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Adjustable robust optimization through multi-parametric programming

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Abstract
Adjustable robust optimization (ARO) involves recourse decisions (i.e. reactive actions after the realization of the uncertainty, 'wait-and-see') as functions of the uncertainty, typically posed in a two-stage stochastic setting. Solving the general ARO problems is challenging, therefore ways to reduce the computational effort have been proposed, with the most popular being the affine decision rules, where 'wait-and-see' decisions are approximated as affine adjustments of the uncertainty. In this work we propose a novel method for the derivation of generalized affine decision rules for linear mixed-integer ARO problems through multi-parametric programming, that lead to the exact and global solution of the ARO problem. The problem is treated as a multi-level programming problem and it is then solved using a novel algorithm for the exact and global solution of multi-level mixed-integer linear programming problems. The main idea behind the proposed approach is to solve the lower optimization level of the ARO problem parametrically, by considering 'here-and-now' variables and uncertainties as parameters. This will result in a set of affine decision rules for the 'wait-and-see' variables as a function of 'here-and-now' variables and uncertainties for their entire feasible space. A set of illustrative numerical examples are provided to demonstrate the potential of the proposed novel approach.

Keywords Adjustable robust optimization · Multi-parametric programming · Tri-level programming

1 Introduction

Decisions taken in many disciplines have effects that can extend well into the future, therefore the outcome of such decisions are subject to uncertainties, such as variation in

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customer demands and changes in laws or technological advances. One of the dominant approaches to address decision making under uncertainty is robust optimization (RO).

In RO, uncertainty is described by a distribution-free uncertainty set, typically a bounded convex set [6–8,10,14,15,20,21], and recourse decisions are not allowed, i.e. the decision maker makes all the decisions before the realization of the uncertainty, which can be overly conservative.

Ben-Tal et al. [9] extended classical robust optimization to include adjustable decisions, with this class of problems being referred to as adjustable robust optimization (ARO) problems or two-stage robust optimization problems. A general form of a linear ARO problem is presented below (1).

$$\begin{aligned} \min_{x} \quad & c^T x + \max_{u \in U} \min_{y \in \Omega(x,u)} b^T y \\ \text{s.t.} \quad & Ax \geq d, x \in S_x \\ & \Omega(x,u) = \{y \in S_y \colon Wy \geq h - Tx - Mu\} \end{aligned} \tag{1}$$

where u is the uncertainty set, x are ‘here-and-now’ decisions, i.e. are to be made before the realization of the uncertainty, and y are “wait-and-see” decisions, i.e. are to be made after the realization of the uncertainty.

ARO can be applied to many diverse problems that require decisions to be taken before and after the realization of uncertainties, such as inventory control problems [9, 17], facility location planning [5], and unit commitment [42] among others.

ARO problems are very challenging to be solved, with even the simplest linear continuous case being computationally intractable [43].

A key approach for the approximate solution of ARO problems is to restrict the ‘wait-and-see’ adjustable decisions to be affine functions of the uncertainty [9]. This approach is widely known as affine decision rule approximation and results in computationally tractable problems as the approximated problem can be solved as a static robust optimization problem.

Although ARO has received a lot of attention in the open literature, most of the contributions consider continuous decisions. A very limited number of researchers have tried to tackle the problem of discrete decision rules [11,12,22]. Bertsimas and Caramanis [11] presented an algorithm where integer decision rules are parameterized that provides only probabilistic guarantees on the feasibility of the solution. Bertsimas and Georghiou [12] suggested another binary decision rule structure that offers near-optimal designs at the expense of scalability. Hanasusanto et al. [22] presented an approach that restricts the binary decision rules to a ‘K-adaptable structure’. Most recently, Bertsimas and Georghiou [13] presented linearly parameterized binary decision rule structures that can be used in tandem with different continuous decision rules that appear in the literature. This approach is highly scalable but provides inferior policy designs than Bertsimas and Georghiou [12].

Our key idea for the exact solution of ARO problems came from the observation that the second stage variables (y) are multi-parametric in terms of the first stage variables (x) and uncertainty (u). Therefore, the idea is to solve the lower level problem multi-parametrically considering u and x as parameters. This step would allow us to arrive in a set of generalized (exact) affine decision rules valid for the whole feasible space of uncertainty u and 'here-and-now' decisions x.

