Journal of Theoretical Probability
https://doi.org/10.1007/s10959-019-00898-w

Law of the First Passage Triple of a Spectrally Positive Strictly Stable Process

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Received: 5 February 2018 / Revised: 24 March 2019

Abstract

For a spectrally positive and strictly stable process with index in (1, 2), a series representation is obtained for the joint distribution of the "first passage triple" that consists of the time of first passage and the undershoot and the overshoot at first passage. The result leads to several corollaries, including (1) the joint law of the first passage triple and the pre-passage running supremum, and (2) at a fixed time point, the joint law of the process' value, running supremum, and the time of the running supremum. The representation can be decomposed as a sum of strictly positive functions that allow exact sampling of the first passage triple.

Keywords First passage · Lévy process · Stable · Spectrally positive · Mittag–Leffler · Running supremum · Exact sampling

Mathematics Subject Classification (2010) Primary 60G51; Secondary 60E07

1 Introduction

Let $X = (X_t)_{t\ge0}$ be a Lévy process and $\Pi(\mathrm{d}x)$ its Lévy measure. Denote by

$$\Delta_t = X_t - X_{t-}, \quad \overline{X}_t = \sup_{0\le s\le t} X_s,$$

the jump and running supremum of X at t, respectively. By convention, $X_{0-} = X_0 = 0$. For $c \ge 0$, the first passage time of X at level c is defined as

$$T_c = \inf\{t > 0 : X_t > c\},$$

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while for $x \in \mathbb{R}$, the first hitting time of X at x is defined as

$$\tau_x = \inf\{t > 0 : X_t = x\},$$

where by convention $\inf \emptyset = \infty$.

By definition, a Lévy process is spectrally positive if it only has positive jumps, i.e., its Lévy measure is concentrated on $(0, \infty)$. It is well known that if X is spectrally positive and is not a subordinator, then $(\tau_{-x})_{x\geq 0}$ is a subordinator, possibly killed at an exponential time and for $t, x > 0$, $t\mathbb{P}\{\tau_{-x} \in \mathrm{d}t\}\,\mathrm{d}x = x\mathbb{P}\{X_t \in -\mathrm{d}x\}\,\mathrm{d}t$, which is known as Kendall's identity ([2], Chapter VII). If for each $t > 0$, X_t has a probability density function (p.d.f.) $g_t(x)$, then for each $x > 0$, τ_{-x} has a p.d.f. $f_{-x}(t)$ and Kendall's identity can be written as

$$t f_{-x}(t) = x g_t(-x), \quad x > 0, \ t > 0. \tag{1}$$

In this paper, a p.d.f. is always defined with respect to (w.r.t.) the Lebesgue measure.

Let X be a spectrally positive and strictly stable process with index $\alpha \in (1, 2)$. The first passage of X at a fixed level $c > 0$ is of particular interest and has already drawn a lot of attention. The joint distribution of X_{T_c-} and Δ_{T_c} is known [7] and so is the distribution of T_c [1,19,20]. Related to these random variables, the distribution of τ_x is classical when $x < 0$ [2] and is also known when $x > 0$ [16,19]. On the other hand, the three random variables T_c, X_{T_c-}, and Δ_{T_c} completely describe what happens to X at the moment of first passage. Although some general results are available [7], explicit representations of the joint distribution of the triple have been unknown.

While there may be many different representations, those that allow exact sampling are practically more useful and perhaps conceptually more satisfactory. Ideally, a representation should also allow efficient implementation of the sampling. Although such representations are available for the marginal distributions of X_t, $\overline{X}_t$, T_c, and τ_x [19,20,22], they seem much harder to get for the joint distribution of T_c, X_{T_c-}, and Δ_{T_c}, so we will content ourselves with a representation that allows exact sampling of the triple regardless of efficiency.

The following function will play an important role. For $c > 0$, $x \in (-\infty, c)$, and $t > 0$, define

$$h_c(x, t) = \frac{\mathbb{P}\{X_t \in \mathrm{d}x, \overline{X}_t \leq c\}}{\mathrm{d}x}. \tag{2}$$

Since X has the scaling property, i.e., $(X_{\lambda t})_{t\geq 0} \sim (\lambda^{1/\alpha} X_t)_{t\geq 0}$ for all $\lambda > 0$, one can assume without loss of generality that

$$\mathbb{E}(e^{-qX_t}) = \exp(tq^\alpha), \quad t > 0, \ q \geq 0. \tag{3}$$

Because also by scaling

$$(T_c, X_{T_c-}, \Delta_{T_c}) \sim (c^\alpha T_1, cX_{T_1-}, c\Delta_{T_1}), \tag{4}$$

it suffices to consider $c = 1$.

Theorem 1 *Suppose X is a stable process with index $\alpha \in (1, 2)$ satisfying (3). Then, the triple $(T_1, X_{T_1-}, \Delta_{T_1})$ has a p.d.f. that at each $(t, x, z) \in (0, \infty) \times \mathbb{R} \times [0, \infty)$ takes value*

$$\varrho_1(t, x, z) = \frac{z^{-\alpha-1}}{\Gamma(-\alpha)} \mathbf{1}\{x < 1 < x + z\} h_1(x, t),$$

where for $x \in (-\infty, 1)$,

$$h_1(x, t) = \frac{1}{\pi} \sum_{k,n=1}^{\infty} (-1)^{k+n} \frac{\Gamma(k/\alpha + n)}{\Gamma(\alpha n) k!} \sin(\pi k/\alpha)(1 - x)^k t^{-k/\alpha - n}. \tag{5}$$

The series in (5) converges absolutely for given x and $t > 0$.

Given $c > 0$, by the scaling relation (4), $(T_c, X_{T_c-}, \Delta_{T_c})$ has joint p.d.f.

$$\varrho_c(t, x, z) = c^{-\alpha-2} \varrho_1(c^{-\alpha} t, c^{-1} x, c^{-1} z).$$

Furthermore,

$$h_c(x, t) = c^{-1} h_1(c^{-1} x, c^{-\alpha} t). \tag{6}$$

The core of Theorem 1 is (5), and a key step in its proof is to show

$$h_1(x, t) = \sum_{n=0}^{\infty} \frac{f_{x-1}^{(n)}(t)}{\Gamma(\alpha n + \alpha)}, \tag{7}$$

which can be formally written as

$$h_1(x, \cdot) = E_{\alpha,\alpha}(D) f_{x-1},$$

where D is the differential operator and $E_{\alpha,\alpha}(s)$ is a Mittag–Leffler function ([8,15]; see Sect. 3.1). Many detailed asymptotics of $f_{x-1}^{(n)}(t)$ can be found in [10]. It will be seen that conditionally on $X_{T_1-} = x$, Δ_{T_1} and T_1 are independent, with the latter having p.d.f. $h_1(x, \cdot)/v_1(x)$, where

$$v_1(x) = \int_0^{\infty} h_1(x, t) \,\mathrm{d}t = \frac{1 - (x \vee 0)^{\alpha-1}}{\Gamma(\alpha)}. \tag{8}$$

One may have noticed that when $x \in (0, 1)$, $v_1(x)$ is strictly smaller than $1/\Gamma(\alpha)$, whereas the sum of the term-wise integrals of the series (7) is $1/\Gamma(\alpha)$. The lack of interchangeability of summation and integration reflects the high oscillations of $f_{x-1}^{(n)}(t)$ as functions of t, which are tricky to tackle directly. In this paper, (7) will be first established for $x < a$, where $a \le 0$ is a certain constant, and then, it will be established for all $x < 1$ by analytic extension.

Several results can be derived from Theorem 1. First, an integral representation of $h_1(x, t)$.

Corollary 2 *Under the same condition as above,*

$$h_1(x,t) = \frac{1}{\pi}\int_0^\infty e^{-st-(1-x)s^{1/\alpha}\cos(\pi/\alpha)} \sin((1-x)s^{1/\alpha}\sin(\pi/\alpha))E_{\alpha,\alpha}(-s)\,\mathrm{d}s.$$

The next result on the support of $h_1(x, t)$ will be used later and is of interest in its own right.

Corollary 3 $h_1(x, t) > 0$ *for all* $x < 1$ *and* $t > 0$.

In the last two corollaries, $h_c(x, t)$ is regarded as a function of t and x with $c = 1$ being fixed. When t is fixed and c and x are treated as variables, $h_c(x, t)$ provides the joint distribution of X_t and $\overline{X}_t$. Specifically, from (5) and scaling, the following result obtains. Since $(X_t, \overline{X}_t) \sim (t^{1/\alpha}X_1, t^{1/\alpha}\overline{X}_1)$, it suffices to consider $t = 1$.

