatorname{res}(a) \mapsto \operatorname{res}(a + \dot{\smallo})$ for $a \in \mathcal{O}$. The following is [**2**, Corollary 4.4.4]:

LEMMA 1.17. *If $\partial\smallo \subseteq \smallo$, then $\partial\dot{\smallo} \subseteq \dot{\smallo}$.*

Suppose that the derivation $\partial$ of $K$ is small. Then $\partial$ is also small as a derivation of $K_\Delta$, and the derivation on $\dot{K}$ induced by the derivation of $K_\Delta$ is small as well. This derivation on $\dot{K}$ then induces the same derivation on $\operatorname{res}(\dot{K})$ as $\partial$ on $K$ induces on $\operatorname{res}(K)$. The operation of coarsening commutes with compositional conjugation: $(K^\phi)_\Delta$ and $(K_\Delta)^\phi$ are the same valued differential field, to be denoted by $K^\phi_\Delta$.

The next lemma describes the downward closed subset $\dot{\Gamma}(\partial)$ of $\dot{\Gamma}$ almost completely in terms of $\Gamma(\partial)$ and the canonical map $\pi\colon \Gamma \to \dot{\Gamma}$. Let $\alpha$ range over $\Gamma$.

LEMMA 1.18. *If $\partial\dot{\smallo} \subseteq \phi\dot{\smallo}$, then $v\phi \in \bigcap_{\alpha>\Delta} \Gamma(\partial) + \alpha$. As a consequence we have either $\dot{\Gamma}(\partial) = \pi\Gamma(\partial)$, or $\dot{\Gamma}(\partial) = \pi\Gamma(\partial) \cup \{\dot{\mu}\}$ with $\dot{\mu} = \max \dot{\Gamma}(\partial)$.*

*Proof.* Suppose $\partial\dot{\smallo} \subseteq \phi\dot{\smallo}$. Then $\phi^{-1}\partial\dot{\smallo} \subseteq \dot{\smallo}$, so $\phi^{-1}\partial\dot{\mathcal{O}} \subseteq \dot{\mathcal{O}}$, hence $\partial\dot{\mathcal{O}} \subseteq \phi\dot{\mathcal{O}}$. For $a \in K^\times$ with $\alpha = va > \Delta$ we have $a\dot{\mathcal{O}} \subseteq \smallo$, so $\partial\dot{\mathcal{O}} \subseteq \phi\dot{\mathcal{O}} \subseteq \phi a^{-1}\smallo$, and thus $\partial\smallo \subseteq \phi a^{-1}\smallo$, which gives $v\phi - \alpha \in \Gamma(\partial)$. We conclude that

$$v\phi \in \bigcap_{\alpha>\Delta} \Gamma(\partial) + \alpha.$$

It follows from Lemma 1.17 that $\pi\Gamma(\partial) \subseteq \dot{\Gamma}(\partial)$. Suppose $v\phi > \Gamma(\partial) + \Delta$. Then by the above, $\dot{v}\phi - \dot{\alpha} \in \pi\Gamma(\partial)$ for all $\dot{\alpha} \in \dot{\Gamma}^>$. If $\pi\Gamma(\partial)$ has no largest element, then we get $\dot{v}\phi = \sup \pi\Gamma(\partial)$. If $\pi\Gamma(\partial)$ has a largest element, then $\dot{v}\phi - \max \pi\Gamma(\partial)$ must be the least positive element of $\dot{\Gamma}^>$, and $\dot{\Gamma}(\partial) = \pi\Gamma(\partial) \cup \{\dot{v}\phi\}$. □

The following will be needed in deriving Proposition 7.2:

LEMMA 1.19. *Let $L$ be an immediate strict extension of $K$ such that $\operatorname{res}(L_\Delta) = \operatorname{res}(K_\Delta)$. Then $L_\Delta$ is a strict extension of $K_\Delta$.*

*Proof.* Note that $L_\Delta$ is an immediate extension of $K_\Delta$. Suppose $\partial\dot{\smallo} \subseteq \phi\dot{\smallo}$; applying Lemmas 1.5 and 1.6 to the extension $L_\Delta$ of $K_\Delta$, it suffices to derive from this assumption that $\partial\dot{\smallo}_L \subseteq \phi\dot{\mathcal{O}}_L$. The proof of Lemma 1.18 gives $\partial\smallo \subseteq \phi a^{-1}\smallo$ for every $a \in K^\times$ with $va > \Delta$, and so $\partial\smallo_L \subseteq \phi a^{-1}\smallo_L$ for such $a$. Thus for $f \in \smallo_L$ we have $v(f') > v\phi - \alpha$ for all $\alpha > \Delta$, so $v(f'/\phi) > -\alpha$ for all $\alpha > \Delta$, that is, $f'/\phi \in \dot{\mathcal{O}}_L$, so $f' \in \phi\dot{\mathcal{O}}_L$. This shows $\partial\dot{\smallo}_L \subseteq \partial\smallo_L \subseteq \phi\dot{\mathcal{O}}_L$, as desired. □

*Dominant degree*

We summarize here from [**2**, Section 6.6] what we need about the dominant part and dominant degree of a differential polynomial and its behavior under additive and multiplicative conjugation. We give the definitions, but refer to [**2**, Section 6.6] for the proofs. *In this subsection we assume that the derivation $\partial$ of $K$ is small, and we choose for every $P \in K\{Y\}^{\neq}$ an element $\mathfrak{d}_P \in K^\times$ with $\mathfrak{d}_P \asymp P$, such that $\mathfrak{d}_P = \mathfrak{d}_Q$ whenever $P \sim Q$, $P, Q \in K\{Y\}^{\neq}$. Let $P \in K\{Y\}^{\neq}$.*

We have $\mathfrak{d}_P^{-1} P \asymp 1$, in particular, $\mathfrak{d}_P^{-1} P \in \mathcal{O}\{Y\}$, and we define the *dominant part* $D_P \in \operatorname{res}(K)\{Y\}^{\neq}$ to be the image of $\mathfrak{d}_P^{-1} P$ under the natural differential ring morphism $\mathcal{O}\{Y\} \to \operatorname{res}(K)\{Y\}$. Note that $\deg D_P \leqslant \deg P$. The *dominant degree* of $P$ is defined to be the natural number $\operatorname{ddeg} P := \deg D_P$; unlike $D_P$ it does not depend on the choice of the elements $\mathfrak{d}_P \in K^\times$. Given also $Q \in K\{Y\}^{\neq}$ we have $\operatorname{ddeg} PQ = \operatorname{ddeg} P + \operatorname{ddeg} Q$. If $f \preccurlyeq 1$ in an extension $L$ of $K$ with small derivation satisfies $P(f) = 0$, then $D_P(f + \smallo_L) = 0$ and thus $\operatorname{ddeg} P \geqslant 1$.

LEMMA 1.20. *If $a \in K$ and $a \preccurlyeq 1$, then* $\operatorname{ddeg} P_{+a} = \operatorname{ddeg} P$.

LEMMA 1.21. *Let $a, b \in K$, $g \in K^\times$ be such that $a - b \preccurlyeq g$. Then*
$$\operatorname{ddeg} P_{+a,\times g} = \operatorname{ddeg} P_{+b,\times g}.$$

LEMMA 1.22. *If $g, h \in K^\times$ and $g \preccurlyeq h$, then* $\operatorname{ddeg} P_{\times g} \leqslant \operatorname{ddeg} P_{\times h}$.

For these facts, see [**2**, Lemma 6.6.5(i), Corollary 6.6.6, Corollary 6.6.7].

## 2. *Eventual behavior*

*In this section $\Gamma \neq \{0\}$. We let $\phi$ range over $K^\times$, and $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ over $\mathbb{N}^*$. We also fix a differential polynomial $P \in K\{Y\}^{\neq}$.* Here we generalize parts of [**2**, Sections 11.1, 11.2] by dropping the assumption there that $K$ is asymptotic. The condition $v\phi < (\Gamma^>)'$ there becomes the condition $v\phi \in \Gamma(\partial)$ here.

*Behavior of $vF^n_k(\phi)$*

The differential polynomials $F^n_k(X) \in \mathbb{Q}\{X\} \subseteq K\{X\}$ for $0 \leqslant k \leqslant n$ were introduced in [**2**, Section 5.7] in connection with compositional conjugation: There we considered the $K$-algebra morphism
$$Q \mapsto Q^\phi \colon K\{Y\} \to K^\phi\{Y\}$$
defined by requiring that $Q(y) = Q^\phi(y)$ for $Q \in K\{Y\}$ and all $y$ in all differential field extensions of $K$. The $F^n_k(X)$ $(1 \leqslant k \leqslant n)$ satisfy
$$(Y^{(n)})^\phi = F^n_n(\phi) Y^{(n)} + F^n_{n-1}(\phi) Y^{(n-1)} + \cdots + F^n_1(\phi) Y'$$
and $F^0_0 = 1$, $F^n_0 = 0$ for $n \geqslant 1$. (For example, $F^1_1 = X$ and $F^2_2 = X^2$, $F^2_1 = X'$.) We also recall from there that for $\boldsymbol{\tau} = \tau_1 \cdots \tau_d \geqslant \boldsymbol{\sigma} = \sigma_1 \cdots \sigma_d$,
$$F^{\boldsymbol{\tau}}_{\boldsymbol{\sigma}} := F^{\tau_1}_{\sigma_1} \cdots F^{\tau_d}_{\sigma_d}.$$
In order to better understand $v(P^\phi)$ as a function of $\phi$ we use from [**2**, Lemma 5.7.4] and its proof the identities
$$(Y^{[\boldsymbol{\tau}]})^\phi = \sum_{\boldsymbol{\sigma} \leqslant \boldsymbol{\tau}} F^{\boldsymbol{\tau}}_{\boldsymbol{\sigma}}(\phi) Y^{[\boldsymbol{\sigma}]}, \quad (P^\phi)_{[\boldsymbol{\sigma}]} = \sum_{\boldsymbol{\tau} \geqslant \boldsymbol{\sigma}} F^{\boldsymbol{\tau}}_{\boldsymbol{\sigma}}(\phi) P_{[\boldsymbol{\tau}]}. \tag{1}$$
The next two lemmas have the same proof as [**2**, Lemmas 11.1.1, 11.1.2].

LEMMA 2.1. *If $\partial\mathcal{O} \subseteq \mathcal{O}$ and $\phi \preccurlyeq 1$, then $v(P^\phi) \geqslant v(P)$, with equality if $\phi \asymp 1$.*

We set $\delta = \phi^{-1}\partial$ in the next two results.

LEMMA 2.2. *Suppose $\delta\smallo \subseteq \smallo$, and let $0 \leqslant k \leqslant n$.*

(i) *If $\phi^\dagger \preccurlyeq \phi$, then $F^n_k(\phi) \preccurlyeq \phi^n$ and $F^n_n(\phi) = \phi^n$.*
(ii) *If $\phi^\dagger \prec \phi$ and $k < n$, then $F^n_k(\phi) \prec \phi^n$.*

COROLLARY 2.3. *Suppose $\delta\smallo \subseteq \smallo$ and $\phi^\dagger \preccurlyeq \phi$, and $\boldsymbol{\tau} \geqslant \boldsymbol{\sigma}$. Then $F^{\boldsymbol{\tau}}_{\boldsymbol{\sigma}}(\phi) \preccurlyeq \phi^{\|\boldsymbol{\tau}\|}$ and $F^{\boldsymbol{\tau}}_{\boldsymbol{\tau}}(\phi) = \phi^{\|\boldsymbol{\tau}\|}$. If $\phi^\dagger \prec \phi$ and $\boldsymbol{\tau} > \boldsymbol{\sigma}$, then $F^{\boldsymbol{\tau}}_{\boldsymbol{\sigma}}(\phi) \prec \phi^{\|\boldsymbol{\tau}\|}$.*

Let $P$ have order $\leqslant r$, so $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ with $\boldsymbol{i}$ ranging over $\mathbb{N}^{1+r}$. We define the *dominant degree* $\operatorname{ddeg} P \in \mathbb{N}$ and the *dominant weight* $\operatorname{dwt} P \in \mathbb{N}$ by

$$\operatorname{ddeg} P = \max\big\{|\boldsymbol{i}| :\ P_{\boldsymbol{i}} \asymp P\big\}, \quad \operatorname{dwt} P = \max\big\{\|\boldsymbol{i}\| :\ P_{\boldsymbol{i}} \asymp P\big\}.$$

Thus if $K$ has small derivation, then $\operatorname{ddeg} P = \deg D_P$ as in the previous section, and $\operatorname{dwt} P = \operatorname{wt} D_P$, agreeing with the dominant weight from [2, Sections 4.5, 6.6].

LEMMA 2.4. *Suppose $\partial\mathcal{O} \subseteq \mathcal{O}$ and $\phi \asymp 1$. Then $\operatorname{ddeg} P^\phi = \operatorname{ddeg} P$.*

*Proof.* Set

$$d := \operatorname{ddeg} P, \quad I_d := \big\{\boldsymbol{i} :\ P_{\boldsymbol{i}} \asymp P,\ |\boldsymbol{i}| = d\big\}, \quad I_{<d} := \big\{\boldsymbol{i} :\ P_{\boldsymbol{i}} \asymp P,\ |\boldsymbol{i}| < d\big\}.$$

Then

$$P = Q + R + S \quad \text{with } Q := \sum_{\boldsymbol{i} \in I_d} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \quad R := \sum_{\boldsymbol{i} \in I_{<d}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}},$$

and so

$$P^\phi = Q^\phi + R^\phi + S^\phi, \quad P^\phi \asymp P, \quad Q^\phi \asymp Q \asymp P, \quad R^\phi \asymp R, \quad S^\phi \asymp S \prec P,$$

by Lemma 2.1, and $R \asymp P$ if $R \neq 0$. Also $\deg Q^\phi = \deg Q = d$ and $\deg R^\phi = \deg R < d$ by [2, Corollary 5.7.5], and thus $\operatorname{ddeg} P^\phi = d$. □

It is convenient to introduce two operators $\mathrm{D}, \mathrm{W} \colon K\{Y\}^{\neq} \to K\{Y\}^{\neq}$:

$$\mathrm{D}(P) \ := \ \sum_{\boldsymbol{i} \in I} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \quad I := \{\boldsymbol{i} :\ P_{\boldsymbol{i}} \asymp P\},$$

$$\mathrm{W}(P) \ := \ \sum_{\boldsymbol{i} \in J} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}, \quad J := \big\{\boldsymbol{i} \in I :\ \|\boldsymbol{i}\| = \operatorname{dwt} P\big\}.$$

Thus $\mathrm{D}(P)$ and $\mathrm{W}(P)$ are of degree $\operatorname{ddeg} P$, and every monomial $Y^{\boldsymbol{i}}$ occurring in $\mathrm{W}(P)$ has weight $\|\boldsymbol{i}\| = \operatorname{dwt} P$. Note that $P \asymp \mathrm{D}(P) \asymp \mathrm{W}(P)$. If $K$ has small derivation, then the nonzero coefficients of $\mathrm{D}(P)$ are $\asymp \mathfrak{d}_P$, and the image of $\mathfrak{d}_P^{-1}\,\mathrm{D}(P)$ under the natural differential ring morphism $\mathcal{O}\{Y\} \to \operatorname{res}(K)\{Y\}$ equals the dominant part $D_P$ of $P$.