The rest of this paper is organized as follows. Section 2 presents a sub-class of classical ARO problems that allows 'here-and-now' decision to constrain the feasible space of the uncertainty, Sect. 3 presents a tri-level solution algorithm for the solution of the ARO problems, Sect. 4 presents four numerical examples to illustrate the applicability of the algorithm, Sect. 5 presents the computational performance of the algorithm through a set of randomly generated problems, and Sect. 6 concludes this paper with key overall remarks.

2 ARO with a decision dependent uncertainty set

Uncertainty parameters can be classified as exogenous and endogenous. Exogenous uncertainties are uncertainties that cannot be affected by decisions made, whereas endogenous uncertainties are ones that their realization or ability to be observed is affected by the decision maker [26]. Both RO and ARO literature has mainly considered exogenous uncertainties, with few recent publications considering endogenous uncertainty sets in an RO setting [26,29,32] and even fewer in ARO settings [25,27].

The 'here and now' decisions chosen might not always be operational for all realizations of an endogenous uncertainty. For example, a processing plant might be more profitable if it is designed for a maximum capacity ('here-and-now' variable) that is less than the demand of some customers (uncertainty). When the uncertainty is defined as a function of 'here-and-now' variables (x), instead of being free this would result into problem (2), an optimization problem that can take into consideration 'here-and-now' decisions that constrain the feasible space of the uncertainty.

$$
\begin{aligned}
\min_{x} \quad & c^T x + \max_{u \in U(x)} \min_{y \in \Omega(x,u)} b^T y \\
\text{s.t.} \quad & Ax \geq d, x \in S_x \\
& \Omega(x,u) = \{y \in S_y : Wy \geq h - Tx - Mu\}.
\end{aligned} \tag{2}
$$

Key approaches for the solution of classical mixed-integer ARO problems, such as Zeng and Zhao [41], are based on the enumeration of extreme points of the polyhedral set of U. These approaches will fail for problems that the uncertainty set is dependent on the decision variables, as the optimum is no more guaranteed to lie on the extreme points of the polyhedral set of U. Therefore, such approaches will result to an overly conservative solution.

Problem (2) can be directly reformulated as a tri-level problem (3).

$$
\begin{aligned}
\min_{x} \quad & c^T x + b^T y \\
\text{s.t.} \quad & \min_{u \in U(x)} \quad -b^T y \\
& \quad \text{s.t.} \quad \min_{y \in \Omega(x,u)} \quad b^T y \\
& \quad \quad \quad \text{s.t.} \quad Ax \geq d, x \in S_x \\
& \Omega(x,u) = \{y \in S_y : Wy \geq h - Tx - Mu\}.
\end{aligned} \tag{3}
$$

Multi-level programming problems are very challenging to solve even for the case of two linear continuous decision levels, which has been shown to be NP-hard by Hansen et al. [23]. The computational complexity of multi-level problems with more than two decision levels was discussed by Blair [16] who noted that multi-level linear problems are NP-hard and their complexity increases significantly when the number of levels increases to more than two.

Solution approaches presented in the literature for tri-level problems have addressed a very restricted class of problems, mainly linear continuous problems [4,19,24,33, 35,37–40], with only a few attempts to solve problems containing only integer variables [34,36] with no guarantee of optimality.

Despite the challenges in solving mixed-integer tri-level problems, a recent algorithm for the global solution of mixed-integer linear tri-level problems has been proposed by Avraamidou and Pistikopoulos [2] and can be adopted for the solution of mixed-integer ARO (MI-ARO) problems with a decision dependent uncertainty set. The algorithm is summarized in the next section.

3 Tri-level programming through multi-parametric optimization

As discussed in Sect. 2, when considering the ARO problem as a tri-level problem, the feasible set of the third-level ('wait-and-see') decision variables are parametric in terms of the second (uncertainty) and first level ('here-and-now') decision variables, and the second level (uncertainty) decision variables are parametric in terms of the first level ('here-and-now') variables.

Faisca et al. [19] presented an algorithm based on multi-parametric programming that can address continuous multi-level programming problems. Expanding on their work, the approach used was presented in Avraamidou and Pistikopoulos [2] and is based upon the recently proposed bi-level mixed-integer linear and quadratic algorithms [3,31]. The algorithm is summarized in Table 1. For more details, information and computational studies on this algorithm the reader is directed to Avraamidou and Pistikopoulos [2].