Corollary 4 X_1 *and* $\overline{X}_1$ *have joint p.d.f.*

$$\frac{\mathbb{P}\{X_1 \in \mathrm{d}x, \overline{X}_1 \in \mathrm{d}c\}}{\mathrm{d}x\,\mathrm{d}c} = \mathbf{1}\{c > (x \vee 0)\}\,\frac{\partial h_c(x,1)}{\partial c}$$

with

$$\frac{\partial h_c(x,1)}{\partial c} = \frac{1}{\pi}\sum_{k,n=1}^{\infty} \frac{\Gamma(k/\alpha+n)}{\Gamma(\alpha n)k!}(-1)^{k+n}\sin(\pi k/\alpha)[kc + (\alpha n - 1)(c-x)](c-x)^{k-1}c^{\alpha n-2}. \tag{9}$$

Remark For a standard Brownian motion W, it is known that ([12], Corollary 3.2.1.2).

$$\frac{\mathbb{P}\{W_1 \in \mathrm{d}x, \sup_{s\le 1} W_s \in \mathrm{d}c\}}{\mathrm{d}x\,\mathrm{d}c} = \mathbf{1}\{c > (x \vee 0)\}\,\frac{2(2c-x)}{\sqrt{2\pi}}\left\{-\frac{(2c-x)^2}{2}\right\}. \tag{10}$$

It will be shown in the "Appendix" that (10) can be deduced from (9). Note that by (3), for $\alpha = 2$, $(X_t)_{t\ge 0} \sim (W_{2t})_{t\ge 0}$.

The next corollary combined with Theorem 1 gives the joint distribution of T_1, X_{T_1-}, Δ_{T_1}, and the pre-passage running supremum $\overline{X}_{T_1-}$.

Corollary 5 *Conditionally on* $T_1 = t$ *and* $X_{T_1-} = x < 1$, Δ_{T_1} *and* $\overline{X}_{T_1-}$ *are independent, such that* Δ_{T_1} *follows a Pareto distribution with*

$$\mathbb{P}\{\Delta_{T_1} \in \mathrm{d}z \mid T_1 = t, X_{T_1-} = x\} = \alpha(1-x)^\alpha z^{-\alpha-1}\mathbf{1}\{z > 1-x\}\,\mathrm{d}z,$$

and for each $c \in [x \vee 0, 1]$,

$$\mathbb{P}\{\overline{X}_{T_1-} \le c \mid T_1 = t, X_{T_1-} = x\} = h_c(x,t)/h_1(x,t).$$

Remark The foundation of conditional probability and conditional p.d.f. is measure theory [3]. In Corollary 5, each can be expressed in terms of a joint p.d.f. For example, if $k(z, t, x)$ is the joint p.d.f. of $\overline{X}_{T_1-}$, T_1, and X_{T_1-}, then $\mathbb{P}\{\overline{X}_{T_1-} \le c \mid T_1 = t, X_{T_1-} = x\} = \int_0^c k(z, t, x)\,\mathrm{d}z / \int_0^1 k(z, t, x)\,\mathrm{d}z$.

By further analysis of $h_c(x, t)$, the joint p.d.f. of X_t, the running supremum $\overline{X}_t$, and the time of the running supremum $\overline{G}_t = \sup\{s < t : X_s = \overline{X}_s\}$ can be obtained. Since by scaling

$$(\overline{G}_t, \overline{X}_t, X_t) \sim (t\overline{G}_1, t^{1/\alpha}\overline{X}_1, t^{1/\alpha}X_1),$$

it suffices to consider $t = 1$. As noted earlier, the distribution of $\overline{X}_1$ is known [1,19,20]. The distribution of $\overline{G}_1$ is also known. Indeed, $\overline{G}_t = \Lambda_{\vartheta_t-}$, where $\vartheta_t = \inf\{s > 0 : \Lambda_s > t\}$ and Λ is the ladder time process of X, which is strictly stable with index $1 - 1/\alpha$ ([2], Lemma VIII.1). Then, by scaling, $\overline{G}_1 \sim \overline{G}_t/t = \Lambda_{\vartheta_t-}/t$ and letting $t \to 0$ yields $\overline{G}_1 \sim \mathrm{Beta}(1 - 1/\alpha, 1/\alpha)$ according to the generalized arcsine law ([2], Theorem III.6). That is, the p.d.f. of $\overline{G}_1$ at $x \in (0, 1)$ is $\pi^{-1}\sin(\pi/\alpha)x^{-1/\alpha}(1 - x)^{1/\alpha-1}$. Also, from the excursion theory ([2], IV.4), conditionally on $\overline{G}_1$, $(X_t)_{t\le\overline{G}_1}$ and $(X_{t+\overline{G}_1} - X_{\overline{G}_1})_{t\le 1-\overline{G}_1}$ are independent. With this background, we have the next result. By $(\underline{G}_1, \underline{X}_1, X_1) \sim (1 - \overline{G}_1, X_1 - \overline{X}_1, X_1)$, where $\underline{X}_t = \inf_{0\le s\le t} X_s$ and $\underline{G}_t = \sup\{s < t : X_s = \underline{X}_s\}$, it also provides the joint p.d.f. of $\underline{G}_1$, $\underline{X}_1$, and X_1.

Corollary 6 *$\overline{G}_1$, $\overline{X}_1$, and X_1 have joint p.d.f.*

$$\frac{\mathbb{P}\{\overline{G}_1 \in \mathrm{d}r, \overline{X}_1 \in \mathrm{d}c, X_1 \in \mathrm{d}x\}}{\mathrm{d}r\,\mathrm{d}c\,\mathrm{d}x} = m(c, r) f_{x-c}(1 - r) \tag{11}$$

for $r \in (0, 1)$ and $c > x \vee 0$, where

$$m(c, r) = \frac{\sin(\pi/\alpha)}{\pi c} \sum_{n=1}^{\infty} \frac{\Gamma(1/\alpha + n)}{\Gamma(\alpha n)} (-1)^{1+n} c^{\alpha n} r^{-1/\alpha-n} \tag{12}$$

$$= \frac{\sin(\pi/\alpha)}{\pi c^2} \int_0^{\infty} s^{1/\alpha} E_{\alpha,\alpha}(-s) e^{-sr/c^{\alpha}}\,\mathrm{d}s > 0. \tag{13}$$

Moreover, conditionally on $\overline{G}_1 = r \in (0, t)$, $\overline{X}_1$ and $\overline{X}_1 - X_1$ are independent, such that

$$\frac{\mathbb{P}\{\overline{X}_1 \in \mathrm{d}c \mid \overline{G}_1 = r\}}{\mathrm{d}c} = \Gamma(1 - 1/\alpha) r^{1/\alpha} m(c, r), \quad c > 0, \tag{14}$$

and

$$\frac{\mathbb{P}\{\overline{X}_1 - X_1 \in \mathrm{d}x \mid \overline{G}_1 = r\}}{\mathrm{d}x} = \frac{\Gamma(1/\alpha) x g_{1-r}(-x)}{(1 - r)^{1/\alpha}}, \quad x > 0.$$

Remark (1) For any $r > 0$,

$$\frac{\mathbb{P}\{\overline{X}_1 \in r^{1/\alpha}\mathrm{d}c \mid \overline{G}_1 = r\}}{\mathrm{d}c} = \Gamma(1-1/\alpha)r^{2/\alpha}m(r^{1/\alpha}c, r) = \Gamma(1-1/\alpha)m(c,1),$$

and therefore, $\overline{X}_1/\overline{G}_1^{1/\alpha}$ is independent of $\overline{G}_1$. This is a special case of the result in [14] that shows the independence for any strictly stable process.

(2) By duality, it is natural to interpret $\Gamma(1/\alpha)xg_1(-x) = \Gamma(1/\alpha)f_{-x}(1)$, $x > 0$, as the conditional p.d.f. of X_1 at $-x$ given $\overline{X}_1 = 0$. Likewise, by letting $r = 1$ in (14) and reading $\mathbb{P}\{\overline{X}_1 \in \mathrm{d}c \mid \overline{G}_1 = 1\}$ as $\mathbb{P}\{\overline{X}_1 \in \mathrm{d}c \mid X_1 = \overline{X}_1\} = \mathbb{P}\{\overline{X}_1 \in \mathrm{d}c \mid \underline{X}_1 = 0\}$, $\Gamma(1-1/\alpha)m(c,1)$ may be interpreted as the conditional p.d.f. of $\overline{X}_1$ at c given $\underline{X}_1 = 0$; see more comments in Sect. 3.

(3) It is worth mentioning that, for a Lévy process X in general, if under its law 0 is regular for $(0,\infty)$ and for $(-\infty,0)$, then for any $t > 0$, X is continuous at $\overline{G}_t$. First, $\overline{G}_t \in (0,t)$ a.s. (see [2], p. 157). Second, given $\epsilon > 0$, any $t_0 \in (0,t)$ where X makes a positive jump of size at least ϵ is a stopping time, so by the regularity of 0 for $(0,\infty)$, there are infinitely many $1 > t_n \downarrow t_0$ with $X_{t_n} > X_{t_0} > X_{t_0-}$. On the other hand, any $t_0 \in (0,t)$ where X makes a negative jump of absolute size at least ϵ is a stopping time, so by duality and the regularity of 0 for $(-\infty,0)$, there are infinitely many $0 < t_n \uparrow t_0$ with $X_{t_n} > X_{t_0-} > X_{t_0}$. Since $\epsilon > 0$ is arbitrary, this implies that $\overline{G}_t$ cannot be a time where X makes a jump, and so X is continuous at $\overline{G}_t$.