LEMMA 2.5. *Suppose that $\Gamma^>$ has no smallest element and $\partial\smallo \subseteq \smallo$. Then there exists an $\alpha \in \Gamma^<$ such that for $w := \operatorname{dwt} P$ we have*

$$\mathrm{D}(P^\phi) \sim \phi^w\, \mathrm{W}(P)$$

*for all $\phi$ with $\alpha < v\phi < 0$, so $\operatorname{ddeg} P^\phi = \operatorname{ddeg} P$ and $\operatorname{dwt} P^\phi = \operatorname{dwt} P$ for such $\phi$.*

*Proof.* For any monomial $Y^{\boldsymbol{i}} = Y^{[\boldsymbol{\tau}]}$ we have $(Y^{[\boldsymbol{\tau}]})^{\phi} = \sum_{\boldsymbol{\sigma}\leqslant\boldsymbol{\tau}} F^{\boldsymbol{\tau}}_{\boldsymbol{\sigma}}(\phi)Y^{[\boldsymbol{\sigma}]}$ by (1). Now let $\phi \succ 1$. Then $\phi^{\dagger} \prec \phi$: this is clear if $\phi' \preccurlyeq \phi$, and follows from [**2**, Lemma 6.4.1(iii)] when $\phi' \succ \phi$. Thus by Corollary 2.3 and using $\|\boldsymbol{i}\| = \|\boldsymbol{\tau}\|$:

$$(Y^{\boldsymbol{i}})^{\phi} \sim \phi^{\|\boldsymbol{i}\|} Y^{\boldsymbol{i}}.$$

Now $P = \mathrm{W}(P) + Q$ with $Q \in K\{Y\}$, and for each monomial $Y^{\boldsymbol{i}}$, either $Q_{\boldsymbol{i}} \prec P$, or $Q_{\boldsymbol{i}} = P_{\boldsymbol{i}} \asymp P$ and $\|\boldsymbol{i}\| < \operatorname{dwt} P$. Then

$$P^{\phi} = \mathrm{W}(P)^{\phi} + Q^{\phi}, \quad \mathrm{W}(P)^{\phi} \sim \phi^{w}\, \mathrm{W}(P) \text{ for } w := \operatorname{dwt} P.$$

Now $\Gamma^{>}$ has no smallest element, so given any $\beta \in \Gamma^{>}$ and $n \geqslant 1$ there is an $\alpha \in \Gamma^{>}$ such that $n\gamma < \beta$ whenever $\gamma \in \Gamma$ and $0 < \gamma < \alpha$. Thus by considering the individual monomials in $Q$ we obtain an $\alpha \in \Gamma^{<}$ such that $Q^{\phi} \prec \phi^{w}\, \mathrm{W}(P)$ whenever $\alpha < v\phi < 0$. Any such $\alpha$ witnesses the property stated in the lemma. $\square$

Corollary 2.6. *If $\Gamma^{>}$ has no least element, $\phi_0 \in K^{\times}$ and $v(\phi_0) \in \Gamma(\partial)$, then there exists $\alpha < v(\phi_0)$ such that $\operatorname{ddeg} P^{\phi_0} = \operatorname{ddeg} P^{\phi}$ whenever $\alpha < v(\phi) < v(\phi_0)$.*

*Proof.* Apply Lemma 2.5 to $K^{\phi_0}$ and $P^{\phi_0}$ in the role of $K$ and $P$. $\square$

*Newton degree*

*In this subsection we assume that $\Gamma^{>}$ has no least element.* Let $P \in K\{Y\}^{\neq}$ have order $\leqslant r \in \mathbb{N}$. For $d \leqslant \deg P$ we define

$$\Gamma(P,d) := \{\gamma \in \Gamma(\partial) :\ \operatorname{ddeg} P^{\phi} = d \text{ for some } \phi \text{ with } v\phi = \gamma\}.$$

Note that in this definition of $\Gamma(P,d)$ we can replace ‘some’ by ‘all’ in view of Lemma 2.4, and hence the nonempty sets among the $\Gamma(P,d)$ with $d \leqslant \deg P$ partition $\Gamma(\partial)$. Note also that if $\gamma \in \Gamma(P,d)$, then $(\gamma - \alpha, \gamma] \subseteq \Gamma(P,d)$ for some $\alpha \in \Gamma^{>}$ by Corollary 2.6, so each convex component of $\Gamma(P,d)$ in $\Gamma$ is infinite.

Lemma 2.7. *The set $\Gamma(P,d)$ has only finitely many convex components in $\Gamma$.*

*Proof.* Let $\boldsymbol{i}$ range over the tuples $(i_0,\dots,i_r) \in \mathbb{N}^{1+r}$ with $|\boldsymbol{i}| \leqslant \deg P$, and likewise for $\boldsymbol{j}$. Let $N$ be the number of pairs $(\boldsymbol{i},\boldsymbol{j})$ with $\boldsymbol{i} \neq \boldsymbol{j}$. We claim that for every $\phi_0 \in K^{\times}$ with $v\phi_0 \in \Gamma(\partial)$ the set $\Gamma(P,d)$ has at most $N+1$ convex components with an element $\leqslant v\phi_0$. (It follows easily from this claim that $\Gamma(P,d)$ has at most $N+1$ convex components.) By renaming $K^{\phi_0}$ and $P^{\phi_0}$ as $K$ and $P$ it suffices to prove the claim for $\phi_0 = 1$. So we assume $\partial\smallo \subseteq \smallo$ and have to show that $\Gamma(P,d)$ has at most $N+1$ components with an element $\leqslant 0$. We now restrict $\boldsymbol{i}$ further by the requirement that $P_{\boldsymbol{i}} \neq 0$, and likewise for $\boldsymbol{j}$. By the proof of Lemma 2.5,

$$\operatorname{ddeg} P^{\phi} = \max\left\{|\boldsymbol{i}| :\ vP_{\boldsymbol{i}} + \|\boldsymbol{i}\| v\phi = \min_{\boldsymbol{j}} vP_{\boldsymbol{j}} + \|\boldsymbol{j}\| v\phi\right\} \quad \text{for } v\phi < 0.$$

For each $\boldsymbol{i}$ we have the function $f_{\boldsymbol{i}} : \mathbb{Q}\Gamma \to \mathbb{Q}\Gamma$ given by $f_{\boldsymbol{i}}(\gamma) = vP_{\boldsymbol{i}} + \|\boldsymbol{i}\|\gamma$. For any $\boldsymbol{i}, \boldsymbol{j}$, either $f_{\boldsymbol{i}} = f_{\boldsymbol{j}}$ or we have a unique $\gamma = \gamma_{\boldsymbol{i},\boldsymbol{j}} \in \mathbb{Q}\Gamma$ with $f_{\boldsymbol{i}}(\gamma) = f_{\boldsymbol{j}}(\gamma)$. Let $\gamma_1 < \cdots < \gamma_M$ with $M \leqslant N$ be the distinct values of $\gamma_{\boldsymbol{i},\boldsymbol{j}} < 0$ obtained in this way, and set $\gamma_0 := -\infty$ and $\gamma_{M+1} := 0$. Then on each interval $(\gamma_m, \gamma_{m+1})$ with $0 \leqslant m \leqslant M$, the functions $f_{\boldsymbol{i}} - f_{\boldsymbol{j}}$ have constant sign: $-$, $0$, or $+$. In view of the above identity for $\operatorname{ddeg} P^{\phi}$ it follows easily that for each $m$ with $0 \leqslant m \leqslant M$ the value of $\operatorname{ddeg} P^{\phi}$ is constant as $v\phi$ ranges over $(\gamma_m, \gamma_{m+1}) \cap \Gamma$. Thus $\Gamma(P,d)$ has at most $M+1$ convex components. $\square$

It follows from Lemmas 2.5 and 2.7 that there exists $d \leqslant \deg P$ and a $\phi_0 \in K^{\times}$ such that $v\phi_0 \in \Gamma(\partial)$, $v\phi_0$ is not maximal in $\Gamma(\partial)$, and $\operatorname{ddeg} P^{\phi} = d$ for all $\phi \preccurlyeq \phi_0$ with $v\phi \in \Gamma(\partial)$. We now define the *Newton degree* $\operatorname{ndeg} P$ of $P$ to be this eventual value $d \in \mathbb{N}$ of $\operatorname{ddeg} P^{\phi}$. Note that if $\Gamma(\partial)$ does have a maximal element $v\phi$, then

$$\operatorname{ndeg} P = \operatorname{ddeg} P^{\phi}.$$

Also, for $f \in K^{\times}$ and $Q \in K\{Y\}^{\neq}$ we have

$$\operatorname{ndeg} P^{f} = \operatorname{ndeg} P, \quad \operatorname{ndeg} PQ = \operatorname{ndeg} P + \operatorname{ndeg} Q.$$

*Newton degree and multiplicative conjugation*

*In this subsection* $\Gamma^{>}$ *has no least element.* Here we consider the behavior of $\operatorname{ndeg} P_{\times g}$ as a function of $g \in K^{\times}$. Indeed, $\operatorname{ndeg} P_{\times g} \geqslant 1$ is a useful necessary condition for the existence of a zero $f \preccurlyeq g$ of $P$ in a strict extension of $K$, as stated in the following generalization of [**2**, Lemma 11.2.1]:

LEMMA 2.8. *Let $g \in K^{\times}$ and suppose that some $f \preccurlyeq g$ in a strict extension of $K$ satisfies $P(f) = 0$. Then $\operatorname{ndeg} P_{\times g} \geqslant 1$.*

*Proof.* For such $f$ we have $f = ag$ with $a \preccurlyeq 1$, and $Q(a) = 0$ for $Q := P_{\times g}$. So $Q^{\phi}(a) = 0$ for all $\phi$ with $v\phi \in \Gamma(\partial)$, hence $\operatorname{ddeg} Q^{\phi} \geqslant 1$ for those $\phi$, and thus $\operatorname{ndeg} Q \geqslant 1$. □

Next some results on *Newton degree* that follow easily from corresponding facts at the end of Section 1 on *dominant degree*, using also that compositional conjugation commutes with additive and multiplicative conjugation by [**2**, Lemma 5.7.1].

LEMMA 2.9. *If $a \in K$ and $a \preccurlyeq 1$, then $\operatorname{ndeg} P_{+a} = \operatorname{ndeg} P$.*

LEMMA 2.10. *Let $a, b \in K$, $g \in K^{\times}$ be such that $a - b \preccurlyeq g$. Then*

$$\operatorname{ndeg} P_{+a,\times g} = \operatorname{ndeg} P_{+b,\times g}.$$

LEMMA 2.11. *If $g, h \in K^{\times}$ and $g \preccurlyeq h$, then $\operatorname{ndeg} P_{\times g} \leqslant \operatorname{ndeg} P_{\times h}$.*

For $g \in K^{\times}$ we set $\operatorname{ndeg}_{\prec g} P := \max\{\operatorname{ndeg} P_{\times f} : f \prec g\}$.

LEMMA 2.12. *For $a, g \in K$ with $a \prec g$ we have $\operatorname{ndeg}_{\prec g} P_{+a} = \operatorname{ndeg}_{\prec g} P$.*

*Proof.* Use that $\operatorname{ndeg} P_{+a,\times f} = \operatorname{ndeg} P_{\times f}$ for $a \preccurlyeq f \prec g$, by Lemma 2.10. □

It will also be convenient to define for $\gamma \in \Gamma$,

$$\operatorname{ndeg}_{\geqslant \gamma} P := \max\{\operatorname{ndeg} P_{\times g} : g \in K^{\times}, vg \geqslant \gamma\}.$$

By Lemma 2.11, $\operatorname{ndeg}_{\geqslant \gamma} P = \operatorname{ndeg} P_{\times g}$ for any $g \in K^{\times}$ with $\gamma = vg$. From Lemmas 2.10 and 2.11 we easily obtain:

COROLLARY 2.13. *Let $a, b \in K$ and $\alpha, \beta \in \Gamma$ be such that $v(b - a) \geqslant \alpha$ and $\beta \geqslant \alpha$. Then $\operatorname{ndeg}_{\geqslant \beta} P_{+b} \leqslant \operatorname{ndeg}_{\geqslant \alpha} P_{+a}$.*

*Newton degree in a cut*

*In this subsection* $\Gamma^{>}$ *has no least element.* We do not need the material here to obtain the main theorem. It is only used in proving Corollaries 4.6 and 4.7, which are of interest for other reasons.

Let $(a_\rho)$ be a pc-sequence in $K$, and put $\gamma_\rho = v(a_{s(\rho)} - a_\rho) \in \Gamma_\infty$, where $s(\rho)$ is the immediate successor of $\rho$. Using Corollary 2.13 in place of [**2**, Corollary 11.2.8] we generalize [**2**, Lemma 11.2.11]:

LEMMA 2.14. *There is an index $\rho_0$ and $d \in \mathbb{N}$ such that for all $\rho > \rho_0$ we have $\gamma_\rho \in \Gamma$ and $\operatorname{ndeg}_{\geqslant \gamma_\rho} P_{+a_\rho} = d$. Denoting this number $d$ by $d\big(P,(a_\rho)\big)$, we have $d\big(P,(a_\rho)\big) = d\big(P,(b_\sigma)\big)$ whenever $(b_\sigma)$ is a pc-sequence in $K$ equivalent to $(a_\rho)$.*

As in [**2**, Section 11.2], we now associate to each pc-sequence $(a_\rho)$ in $K$ an object $c_K(a_\rho)$, the *cut* defined by $(a_\rho)$ in $K$, such that if $(b_\sigma)$ is also a pc-sequence in $K$, then

$$c_K(a_\rho) = c_K(b_\sigma) \iff (a_\rho) \text{ and } (b_\sigma) \text{ are equivalent.}$$

We do this in such a way that the cuts $c_K(a_\rho)$, with $(a_\rho)$ a pc-sequence in $K$, are the elements of a set $c(K)$. Using Lemma 2.14 we define for $\mathbf{a} \in c(K)$ the *Newton degree of $P$ in the cut $\mathbf{a}$* as

$$\operatorname{ndeg}_{\mathbf{a}} P := d\big(P,(a_\rho)\big) = \text{eventual value of } \operatorname{ndeg}_{\geqslant \gamma_\rho} P_{+a_\rho},$$

where $(a_\rho)$ is any pc-sequence in $K$ with $\mathbf{a} = c_K(a_\rho)$. Let $(a_\rho)$ be a pc-sequence in $K$ and $\mathbf{a} = c_K(a_\rho)$. For $y \in K$ the cut $c_K(a_\rho + y)$ depends only on $(\mathbf{a}, y)$, and so we can set $\mathbf{a} + y := c_K(a_\rho + y)$. Likewise, for $y \in K^\times$ the cut $c_K(a_\rho y)$ depends only on $(\mathbf{a}, y)$, and so we can set $\mathbf{a} \cdot y := c_K(a_\rho y)$. We record some basic facts about $\operatorname{ndeg}_{\mathbf{a}} P$:

LEMMA 2.15. *Let $(a_\rho)$ be a pc-sequence in $K$, $\mathbf{a} = c_K(a_\rho)$. Then*

(i) $\operatorname{ndeg}_{\mathbf{a}} P \leqslant \deg P$;
(ii) $\operatorname{ndeg}_{\mathbf{a}} P^f = \operatorname{ndeg}_{\mathbf{a}} P$ *for* $f \in K^\times$;
(iii) $\operatorname{ndeg}_{\mathbf{a}} P_{+y} = \operatorname{ndeg}_{\mathbf{a}+y} P$ *for* $y \in K$;
(iv) *if $y \in K$ and $vy$ is in the width of $(a_\rho)$, then* $\operatorname{ndeg}_{\mathbf{a}} P_{+y} = \operatorname{ndeg}_{\mathbf{a}} P$;
(v) $\operatorname{ndeg}_{\mathbf{a}} P_{\times y} = \operatorname{ndeg}_{\mathbf{a}\cdot y} P$ *for* $y \in K^\times$;
(vi) *if $Q \in K\{Y\}^{\neq}$, then* $\operatorname{ndeg}_{\mathbf{a}} PQ = \operatorname{ndeg}_{\mathbf{a}} P + \operatorname{ndeg}_{\mathbf{a}} Q$;
(vii) *if $P(\ell) = 0$ for some pseudolimit $\ell$ of $(a_\rho)$ in a strict extension of $K$, then* $\operatorname{ndeg}_{\mathbf{a}} P \geqslant 1$;

*Proof.* Most of these items are routine or follow easily from earlier facts. Item (iv) follows from (iii), and (vii) from Lemma 2.8. □

## 3. *Flexibility*

We assume in this section about our valued differential field $K$ that

$$\Gamma \neq \{0\}, \quad \partial \neq 0.$$

After the first three lemmas we introduce the useful condition of *flexibility*, which plays a key role in the rest of the story.