This approach can solve mixed-integer linear tri-level problems of the general formulation (4) to global optimality.

$$\begin{aligned}
\min_{x_1, y_1} \quad & z(x,y) = c_1{}^T x + d_1{}^T y \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& \min_{x_2, y_2} \quad u(x,y) = c_2{}^T x + d_2{}^T y \\
& \text{s.t.} \quad A_2 x + B_2 y \leq b_2 \\
& \qquad \min_{x_3, y_3} v(x,y) = c_3{}^T x + d_3{}^T y \\
& \qquad \text{s.t.} \quad A_3 x + B_3 y \leq b_3 \\
& x = [x_1^T x_2^T x_3^T]^T, \quad y = [y_1^T y_2^T y_3^T]^T \\
& x \in \mathbb{R}^n, \quad y \in \mathbb{Z}_2^p
\end{aligned} \tag{4}$$

where $c_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}^p$, $A_i \in \mathbb{R}^{m_i \times n}$, $B_i \in \mathbb{R}^{m_i \times p}$, $b_i \in \mathbb{R}^{m_i}$, x_1 and x_2 are compact (closed and bounded), x is a vector of the continuous problem variables and y is a

Table 1 Multi-parametric based algorithm for the solution of tri-level mixed-integer linear programming (T-MILP) problems

Step 1	Establish integer and continuous variable bounds, and transform the integer variables into binary variables
Step 2	Recast the third level problem as a multi-parametric mixed-integer linear programming (mp-MILP) problem, in which the optimization variables of the second and first level problems are considered as parameters
Step 3	Solve the resulting mp-MILP problem to obtain the optimal solution of the lower level problem as explicit functions of the second and first level decision variables
Step 4	Substitute each multi-parametric solution into the second level problem to formulate k mp-MILP problems, considering the first level decision variables as parameters
Step 5	Solve the resulting k[a] mp-MILP problems to obtain the optimal solution of the second level problem as explicit functions of the first level decision variables
Step 6	Substitute each multi-parametric solution into the first level problem to formulate single-level MILP problems
Step 7	Solve all single level problems using CPLEX® MILP solver
Step 8	Solve the comparison optimization problem (2.6) to select the exact and global optimum solution

[a]The maximum number of critical regions is bounded by $k \leq \Sigma_{c=0}^{n-1} c!l^c$ where $n = \Sigma_{i=0}^{h} l!/(l-i)!i!$, h is the number of optimization variables and l is the number of constraints. More information about this can be found in Dua et al. [18]

vector of the discrete problem variables. The subscript numbers (1, 2 and 3) indicate the optimization level the constant coefficient matrices and the decision variables belong to.

The presented algorithm has been implemented as part of B-POP® toolbox, a multi-parametric toolbox for the solution of multi-level programming problems. The toolbox features (1) bi-level and tri-level programming solvers for linear and quadratic programming problems and their mixed-integer counter-parts, (2) a versatile problem generator capable of creating random bi-level and tri-level problems of arbitrary size, and (3) a library of bi-level and tri-level programming test problems.

This algorithm can yield fully adaptive second stage policies for problems with the general formulation of (1), and fully adaptive second stage policies along with the globally optimal solution for the general class of problems formulated as (2).

Although this approach can be easily extended to quadratic programming problems, as B-POP can also solve quadratic mixed-integer tri-level problems, in this paper we focus on the linear formulation (2).

4 Numerical examples

To illustrate the use and benefits of using the presented algorithm for the solution of the MI-ARO problems with decision-depended uncertainty set, four small scale numerical examples are solved using three different computational methods, (a) affine decision rule approximation, (b) column-and-constraint generation algorithm, and (c) B-POP® toolbox.

(a) *Affine decision rules* (*ADR*), Ben-Tal et al. [9]
Most popular among the methods derived for the solution of classical two-stage linear robust optimization problems that contain only continuous decisions and the uncertainty is a polyhedron. ‘Wait-and-see’ decisions are approximated as affine adjustments of the uncertainty. Affine decision rules can be optimal in a number of instances (e.g. 4.1 Example 1), but their simple structure generally sacrifices optimality for tractability and scalability.
(b) *Column-and-constraint generation algorithm (C&C)*, Zeng and Zhao [41]
This approach was developed for the solution of mixed integer ARO problems. The basic idea behind this algorithm is to reduce the ARO problem to a single-level optimization problem by enumerating significant extreme points of the polyhedral set U on-the-fly in a decomposition framework. This method scales very well in terms of variables but not in terms of the number of constraints.
(c) *Tri-level optimization through multi-parametric programming* (*B-POP*), Avraamidou and Pistikopoulos [1,2]; Faisca et al. [19]
A MATLAB® based toolbox for the exact global solution of different classes of mixed-integer multi-level programming problems through multi-parametric programming algorithms [30]. B-POP® is available for download at *parametric.tamu.edu*.