It can be seen that $m(c,t)\,\mathrm{d}t\,\mathrm{d}c$ is the renewal measure of the bivariate ascending ladder (time and height) process of X, by using the quintuple law for first passage in [7] or more directly, by using $\mathbb{E}(e^{-\beta\overline{X}_\tau}) = \kappa(q,0)/\kappa(q,\beta)$, $q > 0$, $\beta > 0$, where $\kappa(\lambda,\beta)$ is the characteristic exponent of the ladder process, and τ is a random variable with p.d.f. $qe^{-qx}\mathbf{1}\{x > 0\}$ independent of X ([2], p. 163). First, by (14) and $\overline{G}_1 \sim \mathrm{Beta}(1-1/\alpha, 1/\alpha)$, the joint p.d.f. of $(\overline{G}_1, \overline{X}_1)$ can be written down. Then, by scaling and (13), for each $t > 0$, $(\overline{G}_t, \overline{X}_t)$ has joint p.d.f.

$$\frac{t^{-1-1/\alpha}m(ct^{-1/\alpha}, r/t)(1-r/t)^{1/\alpha-1}\mathbf{1}\{0 < r < t\}}{\Gamma(1/\alpha)} = \frac{m(c,r)(t-r)^{1/\alpha-1}\mathbf{1}\{0 < r < t\}}{\Gamma(1/\alpha)}.$$

Then,

$$\begin{aligned}\mathbb{E}(e^{-\beta\overline{X}_\tau}) &= \frac{1}{\Gamma(1/\alpha)}\int_{c>0,t>r>0} m(c,t)(t-r)^{1/\alpha-1}e^{-\beta c}\,\mathrm{d}r\,\mathrm{d}c \times (qe^{-qt})\,\mathrm{d}t \\ &= q^{1-1/\alpha}\int_{c>0,r>0} m(c,r)e^{-qr-\beta c}\,\mathrm{d}r\,\mathrm{d}c.\end{aligned}$$

On the other hand, $\kappa(q,0) = q^{1-1/\alpha}$ ([2], p. 218). Therefore,

$$\kappa(q,\beta) = \left(\int_{c>0,r>0} m(c,r)e^{-qr-\beta c}\,\mathrm{d}r\,\mathrm{d}c\right)^{-1}, \tag{15}$$

and so $m(c, r)$ is the density of the renewal measure of the ladder process. From (13),

$$\int_0^\infty m(c,r)e^{-qr}\,\mathrm{d}r = \frac{\sin(\pi/\alpha)}{\pi c^2}\int_0^\infty \frac{s^{1/\alpha}E_{\alpha,\alpha}(-s)}{q+s/c^\alpha}\,\mathrm{d}s.$$

The integral representation does not seem to provide an easy path to an explicit formula for $\kappa(q, \beta)$. On the other hand, it can be shown that for $q \ge 0$, $\beta \ge 0$,

$$\kappa(q,\beta) = \begin{cases} \dfrac{\beta^\alpha - q}{\beta - q^{1/\alpha}} & \text{if } \beta \ne q^{1/\alpha}, \\ [2ex]\alpha\beta^{\alpha-1} & \text{else.} \end{cases} \tag{16}$$

The formula can be derived from a series expansion of $\kappa(q, \beta)$ in [11], which holds for any non-monotone strictly stable process with index in a dense subset $\mathcal{A}$ of $(0, 2) \setminus \mathbb{Q}$. In the case of X, provided $\alpha \in (1, 2) \cap \mathcal{A}$, the series can be reduced to the closed form in (16). Then, by continuity, (16) holds for all $\alpha \in (1, 2)$. In the "Appendix," we will give an alternative proof of (16) without relying on the continuity argument.

In the next section, as a preparation, some general results on first passage of a Lévy process are derived. This section also collects some standard results on stable processes. In Sect. 3, Theorem 1 and its corollaries are proved. In Sect. 4, we show that $(T_1, X_{T_1-}, \Delta T_1)$ can be sampled exactly. It will be seen that the main issue is the sampling of $h_1(x, \cdot)/v_1(x)$ for any fixed $x < 1$, which is the conditional p.d.f. of T_1 given $X_{T_1-} = x$. The key is to show that $h_1(x, t)$ can be decomposed as the sum of positive functions $\phi_1(t), \phi_2(t), \ldots$. Even though these functions do not have a closed form, given $t > 0$, each can be evaluated in a finite number of steps, and for the exact sampling, only a finite number of them have to be evaluated. It is important to keep in mind that these functions are constructed with the value of $h_1(x, t)$ being intractable. The decomposition then allows the conditional p.d.f. of T_1 to be sampled by the rejection sampling method.

2 Some General Distributional Results

We first consider Lévy processes in general and then specialize to spectrally positive ones.

2.1 Properties of First Passage by a General Lévy Process

Proposition 7 *Let X be a Lévy process and $\Pi(\mathrm{d}x)$ its Lévy measure.*

(a) (Distribution when X jumps over a level). *For every $c \ge 0$, $t > 0$, $x \in \mathbb{R}$, $w \in \mathbb{R}$, and $y > c$,*

$$\begin{aligned}&\mathbb{P}\{T_c \in \mathrm{d}t, X_{T_c-} \in \mathrm{d}x, X_{T_c} \in \mathrm{d}y, \overline{X}_{T_c-} \in \mathrm{d}w\} \\ &= \mathbf{1}\{x \vee 0 \le y \le c\}\,\mathrm{d}t\,\Pi(\mathrm{d}y - x)\,\mathbb{P}\{X_t \in \mathrm{d}x, \overline{X}_t \in \mathrm{d}w\}.\end{aligned} \tag{17}$$

(b) *For every* $c \ge 0$, $\mathbb{P}\{X_{T_c-} < X_{T_c} = c\} = 0$.

Remark Part b) is known when X is strictly stable with index $\alpha > 1$ ([2], Proposition VIII.8).

Proof (a) The proof is standard so we only give a sketch of it (cf. [2], p. 76). Given a Borel function $f(t, x, y, w) \ge 0$, $f(T_c, X_{T_c-}, X_{T_c}, \overline{X}_{T_c-})\mathbf{1}\{X_{T_c} > c\} = \sum_{t:\Delta_t \ne 0} H_t(\Delta_t)$, where $H_t(z) = f(t, X_{t-}, X_{t-} + z, \overline{X}_{t-})\mathbf{1}\{z > c - X_{t-} \ge 0, \overline{X}_{t-} \le c\}$. Then, by the compensation formula ([2], p. 7),

$$\int f(t, x, y, w)\mathbf{1}\{y > c\}\,\mathbb{P}\{T_c \in \mathrm{d}t, X_{T_c-} \in \mathrm{d}x, X_{T_c} \in \mathrm{d}y, \overline{X}_{T_c-} \in \mathrm{d}w\}$$
$$= \int \mathbb{E}[H_t(z)]\,\mathrm{d}t\,\Pi(\mathrm{d}z).$$

However, $\mathbb{E}[H_t(z)] = \int f(t, x, x + z, w)\mathbf{1}\{z > c - x \ge 0, x \vee 0 \le w \le c\}$ $\mathbb{P}\{X_t \in \mathrm{d}x, \overline{X}_t \in \mathrm{d}w\}$. Plug the equation into the right-hand side (r.h.s.) of the display. Since f is arbitrary, by comparing he integrals on both sides, (17) follows.

(b) If 0 is not regular for $(0, \infty)$, then by the strong Markov property of X, there is a random $\epsilon > 0$, such that $X_t \le X_{T_c}$ for $t \in (T_c, T_c + \epsilon)$, implying $X_{T_c} > c$. Now suppose 0 is regular for $(0, \infty)$. If $X_{T_c} = c$, then $T_c \ge \tau := \inf\{t : X_t = c, X_s < c\,\forall s < t\}$. However, by the regularity of 0 and strong Markov property, $X_{t_n} > X_\tau = c$ for an infinite sequence $t_n \downarrow \tau$, implying $T_c = \tau$. Then, $\mathbf{1}\{X_{T_c} = c > X_{T_c-}\} \le \sum_{t:\Delta_t>0} \mathbf{1}\{X_t = c, X_s < c\,\forall s < t\}$. Then, by following the argument for Proposition III.2(ii) in [2] and noting that X is not compound Poisson, the claim follows.

□

In the next preliminary result, denote $\overline{\Pi}(x) = \Pi((x, \infty))$.

Proposition 8 *Suppose* $\overline{\Pi}(0) > 0$ *and each* X_t *has a p.d.f. Fix* $c > 0$ *and define*

$$v_c(x) = \int_0^\infty h_c(x, t)\,\mathrm{d}t, \tag{18}$$

where $h_c(x, t)$ *is as in* (2). *Let* $D_c = \{\Delta_{T_c} > 0\}$*, i.e., the event that* X *has a jump at the first passage at level* c.