LEMMA 3.1. *Let $P \in K\{Y\}^{\neq}$ be such that $\deg P \geqslant 1$. Suppose that $\partial$ is small and the derivation of $\operatorname{res}(K)$ is nontrivial. Then the set*

$$\big\{vP(y) : \ y \in K,\ P(y) \neq 0\big\} \subseteq \Gamma$$

*is coinitial in* $\Gamma$.

*Proof.* Given $Q \in K\{Y\}$, the gaussian valuation $v(Q_{\times f})$ of $Q_{\times f}$ for $f \in K$ depends only on $v(f)$ by [**2**, Lemma 4.5.1(ii)], and so we obtain a function $v_Q \colon \Gamma_\infty \to \Gamma_\infty$ with

$v_Q(vf) = v(Q_{\times f})$ for $f \in K$. We have $v_P(\gamma) = \min_d v_{P_d}(\gamma) \in \Gamma$ for $\gamma \in \Gamma$, and by [**2**, Corollary 6.1.3], $v_{P_d}(\gamma) = v(P_d) + d\gamma + o(\gamma)$ if $P_d \neq 0$ and $\gamma \in \Gamma^{\neq}$. Using also $\deg P \geqslant 1$, it follows that $v_P(\Gamma)$ is a coinitial subset of $\Gamma$. By [**2**, Lemma 4.5.2], there is for each $\beta \in v_P(\Gamma)$ a $y \in K$ with $vP(y) = \beta$. □

LEMMA 3.2. *Let $P \in K\{Y\}^{\neq}$ be such that $\deg P \geqslant 1$. Then the set*

$$\{vP(y) : \ y \in K\} \subseteq \Gamma_\infty$$

*is infinite.*

*Proof.* By compositional conjugation we arrange that $\partial$ is small. Take an elementary extension $L$ of $K$ such that $\Gamma_L$ contains an element $> \Gamma$. Let $\Delta$ be the convex hull of $\Gamma$ in $\Gamma_L$, and let $L_\Delta$ be the $\Delta$-coarsening of $L$ with valuation $\dot{v}$ and (nontrivial) value group $\dot{\Gamma}_L = \Gamma_L/\Delta$. By Lemma 1.17, the derivation of $L_\Delta$ remains small, and since $\partial \neq 0$, the derivation of $\operatorname{res}(L_\Delta)$ is nontrivial. So by the preceding lemma, the set $\{\dot{v}P(y) : \ y \in L, \ P(y) \neq 0\}$ is coinitial in $\dot{\Gamma}_L$. Hence the set $\{vP(y) : \ y \in L, \ P(y) \neq 0\}$ is coinitial in $\Gamma_L$. Thus $\{vP(y) : \ y \in K, \ P(y) \neq 0\}$ is coinitial in $\Gamma$, and hence infinite. □

LEMMA 3.3. *Suppose that $\Gamma^>$ has no least element and $S(\partial) = \{0\}$. Let $P \in K\{Y\}^{\neq}$ be such that* $\operatorname{ndeg} P \geqslant 1$*, and let $\beta \in \Gamma^>$. Then the set*

$$\{vP(y) : \ y \in K, \ |vy| < \beta\} \subseteq \Gamma_\infty$$

*is infinite.*

*Proof.* Let $\gamma \in \Gamma(\partial)$ and $\delta \in \Gamma \setminus \Gamma(\partial)$; then there are $a, g \in K$ such that

$$a \prec 1, \quad vg = \gamma, \quad 0 < v(g^{-1}a') \leqslant \delta - \gamma.$$

To see this, take $a, d \in K$ such that $a \prec 1$, $vd = \delta$, and $d^{-1}a' \succcurlyeq 1$. Take $g \in K$ with $vg = \gamma$. Then $a' \succcurlyeq d$, and so $g^{-1}a' \succcurlyeq g^{-1}d$. It remains to note that $g^{-1}a' \prec 1$.

This fact and Lemma 1.16 yield an elementary extension $L$ of $K$, with elements $\phi \in L^\times$ and $a \in \mathcal{o}_L$ such that $v\phi \in \Gamma_L(\partial_L)$, $v\phi \geqslant \Gamma(\partial)$ and $0 < v(\phi^{-1}a') < \Gamma^>$. Let $\Delta$ be the convex subgroup of $\Gamma_L$ consisting of the $\varepsilon \in \Gamma_L$ with $|\varepsilon| < \Gamma^>$. Then $\operatorname{res}(L^\phi_\Delta)$ has nontrivial derivation with value group $\Delta \neq \{0\}$. Take a nonzero $f \in L$ such that $f^{-1}P^\phi \asymp 1$ in $L^\phi\{Y\}$. Let $P_\Delta \in \operatorname{res}(L^\phi_\Delta)\{Y\}$ be the image of $f^{-1}P^\phi \in \mathcal{O}_{L^\phi}\{Y\}$ under the natural map $\mathcal{O}_{L^\phi}\{Y\} \to \operatorname{res}(L^\phi_\Delta)\{Y\}$. From $\operatorname{ndeg} P \geqslant 1$ it follows that $\deg P_\Delta \geqslant 1$. Now apply Lemma 3.2 to $\operatorname{res}(L^\phi_\Delta)$ and $P_\Delta$ in the role of $K$ and $P$. □

Recall that $a^\sim$ is the equivalence class of $a \in K^\times$ with respect to the equivalence relation $\sim$ on $K^\times$. We define $K$ to be *flexible* if $\Gamma^>$ has no least element and for all $P \in K\{Y\}^{\neq}$ with $\operatorname{ndeg} P \geqslant 1$ and all $\beta \in \Gamma^>$ the set

$$\{P(y)^\sim : \ y \in K, \ |vy| < \beta, \ P(y) \neq 0\}$$

is infinite. Flexibility is an elementary condition on valued differential fields, in the sense of being expressible by a set of sentences in the natural first-order language for these structures. Flexibility is invariant under compositional conjugation. By Lemma 3.3 we have:

COROLLARY 3.4. *If $\Gamma^>$ has no least element and $S(\partial) = \{0\}$, then $K$ is flexible.*

Combined with earlier results on $S(\partial)$ this gives large classes of valued differential fields that are flexible. For example, if $\Gamma^>$ has no least element, then $K$ is flexible whenever $K$ is asymptotic or $\Gamma$ is archimedean. If $\Gamma^{>}$ has no least element and $K$ is flexible, does it follow that $S(\partial) = \{0\}$? We do not know.

REMARK. In [**1**, p. 292] we defined a less 'flexible' notion of flexibility. We stated there as Theorem 4.1, without proof, that every real closed $H$-field has a spherically complete immediate $H$-field extension, and mentioned that we used flexibility in handling the case where the real closed $H$-field has no asymptotic integration. It turned out that for that case 'Theorem 4.1' was not needed in [**2**], and so it was not included there. As we saw in the introduction, [**1**, Theorem 4.1] is now available, even without the *real closed* assumption, as a special case of the main theorem of the present paper.

## 4. *Lemmas on flexible valued differential fields*

*In this section we assume about $K$ that $\partial \neq 0$, $\Gamma \neq \{0\}$, and $\Gamma^{>}$ has no least element.* (Flexibility is only assumed in Lemmas 4.4 and 4.5.) We let $a$, $b$, $y$ range over $K$ and $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{d}$, $\mathfrak{v}$, $\mathfrak{w}$ over $K^{\times}$. Also, $P$ and $Q$ range over $K\{Y\}^{\neq}$.

Using strict extensions and flexibility we now adapt the subsection 'Vanishing' of [**2**, Section 11.4] to our more general setting.

Let $\ell$ be an element in an extension $L$ of $K$ such that $\ell \notin K$ and

$$v(K-\ell) := \big\{v(a-\ell) :\ a \in K\big\}$$

has no largest element. Recall that then $\ell$ is a pseudolimit of a divergent pc-sequence in $K$ and $v(K-\ell) \subseteq \Gamma$.

We say that $P$ *vanishes at* $(K, \ell)$ if for all $a$ and $\mathfrak{v}$ with $a - \ell \prec \mathfrak{v}$ we have $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} \geqslant 1$, that is $\operatorname{ndeg} P_{+a,\times b} \geqslant 1$ for some $b \prec \mathfrak{v}$. By Lemma 2.8, if $L$ is an immediate strict extension of $K$ and $P(\ell) = 0$, then $\operatorname{ndeg} P_{+a,\times b} \geqslant 1$ whenever $\ell - a \preccurlyeq b$, hence $P$ vanishes at $(K, \ell)$. Let $Z(K, \ell)$ be the set of all $P$ that vanish at $(K, \ell)$. Here are some frequently used basic facts:

(1) $P \in Z(K,\ell) \Longleftrightarrow P_{+b} \in Z(K, \ell - b)$;
(2) $P \in Z(K,\ell) \Longleftrightarrow P_{\times\mathfrak{m}} \in Z(K, \ell/\mathfrak{m})$;
(3) $P \in Z(K,\ell) \Longrightarrow PQ \in Z(K,\ell)$ for all $Q$;
(4) $P \in K^{\times} \Longrightarrow P \notin Z(K,\ell)$.

Moreover, if $P \notin Z(K,\ell)$, we have $a$, $\mathfrak{v}$ with $a - \ell \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} = 0$, and then also $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+b} = 0$ for any $b$ with $b - \ell \prec \mathfrak{v}$, by Lemma 2.12. (In general, $Z(K,\ell) \cup \{0\}$ is not closed under addition, see the remark following the proof of Corollary 4.6 below.)

LEMMA 4.1. $Y - b \notin Z(K, \ell)$.

*Proof.* Take $a$ and $\mathfrak{v}$ such that $a - \ell \prec \mathfrak{v} \asymp b - \ell$. Then for $P := Y - b$ and $\mathfrak{m} \prec \mathfrak{v}$ we have $P_{+a,\times\mathfrak{m}} = \mathfrak{m}Y + (a-b)$ and $\mathfrak{m} \prec a - b$, so $D_{P_{+a,\times\mathfrak{m}}} \in \operatorname{res}(K)^{\times}$. It follows that $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} = 0$. ☐

LEMMA 4.2. *Suppose $P \notin Z(K,\ell)$, and let $a$, $\mathfrak{v}$ be such that $a - \ell \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} = 0$. Then $P(f) \sim P(a)$ for all $f$ in all strict extensions of $K$ with $f - a \asymp \mathfrak{m} \prec \mathfrak{v}$ for some $\mathfrak{m}$.* (*Recall: $\mathfrak{m} \in K^{\times}$ by convention.*)

*Proof.* Let $f$ in a strict extension $E$ of $K$ satisfy $f - a \asymp \mathfrak{m} \prec \mathfrak{v}$, so $f = a + \mathfrak{m}u$ with $u \asymp 1$ in $E$. Now

$$P_{+a,\times\mathfrak{m}} = P(a) + R \quad \text{ with } R \in K\{Y\},\ R(0) = 0,$$

so for $\phi \in K^\times$ we have

$$P^{\phi}_{+a,\times\mathfrak{m}} = P(a) + R^{\phi}.$$

From $\operatorname{ndeg} P_{+a,\times\mathfrak{m}} = 0$, we get $\phi \in K^\times$ with $\partial\mathcal{O} \subseteq \phi\mathcal{O}$ and $R^{\phi} \prec P(a)$. Thus

$$P(f) = P_{+a,\times\mathfrak{m}}(u) = P^{\phi}_{+a,\times\mathfrak{m}}(u) = P(a) + R^{\phi}(u) \quad \text{in } E^{\phi},$$

with $R^{\phi}(u) \preccurlyeq R^{\phi} \prec P(a)$ in $E^{\phi}$, so $P(f) \sim P(a)$. $\square$

Suppose that $L$ is a strict extension of $K$. Then the conclusion applies to $f = \ell$, and so for $P$ and $a$, $\mathfrak{v}$ as in the lemma we have $P(\ell) \sim P(a)$, hence $P(\ell) \neq 0$. Thus for $P$, $a$, $\mathfrak{v}$ as in the lemma we have $P(f) \sim P(a) \sim P(\ell)$ for all $f \in K$ with $f - \ell \prec \mathfrak{v}$.