4.1 Example 1: Linear ARO problem

A simple problem taken from Ning and You [28], (5), is chosen as the first numerical example to illustrate in detail the steps of the proposed algorithm.

$$\begin{aligned} \min_{x_1,x_2} \quad & 3x_1 + 5x_2 + \max_{u \in U_{box}} \min_{y_1,y_2} \; 6y_1 + 10y_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 100 \\ & x_1 + y_1 \ge u_1 \\ & x_2 + y_2 \ge u_2 \\ & x_i, y_i \ge 0, \quad i = 1, 2 \\ & U_{box} = \{u_1, u_2 | 5.5 \le u_1 \le 52.1, 9.5 \le u_2 \le 54.8\}. \end{aligned} \tag{5}$$

The first step is to recast the lowest optimization level into a multi-parametric problem, by considering x and u variables as parameters.

$$\begin{aligned} \min_{y_1,y_2} \quad & 6y_1 + 10y_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 100 \\ & x_1 + y_1 \ge u_1 \\ & x_2 + y_2 \ge u_2 \\ & x_i, y_i \ge 0, \quad i = 1, 2 \\ & U_{box} = \{u_1, u_2 | 5.5 \le u_1 \le 52.1, 9.5 \le u_2 \le 54.8\}. \end{aligned} \tag{6}$$

Table 2 Parametric solution of problem (6)

CR	Definition	Variables
1	$x_1 - u_1 \leq 0$	$y_1 = -x_1 + u_1$
	$x_2 - u_2 \leq 0$	$y_2 = -x_2 + u_2$
	$x_1 + x_2 \leq 100$	
2	$-x_1 + u_1 \leq 0$	$y_1 = 0$
	$x_2 - u_2 \leq 0$	$y_2 = -x_2 + u_2$
	$x_1 + x_2 \leq 100$	
3	$x_1 - u_1 \leq 0$	$y_1 = -x_1 + u_1$
	$-x_2 + u_2 \leq 0$	$y_2 = 0$
	$x_1 + x_2 \leq 100$	
4	$-x_1 + u_1 \leq 0$	$y_1 = 0$
	$-x_2 + u_2 \leq 0$	$y_2 = 0$
	$x_1 + x_2 \leq 100$	

Problem (6) is then solved using POP® toolbox, to get the optimal parametric solution of y as a set of affine functions of the rest of the optimization variables (x_1, x_2, u_1, u_2) in different critical regions (Table 2).

As a next step, 4 multi-parametric problems are formulated, each corresponding to one critical region, by adding the critical region definitions to the set of constraints of the next level problem, substituting in the affine functions for y_1, y_2, and considering the higher level variables (x_1, x_2) as parameters. The four problems created are presented below (7–10).

$$\begin{aligned} \min_{u} \quad & -6u_1 - 10u_2 + 6x_1 + 10x_2 \\ \text{s.t.} \quad & x_1 - u_1 \leq 0 \\ & x_2 - u_2 \leq 0 \\ & x_2 + y_2 \geq u_2 \\ & x_1 + x_2 \leq 100 \\ & 5.5 \leq u_1 \leq 52.1 \\ & 9.5 \leq u_2 \leq 54.8 \end{aligned} \tag{7}$$

$$\begin{aligned} \min_{u} \quad & -10u_2 + 10x_2 \\ \text{s.t.} \quad & -x_1 + u_1 \leq 0 \\ & x_2 - u_2 \leq 0 \\ & x_2 + y_2 \geq u_2 \\ & x_1 + x_2 \leq 100 \\ & 5.5 \leq u_1 \leq 52.1 \\ & 9.5 \leq u_2 \leq 54.8 \end{aligned} \tag{8}$$

$$\begin{aligned} \min_{u} \quad & -6u_1 + 6x_1 \\ \text{s.t.} \quad & x_1 - u_1 \leq 0 \\ & -x_2 + u_2 \leq 0 \\ & x_2 + y_2 \geq u_2 \\ & x_1 + x_2 \leq 100 \\ & 5.5 \leq u_1 \leq 52.1 \\ & 9.5 \leq u_2 \leq 54.8 \end{aligned} \tag{9}$$