(a) $v_c(x) < \infty$ *for a.e.* $x \le c$ *(in Lebesgue measure).*
(b) *Conditionally on* D_c, X_{T_c-} *is concentrated on* $\Omega_c = \{x \le c : \overline{\Pi}(c-x)v_c(x) > 0\}$. *Moreover, conditionally on* D_c *and* $X_{T_c-} = x \in \Omega_c$, $(T_c, \overline{X}_{T_c-})$ *and* Δ_{T_c} *are independent, such that*

$$\mathbb{P}\{\Delta_{T_c} \in \mathrm{d}z \mid D_c, X_{T_c-} = x\} = \frac{\mathbf{1}\{z > c - x\}\,\Pi(\mathrm{d}z)}{\overline{\Pi}(c - x)},$$
$$\mathbb{P}\{T_c \in \mathrm{d}t \mid D_c, X_{T_c-} = x\} = \frac{h_c(x, t)}{v_c(x)},$$

and for $w \in [x \vee 0, c]$

$$\mathbb{P}\{\overline{X}_{T_c-} \le w \,|\, T_c = t, D_c, X_{T_c-} = x\} = \frac{h_w(x,t)}{h_c(x,t)}.$$

Proof (a) Fix $-\infty < a < b \le c$. By Fubini theorem,

$$\int_a^b v_c(x)\,\mathrm{d}x \le \int_a^b \mathrm{d}x \int_0^\infty \frac{\mathbb{P}\{X_t \in \mathrm{d}x\}}{\mathrm{d}x}\,\mathrm{d}t = \int_0^\infty \mathbb{P}\{a \le X_t \le b\}\,\mathrm{d}t.$$

By definition, if X is transient, then the last integral is finite ([2]. p. 32) and so $\int_a^b v_c < \infty$. Since a and b are arbitrary, $v_c(x) < \infty$ for a.e. $x < c$. If X is not transient, then it is recurrent, so $\overline{X}_t \to \infty$ and $\underline{X}_t \to -\infty$ a.s. ([2], p. 167–168). Given $r > 0$, let τ be an exponentially distributed random variable with mean $1/r$ and independent of X. Then,

$$\begin{aligned}\int_a^b \mathrm{d}x \int_0^\infty e^{-rt} h_c(x,t)\,\mathrm{d}t &= \int_0^\infty e^{-rt}\,\mathrm{d}t \int_a^b \mathbb{P}\{X_t \in \mathrm{d}x, \overline{X}_t \le c\} = r^{-1}\mathbb{P}\{\overline{X}_\tau \le c, X_\tau \in [a,b]\}\\ &\overset{(*)}{=} r^{-1}\int \mathbf{1}\{0 \le s \le c, y \ge 0, a \le s-y \le b\}\,\mathbb{P}\{\overline{X}_\tau \in \mathrm{d}s\}\mathbb{P}\{-\underline{X}_\tau \in \mathrm{d}y\}\\ &\le r^{-1}\mathbb{P}\{\overline{X}_\tau \in [0,c]\}\mathbb{P}\{-\underline{X}_\tau \in [(-b)\vee 0, c-a]\},\end{aligned}$$

where $(*)$ is due to $\overline{X}_\tau$ and $X_\tau - \overline{X}_\tau \sim \underline{X}_\tau$ being independent ([2], Theorem VI.5 and Proposition VI.3). As in the proof of Theorem VI.20 in [2] or Theorem 3 in [7], let $r \downarrow 0$. By monotone convergence, $\int_a^b v_c \le \mathcal{U}([0,c])\widehat{\mathcal{U}}([(-b)\vee 0, c-a])$, where $\mathcal{U}$ (resp. $\widehat{\mathcal{U}}$) is the renewal measure of the ascending (resp. descending) ladder height process of X. Since both ladder processes are transient, the r.h.s. is finite, again yielding $v_c(x) < \infty$ for a.e. x.

(b) By Proposition 7(b), for $t > 0$, $x \le c$, $x \vee 0 \le w \le c$, and $z > 0$,

$$\mathbb{P}\{T_c \in \mathrm{d}t, X_{T_c-} \in \mathrm{d}x, \overline{X}_{T_c-} \le w, \Delta_{T_c} \in \mathrm{d}z\} = \mathbf{1}\{z > c-x\}\,\mathrm{d}t\, h_w(x,t)\,\mathrm{d}x\,\Pi(\mathrm{d}z).$$

Integrating over t and z yields $\mathbb{P}\{X_{T_c-} \in \mathrm{d}x, \overline{X}_{T_c-} \le w, D_c\} = \overline{\Pi}(c-x)v_w(x)\,\mathrm{d}x$. In particular, letting $w = c$ gives $\mathbb{P}\{X_{T_c-} \in \mathrm{d}x, \Delta_{T_c} > 0\} = \overline{\Pi}(c-x)v_c(x)\,\mathrm{d}x$. This shows that conditionally on D_c, X_{T_c-} is concentrated on Ω_c and, together last display, also shows that for $x \in \Omega_c$,

$$\begin{aligned}&\mathbb{P}\{T_c \in \mathrm{d}t, \overline{X}_{T_c-} \le w, \Delta_{T_c} \in \mathrm{d}z \,|\, X_{T_c-} \in \mathrm{d}x, \Delta_{T_c} > 0\}\\ &= \frac{h_w(x,t)\,\mathrm{d}t}{h_c(x,t)} \times \frac{h_c(x,t)}{v_c(x)} \times \frac{\mathbf{1}\{z > c-x\}\,\Pi(\mathrm{d}z)}{\overline{\Pi}(c-x)}.\end{aligned}$$

Then, the rest of the claim easily follows.

□

2.2 The Spectrally Positive Case

Let X be a spectrally positive Lévy process that is not a subordinator. Then, single points are not essentially polar for X, whether the process has bounded variation ([18], Theorem 43.13) or not ([2], Corollary VII.5). From potential theory ([2], Section II.5), it follows that X has a bounded q-coexcessive version of resolvent density $u^q(x)$ that satisfies

$$u^q(x) = \mathbb{E}[e^{-q\tau_x}]u^q(0) \tag{19}$$

for $q > 0$ and $x \in \mathbb{R}$, and if for every $t > 0$, X_t has a p.d.f. g_t, then

$$u^q(x) = \int_0^\infty e^{-qt} g_t(x)\, \mathrm{d}t, \tag{20}$$

which can be extended to $q = 0$ when X is transient. It will always be assumed that g_t is the unique version of p.d.f. that satisfies

$$\int g_s(y) g_t(x-y)\, \mathrm{d}y = g_{s+t}(x)$$

for all $x, y \in \mathbb{R}$ and $s, t > 0$. Equation (20) is stated in Remark 41.20 of [18] under the assumption that g_t is bounded and continuous. It is probably known that (20) holds in general. However, we could not find an explicit proof in the literature, so for convenience, one is given in the "Appendix."

To evaluate $h_c(x,t)$ defined in (2) for stable processes, the following result will be used.

Proposition 9 *Suppose that each X_t has a p.d.f. g_t. Then, given $x < c$,*

$$h_c(x,t) = g_t(x) - \int_0^t f_c(s) g_{t-s}(x-c)\, \mathrm{d}s \tag{21}$$

$$= g_t(x) - \int_0^t f_{x-c}(s) g_{t-s}(c)\, \mathrm{d}s. \tag{22}$$

Furthermore, if $x > 0$, then $h_c(-x,\cdot)$ is the convolution of $h_c(0,\cdot)$ and f_{-x}, i.e.,

$$h_c(-x,\cdot) = h_c(0,\cdot) * f_{-x}. \tag{23}$$

Proof Since X has no negative jumps and $\tau_c > T_c$ a.s. ([2], Proposition VIII.8(ii)), for each $A \subset (-\infty, c)$, $\mathbf{1}\{X_t \in A, \overline{X}_t > c\} = \mathbf{1}\{X_t \in A, \tau_c < t\}$ a.s. Then, by the strong Markov property of X, for any bounded continuous function $k(x) \ge 0$ with support in $(-\infty, c)$,

$$\mathbb{E}[k(X_t)\mathbf{1}\{\overline{X}_t > c\}] = \mathbb{E}[k(X_t)\mathbf{1}\{\tau_c < t\}] = \int_0^t \mathbb{E}[k(X_{t-s}+c)]\mathbb{P}\{\tau_c \in \mathrm{d}s\}$$

$$= \int_0^t \left[\int k(x+c) g_{t-s}(x)\,\mathrm{d}x\right] f_c(s)\,\mathrm{d}s$$
$$= \int k(x) \left[\int_0^t f_c(s) g_{t-s}(x-c)\,\mathrm{d}s\right] \mathrm{d}x.$$

On the other hand,

$$\int k(x) h_c(x,t)\,\mathrm{d}x = \mathbb{E}[k(X_t)\mathbf{1}\{\overline{X}_t \le c\}] = \mathbb{E}[k(X_t)] - \mathbb{E}[k(X_t)\mathbf{1}\{\overline{X}_t > c\}].$$