LEMMA 4.3. *Suppose $P, Q \notin Z(K, \ell)$. Then $PQ \notin Z(K, \ell)$.*

*Proof.* Take $a$, $b$, $\mathfrak{v}$, $\mathfrak{w}$ such that $a - \ell \prec \mathfrak{v}$, $b - \ell \prec \mathfrak{w}$ and

$$\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} = \operatorname{ndeg}_{\prec\mathfrak{w}} Q_{+b} = 0.$$

We can assume $a - \ell \preccurlyeq b - \ell$. Take $\mathfrak{n} \asymp a - \ell$ and $d \in K$ with $d - \ell \prec \mathfrak{n}$. Then $d - \ell \prec \mathfrak{v}$ and $d - \ell \prec \mathfrak{w}$, so $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} = \operatorname{ndeg}_{\prec\mathfrak{v}} P_{+d} = 0$, and so $\operatorname{ndeg}_{\prec\mathfrak{n}} P_{+d} = 0$. Likewise, $\operatorname{ndeg}_{\prec\mathfrak{n}} Q_{+d} = 0$, so $\operatorname{ndeg}_{\prec\mathfrak{n}} (PQ)_{+d} = 0$. $\square$

LEMMA 4.4. *Assume that $K$ is flexible. Let $P \in Z(K, \ell)$, and let any $b$ be given. Then there exists an $a$ such that $a - \ell \prec b - \ell$ and $P(a) \neq 0$, $P(a) \not\sim P(b)$.*

*Proof.* Take $\mathfrak{v} \asymp b - \ell$ and $a_1 \in K$ with $a_1 - \ell \prec \mathfrak{v}$, so $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a_1} \geqslant 1$, which gives $\mathfrak{m} \prec \mathfrak{v}$ with $\operatorname{ndeg} P_{+a_1,\times\mathfrak{m}} \geqslant 1$. By flexibility of $K$, the set

$$\{P(a_1 + \mathfrak{m}y)^{\sim} : \ |vy| < \beta, \ P(a_1 + \mathfrak{m}y) \neq 0\}$$

is infinite, for each $\beta \in \Gamma^{>}$, so we can take $y$ such that $a_1 + \mathfrak{m}y - \ell \prec \mathfrak{v}$ and $0 \neq P(a_1 + \mathfrak{m}y) \not\sim P(b)$. Then $a := a_1 + \mathfrak{m}y$ has the desired property. $\square$

LEMMA 4.5. *Assume that $K$ is flexible, $L$ is a strict extension of $K$, $P, Q \notin Z(K, \ell)$ and $P - Q \in Z(K, \ell)$. Then $P(\ell) \sim Q(\ell)$.*

*Proof.* By Lemma 4.3 we have $b$ and $\mathfrak{v}$ such that

$$\ell - b \prec \mathfrak{v}, \quad \operatorname{ndeg}_{\prec\mathfrak{v}} P_{+b} = \operatorname{ndeg}_{\prec\mathfrak{v}} Q_{+b} = 0.$$

Replacing $\ell$ by $\ell - b$ and $P, Q$ by $P_{+b}, Q_{+b}$ we arrange $b = 0$, that is,

$$\ell \prec \mathfrak{v}, \quad \operatorname{ndeg}_{\prec\mathfrak{v}} P = \operatorname{ndeg}_{\prec\mathfrak{v}} Q = 0,$$

so $P(0) \neq 0$ and $Q(0) \neq 0$. If $a \prec \mathfrak{v}$, then by the remark preceding Lemma 4.3,

$$P(a) \sim P(0) \sim P(\ell), \quad Q(a) \sim Q(0) \sim Q(\ell).$$

If $P(\ell) \not\sim Q(\ell)$, then $P(0) \not\sim Q(0)$, so $(P - Q)(a) \sim (P - Q)(0)$ for all $a \prec \mathfrak{v}$, contradicting $P - Q \in Z(K, \ell)$ by Lemma 4.4. Thus $P(\ell) \sim Q(\ell)$. $\square$

### *Relation to the Newton degree in a cut*

Let $(a_\rho)$ be a divergent pc-sequence in $K$ with pseudolimit $\ell$. The following generalizes [**2**, Lemma 11.4.11], with the same proof except for using Lemma 4.2 instead of [**2**, Lemma 11.4.3].

COROLLARY 4.6. *If $P(a_\rho) \rightsquigarrow 0$, then $P \in Z(K,\ell)$.*

*Proof.* Suppose $P \notin Z(K,\ell)$. Take $a$ and $\mathfrak{v}$ such that $a-\ell \prec \mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} = 0$. Now $v(a-a_\rho) = v(a-\ell)$, eventually, so by Lemma 4.2 we have $P(a_\rho) \sim P(a)$ eventually, so $v\big(P(a_\rho)\big) = v\big(P(a)\big) \neq \infty$ eventually. □

In particular, if $P(a_\rho) \rightsquigarrow 0$, then $P(Y)+\varepsilon \in Z(K,\ell)$ for all $\varepsilon \in K$ such that $\varepsilon \prec P(a_\rho)$ eventually. We now connect the notion of $P$ vanishing at $(K,\ell)$ with the Newton degree $\operatorname{ndeg}_{\mathbf{a}} P$ of $P$ in the cut $\mathbf{a} = c_K(a_\rho)$ (introduced in the last subsection of Section 2) generalizing [**2**, Lemma 11.4.12]. The proof is the same, except for using Lemma 2.12 and Corollary 2.13 above instead of [**2**, Lemma 11.2.7]:

COROLLARY 4.7. $\operatorname{ndeg}_{\mathbf{a}} P \geqslant 1 \iff P \in Z(K,\ell)$. *More precisely,*

$$\operatorname{ndeg}_{\mathbf{a}} P = \min\big\{\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} : a-\ell \prec \mathfrak{v}\big\}.$$

*Proof.* We may assume $v(\ell-a_\rho)$ is strictly increasing with $\rho$. Given any index $\rho$, take $\mathfrak{v} \asymp \ell - a_\rho$, take $\rho' > \rho$, and set $a := a_{\rho'}$. Then $a-\ell \prec \mathfrak{v}$. Now $\gamma_\rho := v(\ell-a_\rho) = v(\mathfrak{v}) = v(a-a_\rho)$, and thus (using Corollary 2.13 for the last inequality):

$$\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} \leqslant \operatorname{ndeg}_{\preccurlyeq\mathfrak{v}} P_{+a} = \operatorname{ndeg}_{\geqslant\gamma_\rho} P_{+a} \leqslant \operatorname{ndeg}_{\geqslant\gamma_\rho} P_{+a_\rho}.$$

It follows that $\min\big\{\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} : a-\ell \prec \mathfrak{v}\big\} \leqslant \operatorname{ndeg}_{\mathbf{a}} P$. For the reverse inequality, let $a$ and $\mathfrak{v}$ be such that $a-\ell \prec \mathfrak{v}$. Let $\rho$ be such that $\ell - a_\rho \preccurlyeq \ell - a$. Then $a_\rho - a \prec \mathfrak{v}$ and $\gamma_\rho = v(\ell-a_\rho) > v(\mathfrak{v})$, so by Lemma 2.12:

$$\operatorname{ndeg}_{\geqslant\gamma_\rho} P_{+a_\rho} \leqslant \operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a_\rho} = \operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a}.$$

Therefore $\operatorname{ndeg}_{\mathbf{a}} P \leqslant \min\big\{\operatorname{ndeg}_{\prec\mathfrak{v}} P_{+a} : a-\ell \prec \mathfrak{v}\big\}$. □

## 5. *Constructing immediate extensions*

Our goal in this section is to establish the following:

THEOREM 5.1. *Suppose $\partial \neq 0$, $\Gamma \neq \{0\}$, $\Gamma^{>}$ has no least element, and $S(\partial) = \{0\}$. Then $K$ has the Krull property.*

Much of this section is very similar to the subsection 'Constructing immediate extensions' of [**2**, Section 11.4], but there are some differences that make it convenient to give all details. In the next section we show how to derive our main theorem from Theorem 5.1 by constructions involving coarsening by $S(\partial)$.

*In the rest of this section we assume about $K$ that $\partial \neq 0$, $\Gamma \neq \{0\}$, and $\Gamma^{>}$ has no least element.* We also keep the notational conventions of the previous section, and assume that $\ell$ is an element of a *strict* extension $L$ of $K$.

LEMMA 5.2. *Suppose $Z(K,\ell) = \emptyset$. Then $P(\ell) \neq 0$ for all $P$, and $K\langle\ell\rangle$ is an immediate strict extension of $K$. Suppose also that $M$ is a strict extension of $K$ and $g \in M$ satisfies $v(a-g) = v(a-\ell)$ for all $a$. Then there is a unique valued differential field embedding $K\langle\ell\rangle \to M$ over $K$ that sends $\ell$ to $g$.*

*Proof.* Clearly $P(\ell) \neq 0$ for all $P$. Let any nonzero element $f = P(\ell)/Q(\ell)$ of the extension $K\langle\ell\rangle$ of $K$ be given. Lemma 4.3 gives $a$ and $\mathfrak{v}$ such that

$$a-\ell \prec \mathfrak{v}, \quad \operatorname{ddeg}_{\prec\mathfrak{v}} P_{+a} = \operatorname{ddeg}_{\prec\mathfrak{v}} Q_{+a} = 0,$$

and so $P(\ell) \sim P(a)$ and $Q(\ell) \sim Q(a)$ by Lemma 4.2, and thus $f \sim P(a)/Q(a)$. It follows that $K\langle \ell \rangle$ is an immediate extension of $K$.

It is clear that $Z(K,g) = Z(K,\ell) = \emptyset$, so $g$ is differentially transcendental over $K$ and $K\langle g \rangle$ is an immediate extension of $K$, by the first part of the proof. Given any $P$ we take $a$ and $\mathfrak{v}$ such that $a - \ell \prec \mathfrak{v}$ and $\operatorname{ddeg}_{\prec \mathfrak{v}} P_{+a} = 0$. Then $P(a) \sim P(g)$ and $P(a) \sim P(\ell)$, and thus $vP(g) = vP(\ell)$. Hence the unique differential field embedding $K\langle \ell \rangle \to M$ over $K$ that sends $\ell$ to $g$ is also a valued field embedding. $\square$

LEMMA 5.3. *Suppose that $K$ is flexible, $Z(K,\ell) \neq \emptyset$, and $P$ is an element of $Z(K,\ell)$ of minimal complexity. Then $K$ has an immediate strict extension $K\langle f \rangle$ such that $P(f) = 0$ and $v(a - f) = v(a - \ell)$ for all $a$, and such that if $M$ is any strict extension of $K$ and $s \in M$ satisfies $P(s) = 0$ and $v(a - s) = v(a - \ell)$ for all $a$, then there is a unique valued differential field embedding $K\langle f \rangle \to M$ over $K$ that sends $f$ to $s$.*

*Proof.* Let $P$ have order $r$ and take $p \in K[Y_0, \dots, Y_r]$ such that

$$P = p(Y, Y', \dots, Y^{(r)}).$$

Then $p$ is irreducible by $P$ having minimal complexity in $Z(K,\ell)$ and Lemma 4.3. Thus we have an integral domain

$$K[y_0, \dots, y_r] = K[Y_0, \dots, Y_r]/(p), \quad y_i = Y_i + (p) \text{ for } i = 0, \dots, r,$$

with fraction field $K(y_0, \dots, y_r) = K(y_0, \dots, y_{r-1})[y_r]$ where $y_0, \dots, y_{r-1}$ are algebraically independent over $K$. Let $s \in K(y_0, \dots, y_r)^\times$, so

$$s = g(y_0, \dots, y_r)/h(y_0, \dots, y_{r-1})$$

where $g \in K[Y_0, \dots, Y_r]^{\neq}$, $h \in K[Y_0, \dots, Y_{r-1}]^{\neq}$, and $g(Y, Y', \dots, Y^{(r)}) \notin Z(K,\ell)$. (This non-membership in $Z(K,\ell)$ can be arranged by taking $g$ of lower degree in $Y_r$ than $p$.) The comment following the proof of Lemma 4.2 gives an $a$ such that

$$g(\ell, \ell', \dots, \ell^{(r)}) \sim g(a, a', \dots, a^{(r)}), \quad h(\ell, \dots, \ell^{(r-1)}) \sim h(a, \dots, a^{(r-1)}),$$

so $vg(\ell, \ell', \dots, \ell^{(r)}),\ vh(\ell, \dots, \ell^{(r-1)}) \in \Gamma$. We claim that

$$vg(\ell, \ell', \dots, \ell^{(r)}) - vh(\ell, \dots, \ell^{(r-1)})$$

depends only on $s$ and not on the choice of $g$ and $h$. To see this, let $g_1 \in K[Y_0, \dots, Y_r]$, $h_1 \in K[Y_0, \dots, Y_{r-1}]$ be such that $g_1(Y, \dots, Y^{(r)}) \notin Z(K,\ell)$, $h_1 \neq 0$, and $s = g_1(y_0, \dots, y_r)/h_1(y_0, \dots, y_{r-1})$. Then

$$gh_1 - g_1 h \in pK[Y_0, \dots, Y_r], \quad (gh_1)(Y, \dots, Y^{(r)}), (g_1 h)(Y, \dots, Y^{(r)}) \notin Z(K,\ell),$$

which yields the claim by Lemma 4.5. We now set, for $g$, $h$ as above,

$$vs := vg(\ell, \ell', \dots, \ell^{(r)}) - vh(\ell, \dots, \ell^{(r-1)}),$$

or more suggestively,

$$vs = v\big(G(\ell)/H(\ell)\big) \in \Gamma, \quad \text{with } G = g(Y, \dots, Y^{(r)}),\ H = h(Y, \dots, Y^{(r-1)}).$$

We thus have extended $v\colon K^\times \to \Gamma$ to a map

$$v\colon K(y_0, \dots, y_r)^\times \to \Gamma.$$

Let $s \in K(y_0, \dots, y_r)^\times$ and take $g \in K[Y_0, \dots, Y_r]$, $h \in K[Y_0, \dots, Y_{r-1}]$ with $g(Y, Y', \dots, Y^{(r)}) \notin Z(K,\ell)$ and $h \neq 0$ such that $s = g(y_0, \dots, y_r)/h(y_0, \dots, y_{r-1})$. Let $s_1, s_2 \in K(y_0, \dots, y_r)^\times$. Then $v(s_1 s_2) = vs_1 + vs_2$ follows easily by means of Lemma 4.3. Next, assume also $s_1 + s_2 \neq 0$; to prove that $v\colon K(y_0, \dots, y_r)^\times \to \Gamma$ is a valuation it remains to show that then $v(s_1+s_2)\geqslant \min(vs_1, vs_2)$. For $i=1,2$ we have $s_i = g_i(y_0,\dots,y_r)/h_i(y_0,\dots,y_{r-1})$ where

$$0\neq g_i\in K[Y_0,\dots,Y_r],\quad 0\neq h_i\in K[Y_0,\dots,Y_{r-1}],$$

and $g_i$ has lower degree in $Y_r$ than $p$. Then for $s:=s_1+s_2$ we have

$$s=g(y_0,\dots,y_r)/h(y_0,\dots,y_{r-1}),\quad g:=g_1h_2+g_2h_1,\quad h=h_1h_2,$$

and so $g\neq 0$ (because $s\neq 0$) and $g$ has also lower degree in $Y_r$ than $p$. In particular, $g(Y,\dots,Y^{(r)})\notin Z(K,\ell)$, hence $vs = v\big(g(\ell,\dots,\ell^{(r)})/h(\ell,\dots,\ell^{(r-1)})\big)$, and so by working in the valued field $K\langle\ell\rangle$ we see that $vs\geqslant\min(vs_1,vs_2)$, as promised. Thus we now have $K(y_0,\dots,y_r)$ as a valued field extension of $K$. To show that $K(y_0,\dots,y_r)$ has the same residue field as $K$, consider an element $s=g(y_0,\dots,y_r)\notin K$ with nonzero $g\in K[Y_0,\dots,Y_r]$ of lower degree in $Y_r$ than $p$; it suffices to show that $s\sim b$ for some $b$. Set $G:=g(Y,\dots,Y^{(r)})$ and take $a$ and $\mathfrak{v}$ with $a-\ell\prec\mathfrak{v}$ and $\operatorname{ndeg}_{\prec\mathfrak{v}} G_{+a}=0$. Then $G(\ell)\sim G(a)$ by Lemma 4.2, so for $b:=G(a)$ we have $v(s-b)=v\big(g(y_0,\dots,y_r)-b\big)=v(G(\ell)-b)>vb$, that is, $s\sim b$. This finishes the proof that the valued field $F:=K(y_0,\dots,y_r)$ is an immediate extension of $K$.