Table 3 Parametric solution of the middle level problem

CR	Definition	Variables	Objective
1.1	$0 \leq x_1 \leq 52.1$	$u_1 = 52.1$	$6x_1 + 10x_2 - 860.6$
	$0 \leq x_2 \leq 54.8$	$u_2 = 54.8$	
	$x_1 + x_2 \leq 100$		
2.1	$5.5 \leq x_1$	$u_1 = 5.5$	$10x_2 - 548$
	$0 \leq x_2$	$u_2 = 54.8$	
	$x_1 + x_2 \leq 100$		
3.1	$0 \leq x_1$	$u_1 = 52.1$	$6x_1 - 312.6$
	$9.5 \leq x_2$	$u_2 = 9.5$	
	$x_1 + x_2 \leq 100$		
4	$x_1 + x_2 \leq 100$		0

$$
\begin{aligned}
\min_{u} \quad & 0 \\
\text{s.t.} \quad & -x_1 + u_1 \leq 0 \\
& -x_2 + u_2 \leq 0 \\
& x_2 + y_2 \geq u_2 \\
& x_1 + x_2 \leq 100 \\
& 5.5 \leq u_1 \leq 52.1 \\
& 9.5 \leq u_2 \leq 54.8
\end{aligned} \tag{10}
$$

The resulting four multi-parametric problems are then solved using POP® toolbox. The solution of all of the problems is presented in Table 3.

The next step of the algorithm is to substitute the optimal solutions in Table 2 into the higher level problem and solve the resulting single-level problems to get four different solution strategies.

The final step is to compare those strategies and choose the optimal one to be the exact and global solution of the original tri-level programming problem.

The global solution of the original tri-level problem is $x_1 = 45.2, x_2 = 54.8$ and the objective is 451. Note that this solution is the same as the one presented in Ning and You [28]. The optimal affine decision rules developed through the tri-level multi-parametric programming method are:

$$
\begin{aligned}
y_1 &= -x_1 + u_1 \\
y_2 &= -x_2 + u_2
\end{aligned} \tag{11}
$$

For this ARO problem, using the affine decision rules method results in the same solution as the multi-parametric method and the column and constraint generation algorithm. Even though the solution is the same, the affine decision rules developed through the affine decision rule method are different from the multi-parametric method and are presented below (12).

$$
\begin{aligned}
y_1 &= -0.81438 + 0.14807u_1 \\
y_2 &= 0.
\end{aligned} \tag{12}
$$

Table 4 ARO Example 2 solutions

	Affine decision rules	Column-and-constraint	B-POP
Objective	0	6600	6600
First-stage decisions	$v_1 = 0, v_2 = 0$	$v_1 = 1, v_2 = 0$	$v_1 = 1, v_2 = 0$
	$x_1 = 0, x_2 = 0$	$x_1 = 24{,}000, x_2 = 0$	$x_1 = 24{,}000, x_2 = 0$
Second stage decision rules	$y_{11} = 0, y_{12} = 0$		$y_{11} = -18{,}000\delta_1 + 20{,}000$
	$y_{13} = 0, y_{21} = 0$		$y_{12} = -x_2 - 18{,}000\delta_2 + 20{,}000$
	$y_{22} = 0, y_{23} = 0$		$y_{13} = x_1 + x_2 + 18{,}000\delta_1 + 18{,}000\delta_2 - 40{,}000$
			$y_{21} = 0, y_{22} = x_2, y_{23} = 0$

4.2 Example 2: Mixed integer linear ARO problem

Example problem (13) was formulated to show how the three approaches considered perform when the first stage decisions contain both binary and continuous variables.

Using the three different approaches the solutions presented in Table 4 were obtained. In this instance, both the column-and-constraint generation algorithm and B-POP return the exact optimal solution of the MI-ARO problem, whereas the solution generated through the affine decision rule approach is suboptimal and overly conservative.