The two displays together with $\mathbb{E}[k(X_t)] = \int k(x) g_t(x)\,\mathrm{d}x$ give (21). Equation (22) is essentially shown on p. 4/10 of [13] (also see [16]). Given $x > 0$, write $m_{-x}(t) = g_t(-x)$ as a function of t and $\mathcal{L}[m_{-x}]$ its Laplace transform. By (19) and (20), $\mathcal{L}[m_{-x}] = u^q(-x) = \mathcal{L}[f_{-x}]u^q(0) = \mathcal{L}[f_{-x}]\mathcal{L}[m_0]$. Then, $m_{-x} = m_0 * f_{-x}$. Also, $f_{-x-c} = f_{-c} * f_{-x}$. Plugging the two equations into (22) yields (23). □

2.3 Preliminaries on Stable Processes

From now on, let X be a spectrally positive and strictly stable process with index $\alpha \in (1,2)$ satisfying (3). Then, the Lévy measure of X is

$$\Pi(\mathrm{d}x) = \frac{\mathbf{1}\{x > 0\}\, x^{-\alpha-1}}{\Gamma(-\alpha)}\,\mathrm{d}x \tag{24}$$

([9], p. 570). By scaling and [18], p. 88, g_t has power series expansion on $\mathbb{R}$,

$$g_t(x) = t^{-1/\alpha} g_1(t^{-1/\alpha}x) = \frac{1}{\alpha\pi}\sum_{k=1}^\infty \frac{\Gamma(k/\alpha)}{(k-1)!}\sin(k\pi/\alpha) t^{-k/\alpha} x^{k-1}. \tag{25}$$

By [2], Theorem VII.1, $(\tau_{-x})_{x\ge 0}$ is strictly stable with index $1/\alpha$, such that

$$\mathbb{E}(e^{-q\tau_{-x}}) = \exp(-xq^{1/\alpha}), \quad q \ge 0. \tag{26}$$

By scaling and [18], p. 88, or by Kendall's identity, for $x > 0$ and $t > 0$,

$$f_{-x}(t) = x^{-\alpha} f_{-1}(x^{-\alpha}t) = \frac{1}{\pi}\sum_{k=1}^\infty (-1)^{k-1}\frac{\Gamma(k/\alpha+1)}{k!}\sin(\pi k/\alpha) x^k t^{-k/\alpha-1}. \tag{27}$$

From (25), $g_t(x)$ as a function of (x,t) can be extended from $\mathbb{R}\times(0,\infty)$ to $\mathbb{C}\times(\mathbb{C}\setminus(-\infty,0])$, such that for each fixed $x\in\mathbb{C}$, the extension is an analytic function of $t\in\mathbb{C}\setminus(-\infty,0]$, and for each fixed $t\in\mathbb{C}\setminus(-\infty,0]$, it is an analytic function of $x\in\mathbb{C}$. By (27), $f_{-x}(t)$ can be similarly extended from $(0,\infty)\times(0,\infty)$ to $\mathbb{C}\times(\mathbb{C}\setminus(-\infty,0])$. However, the extension is not the same as $f_{-x}(t)$ for $(x,t)\in(-\infty,0)\times(0,\infty)$. Indeed, for $x<0$ and $t>0$, the extension necessarily has the power series expansion (27). On the other hand, for $x < 0$, the power series of $f_{-x}(t)$ is quite different ([20], Proposition 3).

Finally, for $s \in \mathbb{R}$ ([21], Section 5.6),

$$\int_0^\infty x^s g_1(x)\,\mathrm{d}x = \begin{cases} \dfrac{\Gamma(s)\Gamma(1-s/\alpha)}{\Gamma(s(1-1/\alpha))\Gamma(1-s(1-1/\alpha))} & \text{if } s \in (-1,\alpha) \\ \infty & \text{else} \end{cases} \tag{28}$$

and

$$\int_0^\infty t^s f_{-1}(t)\,\mathrm{d}t = \begin{cases} \dfrac{\Gamma(1-s\alpha)}{\Gamma(1-s)} & \text{if } s < 1/\alpha \\ \infty & \text{else.} \end{cases} \tag{29}$$

3 Proof of Main Results

3.1 Initial Deduction by Laplace Transform

We need the following formulas from [20]. Given $x > 0$,

$$\mathbb{E}(e^{-q\tau_x}) = \mathbb{E}(e^{-qx^\alpha \tau_1}) = F_1(q^{1/\alpha}x) - \alpha F'_\alpha(q^{1/\alpha}x), \tag{30}$$

where $F_a(x) = E_a(x^a) := E_{a,1}(x^a)$ and for fixed $a > 0$ and $d \in \mathbb{C}$, the following function of $z \in \mathbb{C}$

$$E_{a,d}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(an+d)}$$

is known as the Mittag–Leffler function. Then, $F_1(q^{1/\alpha}x) = E_1(q^{1/\alpha}x) = e^{q^{1/\alpha}x}$ and from

$$F'_\alpha(z) = \left(\sum_{n=0}^\infty \frac{z^{\alpha n}}{\Gamma(1+\alpha n)}\right)' = \sum_{n=1}^\infty \frac{z^{\alpha n-1}}{\Gamma(\alpha n)},$$

it follows that

$$\alpha F'_\alpha(q^{1/\alpha}x) = \alpha \sum_{n=1}^\infty \frac{q^{n-1/\alpha}x^{\alpha n-1}}{\Gamma(\alpha n)}.$$

Fix $x < 1$. We seek the Laplace transform of $h_1(x,\cdot)$. For brevity, put

$$\hbar(t) = h_1(x,t).$$

Proposition 10 *For* $q > 0$,

$$\mathcal{L}[\hbar](q) = e^{-q^{1/\alpha}(1-x)} \sum_{n=1}^{\infty} \frac{q^{n-1}}{\Gamma(\alpha n)} - \sum_{n=1}^{\infty} \frac{q^{n-1}(x \vee 0)^{\alpha n-1}}{\Gamma(\alpha n)}. \tag{31}$$

Remark By (18), $v_1(x) = \mathcal{L}[\hbar](0+)$, which together with (31) yields (8).

Proof of Proposition 10 By (21) in Proposition 9, the Laplace transform of $\hbar$ is

$$\begin{aligned}\mathcal{L}[\hbar](q) &= \int_0^{\infty} e^{-qt}\left[g_t(x) - \int_0^t f_1(s) g_{t-s}(x-1)\,\mathrm{d}s\right]\mathrm{d}t \\ &= u^q(x) - \mathbb{E}(e^{-q\tau_1})u^q(x-1).\end{aligned}$$

By (20), scaling, and (25),

$$u^q(0) = \int_0^{\infty} e^{-qt} g_t(0)\,\mathrm{d}t = g_1(0)\int_0^{\infty} e^{-qt} t^{-1/\alpha}\,\mathrm{d}t = \alpha^{-1} q^{1/\alpha-1}.$$

Then, by (19),

$$\mathcal{L}[\hbar](q) = \alpha^{-1} q^{1/\alpha-1}[\mathbb{E}(e^{-q\tau_x}) - \mathbb{E}(e^{-q\tau_{x-1}})\mathbb{E}(e^{-q\tau_1})]. \tag{32}$$

Since $x - 1 < 0$, by (26), $\mathbb{E}(e^{-q\tau_{x-1}}) = e^{(x-1)q^{1/\alpha}}$. If $x \le 0$, then $\mathbb{E}(e^{-q\tau_x}) = e^{xq^{1/\alpha}}$ as well, and so applying (30) to $\mathbb{E}(e^{-q\tau_1})$,

$$\begin{aligned}\mathbb{E}(e^{-q\tau_x}) - \mathbb{E}(e^{-q\tau_{x-1}})\mathbb{E}(e^{-q\tau_1}) &= e^{q^{1/\alpha}x} - e^{q^{1/\alpha}(x-1)}\left(e^{q^{1/\alpha}} - \alpha\sum_{n=1}^{\infty}\frac{q^{n-1/\alpha}}{\Gamma(\alpha n)}\right) \\ &= \alpha e^{q^{1/\alpha}(x-1)}\sum_{n=1}^{\infty}\frac{q^{n-1/\alpha}}{\Gamma(\alpha n)}.\end{aligned}$$

On the other hand, if $x > 0$, then applying (30) to both $\mathbb{E}(e^{-q\tau_x})$ and $\mathbb{E}(e^{-q\tau_1})$,

$$\begin{aligned}&\mathbb{E}(e^{-q\tau_x}) - \mathbb{E}(e^{-q\tau_{x-1}})\mathbb{E}(e^{-q\tau_1}) \\ &= e^{q^{1/\alpha}x} - \alpha\sum_{n=1}^{\infty}\frac{q^{n-1/\alpha}x^{\alpha n-1}}{\Gamma(\alpha n)} - e^{q^{1/\alpha}(x-1)}\left(e^{q^{1/\alpha}} - \alpha\sum_{n=1}^{\infty}\frac{q^{n-1/\alpha}}{\Gamma(\alpha n)}\right) \\ &= \alpha e^{q^{1/\alpha}(x-1)}\sum_{n=1}^{\infty}\frac{q^{n-1/\alpha}}{\Gamma(\alpha n)} - \alpha\sum_{n=1}^{\infty}\frac{q^{n-1/\alpha}x^{\alpha n-1}}{\Gamma(\alpha n)}.\end{aligned}$$

The above two identities for $\mathbb{E}(e^{-q\tau_x}) - \mathbb{E}(e^{-q\tau_{x-1}})\mathbb{E}(e^{-q\tau_1})$ combined with (32) then lead to (31). □

Given $x < 1$, f_{x-1} belongs to $C_0^\infty([0,\infty))$, the family of infinitely differentiable functions on $[0,\infty)$ with derivative of any order equal to zero at 0 and ∞. Since $e^{-q^{1/\alpha}(1-x)}$ is the Laplace transform of f_{x-1}, in view of (31) and the relationship between Laplace transform and differentiation, if $x < 0$, then it is possible to show (7) by interchanging Laplace transform and the infinite summation on the r.h.s. of (31). However, as noted in the introduction, for $x > 0$, the approach fails to work. In our proof, (7) is first established for $x < a$ and $t > 0$, where $a < 0$ is some constant. The argument based on interchanging Laplace transform and infinite summation is carried out. Then, the general case is resolved by analytic extension.