Next we equip $F$ with the derivation extending the derivation of $K$ such that $y_i'=y_{i+1}$ for $0\leqslant i<r$. Setting $f:=y_0$ we have $f^{(i)}=y_i$ for $i=0,\dots,r$, $F=K\langle f\rangle=K(y_0,\dots,y_r)$, and $P(f)=0$. Note that $v(G(f))=v(G(\ell))$ for every nonzero $G\in K[Y,\dots,Y^{(r)}]$ of lower degree in $Y^{(r)}$ than $P$, in particular, $v(f-a)=v(\ell-a)$ for all $a$. We now show that the derivation of $F$ is continuous and that $F$ is a strict extension of $K$.

Let $\phi\in K^\times$ and $v\phi\in\Gamma(\partial)$. To get $\partial\smallo_F\subseteq\phi\smallo_F$, we set

$$S:=\big\{H(f):\ H\in K\big[Y,\dots,Y^{(r-1)}\big],\ H(f)\preccurlyeq 1\big\}.$$

(If $r=0$, then we have $K\big[Y,\dots,Y^{(r-1)}\big]=K$, so $S=\mathcal{O}$.) By Lemma 1.5 and by [**2**, Lemma 6.2.3] applied to $K\big(f,\dots,f^{(r-1)}\big)$ in the role of $E$ and with $F=L$, it is enough to show that $\partial S\subseteq\phi\mathcal{O}_F$ and $\partial(S\cap\smallo_F)\subseteq\phi\smallo_F$. We prove the first of these inclusions. The second follows in the same way.

Let $H\in K\big[Y,\dots,Y^{(r-1)}\big]\setminus K$ with $H(f)\preccurlyeq 1$; we have to show $H(f)'\preccurlyeq\phi$. We can assume $H(f)'\neq 0$. Take $H_1(Y),H_2(Y)\in K\big[Y,\dots,Y^{(r-1)}\big]$ such that

$$H'=H(Y)'=H_1(Y)+H_2(Y)Y^{(r)}\quad\text{in } K\{Y\}.$$

Then

$$H'(f)=H(f)'=H_1(f)+H_2(f)f^{(r)},$$

and for all $a$,

$$H'(a)=H(a)'=H_1(a)+H_2(a)a^{(r)}.$$

We now distinguish two cases:

*Case* 1: *$P$ has degree $>1$ in $Y^{(r)}$, or $H_2=0$.* Then $H'$ has lower degree in $Y^{(r)}$ than $P$, so we can take $a,\mathfrak{v}$ with $a-\ell\prec\mathfrak{v}$, $\operatorname{ndeg}_{\prec\mathfrak{v}} H_{+a}=0$, and $\operatorname{ndeg}_{\prec\mathfrak{v}} H'_{+a}=0$, so $H(a)\sim H(f)\preccurlyeq 1$ and $H'(a)\sim H'(f)$. Hence $H(f)'\sim H(a)'\preccurlyeq\phi$.

*Case* 2: *$P$ has degree 1 in $Y^{(r)}$ and $H_2\neq 0$.* Then

$$H'=\frac{G_1P+G_2}{G},\quad G_1,G_2,G\in K[Y,\dots,Y^{(r-1)}],\ G_1,G\neq 0,$$

so $0\neq H(f)'=G_2(f)/G(f)$, so $G_2\neq 0$. By Lemma 4.3 there is a $\mathfrak{v}$ such that for some $a$ we have $a-\ell\prec\mathfrak{v}$ and

$$\operatorname{ndeg}_{\prec\mathfrak{v}} H_{+a}=\operatorname{ndeg}_{\prec\mathfrak{v}}(G_1)_{+a}=\operatorname{ndeg}_{\prec\mathfrak{v}}(G_2)_{+a}=\operatorname{ndeg}_{\prec\mathfrak{v}} G_{+a}=0.$$

Fix such $\mathfrak{v}$, and let $A \subseteq K$ be the set of all $a$ satisfying the above. Then for $a \in A$ we have $G(f) \sim G(a)$ and $H(f) \sim H(a)$, so $H(a)' \preccurlyeq \phi$. Also $G_1(f) \sim G_1(a)$ and

$$G(f)H(f)' = G_2(f) \sim G_2(a),$$
$$G(a)H(a)' = G_1(a)P(a) + G_2(a).$$

We now make crucial use of Lemma 4.4 to arrange that

$$v\big(G_1(a)P(a) + G_2(a)\big) = \min\big(v(G_1(a)P(a)),\ v(G_2(a)\big)$$

by changing $a$ if necessary. Hence $G_2(a) \preccurlyeq G_1(a)P(a) + G_2(a) = G(a)H(a)'$, so

$$G(f)H(f)' \sim G_2(a) \preccurlyeq G(a)H(a)' \sim G(f)H(a)' \preccurlyeq G(f)\phi$$

and thus $H(f)' \preccurlyeq \phi$. This concludes the proof that $F$ is a strict extension of $K$.

Suppose that $s$ in a strict extension $M$ of $K$ satisfies $P(s) = 0$ and $v(a - s) = v(a - \ell)$ for all $a$. By Lemma 4.2 and the remarks following its proof we have $vQ(s) = vQ(f)$ for all $Q \notin Z(K, \ell)$, in particular, $Q(s) \neq 0$ for all $Q$ of lower complexity than $P$. Thus we have a differential field embedding $K\langle f\rangle \to M$ over $K$ sending $f$ to $s$, and this is also a valued field embedding. □

*Proof of Theorem* 5.1. Assume $S(\partial) = \{0\}$; we show that $K$ has an immediate strict extension that is maximal as a valued field. We can assume that $K$ itself is not yet maximal, and it is enough to show that then $K$ has a proper immediate strict extension, since by Lemma 1.10 the property $S(\partial) = \{0\}$ is preserved by immediate strict extensions. As $K$ is not maximal, we have a divergent pc-sequence in $K$, which pseudoconverges in an elementary extension of $K$, and thus has a pseudolimit $\ell$ in a strict extension of $K$. If $Z(K, \ell) = \emptyset$, then Lemma 5.2 provides a proper immediate strict extension of $K$, and if $Z(K, \ell) \neq \emptyset$, then Lemma 5.3 provides such an extension. This concludes the proof of Theorem 5.1. □

## 6. *Coarsening and* $S(\partial)$

In this section we finish the proof of the main theorem stated in the introduction.

### *Making* $S(\partial)$ *vanish*

*In this subsection we set* $\Delta := S(\partial)$ *and assume* $\Delta \neq \{0\}$. Then $\Gamma(\partial)$ has no largest element, and so $v(\partial\mathcal{O}) > \Gamma(\partial)$ by Lemma 1.8. The next lemma says much more. Let $K_\Delta$ be the $\Delta$-coarsening, with valuation ring $\dot{\mathcal{O}}$.

LEMMA 6.1. $v(\partial\dot{\mathcal{O}}) > \Gamma(\partial)$.

*Proof.* Let $a \in \dot{\mathcal{O}}$. If $va \geqslant 0$, then $va' > \Gamma(\partial)$ by the above. If $va < 0$, then $va \in \Delta$ and $va' - 2va = v((1/a)') > \Gamma(\partial)$, so $va' > \Gamma(\partial) + 2va = \Gamma(\partial)$. □

It follows in particular that if $\partial$ is small, then the derivation of $\operatorname{res}(K_\Delta)$ is trivial. Let $\pi\colon \Gamma \to \dot{\Gamma} := \Gamma/\Delta$ be the canonical map, so $\pi\Gamma(\partial) \subseteq \dot{\Gamma}$. We also have $\dot{\Gamma}(\partial) := \dot{\Gamma}_{K_\Delta}(\partial) \subseteq \dot{\Gamma}$, with $\pi\Gamma(\partial) \subseteq \dot{\Gamma}(\partial)$ by Lemma 1.18.

LEMMA 6.2. $S_{K_\Delta}(\partial) = \{0\} \subseteq \dot{\Gamma}$.

*Proof.* If $\pi\Gamma(\partial) = \dot{\Gamma}(\partial)$, then clearly $S_{K_\Delta}(\partial) = \{0\}$. Suppose $\pi\Gamma(\partial) \neq \dot{\Gamma}(\partial)$. (We do not know if this can happen.) Then Lemma 1.18 tells us that $\dot{\Gamma}(\partial)$ has a largest element, and so $S_{K_\Delta}(\partial) = \{0\}$ by Lemma 1.15. □

*Lifting strictness*

Let $K$ have small derivation and let $L$ be an immediate extension of $K$ with small derivation. Let $\Delta$ be a convex subgroup of $\Gamma$, giving rise to the extension $L_\Delta$ of $K_\Delta$, both with value group $\dot{\Gamma} = \Gamma/\Delta$. Note that if $\phi \in K^\times$ and $v\phi \in \Gamma_K(\partial)$, then $\phi^{-1}\partial$ is small with respect to $v$, and thus small with respect to $\dot{v}$ by Lemma 1.17, so $\dot{v}\phi \in \dot{\Gamma}_{K_\Delta}(\partial)$. We show that under various assumptions strictness of $L_\Delta \supseteq K_\Delta$ yields strictness of $L \supseteq K$:

Lemma 6.3. *Suppose that $L_\Delta$ strictly extends $K_\Delta$ and* $\operatorname{res}(L_\Delta) = \operatorname{res}(K_\Delta)$. *Then $L$ strictly extends $K$.*

*Proof.* Let $\phi \in K^\times$, $v\phi \in \Gamma(\partial)$, and $0 \neq f \in \mathcal{o}_L$. Then $f = g(1+\varepsilon)$ with $g \in K^\times$ and $\dot{v}(\varepsilon) > 0$, so $vf = vg$ and $f' = g'(1+\varepsilon) + g\varepsilon'$. Now $v(g') > v(\phi)$. Since $L_\Delta$ strictly extends $K_\Delta$ we have $\dot{v}(\varepsilon') > \dot{v}(\phi)$, so $v(\varepsilon') > v(\phi)$. Hence $v(f') > v(\phi)$. □

Lemma 6.4. *Suppose that $L_\Delta$ strictly extends $K_\Delta$ and $\Delta = S(\partial) \neq \{0\}$. Then $L$ strictly extends $K$.*

*Proof.* Let $0 \neq f \in \mathcal{o}_L$. Then $f = gu$ with $g \in K$ and $v(u) = 0$, so $g \in \mathcal{o}$ and $f' = g'u + gu'$. We have $v(g'u) = v(g') > \Gamma(\partial)$. By Lemma 6.1 we have $\dot{v}(\partial\dot{\mathcal{O}}) > \dot{\gamma}$ for every $\gamma \in \Gamma(\partial)$. Since $L_\Delta$ strictly extends $K_\Delta$, this gives $\dot{v}(\partial\dot{\mathcal{O}}_L) > \dot{\gamma}$ for every $\gamma \in \Gamma(\partial)$, hence $v(\partial\dot{\mathcal{O}}_L) > \Gamma(\partial)$, and so $v(u') > \Gamma(\partial)$. This gives $v(f') > \Gamma(\partial)$. □

*Building strict extensions by extending the residue field*

Suppose that the derivation of $K$ is small. Let $f \in \mathcal{o}$, and let $a$ be an element in a field extension of $K$, transcendental over $K$. We extend the derivation of $K$ to the derivation on $K(a)$ such that $a' = f$. We equip $K(a)$ with the gaussian extension of the valuation of $K$ [**2**, Lemma 3.1.31]: The unique valuation on $K(a)$ extending the valuation of $K$ such that $a \preccurlyeq 1$ and $\operatorname{res} a$ is transcendental over $\operatorname{res}(K)$. So for $b = P(a)/Q(a) \in K(a)$ where $0 \neq P, Q \in K[Y]$, we have $vb = vP - vQ$; in particular, $\Gamma_{K(a)} = \Gamma$ and $\operatorname{res}\big(K(a)\big) = \operatorname{res}(K)(\operatorname{res} a)$.

Lemma 6.5. *The derivation of $K\langle a\rangle$ is small. If $\partial\mathcal{O} \subseteq \mathcal{o}$, then $\partial\mathcal{O}_{K(a)} \subseteq \mathcal{o}_{K(a)}$.*

*Proof.* Given $P = P_dY^d + \cdots + P_0 \in K[Y]$ (where $P_0, \ldots, P_d \in K$), we have $P(a)' = P_d'a^d + \cdots + P_0' + f \cdot (\partial P/\partial Y)(a)$, hence $P(a) \prec 1 \Rightarrow P(a)' \prec 1$, and $P(a) \preccurlyeq 1 \Rightarrow P(a)' \preccurlyeq 1$. Let $b \in \mathcal{o}_{K\langle a\rangle}$. Then $b = P(a)/Q(a)$ where $P, Q \in K[Y]$ and $P(a) \prec 1 \asymp Q(a)$, so $P(a)' \prec 1$ and $Q(a)' \preccurlyeq 1$, hence

$$b' = \frac{P(a)'Q(a) - P(a)Q(a)'}{Q(a)^2} \prec 1.$$

Thus $\partial\mathcal{o}_{K\langle a\rangle} \subseteq \mathcal{o}_{K\langle a\rangle}$. Similarly one shows that if $\partial\mathcal{O} \subseteq \mathcal{o}$, then $\partial\mathcal{O}_{K(a)} \subseteq \mathcal{o}_{K(a)}$. □

Lemma 6.6. *Suppose $vf > \Gamma(\partial)$. Then $L := K(a)$ is a strict extension of $K$.*

*Proof.* Let $\phi \in K^\times$ and $\partial\mathcal{o} \subseteq \phi\mathcal{o}$. Then the derivation of $K^\phi$ is small and $L^\phi = K^\phi(a)$ where $\phi^{-1}\partial(a) = \phi^{-1}f \prec 1$. Hence by the preceding lemma applied to $K^\phi$, $\phi^{-1}f$ instead of $K$, $f$, we have $\phi^{-1}\partial o_L \subseteq o_L$ and hence $\partial o_L \subseteq \phi o_L$. In the same way we show that if $\partial\mathcal{O} \subseteq \phi o$, then $\partial\mathcal{O}_L \subseteq \phi o_L$. $\square$

This leads to the following variant of [**2**, Corollary 6.3.3]:

Corollary 6.7. *Suppose $\partial\mathcal{O} \subseteq o$ and let $E$ be a field extension of* $\operatorname{res}(K)$*. Then there is a strict extension $L$ of $K$ such that $\Gamma_L = \Gamma$, the derivation of* $\operatorname{res}(L)$ *is trivial, and* $\operatorname{res}(L)$ *is, as a field, isomorphic to $E$ over* $\operatorname{res}(K)$*.*