Furthermore, B-POP has the capability to also generate the second-stage decision rules for the entire feasible space of the uncertainty and first-stage variables, giving the decision maker more insights into the dynamics of the problem.

$$
\begin{aligned}
\max_{x,v} \quad & -0.6x_1 - 0.6x_2 - 100,000v_1 - 100,00v_2 \\
& \quad + \min_{\delta \in U(x,v)} \quad \max_{y \in \Omega(x,v,\delta)} \quad 5.9y_{11} + 5.6y_{12} + 4.9y_{13} \\
& \qquad\qquad\qquad + 5.6y_{21} + 5.9y_{22} + 4.9y_{23} \\
\text{s.t.} \quad & y_{11} + y_{21} \le 20,000 - 18,000\delta_1 \\
& y_{12} + y_{22} \le 20,000 - 18,000\delta_2 \\
& y_{13} + y_{23} \le 20,000 - 18,000\delta_3 \\
& y_{11} + y_{12} + y_{13} \le x_1 \\
& y_{21} + y_{22} + y_{23} \le x_2 \\
& x_1 \le 130,000v_1 \\
& x_2 \le 130,000v_2 \\
& \delta_1 + \delta_2 + \delta_3 \le 2 \\
& x_1, x_2 \ge 0 \\
& y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \ge 0 \\
& \delta_1, \delta_2, \delta_3 \ge 0 \\
& \delta_1, \delta_2, \delta_3 \le 1 \\
& v_1, v_2 \in \{0, 1\}^2.
\end{aligned}
\tag{13}
$$

4.3 Example 3: Mixed integer linear ARO problem

Example problem (14) was formulated in order to investigate how the three approaches considered perform when the first-stage decision variables can directly constrain the feasible region of the uncertainty through the problem's constraints. This example problem contains both continuous and integer 'here-and-now' optimization variables.

Following the same three different approaches, the solutions in Table 5 were obtained. In this instance, where the first-stage decision variables can constrain the induced feasible region of the uncertainty and second stage variables, enumeration of the extreme points of the uncertainty set may not result in the optimal solution. For this reason the column-and-constraint generation algorithm along with the affine decision rules gave a sub-optimal solution to the original ARO problem.

$$\begin{aligned}
\min_{x,z}\quad & 400x_1 + 414x_2 + 326x_3 + 18z_1 + 25z_2 + 20z_3 \\
& + \max_{g\in G(x,z)} \min_{y} \quad 22y_{11} + 33y_{12} + 24y_{13} + 33y_{21} + 23y_{22} \\
& \qquad\qquad\qquad\qquad +30y_{23} + 20y_{31} + 25y_{32} + 27y_{33} \\
\text{s.t.}\quad & z_1 + z_2 + z_3 \geq 700 + 40(g_1 + g_2 + g_3) \\
& z_1 \leq 800x_1 \\
& z_2 \leq 800x_2 \\
& z_3 \leq 800x_3 \\
& y_{11} + y_{12} + y_{13} \leq z_1 \\
& y_{21} + y_{22} + y_{23} \leq z_2 \\
& y_{31} + y_{32} + y_{33} \leq z_3 \\
& y_{11} + y_{21} + y_{31} \geq 206 + 40g_1 \\
& y_{12} + y_{22} + y_{32} \geq 274 + 40g_2 \\
& y_{13} + y_{23} + y_{33} \geq 220 + 40g_3 \\
& g_1 + g_2 + g_3 \leq 1.8 \\
& z_1, z_2, z_3 \geq 0 \\
& y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33} \geq 0 \\
& g_1, g_2, g_3 \geq 0 \\
& g_1, g_2, g_3 \leq 1 \\
& x_1, x_2, x_3 \in \{0, 1\}^3.
\end{aligned} \tag{14}$$

4.4 Example 4: Mixed integer linear ARO problem

The last example problem formulated contains both continuous and integer 'here-and-now' and 'wait-and-see' optimization variables and both continuous and integer uncertainties.

Table 5 ARO Example 3 solutions

	Affine decision rules	Column-and-constraint	B-POP
Objective	33,680	33,680	30,536
First stage decisions	$x_1 = 1, x_2 = 0$	$x_1 = 1, x_2 = 0$	$x_1 = 1, x_2 = 0$
	$x_3 = 1, z_1 = 458$	$x_3 = 1, z_1 = 458$	$x_3 = 1, z_1 = 220$
	$z_2 = 0, z_3 = 314$	$z_2 = 0, z_3 = 314$	$z_2 = 0, z_3 = 480$
Second stage decision rules	$y_{11} = 166 + 40g_1 + 40g_2$		$y_{11} = 0, y_{12} = 0, y_{13} = z_1$
	$y_{12} = 0$		$y_{21} = 0, y_{22} = z_2, y_{23} = 0$
	$y_{13} = 220 + 40g_3$		$y_{31} = 40g_1 + 206$
	$y_{21} = 0, y_{22} = 0, y_{23} = 0$		$y_{32} = -z_2 + 40g_2 + 274$
	$y_{31} = 40 - 40g_2$		$y_{33} = -z_1 + 40g_3 + 220$
	$y_{32} = 274 + 40g_2, y_{33} = 0$		