3.2 Proof of Theorem

Fixing $t > 0$, regard $h_1(1-x,t)$ as a function of x. We need a preliminary estimate of the domain it can be analytically extended to. Recall that a domain is a connected open set in $\mathbb{C}$.

Lemma 11 *Given $t > 0$, the mapping $x \to h_1(1-x,t)$ can be analytically extended from $(0,\infty)$ to $\Omega := \{z \in \mathbb{C} : |\arg z| < \pi/2 - \pi/(2\alpha)\}$.*

Proof The following fact will be used. Let $D \subset \mathbb{C}$ be a domain and $J \subset \mathbb{R}$. Suppose $m(z,\lambda)$ is a measurable function on $D \times J$ and ν is a measure on J. If $m(\cdot,\lambda)$ is analytic in D for each $\lambda \in J$, and the mapping $z \to \int |m(z,\lambda)|\nu(\mathrm{d}\lambda)$ is bounded on any compact subset of D, then by Fubini's theorem and Morera's theorem ([17], p. 208), $M(z) = \int m(z,\lambda)\nu(\mathrm{d}\lambda)$ is analytic on D.

Given $t > 0$, by Proposition 9, $h_1(1-x,t) = g_t(1-x) - \int_0^t f_1(t-s)g_s(-x)\,\mathrm{d}s$. Since g_t can be analytically extended to $\mathbb{C}$, it suffices to show that $x \to \int_0^t g_s(-x)f_1(t-s)\,\mathrm{d}s$ can be analytically extended to Ω. By Kendall's identity

$$\int_0^t g_s(-x)f_1(t-s)\,\mathrm{d}s = \frac{1}{x}\int_0^t sf_{-x}(s)f_1(t-s)\,\mathrm{d}s.$$

The Fourier transform of f_{-x} is $\widehat{f}_{-x}(\lambda) = \mathcal{L}[f_{-x}](-\mathrm{i}\lambda) = e^{-(-\mathrm{i}\lambda)^{1/\alpha}x}$, $\lambda \in \mathbb{R}$, where $-\pi < \arg(-\mathrm{i}\lambda) \le \pi$. Then, $|\widehat{f}_{-x}(\lambda)| = e^{-\mathrm{Re}(-\mathrm{i}\lambda)^{1/\alpha}x} = e^{-|\lambda|^{1/\alpha}x\cos a}$ with $a = \pi/(2\alpha)$. As $\cos a > 0$, Fourier inversion can be applied to get

$$f_{-x}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}\lambda s-(-\mathrm{i}\lambda)^{1/\alpha}x}\,\mathrm{d}\lambda$$

and Fubini theorem can be applied to get

$$\int_0^t g_s(-x)f_1(t-s)\,\mathrm{d}s = \frac{1}{2\pi x}\int_{-\infty}^{\infty} \psi(\lambda)e^{-(-\mathrm{i}\lambda)^{1/\alpha}x}\,\mathrm{d}\lambda,$$

where $\psi(\lambda) = \int_0^t se^{-\mathrm{i}\lambda s}f_1(t-s)\,\mathrm{d}s$ is bounded. Given a compact $C \subset \Omega$, $\theta_0 := \sup_{z\in C}|\arg(z)| < \pi/2 - a$ and $r_0 := \inf_{z\in C}|z| > 0$. For $z = re^{\mathrm{i}\theta} \in C$,

$\mathrm{Re}((-\mathrm{i}\lambda)^{1/\alpha} z) = \lambda^{1/\alpha} r \cos(\theta \pm a)$, where the sign of a is opposite to that of λ. By $|\theta \pm a| \le \theta_0 + a < \pi/2$, $\mathrm{Re}((-\mathrm{i}\lambda)^{1/\alpha} z) \ge c\lambda^{1/\alpha}$ with $c = r_0 \cos(\theta_0 + a) > 0$. Then, $\int_{-\infty}^{\infty} \psi(\lambda) e^{-(-\mathrm{i}\lambda)^{1/\alpha} z}\, \mathrm{d}\lambda$ is bounded on C. As remarked at the beginning, this yields the proof. □

Lemma 12 *For every $x < 1$ and $t > 0$, the series in (5) and (7) converge absolutely and are equal to each other, and as functions of (x, t) can be extended to $\mathbb{C} \times (\mathbb{C} \setminus (-\infty, 0])$, such that for each fixed $x \in \mathbb{C}$, the extended function is analytic in $t \in \mathbb{C} \setminus (-\infty, 0]$, and for each fixed $t \in \mathbb{C} \setminus (-\infty, 0]$, the extended function is analytic in $x \in \mathbb{C}$.*

Proof By (27), the series in (7) is

$$\sum_{n=1}^{\infty} \frac{1}{\Gamma(\alpha n)} \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \left(\frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k/\alpha + 1)}{k!} \sin(\pi k/\alpha)(1-x)^k t^{-k/\alpha - 1} \right)$$
$$= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\Gamma(\alpha n)} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k/\alpha + n)}{k!} \sin(\pi k/\alpha)(1-x)^k t^{-k/\alpha - n}.$$

Then, to show the entire lemma, it suffices to show that series on (5) converges absolutely. Letting $M = [1 - (x \wedge 0)]t^{-1}$, the sum of the absolute values of the terms in the series is less than

$$\sum_{k,n=1}^{\infty} \frac{\Gamma(k/\alpha + n)}{\Gamma(\alpha n) k!} M^{k/\alpha + n} = \int_0^{\infty} \sum_{k,n=1}^{\infty} \frac{s^{k/\alpha + n - 1}}{\Gamma(\alpha n) k!} e^{-s/M}\, \mathrm{d}s$$
$$= \int_0^{\infty} \left(\sum_{n=1}^{\infty} \frac{s^{n-1}}{\Gamma(\alpha n)} \right) \left(\sum_{k=1}^{\infty} \frac{s^{k/\alpha}}{k!} \right) e^{-s/M}\, \mathrm{d}s \le \int_0^{\infty} E_{\alpha,\alpha}(s) e^{s^{1/\alpha} - s/M}\, \mathrm{d}s. \tag{33}$$

From (22) on p. 210 of [8], as $s \to \infty$, $E_{\alpha,\alpha}(s) e^{s^{1/\alpha} - s/M} = O(e^{2s^{1/\alpha} - s/M})$. Therefore, the last integral is finite, yielding the desired absolute convergence. □

Given $x > 0$, denote

$$h_x(t) = h_1(1 - x, t)$$

and regard it as a function of $t > 0$. A key ingredient of the proof of Theorem 1 is to show that $h_x(t)$ is identical to

$$\omega_x(t) = \sum_{n=0}^{\infty} \frac{f_{-x}^{(n)}(t)}{\Gamma(\alpha n + \alpha)}. \tag{34}$$

For this purpose, the following lemmas are needed.

Lemma 13 *Given $x > 0$, $h_x(t)$ is a bounded and continuous function of $t > 0$*

Proof By Proposition 9, $h_x(t) \le m_x(t) := g_t(x)$. From (25), $m_x(t)$ is continuous in $t > 0$ and is bounded on $[a, \infty)$ for any $a > 0$. On the other hand, by scaling and $g_1(z) = O(z^{-1-\alpha})$ as $z \to \infty$, $m_x(t) = g_t(x) = t^{-1/\alpha} g_1(t^{-1/\alpha} x) = O(t)$ as $t \to 0$. Therefore, $m_x(t)$ is bounded on $(0, \infty)$ and so is $h_x(t)$. Next, observe that both $f_{-x}(t)$ and $m_1(t)$ can be extended into uniformly continuous and integrable functions on the entire $\mathbb{R}$ with values on $(-\infty, 0]$ equal to 0. As a result, $f_{-x} * m_1$ is continuous. Then, by Proposition 9, $h_x(t) = m_x(t) - (f_{-x} * m_1)(t)$ is continuous. □

Lemma 14 *Let* $x_0 = 1/\cos(\pi/(2\alpha))$ *and fix* $x > x_0$.