*Proof.* We can reduce to the case $E = \operatorname{res}(K)(y)$. If $y$ is transcendental over $\operatorname{res}(K)$, then the corollary holds with $L = K(a)$ as defined above with $f = 0$, by Lemma 6.6. Next, suppose that $y$ is algebraic over $\operatorname{res}(K)$, with minimum polynomial $\overline{F}(Y) \in \operatorname{res}(K)[Y]$ over $\operatorname{res}(K)$. Take monic $F \in \mathcal{O}[Y]$ with image $\overline{F}$ in $\operatorname{res}(K)[Y]$. Then $F$ is irreducible in $K[Y]$. Take a field extension $L = K(a)$ of $K$ where $a$ is algebraic over $K$ with minimum polynomial $F$ over $K$. Then there is a unique valuation $v_L\colon L^\times \to \Gamma$ that extends the valuation of $K$; see [**2**, Lemma 3.1.35]. Then $L$ with this valuation and the unique derivation extending the derivation of $K$ has the desired property, by Lemma 1.4 and the remark following its proof. $\square$

For future reference we also state [**2**, Corollary 6.3.3] itself:

Lemma 6.8. *Let $E$ be a differential field extension of* $\operatorname{res}(K)$*. Then there is an extension $L$ of $K$ with small derivation having the same value group as $K$ and differential residue field isomorphic to $E$ over* $\operatorname{res}(K)$*.*

*Further generalities about coarsening*

In this subsection we suspend our convention that $K$ denotes a valued *differential* field, and just assume that it is a valued field, not necessarily of characteristic zero. Notations not involving $\partial$ keep their usual meaning; in particular, the valuation of $K$ is $v\colon K^\times \to \Gamma = v(K^\times)$. Let $\Delta$ be a convex subgroup of $\Gamma$. Then the coarsening $K_\Delta$ of $K$ by $\Delta$ is the valued field with the same underlying field as $K$, but with valuation $\dot{v} = v_\Delta\colon K^\times \to \dot{\Gamma} = \Gamma/\Delta$. The residue field $\operatorname{res}(K_\Delta)$ of $K_\Delta$ is turned into a valued field with value group $\Delta$ and residue field $\operatorname{res}(K)$ as described in the subsection on coarsening of Section 1. The following well-known fact is [**2**, Corollary 3.4.6], and is used several times below:

Lemma 6.9. *The valued field $K$ is spherically complete if and only if the valued fields $K_\Delta$ and* $\operatorname{res}(K_\Delta)$ *are spherically complete.*

Let $F$ be a valued field extension of $K_\Delta$ with value group $v_F(F^\times) = \Gamma/\Delta$. Let also $\operatorname{res}(F)$ be given a valuation $w\colon \operatorname{res}(F)^\times \to \Delta$ that extends the valuation $v\colon \operatorname{res}(K_\Delta)^\times \to \Delta$. Then we can extend $v\colon K^\times \to \Gamma$ to a map $v\colon F^\times \to \Gamma$ as follows. For $f \in F^\times$, take $g \in K^\times$ and $u \in F^\times$ such that $f = gu$ and $v_F(u) = 0$; then $\operatorname{res} u \in \operatorname{res}(F)^\times$, so $w(\operatorname{res} u) \in \Delta$; it is easy to check that $v(g) + w(\operatorname{res} u) \in \Gamma$ depends only on $f$ and not on the choice of $g$, $u$; now put $v(f) := v(g) + w(\operatorname{res} u)$.

Lemma 6.10. *$v\colon F^\times \to \Gamma$ is a valuation on $F$ with $\Delta$-coarsening $v_\Delta = v_F$.*

*Proof.* Clearly $v\colon F^\times \to \Gamma$ is a group morphism with $v_F(f) = v(f) + \Delta \in \Gamma/\Delta$ for $f \in F^\times$. Also, if $f \in F^\times$ and $v_F(f) > 0$, then $vf > 0$ and $v(1+f) = 0$. Next, for $f_1, f_2 \in F^\times$ with $f_1 + f_2 \neq 0$ one shows that $v(f_1 + f_2) \geqslant \min\{vf_1, vf_2\}$ by distinguishing the cases $v_F(f_1) = v_F(f_2)$ and $v_F(f_1) < v_F(f_2)$. $\square$

Let $L$ be the valued field extension of $K$ that has the same underlying field as $F$ and has valuation $v$ as above. Then the lemma above says that $L_\Delta = F$, and the valuation $w$ on $\operatorname{res}(F)$ equals the valuation $v\colon \operatorname{res}(L_\Delta)^\times \to \Delta$ induced by $v\colon L^\times \to \Gamma$ and $\Delta$. If $\operatorname{res}(L_\Delta)$ is an immediate extension of $\operatorname{res}(K_\Delta)$, then $L$ is an immediate extension of $K$. See the following diagram, where arrows like $\dashrightarrow$ indicate partial maps; for example, the residue map of $K_\Delta$ is defined only on $\dot{\mathcal{O}}$.

$$\begin{tikzcd}
K_\Delta \arrow[r, hook] \arrow[d, dashed] & F = L_\Delta \arrow[d, dashed] \\
\operatorname{res}(K_\Delta) \arrow[r, hook] \arrow[dr, dashed, "v"'] & \operatorname{res}(F) \arrow[d, dashed, "w"] \\
& \Delta
\end{tikzcd}$$

In the situation above, assume that $K$ is of characteristic zero and is equipped with a small derivation (with respect to $v$), and $F$ is equipped with a small derivation (with respect to $v_F$) that makes it a valued differential field extension of $K_\Delta$. Assume also that the induced derivation on $\operatorname{res}(F)$ is small with respect to $w$. Then the derivation of $F$ is small as a derivation of $L$ (with respect to the valuation $v$ of $L$).

*Putting it all together*

First one more special case of the main theorem:

PROPOSITION 6.11. *Suppose that $\partial$ is small and the derivation of* $\operatorname{res}(K)$ *is nontrivial. Then $K$ has the Krull property.*

In view of Lemma 1.3, this is just [**2**, Corollary 6.9.5]. We have not yet completely settled the case $S(\partial) = \{0\}$ of the main theorem, but we can now take care of this:

PROPOSITION 6.12. *Suppose $S(\partial) = \{0\}$ and $\Gamma^>$ has a least element. Then $K$ has the Krull property.*

*Proof.* Let $1$ denote the least element of $\Gamma^>$. We first note that $\Gamma(\partial)$ has a largest element: otherwise, $\Gamma(\partial)$ would be closed under adding $1$, and so $1 \in S(\partial)$, a contradiction. Thus by compositional conjugation we can arrange that $\Gamma(\partial) = \Gamma^{\leqslant}$, so the derivation of $K$ is small. We have the convex subgroup $\Delta := \mathbb{Z}1$ of $\Gamma$, so the valuation of the differential residue field $\operatorname{res}(K_\Delta)$ of the coarsening $K_\Delta$ is discrete. The completion $\operatorname{res}(K_\Delta)^{\mathrm{c}}$ of the valued field $\operatorname{res}(K_\Delta)$ is a spherically complete immediate extension of $\operatorname{res}(K_\Delta)$. Since the derivation of $K$ is small, so is that of $K_\Delta$ and hence that of $\operatorname{res}(K_\Delta)$. (See the remarks after Lemma 1.17.) The derivation of $\operatorname{res}(K_\Delta)$ is nontrivial: With $\phi \in K$ satisfying $v\phi = 1$ we have $\partial\smallo \not\subseteq \phi\smallo$, since $\Gamma(\partial) = \Gamma^{\leqslant}$, so we can take $g \in \smallo$ with $v(g') \leqslant v\phi = 1$, and then $v_\Delta(g) \geqslant 0 = v_\Delta(g')$. This derivation extends uniquely to a continuous derivation on $\operatorname{res}(K_\Delta)^{\mathrm{c}}$, and $\operatorname{res}(K_\Delta)^{\mathrm{c}}$ equipped with this derivation is a strict extension of the valued differential field $\operatorname{res}(K_\Delta)$.

$$\begin{tikzcd}
K_\Delta \arrow[r, hook] \arrow[d, dashed] & F = L_\Delta \arrow[d, dashed] \\
\operatorname{res}(K_\Delta) \arrow[r, hook] & \operatorname{res}(F) \cong \operatorname{res}(K_\Delta)^{\mathrm{c}}
\end{tikzcd}$$

By applying Lemma 6.8 to the differential field extension $\operatorname{res}(K_\Delta)^{\mathrm{c}} \supseteq \operatorname{res}(K_\Delta)$ we obtain an extension $F$ of $K_\Delta$ with small derivation, the same value group $v_F(F^\times) = \Gamma/\Delta$ as $K_\Delta$, and with differential residue field $\operatorname{res}(F)$ isomorphic to $\operatorname{res}(K_\Delta)^c$ over $\operatorname{res}(K_\Delta)$. Extending $F$ further using Proposition 6.11, if necessary, we arrange also that $F$ is spherically complete.

Next we equip $\operatorname{res}(F)$ with a valuation $w\colon \operatorname{res}(F)^\times \to \Delta$ that makes $\operatorname{res}(F)$ isomorphic as a valued differential field to $\operatorname{res}(K_\Delta)^c$ over $\operatorname{res}(K_\Delta)$. This places us in the situation of the previous subsection, and so we obtain an extension $L$ of $K$ with the same value group $\Gamma$ such that $L_\Delta = F$ (so $L$ and $F$ have the same underlying differential field), the valuation induced by $L$ and $\Delta$ on $\operatorname{res}(L_\Delta) = \operatorname{res}(F)$ equals $w$, and the derivation of $L$ is small. It follows easily that $L$ is an immediate extension of $K$. Since $F = L_\Delta$ and $\operatorname{res}(L_\Delta)$ are spherically complete, $L$ is spherically complete by Lemma 6.9. Since the derivation of $L$ is small and $\Gamma_K(\partial)$ has largest element 0, the extension $L$ of $K$ is strict, by Lemma 1.5. □

We can now finish the proof of our main theorem. We are given $K$ and have to show that $K$ has a spherically complete immediate strict extension. We already did this in several cases, and by Theorems 5.1 and Proposition 6.12 it only remains to consider the case $\Delta := S(\partial) \neq \{0\}$. We assume this below and also arrange by compositional conjugation that the derivation is small. By Lemma 6.1 we have $\partial\dot{\mathcal{O}} \subseteq \dot{o}$, and so the derivation of $\operatorname{res}(K_\Delta)$ is trivial. Take a spherically complete immediate valued field extension $E$ of the valued field $\operatorname{res}(K_\Delta)$. By Corollary 6.7 applied to $K_\Delta$ we obtain a strict extension $F$ of $K_\Delta$ with value group $v_F(F^\times) = \Gamma/\Delta$, the derivation of $\operatorname{res}(F)$ is trivial, and $\operatorname{res}(F)$, as a field, is isomorphic to $E$ over $\operatorname{res}(K_\Delta)$. We equip $\operatorname{res}(F)$ with a valuation $w\colon \operatorname{res}(F)^\times \to \Delta$ that makes $\operatorname{res}(F)$ isomorphic as a valued field to $E$ over $\operatorname{res}(K_\Delta)$. We are now in the situation of the previous subsection, and so we obtain an extension $L$ of $K$ with the same value group $\Gamma$ as $K$ such that $L_\Delta = F$ (so $L$ and $F$ have the same underlying differential field), the valuation induced by $L$ and $\Delta$ on $\operatorname{res}(L_\Delta) = \operatorname{res}(F)$ equals $w$, and the derivation of $L$ is small. Now $\operatorname{res}(L_\Delta)$ is an immediate extension of $\operatorname{res}(K_\Delta)$, hence $L$ is an immediate extension of $K$, and so $L$ strictly extends $K$ by Lemma 6.4.

$$\begin{tikzcd}
K_\Delta \arrow[r, hook, "\text{strict}"] \arrow[d, dashed] & L_\Delta \arrow[r, hook, "\text{strict}"] \arrow[d, dashed] & M_\Delta \arrow[d, dashed] \\
\operatorname{res}(K_\Delta) \arrow[r, hook] & \operatorname{res}(L_\Delta) \arrow[r, equal] & \operatorname{res}(M_\Delta)
\end{tikzcd}$$

Lemma 6.2 yields $S_{K_\Delta}(\partial) = \{0\}$, and so $S_{L_\Delta}(\partial) = \{0\}$ by Lemma 1.10. Then Theorem 5.1 and Proposition 6.12 yield a spherically complete immediate strict extension $G$ of $L_\Delta$. This places us again in the situation of the previous subsection, with $L$ and $G$ in the role of $K$ and $F$. Hence we obtain an extension $M$ of $L$ with the same value group $\Gamma$ as $L$ such that $M_\Delta = G$ (so $M$ and $G$ have the same underlying differential field), the valuation induced by $M$ and $\Delta$ on $\operatorname{res}(M_\Delta) = \operatorname{res}(G) = \operatorname{res}(F)$ equals $w$, and the derivation of $L$ is small. Therefore $M$ is an immediate extension of $L$ and thus of $K$. Since $M_\Delta$ and $\operatorname{res}(M_\Delta)$ are spherically complete, $M$ is spherically complete by Lemma 6.9. The extension $M$ of $L$ is strict by Lemma 6.3. Thus $M$ is a spherically complete immediate strict extension of $K$ as required. This concludes the proof of the main theorem. □

## 7. *Uniqueness*

Let us say that $K$ *has the uniqueness property* if it has up to isomorphism over $K$ a unique spherically complete immediate strict extension. If $\Gamma = \{0\}$ and more generally, if $K$ is spherically complete, then $K$ clearly has the uniqueness property. If $\partial = 0$, then the derivation of any immediate strict extension of $K$ is also trivial, so $K$ has the uniqueness property. The next result describes a more interesting situation where $K$ has the uniqueness property.

PROPOSITION 7.1. *Suppose $\Gamma = \mathbb{Z}$. Then $K$ has the uniqueness property.*

*Proof.* Let $\widehat{K}$ be the completion of the discretely valued field $K$. Then the unique extension of $\partial$ to a continuous function $\widehat{K} \to \widehat{K}$ is a derivation on $\widehat{K}$ that makes $\widehat{K}$ an immediate strict extension of $K$. If $L$ is any spherically complete immediate extension of $K$, then we have a unique valued field embedding $\widehat{K} \to L$ over $K$, and this embedding is clearly an isomorphism of valued differential fields. $\square$

PROPOSITION 7.2. *Suppose that $\Delta$ is a convex subgroup of $\Gamma$ and $\operatorname{res}(K_\Delta)$ is spherically complete. If $K_\Delta$ has the uniqueness property, then so does $K$.*

*Proof.* Let $L$ and $M$ be spherically complete immediate strict extensions of $K$. Then $\operatorname{res}(L_\Delta)$ and $\operatorname{res}(M_\Delta)$ are immediate valued field extensions of $\operatorname{res}(K_\Delta)$ and thus equal to $\operatorname{res}(K_\Delta)$. Hence $L_\Delta$ and $M_\Delta$ are spherically complete immediate extensions of $K_\Delta$, and $L_\Delta$ and $M_\Delta$ are strict extensions of $K_\Delta$ by Lemma 1.19. Next, let $i\colon L_\Delta \to M_\Delta$ be an isomorphism over $K_\Delta$; it is enough to show that then $i\colon L \to M$ is an isomorphism over $K$. For $a \in L^\times$ we have $a = b(1+\varepsilon)$ with $b \in K^\times$ and $\varepsilon \in \dot{o}_L$, so $i(a) = b(1+i(\varepsilon))$ and $i(\varepsilon) \in \dot{o}_M$, hence $va = vb = vi(a)$. $\square$

One could try to use this last result inductively, but at this stage we do not even know if uniqueness holds when $\Gamma = \mathbb{Z}^2$, lexicographically ordered.