Table 6 ARO Example 4 solutions

	B-POP
Objective	-27.1784
First-stage decisions	$x_1 = 10, y_1 = 0$
Second-stage	$x_2 = -165x_1 - 10y_1 - 205u_1 - 40u_2 + 50$
Decision rules	$y_2 = 0, y_3 = 0$

$$
\begin{aligned}
\min_{x_1,y_1}\ & 9x_1 + \max_{u_1,u_2} \min_{x_2,y_2,y_3} 2x_1 + y_1 + 2u_1 + u_2 - 4x_2 + 2y_2 + 10y_3 \\
\text{s.t.}\ & \ 6.4x_1 + 7.2u_1 + 2.5x_2 \le 11.5 \\
& -8x_1 - 4.9u_1 - 3.2x_2 \le 5 \\
& 3.3x_1 + 4.1u_1 + 0.02x_2 + 0.2y_1 + 0.8u_2 + 4y_2 + 4.5y_3 \le 1 \\
& y_1 + u_2 + y_2 + y_3 \ge 1 \\
& -10 \le x_1 \le 10 \\
& -10 \le u_1 \le 10 \\
& x_1, x_2 \in \mathbb{R}^2, \quad y_1, y_2, y_3 \in \{0, 1\}^3 \\
& u_1 \in \mathbb{R}, \quad u_2 \in \{0, 1\}.
\end{aligned}
\tag{15}
$$

Both ADR and C&C solution approaches cannot be used for classes of problems that contain integer variables in the second stage, like problem (13). Therefore, in Table 6 we present only the exact optimal solution obtained using B-POP® toolbox.

5 Computational studies

The algorithm presented in this paper has been implemented in a toolbox, B-POP, a MATLAB based toolbox for multi-level programming. The toolbox features (1) multi-level programming solvers for linear and quadratic programming problems and their mixed-integer counter-parts, (2) a versatile problem generator capable of creating random multi-level problems of arbitrary size, and (3) a library of multi-level programming test problems. The reader is also directed to Avraamidou and Pistikopoulos [2] where the efficiency and performance of the toolbox for the solution of mixed-integer tri-level problems were presented and discussed.

In this section, we will assess the performance and efficiency of the toolbox for the solution of two classes of ARO problems.

(a) **Class 1** This class considers the mixed-integer version of the ARO problem with decision dependent uncertainty set presented as (2). Formulation (2) is expanded here to include both integer and continuous variables.

$$\begin{aligned}
\min_{x_1,y_1} \quad & c^T\omega_1 + \max_{u,v} \quad \min_{x_2,y_2} b^T\omega_2 \\
\text{s.t.} \quad & W\omega_2 \geq h - T\omega_1 - M\omega_u \\
& \omega_1 = [x_1^T y_1^T]^T \\
& \omega_2 = [x_2^T y_2^T]^T \\
& \omega_u = [u^T v^T]^T \\
& \omega_1 \in S_{\omega_1}, \quad x_1 \in \mathbb{R}^\alpha, \quad y_1 \in \mathbb{Z}_2^k \\
& \omega_u \in U(\omega_1), \quad u \in \mathbb{R}^\beta, \quad v \in \mathbb{Z}_2^n \\
& \omega_2 \in S_{\omega_2}(\omega_1, \omega_u), \quad x_2 \in \mathbb{R}^\gamma, \quad y_2 \in \mathbb{Z}_2^m.
\end{aligned} \tag{16}$$

(b) **Class 2** This class considers a more general mixed-integer ARO problem where uncertainties can directly affect the objective of the problem. This class is formulated as (17).