(a) *The following function is bounded in* $t > 0$,

$$\varsigma_x(t) = \sum_{n=0}^{\infty} \frac{|f_{-x}^{(n)}(t)|}{\Gamma(\alpha n + \alpha)}.$$

(b) $\omega_x(t)$ *is a bounded and continuous function of* $t > 0$.

(c) $\mathcal{L}[\omega_x](q) = \mathcal{L}[h_x](q)$.

Proof (a) From the bound on $|\widehat{f}_{-1}(\lambda)|$ in the proof of Lemma 11, it follows that for any $n \ge 0$, $\int_{-\infty}^{\infty} |\lambda|^n |\widehat{f}_{-1}(\lambda)| \,\mathrm{d}\lambda < \infty$, so by Fourier inversion

$$f_{-1}^{(n)}(t) = \frac{(-\mathrm{i})^n}{2\pi} \int_{-\infty}^{\infty} \lambda^n e^{-\mathrm{i}\lambda t} \widehat{f}_{-1}(\lambda) \,\mathrm{d}\lambda$$

and hence

$$\begin{aligned} \sup_t |f_{-1}^{(n)}(t)| &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} |\lambda|^n |\widehat{f}_{-1}(\lambda)| \,\mathrm{d}\lambda = \frac{1}{\pi} \int_0^{\infty} \lambda^n e^{-\lambda^{1/\alpha}/x_0} \,\mathrm{d}\lambda \\ &= \frac{\alpha}{\pi} \int_0^{\infty} \lambda^{\alpha n + \alpha - 1} e^{-\lambda/x_0} \,\mathrm{d}\lambda = \frac{\alpha \Gamma(\alpha n + \alpha) x_0^{\alpha n + \alpha}}{\pi}. \end{aligned}$$

Since $f_{-x}(t) = x^{-\alpha} f_{-1}(x^{-\alpha} t)$, then $f_{-x}^{(n)}(t) = x^{-\alpha n - \alpha} f_{-1}^{(n)}(x^{-\alpha} t)$. As a result,

$$\varsigma_x(t) = \sum_{n=0}^{\infty} \frac{x^{-\alpha n - \alpha} |f_{-1}^{(n)}(x^{-\alpha} t)|}{\Gamma(\alpha n + \alpha)} \le \frac{\alpha}{\pi} \sum_{n=0}^{\infty} (x_0/x)^{\alpha n + \alpha},$$

and hence for $x > x_0$, $\varsigma_x(t)$ is bounded in $t > 0$.

(b) From (a), it follows that $\omega_x(t)$ is bounded. The continuity of $\omega_x(t)$ is implied in Lemma 12.

(c) By (a), for $x > x_0$, the summation and integration can interchange in the calculation of $\mathcal{L}[\omega_x](q)$ to yield

$$\mathcal{L}[\omega_x](q) = \sum_{n=0}^{\infty} \frac{\mathcal{L}[f_{-x}^{(n)}](q)}{\Gamma(\alpha n + \alpha)} = \sum_{n=0}^{\infty} \frac{q^n \mathcal{L}[f_{-x}](q)}{\Gamma(\alpha n + \alpha)} = e^{-x q^{1/\alpha}} \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(\alpha n + \alpha)}.$$

Since $x > 1$, from Proposition 10, the r.h.s. is the Laplace transform of $h_x(\cdot) = h_1(1-x,\cdot)$.

□

Proof of Theorem 1 Again write $h_x(t) = h_1(1-x,t)$. By Lemma 14, for $x > x_0 > 1$, $\omega_x(t)$ is bounded and $\mathcal{L}[\omega_x](q) = \mathcal{L}[h_x](q)$ for all $q > 0$. Then, by the one-to-one correspondence between bounded continuous functions and their Laplace transforms, $h_x(t) = \omega_x(t)$. Thus,

$$h_1(1-x,t) = \sum_{n=0}^{\infty} \frac{f_{-x}^{(n)}(t)}{\Gamma(\alpha n + \alpha)}$$
$$= \frac{1}{\pi} \sum_{k,n=1}^{\infty} \frac{\Gamma(k/\alpha + n)}{\Gamma(\alpha n) k!} (-1)^{k+n} x^k \sin(\pi k/\alpha) t^{-k/\alpha - n}.$$

Fix $t > 0$ and treat x as the only variable. By Lemma 11, $h_1(1-x,t)$ can be analytically extended from $(0,\infty)$ to a domain $\Omega \subset \mathbb{C}$ containing $(0,\infty)$, while by Lemma 12, the two series in the display converge absolutely and can be analytically extended to the entire $\mathbb{C}$. Since $h_1(1-x,t)$ and the two series agree on (x_0,∞), they must be equal on Ω, in particular, on the entire $(0,\infty)$. It follows that for every fixed $t > 0$, (5) and (7) hold for all $x < 1$. This completes the proof of (5) and (7). The rest of the theorem follows by combining (5) and (7) with Proposition 7 and (24). □

3.3 Proofs of Corollaries

Proof of Corollary 2 From $\Gamma(z) = \int_0^\infty s^{z-1} e^{-s}\,\mathrm{d}s$ and the absolute convergence of (7),

$$h_1(x,t) = \frac{1}{\pi} \int_0^\infty e^{-s} \sum_{k,n=1}^{\infty} \frac{s^{k/\alpha+n-1}}{\Gamma(\alpha n) k!} (-1)^{k+n} (1-x)^k \sin(\pi k/\alpha) t^{-k/\alpha-n}\,\mathrm{d}s$$
$$= \frac{1}{\pi} \int_0^\infty e^{-s} \left[\sum_{k=1}^{\infty} \frac{((x-1)(s/t)^{1/\alpha})^k \sin(\pi k/\alpha)}{k!}\right] \left[\sum_{n=1}^{\infty} \frac{s^{n-1}(-1/t)^n}{\Gamma(\alpha n)}\right] \mathrm{d}s$$
$$= \frac{1}{\pi} \int_0^\infty \left(\Im e^{-s-(1-x)(-s/t)^{1/\alpha}}\right) (-1/t) E_{\alpha,\alpha}(-s/t)\,\mathrm{d}s.$$

By change of variable, the integral representation follows. □

To prove the other corollaries, $h_c(x,t)$ is treated as a function of c as well as x and t. From (5) and the scaling relationship (6),

$$h_c(x,t) = \frac{1}{c}\sum_{n=0}^{\infty} \frac{c^{\alpha n+\alpha} f^{(n)}_{x-c}(t)}{\Gamma(\alpha n+\alpha)} \tag{35}$$

$$= \frac{1}{\pi}\sum_{k,n=1}^{\infty} \frac{\Gamma(k/\alpha+n)}{\Gamma(\alpha n)k!}(-1)^{k+n}(c-x)^k c^{\alpha n-1}\sin(\pi k/\alpha)t^{-k/\alpha-n}. \tag{36}$$

Both series converge absolutely for given $t > 0$.

Proof of Corollary 3 Fix $x < 1$ and $t > 0$. Put $\varphi(c) = h_c(x,t) > 0$ and $c_0 = x \vee 0$. We shall show that $\varphi(c) > 0$ for all $c > c_0$. From its definition in (2), $\varphi(c)$ is increasing on (c_0, ∞). By Proposition 9, $\varphi(c) = g_t(x) - \int_0^t k_c(s)\,\mathrm{d}s$, where $k_c(s) = f_{x-c}(s)g_{t-s}(c)$. For $c > c_0 + 1$, $f_{x-c}(s) = (c-x)^{-\alpha} f_{-1}((c-x)^{-\alpha}s) \le \sup f_{-1}$ and as $s \uparrow t$, $g_{t-s}(c) = (t-s)^{-1/\alpha} g_1(c(t-s)^{-1/\alpha}) = (t-s)^{-1/\alpha} O((t-s)^{1+1/\alpha}) = O(t)$. As $c \to \infty$, $k_c(s) \to 0$ for $s \in (0,t)$. Then, by dominated convergence, $\varphi(c) \to g_t(x) > 0$. On the other hand, from (36), $\varphi(c)$ can be analytically extended to $\mathbb{C} \setminus (-\infty, 0]$. If $\varphi(c) = 0$ for some $c > c_0$, then by monotonicity, $\varphi(z) = 0$ for all $z \in (c_0, c)$. Then, by analyticity, $\varphi(z) = 0$ for all $z > c_0$, yielding $\varphi(z) \to 0$ as $z \to \infty$, a contradiction. □

Proof of Corollary 4 Fix $t = 1$ and x. Then, each term in the series (36) is a function of c, denoted $d_{k,n}(c)$. It is seen that $d'_{k,n}(c)$ is the $(k,n)^{\text{th}}$ term in the series (9). For each bounded interval $I = [a,b] \subset (x \vee 0, \infty)$, letting $M = b + |x|$, for all $c \in I$ and $k, n \ge 1$,

$$|d'_{k,n}(c)| \le D_{k,n} := \frac{\Gamma(k/\alpha+n)}{\Gamma(\alpha n)k!}(k+\alpha n-1)M^{k+\alpha n-1}/a.$$