*The role of linear surjectivity*

In the next section we give an example of an $H$-field $K$ that does not have the uniqueness property. This has to do with the fact that certain linear differential equations over this $K$ have no solution in $K$. Here we focus on the opposite situation: as in [2, Section 5.1] a differential field $E$ of characteristic zero is said to be *linearly surjective* if for all $a_1, \dots, a_n, b \in E$ the linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b$$

has a solution in $E$. For valued differential fields this property is related to differential-henselianity: We say that $K$ is *differential-henselian* (for short: d-henselian) if $K$ has small derivation and every differential polynomial $P \in \mathcal{O}\{Y\} = \mathcal{O}[Y, Y', Y'', \dots]$ whose reduction $\overline{P} \in \operatorname{res}(K)\{Y\}$ has degree 1 has a zero in $\mathcal{O}$; (cf. [2, Chapter 7]). If $K$ is d-henselian, then its differential residue field $\operatorname{res}(K)$ is clearly linearly surjective. Here is a differential analogue of Hensel's Lemma:

*If $K$ has small derivation, $\operatorname{res}(K)$ is linearly surjective, and $K$ is spherically complete, then $K$ is* d-*henselian.* This is [2, Corollary 7.0.2]; the case where $K$ is monotone goes back to Scanlon [7].

CONJECTURE. If $K$ has small derivation and $\operatorname{res}(K)$ is linearly surjective, then $K$ has the uniqueness property.

For monotone $K$ this conjecture has been established [2, Theorem 7.4]. It has also been proved for $K$ whose value group has finite archimedean rank and some related cases in [3]. Recently, Nigel Pynn-Coates has proved the conjecture in the case of most interest to us, namely for asymptotic $K$. This is part of work in progress.

## 8. *Nonuniqueness*

We begin with a general remark. Let $A \in K[\partial]$ and suppose that the equation $A(y) = 1$ has no solution in any immediate strict extension of $K$. Assume in addition that $a \in K$ is such that the equation $A(y) = a$ has a solution $y_0$ in an immediate strict extension $K_0$ of $K$ and the equation $A(y) = a + 1$ has a solution $y_1$ in an immediate strict extension $K_1$ of $K$. Extending $K_0$ and $K_1$ we arrange that $K_0$ and $K_1$ are spherically complete, and we then observe that $K_0$ and $K_1$ cannot be isomorphic over $K$. Thus $K$ does not have the uniqueness property.

Below we indicate a real closed $H$-field $K$ where the above assumptions hold for a certain $A \in K[\partial]$ of order 1, and so this $K$ does not have the uniqueness property.

The first two subsections contain generalities about solving linear differential equations of order 1 in immediate extensions of d-valued fields. In the last subsection we assume familiarity with [**2**, Sections 5.1, 11.5, 11.6, 13.9, Appendix A].

We recall from [**2**, Section 9.1] that an asymptotic field $K$ is said to be d-*valued* (short for: 'differential-valued') if $\mathcal{O} = C + \smallo$. (So each $H$-field is d-valued.) We also recall that if $K$ is an asymptotic field, then for $f \in K^\times$ with $f \not\asymp 1$, the valuation $v(f^\dagger)$ of the logarithmic derivative of $f$ only depends on $vf$, so we have a function $\psi\colon \Gamma^{\neq} := \Gamma \setminus \{0\} \to \Gamma$ with $\psi(vf) = v(f^\dagger)$ for such $f$. If we want to stress the dependence on $K$ we write $\psi_K$ instead of $\psi$, and for $\gamma \in \Gamma^{\neq}$ we also set $\gamma' := \gamma + \psi(\gamma)$. The pair $(\Gamma, \psi)$ is an *asymptotic couple,* that is (see [**2**, Section 6.5]): $\psi(\alpha + \beta) \geqslant \min\{\psi(\alpha), \psi(\beta)\}$ for all $\alpha, \beta \in \Gamma^{\neq}$ with $\alpha + \beta \neq 0$; $\psi(k\gamma) = \psi(\gamma)$ for $\gamma \in \Gamma^{\neq}$ and $0 \neq k \in \mathbb{Z}$; and

$$\Psi := \{\psi(\gamma) : \gamma \in \Gamma^{\neq}\} < (\Gamma^{>})' := \{\gamma' : \gamma \in \Gamma^{>}\}.$$

### *Slowly varying functions*

In this subsection $K$ is an asymptotic field, $\Gamma \neq \{0\}$, and $A \in K[\partial]$ is of order 1. Proposition 8.4 below is a variant of [**2**, Proposition 9.7.1]. Recall from [**2**, Section 9.7] that for an ordered abelian group $G$ and $U \subseteq G$ a function $\eta\colon U \to G$ is said to be *slowly varying* if $\eta(\alpha) - \eta(\beta) = o(\alpha - \beta)$ for all $\alpha \neq \beta$ in $U$; note that then $\gamma \mapsto \gamma + \eta(\gamma)\colon U \to G$ is strictly increasing. Note also that $\psi\colon \Gamma^{\neq} \to \Gamma$ is slowly varying [**2**, Lemma 6.5.4(ii)].

LEMMA 8.1. *Let $a \in K^\times$ and $s = a^\dagger$. Then there is a slowly varying function $\eta\colon \Gamma \setminus \{va\} \to \Gamma$ such that $v(y^\dagger - s) = \eta(vy)$ for all $y \in K^\times$ with $vy \neq va$.*

*Proof.* We can take $\eta(\gamma) := \psi(\gamma - va)$ for $\gamma \in \Gamma \setminus \{va\}$. □

LEMMA 8.2. *Assume that $K$ is* d-*valued. Let $s \in K$ be such that $v(y^\dagger - s) < (\Gamma^{>})'$ for all $y \in K^\times$. Then there is a slowly varying function $\eta\colon \Gamma \to \Gamma$ such that*

$$\eta(vy) = v(y^\dagger - s) \quad \textit{for all } y \in K^\times.$$

*Proof.* Let $y$ range over $K^\times$. Take a nonzero $\phi$ in an elementary extension $L$ of $K$ such that $\phi^\dagger - s \preccurlyeq y^\dagger - s$ for all $y$; thus $\delta := v(\phi^\dagger - s) < \big(\Gamma_L^{>}\big)'$. From $v(y^\dagger - s) \leqslant v(\phi^\dagger - s)$ we get $y^\dagger - \phi^\dagger \not\sim s - \phi^\dagger$, and thus

$$v(y^\dagger - s) = v\big((y^\dagger - \phi^\dagger) - (s - \phi^\dagger)\big) = \min\big\{v\big((y/\phi)^\dagger\big), \delta\big\} = \min\big\{\psi_L(vy - v\phi), \delta\big\},$$

where in case $y \asymp \phi$ we use that $L$ is d-valued to get the last equality. Thus $v(y^\dagger - s) = \eta(vy)$, where $\eta\colon \Gamma \to \Gamma$ is defined by $\eta(\gamma) := \min\big\{\psi_L(\gamma - v\phi), \delta\big\}$. Next we show that $\eta$ is slowly varying. The function $\gamma \mapsto \psi_L(\gamma - v\phi)\colon \Gamma_L \setminus \{v\phi\} \to \Gamma_L$ is slowly varying, hence so is the restriction of $\eta$ to $\Gamma \setminus \{v\phi\}$. Moreover, if $v\phi \in \Gamma$ and $\gamma \in \Gamma \setminus \{v\phi\}$, then $\eta(v\phi) = \delta$, so

$$\begin{aligned}\eta(\gamma) - \eta(v\phi) &= \min\big\{\psi_L(\gamma - v\phi), \delta\big\} - \delta \\ &= \min\big\{\psi_L(\gamma - v\phi) - \delta, 0\big\} = o(\gamma - v\phi)\end{aligned}$$

by [**2**, Lemma 9.2.10(iv)] applied to the asymptotic couple $(\Gamma_L, \psi_L - \delta)$, which has small derivation. □

Lemma 8.3. *Suppose that $K$ is* d*-valued and* $\{f\in K:\ vf\in(\Gamma^{>})'\}\subseteq(K^{\times})^{\dagger}$. *Then there is a slowly varying function* $\eta\colon\Gamma\setminus v(\ker A)\to\Gamma$ *such that*

$$v\big(A(y)\big)=vy+\eta(vy)\quad\text{for all } y\in K \text{ with } vy\notin v(\ker A).$$

*Proof.* We have $A=a_0+a_1\partial$ with $a_0,a_1\in K$, $a_1\neq 0$; put $s:=-a_0/a_1$. For $y\in K^{\times}$ we get $A(y)=a_1y(y^{\dagger}-s)$, hence $v\big(A(y)\big)=va_1+vy+v(y^{\dagger}-s)$, and the claim follows from Lemmas 8.1 and 8.2. □

We refer to [**2**, Section 11.1] for the definition of the subset $\mathscr{E}^{\mathrm{e}}(A)$ of $\Gamma$, for ungrounded $K$; since $A$ has order 1, this set $\mathscr{E}^{\mathrm{e}}(A)$ has at most one element. Recall also that $K$ is said to be of *$H$-type* or *$H$-asymptotic* if $\psi$ restricts to a decreasing function $\Gamma^{>}\to\Gamma$, and to have *asymptotic integration* if $(\Gamma^{\neq})'=\Gamma$.

Proposition 8.4. *Let $K$ be* d*-valued of $H$-type with asymptotic integration. Then there is a slowly varying function* $\eta\colon\Gamma\setminus\mathscr{E}^{\mathrm{e}}(A)\to\Gamma$ *such that*

$$v\big(A(y)\big)=vy+\eta(vy)\quad\text{for all } y\in K^{\times} \text{ with } vy\notin\mathscr{E}^{\mathrm{e}}(A).$$

*Proof.* By [**2**, Lemma 10.4.3] we have an immediate d-valued extension $L$ of $K$ such that $\{s\in L:\ vs\in(\Gamma_L^{>})'\}\subseteq(L^{\times})^{\dagger}$. Applying Lemma 8.3 to $L$ in place of $K$ yields a slowly varying function $\eta\colon\Gamma\setminus v(\ker_L A)\to\Gamma$ such that

$$v\big(A(y)\big)=vy+\eta(vy)\quad\text{for all } y\in K \text{ with } vy\notin v(\ker_L A).$$

It only remains to note that $v\big((\ker_L A)\setminus\{0\}\big)\subseteq\mathscr{E}^{\mathrm{e}}_L(A)=\mathscr{E}^{\mathrm{e}}(A)$. □

*Application to solving first-order linear differential equations*

In this subsection $K$ is d-valued, $A\in K[\partial]$ has order 1, and $g\in K$ is such that $g\notin A(K)$, so $S:=v\big(A(K)-g\big)\subseteq\Gamma$.

Lemma 8.5. *Suppose that $K$ is henselian of $H$-type with asymptotic integration. Also assume $\mathscr{E}^{\mathrm{e}}(A)=\emptyset$ and $S$ does not have a largest element. Let $L=K(f)$ be a field extension of $K$ with $f$ transcendental over $K$, equipped with the unique derivation extending that of $K$ such that $A(f)=g$. Then there is a valuation of $L$ that makes $L$ an immediate asymptotic extension of $K$.*

*Proof.* Take a well-indexed sequence $(y_\rho)$ in $K$ such that $\big(v\big(A(y_\rho)-g\big)\big)$ is strictly increasing and cofinal in $S$. Proposition 8.4 yields a strictly increasing function $i\colon\Gamma\to\Gamma$ with $v\big(A(y)\big)=i(vy)$ for all $y\in K^{\times}$. Hence for $\rho<\sigma$,

$$v\big(A(y_\rho)-g\big)=v\big((A(y_\rho)-g)-(A(y_\sigma)-g)\big)=v\big(A(y_\rho-y_\sigma)\big)=i\big(v(y_\rho-y_\sigma)\big),$$

so $i\big(v(y_\rho-y_\sigma)\big)<i\big(v(y_\sigma-y_\tau)\big)$ and thus $v(y_\rho-y_\sigma)<v(y_\sigma-y_\tau)$ for $\rho<\sigma<\tau$. Hence $(y_\rho)$ is a pc-sequence. Suppose toward a contradiction that $y_\rho\leadsto y\in K$. Then $v(y_\rho-y)$ is eventually strictly increasing, so $v\big(A(y_\rho)-A(y)\big)=i\big(v(y_\rho-y)\big)$ is eventually strictly increasing, and thus eventually $v\big(A(y_\rho)-g\big)\leqslant v\big(A(y)-g\big)$, contradicting the assumption that $S$ has no largest element. Hence $(y_\rho)$ does not have a pseudolimit in $K$. It remains to use [**2**, Proposition 9.7.6]. □

Here is a situation where the hypothesis about $S$ in Lemma 8.5 is satisfied:

Lemma 8.6. *If $S\subseteq v\big(A(K)\big)$, then $S$ does not have a largest element.*

*Proof.* Let $y \in K$ be given; we need to find $y_{\text{new}} \in K$ with $A(y_{\text{new}}) - g \prec A(y) - g$. Since $v\big(A(y) - g\big) \in v\big(A(K)\big) \cap \Gamma$, we can pick $h \in K^\times$ such that $A(h) \sim A(y) - g$. Set $y_{\text{new}} := y - h$. Then $A(y_{\text{new}}) - g = \big(A(y) - g\big) - A(h) \prec A(y) - g$ as required. □

*Some differential-algebraic lemmas*

In this subsection $E$ is a differential field of characteristic zero and $F$ is a differential field extension of $E$.