$$\begin{aligned}
\min_{x_1,y_1} \quad & c^T\omega + \max_{u,v} \quad \min_{x_2,y_2} b^T\omega \\
\text{s.t.} \quad & W\omega_2 \geq h - T\omega_1 - M\omega_u \\
& \omega = [x_1^T y_1^T u^T v^T x_2^T y_2^T]^T \\
& \omega_1 = [x_1^T y_1^T]^T \\
& \omega_2 = [x_2^T y_2^T]^T \\
& \omega_u = [u^T v^T]^T \\
& \omega_1 \in S_{\omega_1}, \quad x_1 \in \mathbb{R}^\alpha, \quad y_1 \in \mathbb{Z}_2^k \\
& \omega_u \in U(\omega_1), \quad u \in \mathbb{R}^\beta, \quad v \in \mathbb{Z}_2^n \\
& \omega_2 \in S_{\omega_2}(\omega_1, \omega_u), \quad x_2 \in \mathbb{R}^\gamma, \quad y_2 \in \mathbb{Z}_2^m.
\end{aligned} \tag{17}$$

Computational results for problems belonging to Classes 1 and 2 are presented in Table 7. P column lists the problem names, X_1 denotes the total number of continuous first stage variables, Y_1 denotes the total number of binary first stage variables, X_2 denotes the total number of continuous second stage variables, Y_2 denotes the total number of binary second stage variables, U denotes the total number of continuous uncertainties, V denotes the total number of binary uncertainties, S_i denotes the computational time required for the completion of step i of the algorithm (see Table 1

Table 7 Computational results for ARO problems

P	X_1	Y_1	X_2	Y_2	U	V	S_3 (s)	S_5 (s)	S_7 (s)	S_8 (s)	T (s)
P1	2	2	1	1	2	2	0.358	6.279	0.006	0.001	6.644
P2	4	4	5	5	2	2	0.472	0.338	0.002	0.0004	0.812
P3	10	10	2	2	10	10	0.989	0.825	0.002	0.0004	1.816
P4	10	10	10	10	10	10	1.835	2.505	0.002	0.0004	4.342
P5	30	20	30	20	30	20	11.198	3.941	0.040	0.017	15.196
P6	2	2	1	1	2	2	0.182	0.082	0.002	0.0008	0.266
P7	4	4	5	5	2	2	0.448	0.152	0.002	0.0005	0.602
P8	10	10	2	2	10	10	0.910	0.668	0.002	0.0005	1.581
P9	10	10	10	10	10	10	1.550	1.608	0.0028	0.0005	3.162
P10	50	50	5	5	30	30	26.839	7.083	0.033	0.032	33.988

for the steps), and T denotes the total computational time for each test problem. Test problems P1 to P5 belong to Class 1 and P6 to P10 belong to Class 2.

The computations were carried out on a 2-core machine with an Intel Core i7 at 3.6 GHz and 16 GB of RAM, MATLAB R2015a, and IBM ILOG CPLEX Optimization Studio 12.6.2. Note that, the independent problems in Steps 5 and 7 of the algorithm in Table 1 were solved sequentially and not simultaneously (which obviously would have increased the computational performance).

The test problems listed in Table 7 are available through parametric.tamu.edu website, in the POP section under the name 'BPOP_ARO'.

Results in Table 7 show that we can solve problems of class 1 with a total of 90 continuous variables and 60 integer variables in about 15 s, and problems of class 2 with a total of 85 continuous variables and 85 integer variables in about 34 s. As discussed in Avraamidou and Pistikopoulos [2], the limiting steps of the algorithm involve the solution of the multi-parametric problems (S_3 and S_5 on the table).

6 Key overall remarks

Using multi-parametric programming to solve the inner problem for 'wait-and-see' variables, we were able to show that affine rules are mere approximations, and can be sub-optimal for general classes of problems, as different affine decision rules are optimal in different spaces of 'here-and-now' and uncertainty variables feasible space.

Moreover, we developed a theory for generating exact generalized affine rules and solving to global optimality linear ARO problems involving both continuous and integer decision variables in both stages, where the uncertainty can be both integer and continuous and is a function of the first stage variables. This theory can be extended for quadratic problems by the use of multi-parametric mixed-integer quadratic programming algorithms that are already implemented in B-POP® toolbox. Through this methodology, we are able to capture the full set of affine decisions as a function of 'here-and-now' decisions and uncertainty.

One can also observe that even if the first-stage objective function is changed, the affine decision rules generated through this approach are still valid. Furthermore, the decision maker has access to a plethora of strategies to choose from, as opposed to other solution methods that will only generate one strategy.

Future directions include extending this approach for the solution of the classical ARO problem where the uncertainty is not decision dependent, and for quadratic MI-ARO problems.

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