LEMMA 8.7. *Let $F$ be algebraic over $E$, and $f' + af = 1$ with $a \in E$ and $f \in F$. Then $g' + ag = 1$ for some $g \in E$.*

*Proof.* We can assume $n := [F : E] < \infty$. The trace map $\operatorname{tr}_{F|E}\colon F \to E$ is $E$-linear and satisfies $\operatorname{tr}_{F|E}(y') = \operatorname{tr}_{F|E}(y)'$ for all $y \in F$ and $\operatorname{tr}_{F|E}(1) = n$. Thus $g := \frac{1}{n} \operatorname{tr}_{F|E}(f) \in E$ satisfies $g' + ag = 1$. □

LEMMA 8.8. *Let $F = E\langle y \rangle$ where $y$ is differentially transcendental over $E$, and let $a \in E(y)$. Then there is no $f \in F \setminus E$ with $f' + af = 1$.*

*Proof.* This is a special case of [2, Lemma 4.1.5]. □

LEMMA 8.9. *Let $Y$ be an indeterminate over a field $G$ and let $R \in G(Y)$ be such that $R(Y) = R(Y + g)$ for infinitely many $g \in G$. Then $R \in G$.*

*Proof.* We have $R = P/Q$ with $P, Q \in G[Y]$. Let $Z$ be an indeterminate over $G(Y)$. Then $R(Y) = R(Y + g)$ for infinitely many $g \in G$ yields

$$P(Y)Q(Y + Z) = Q(Y)P(Y + Z).$$

Substituting $g - Y$ for $Z$ yields $P(Y)Q(g) = Q(Y)P(g)$ for all $g \in G$. Choosing $g$ such that $Q(g) \neq 0$, we obtain $R(Y) = P(Y)/Q(Y) = P(g)/Q(g) \in G$. □

COROLLARY 8.10. *Let $F = E(y)$ with $y' \in E \setminus \partial E$, and let $a \in E \setminus (E^\times)^\dagger$. Then there is no $f \in F \setminus E$ with $f' + af = 1$.*

*Proof.* By [2, Lemma 4.6.10], $y$ is transcendental over $E$, and by [2, Corollary 4.6.13] there is no $g \in F^\times$ with $g' + ag = 0$. For each $c \in C_E$ we have an automorphism $\sigma_c$ of the differential field $E(y)$, which is the identity on $E$ and sends $y$ to $y + c$. Suppose $f' + af = 1$, $f \in F$. Then $\big(f - \sigma_c(f)\big)' + a\big(f - \sigma_c(f)\big) = 0$ and hence $\sigma_c(f) = f$, for each $c \in C_E$. Hence $f \in F$ by the preceding lemma. □

*Nonisomorphic spherically complete extensions*

We now use the preceding subsections to construct an $H$-field $K$ with two spherically complete immediate $H$-field extensions that are not isomorphic over $K$. Let $\mathfrak{M}$ be the subgroup of the ordered multiplicative group $G^{\mathrm{LE}}$ of LE-monomials generated by the rational powers of $\mathrm{e}^x$ and the iterated logarithms $\ell_n$ of $x$:

$$\mathfrak{M} = \bigcup_n \mathrm{e}^{\mathbb{Q}x}\, \ell_0^{\mathbb{Q}} \cdots \ell_n^{\mathbb{Q}}.$$

We consider the spherically complete ordered valued Hahn field

$$M := \mathbb{R}[[\mathfrak{M}]] \subseteq \mathbb{R}[[G^{\mathrm{LE}}]].$$

Note that $\mathfrak{L} := \bigcup_n \ell_0^{\mathbb{Q}} \cdots \ell_n^{\mathbb{Q}}$ is a convex subgroup of $\mathfrak{M}$ with $\mathfrak{L} \cap \mathrm{e}^{\mathbb{Q}x} = \{1\}$ and $\mathfrak{M} = \mathfrak{L}\,\mathrm{e}^{\mathbb{Q}x}$, and so $M = \mathbb{L}[[\mathrm{e}^{\mathbb{Q}x}]]$ where $\mathbb{L} = \mathbb{R}[[\mathfrak{L}]]$. (Our use of the symbols $\mathfrak{L}$, $\mathbb{L}$ differs slightly from that in [**2**, Section 13.9].) We equip $M$ with the unique strongly $\mathbb{R}$-linear derivation satisfying

$$(\mathrm{e}^{rx})' = r\,\mathrm{e}^{rx}, \quad (\ell_0^r)' = r\,\ell_0^{r-1}, \quad (\ell_{n+1}^r)' = r\,\ell_{n+1}^{r-1}(\ell_0 \cdots \ell_n)^{-1} \quad (r \in \mathbb{Q}).$$

Then $M$ is an $H$-field with constant field $\mathbb{R}$. The element $\lambda \in \mathbb{L}$ is defined by

$$\lambda := \left(\sum_{n=1}^{\infty} \ell_n\right)' = \sum_{n=0}^{\infty} (\ell_0 \cdots \ell_n)^{-1},$$

as in [**2**, Section 13.9]. Consider the real closed $H$-subfield $E := \mathbb{R}\langle \lambda, \ell_0, \ell_1, \dots \rangle^{\mathrm{rc}}$ of $\mathbb{L}$ and the real closed $H$-subfield $K := E[[\mathrm{e}^{\mathbb{Q}x}]]$ of $M$. Note that $\mathbb{L}$ is an immediate extension of $E$ and $M$ is an immediate extension of $K$. Thus $K$ has the same divisible value group $\mathbb{Q}v(\mathrm{e}^x) \oplus \bigoplus_n \mathbb{Q}v(\ell_n)$ as $M$, and $K$ has asymptotic integration. Note also that $(\ell_n)$ is a logarithmic sequence in $K$ in the sense of [**2**, Section 11.5].

We set $A := \partial - \lambda \in E[\partial]$. Let $K^*$ be an immediate $H$-field extension of $K$. By [**2**, Lemma 11.5.13] we have $\ker_{K^*} A = \{0\}$. Moreover, $-\lambda$ creates a gap over $K^*$, by [**2**, Lemma 11.5.14] and so $A(y) \not\asymp 1$ for all $y \in K^*$, by [**2**, Lemma 11.5.12]; in particular $1 \notin A(K^*)$. These remarks apply in particular to $K^* = M$. We are going to show:

Proposition 8.11. *For every $c \in \mathbb{R}$ there is an element $y$ in some immediate $H$-field extension $K_c$ of $K$ with $A(y) = \mathrm{e}^x + c$.*

By Lemma 1.11, any immediate $H$-field extension of $K$ strictly extends $K$. Thus in view of the remark in the beginning of this section and using Proposition 8.11:

Corollary 8.12. *There is a family $(K_c)_{c\in\mathbb{R}}$ of spherically complete immediate strict $H$-field extensions $K_c$ of $K$ that are pairwise nonisomorphic over $K$.*

In particular, $K$ does not have the uniqueness property. Toward the proof of the proposition, we still need two lemmas.

Lemma 8.13. *The elements $\ell_0, \ell_1, \dots$ of $\mathbb{L}$ are algebraically independent over the subfield $\mathbb{R}\langle\lambda\rangle = \mathbb{R}(\lambda, \lambda', \dots)$ of $\mathbb{L}$.*

*Proof.* The element $\lambda$ is differentially transcendental over $\mathbb{R}$ by [**2**, Corollary 13.6.3], and hence over $\mathbb{R}(\ell_0, \ell_1, \dots)$, so $\lambda, \lambda', \lambda'', \dots$ are algebraically independent over $\mathbb{R}(\ell_0, \ell_1, \dots)$. Since $\ell_0, \ell_1, \dots$ are algebraically independent over $\mathbb{R}$,

$$\ell_0, \ell_1, \ell_2, \dots, \lambda, \lambda', \lambda'', \dots$$

are algebraically independent over $\mathbb{R}$. Hence $\ell_0, \ell_1, \dots$ are algebraically independent over $\mathbb{R}(\lambda, \lambda', \dots)$. □

Let $B := \partial + (1 - \lambda) \in E[\partial]$. We have $\lambda \notin (M^\times)^\dagger$ by [**2**, Lemma 11.5.13] and $1 = (\mathrm{e}^x)^\dagger \in (M^\times)^\dagger$, so $1 - \lambda \notin (M^\times)^\dagger$, that is, $\ker_M B = \{0\}$.

Lemma 8.14. *$1 \notin B(E)$.*

*Proof.* Put $L_0 := \mathbb{R}\langle\lambda\rangle$ and $L_{n+1} := \mathbb{R}\langle \lambda, \ell_0, \dots, \ell_n\rangle$, so $L_{n+1} = L_n(\ell_n)$ in view of $\ell_n' = \ell_{n-1}^\dagger \in L_n$ for $n \geqslant 1$, and $\ell_0' = 1 \in L_0$. Note that $E$ is algebraic over $\mathbb{R}\langle\lambda, \ell_0, \ell_1, \dots\rangle = \bigcup_n L_n$. By Lemma 8.7 it suffices that $1 \notin B(L_n)$ for all $n$. The case $n = 0$ follows from Lemma 8.8.

Suppose $1 \notin B(L_n)$. Now $L_{n+1} = L_n(\ell_n)$ and $\ell_n$ is transcendental over $L_n$, by Lemma 8.13, so $1 \notin B(L_{n+1})$ by Corollary 8.10. □

*Proof of Proposition* 8.11. Let $c \in \mathbb{R}$ and $g := \mathrm{e}^x + c \in K$.

*Claim* 1. $A(y) \neq g$ and $A(y) - g \succcurlyeq \mathrm{e}^x$, for all $y \in K$.

This is obvious for $y = 0$, so assume $y \in K^\times$. Let $r$ range over $\mathbb{Q}$ and let the $y_r \in E$ be such that $y = \sum_r y_r \mathrm{e}^{rx}$ with the reverse-well-ordered set $\{r : y_r \neq 0\}$ having largest element $r_0$. Then

$$A(y) = \sum_r \big(y_r' + (r - \lambda) y_r\big)\, \mathrm{e}^{rx}\,.$$

For $r_0 \neq 0$ we have $r_0 - \lambda \asymp 1$, so $r_0 - \lambda \notin (\mathbb{L}^\times)^\dagger$, and thus for $r_0 > 1$,

$$A(y) - g \sim \big(y_{r_0}' + (r_0 - \lambda) y_{r_0}\big)\, \mathrm{e}^{r_0 x} \ \succcurlyeq\ \mathrm{e}^x\,.$$

Next, assume $r_0 = 1$. By Lemma 8.14 we have $y_1' + (1 - \lambda) y_1 - 1 \neq 0$, and thus

$$A(y) - g \sim \big(y_1' + (1 - \lambda) y_1 - 1\big)\, \mathrm{e}^x \succcurlyeq \mathrm{e}^x\,.$$

Finally, if $r_0 < 1$, then $A(y) - g \sim -g \asymp \mathrm{e}^x$.

Since $K$ is an $H$-field with asymptotic integration we can pick for every $f \in K^\times$ an element $If \in K^\times$ with $If \not\asymp 1$ and $(If)' \sim f$.

*Claim* 2. Suppose $f \in K^\times$ and $f \succcurlyeq \mathrm{e}^x$. Then $If \asymp f$.

To prove this, note that $h^\dagger \preccurlyeq 1$ for all $h \in M^\times$, hence $f/If \sim (If)^\dagger \preccurlyeq 1$ and so $f \preccurlyeq If$. If $f \prec If$, then $f' \prec (If)' \sim f$, whereas $f \succcurlyeq \mathrm{e}^x$ means $f^\dagger \asymp (\mathrm{e}^x)^\dagger = 1$, a contradiction. Thus $f \asymp If$, as claimed.

Let $y \in K$ be given, and set $z := A(y) - g$ and $y_{\text{new}} := y - Iz \in K$. Then

$$z_{\text{new}} := A(y_{\text{new}}) - g = z - (Iz)' + \lambda Iz.$$

By Claim 1 we have $z \succcurlyeq \mathrm{e}^x$, so $Iz \asymp z$ by Claim 2, and thus $\lambda Iz \prec z$. Since $z - (Iz)' \prec z$, this yields $z_{\text{new}} \prec z$.

This argument shows that the subset $v\big(A(K) - g\big)$ of $\Gamma$ does not have a largest element. By [**2**, Example at end of Section 11.1, Lemma 11.5.13] we have $\mathscr{E}_K^{\mathrm{e}}(A) = \emptyset$. Thus Proposition 8.11 follows from Lemma 8.5. □

To finish this paper we indicate how the operator $B$ differs in its behavior on $E$ from that on its immediate extension $\mathbb{L}$. This uses the following:

Lemma 8.15. *Let $L$ be an $H$-asymptotic field with asymptotic integration and divisible value group $\Gamma_L$, and let $s \in L$ be such that*

$$S := \big\{v(s - a^\dagger) :\ a \in L^\times\big\} \subseteq \Psi_L^{\downarrow}.$$

*Then the following are equivalent for $g \in L^\times$:*

(i) $vg \notin v\big(D(L)\big)$ *for* $D := \partial - s \in L[\partial]$;
(ii) $g^\dagger - s$ *creates a gap over* $L$.

*Proof.* If $S$ has no largest element, this is [**2**, Lemma 11.6.15]. Suppose that $S$ has a largest element. Then [**2**, Lemma 10.4.6] yields an $H$-asymptotic extension $L(b)$ with $b \neq 0$, $b^\dagger = s$,

$\eta := vb \notin \Gamma_L$, and $\Gamma_{L(b)} = \Gamma \oplus \mathbb{Z}\eta$, and $\Psi_{L(b)} = \Psi_L \cup \{\max S\} \subseteq \Psi_L^{\downarrow}$. The rest of the argument is as in the proof of [2, Lemma 11.6.15]. □

In contrast to Lemma 8.14 we have:

Proposition 8.16. *$B(\mathbb{L}) = \mathbb{L}$; in particular $1 \in B(\mathbb{L})$.*

*Proof.* Set $s := \lambda - 1$. We have $\lambda \prec 1$ and for $a \in \mathbb{L}^\times$ we have $a^\dagger \prec 1$. Thus

$$\{v(s - a^\dagger) : \ a \in \mathbb{L}^\times\} = \{0\} \subseteq \Psi_{\mathbb{L}}^{\downarrow}.$$

Let $g \in \mathbb{L}^\times$. Applying Lemma 8.15 yields:

$$vg \notin v\big(B(\mathbb{L})\big) \Longleftrightarrow g^\dagger - s \ \text{ creates a gap over } \ \mathbb{L}.$$

We have $\lambda_n \rightsquigarrow \lambda$. If $vg \notin v\big(B(\mathbb{L})\big)$, then $\lambda_n \rightsquigarrow s + g^\dagger$ by [2, 11.5.12] and the above equivalence, so $v(1 + g^\dagger) > \Psi_{\mathbb{L}}$ by [2, Lemma 11.5.2]. But $g^\dagger \prec 1$, so $v(1 + g^\dagger) = 0 \in \Psi_{\mathbb{L}}^{\downarrow}$. Thus $v\big(B(\mathbb{L})\big) = v(\mathbb{L}^\times)$. As we saw, $v(g^\dagger - s) \in \Psi_{\mathbb{L}}^{\downarrow}$ for all $g \in \mathbb{L}^\times$, so $\mathscr{E}_{\mathbb{L}}^{\mathrm{e}}(B) = \emptyset$, by [2, Example at end of Section 11.1]. The desired result now follows from Lemmas 8.5 and 8.6 and the spherical completeness of $\mathbb{L}$. □

As a consequence of Proposition 8.16 we have $\mathrm{e}^x \in A(M)$: Taking $y \in \mathbb{L}$ with $B(y) = 1$ gives $A(y\,\mathrm{e}^x) = \mathrm{e}^x$. In view of the remarks just before Proposition 8.11 we also obtain that $\mathrm{e}^x + c \notin A(M)$ for all nonzero $c \in \mathbb{R}$.

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