By argument similar to that for Lemma 12, $\sum_{k,n=1}^{\infty} D_{k,n} < \infty$. As a result, $\partial h_c(x,1)/\partial c = \pi^{-1}\sum_{k,n=1}^{\infty} d'_{k,n}(c)$ for $c \in I$. Since I is arbitrary, then (9) holds for all $c > x \vee 0$, as claimed. □

Proof of Corollary 5 This is immediate from Proposition 8 and the fact that being spectrally positive with infinite variation X does not creep, i.e., $\Delta_{T_c} > 0$ a.s. ([6], p. 64). □

Proof of Corollary 6 Let

$$a(x,c,r) = \frac{\mathbb{P}\{X_1 \in \mathrm{d}x, \overline{X}_1 \in \mathrm{d}c, \overline{G}_1 \le r\}}{\mathrm{d}x\,\mathrm{d}c}, \quad b(x,c,t) = \frac{\partial h_c(x,t)}{\partial c}.$$

Although Corollary 4 provides a series expression of $b(x,c,t)$, it is not very useful here. Instead, by (22), for $x < c$,

$$b(x,c,t) = -\frac{\partial}{\partial c}\left[\int_0^t f_{x-c}(s)g_{t-s}(c)\,\mathrm{d}s\right]$$

$$= -\int_0^t \left[\frac{\partial f_{x-c}(s)}{\partial c} g_{t-s}(c) + f_{x-c}(s)g'_{t-s}(c)\right]\mathrm{d}s, \tag{37}$$

where the interchange of integration and differentiation on the second line is justified by the uniform boundedness of $\partial(f_{x-c}(s)g_{t-s}(c))/\partial c$ as a function of (c, s) on any compact set in $(x \vee 0, \infty) \times [0, \infty)$.

Given $r \in (0, 1)$, for each $\epsilon \in (0, 1-r)$, by conditioning on X_r and $X_{r+\epsilon}$ and the Markov property of X,

$$\begin{aligned}
&\mathbb{P}\{X_1 \in \mathrm{d}x, \overline{X}_1 \in \mathrm{d}c, r < \overline{G}_1 \le r + \epsilon\} \\
&= \int_{u<c, v<c-u} \mathbb{P}\{X_r \in \mathrm{d}u, \overline{X}_r < c\}\, \mathbb{P}\{X_\epsilon \in \mathrm{d}v, \overline{X}_\epsilon \in \mathrm{d}c - u\} \\
&\qquad\qquad \times \mathbb{P}\{X_{1-r-\epsilon} \in \mathrm{d}x - u - v, \overline{X}_{1-r-\epsilon} < c - u - v\} \\
&= \int_{u<c, v<c-u} \mathbb{P}\{X_r \in \mathrm{d}u, \overline{X}_r \le c\}\, \mathbb{P}\{X_\epsilon \in \mathrm{d}v, \overline{X}_\epsilon \in \mathrm{d}c - u\} \\
&\qquad\qquad \times \mathbb{P}\{X_{1-r-\epsilon} \in \mathrm{d}x - u - v, \overline{X}_{1-r-\epsilon} \le c - u - v\},
\end{aligned}$$

where the second equation is due to $\overline{X}_r$ and $\overline{X}_{1-r-\epsilon}$ having continuous distributions according to Corollary 4. Make change of variables $y = c - u$ and $z = c - u - v$. Then, divide both sides of the above display by $\epsilon\, \mathrm{d}x\, \mathrm{d}c$ and use (2) and Corollary 4 to get

$$\begin{aligned}
&\frac{a(x, c, r + \epsilon) - a(x, c, r)}{\epsilon} \\
&= \int_{y>0, z>0} h_c(c - y, r) \frac{b(y - z, y, \epsilon)}{\epsilon} h_z(x - c + z, t - r - \epsilon)\, \mathrm{d}y\, \mathrm{d}z. \qquad (38)
\end{aligned}$$

From (37), it follows that for $y > 0$ and $z > 0$,

$$b(y - z, y, \epsilon) = -\int_0^\epsilon \left[g_{\epsilon-s}(y) \frac{\partial f_{-z}(s)}{\partial z} + g'_{\epsilon-s}(y) f_{-z}(s) \right] \mathrm{d}s.$$

By $f_{-z}(s) = z^{-\alpha} f_{-1}(z^{-\alpha} s)$,

$$\frac{\partial f_{-z}(s)}{\partial z} = -\alpha z^{-\alpha-1} [f_{-1}(z^{-\alpha} s) + s z^{-\alpha} f'_{-1}(z^{-\alpha} s)].$$

Make change of variable $s = \epsilon w$. Then,

$$\begin{aligned}
\frac{b(y - z, y, \epsilon)}{\epsilon} &= \alpha z^{-\alpha-1} \int_0^1 g_{\epsilon(1-w)}(y) [f_{-1}(\epsilon z^{-\alpha} w) + \epsilon z^{-\alpha} w f'_{-1}(\epsilon z^{-\alpha} w)]\, \mathrm{d}w \\
&\quad - z^{-\alpha} \int_0^1 g'_{\epsilon(1-w)}(y) f_{-1}(\epsilon z^{-\alpha} w)\, \mathrm{d}w.
\end{aligned}$$

Put $u = \epsilon^{-1/\alpha} y$ and $v = \epsilon^{-1} z^\alpha$. Then, $g_{\epsilon(1-w)}(y) = \epsilon^{-1/\alpha} g_{1-w}(\epsilon^{-1/\alpha} y) = \epsilon^{-1/\alpha} g_{1-w}(u)$ and $g'_{\epsilon(1-w)}(y) = \epsilon^{-2/\alpha} g'_{1-w}(\epsilon^{-1/\alpha} y) = \epsilon^{-2/\alpha} g'_{1-w}(u)$, and so

$$\begin{aligned}\frac{b(y-z, y, \epsilon)}{\epsilon} &= \alpha \epsilon^{-1/\alpha} z^{-\alpha-1} \int_0^1 g_{1-w}(u)[f_{-1}(w/v) + (w/v) f'_{-1}(w/v)]\, \mathrm{d}w \\ &\quad - \epsilon^{-2/\epsilon} z^{-\alpha} \int_0^1 g'_{1-w}(u) f_{-1}(w/v)\, \mathrm{d}w \\ &= \epsilon^{-2/\alpha-1} v^{-1} I(u, v),\end{aligned}$$

where the second equality is obtained by replace z with $(\epsilon v)^{1/\alpha}$ and

$$\begin{aligned}I(u, v) = \alpha v^{-1/\alpha} &\int_0^1 g_{1-w}(u)[f_{-1}(w/v) + (w/v) f'_{-1}(w/v)]\, \mathrm{d}w \\ &- \int_0^1 g'_{1-w}(u) f_{-1}(w/v)\, \mathrm{d}w.\end{aligned}$$

Combined with (38) and $\mathrm{d}y\, \mathrm{d}z = \epsilon^{2/\alpha} \alpha^{-1} v^{1/\alpha-1}\, \mathrm{d}u\, \mathrm{d}v$, the above display yields

$$\begin{aligned}&\frac{a(x, c, r+\epsilon) - a(x, c, r)}{\epsilon} \\ &= \epsilon^{-2/\alpha-1} \int_{y>0, z>0} h_c(c-y, r) \times v^{-1} I(u, v) h_z(x-c+z, 1-r-\epsilon)\, \mathrm{d}y\, \mathrm{d}z \\ &= \frac{1}{\alpha} \int_{u>0, v>0} \frac{h_c(c-\epsilon^{1/\alpha} u, r)}{\epsilon^{1/\alpha} u} \times (u/v) I(u, v) \\ &\quad \times \frac{h_{(\epsilon v)^{1/\alpha}}((\epsilon v)^{1/\alpha} - (c-x), 1-r-\epsilon)}{(\epsilon v)^{1-1/\alpha}}\, \mathrm{d}u\, \mathrm{d}v.\end{aligned} \tag{39}$$

We need the following two lemmas.

Lemma 15 *The function $(u/v)I(u, v)$ is integrable over $u > 0$ and $v > 0$ with*

$$\int_{u>0, v>0} (u/v) I(u, v)\, \mathrm{d}u\, \mathrm{d}v = \Gamma(\alpha+1).$$

Lemma 16 *The following statements hold.*

(a) *Given $c > 0$ and $t > 0$, $h_c(c-x, t)/x$ as a function of x is bounded on $(0, \infty)$ and*

$$\lim_{x \to 0+} \frac{h_c(c-x, t)}{x} = m(c, t).$$

Furthermore, given $c > 0$, $m(c, t)$ is bounded in $t > 0$.

(b) *Given* $x > 0$ *and* $t > 0$, $h_c(c-x,t)/c^{\alpha-1}$ *as a function of* c *is bounded on* $(0,\infty)$ *and*

$$\lim_{c\to 0}\frac{h_c(c-x,t)}{c^{\alpha-1}}=\frac{f_{-x}(t)}{\Gamma(\alpha)}.$$

Assuming the lemmas are true, let $\epsilon \to 0$ in (39). By the lemmas and dominated convergence, the limit is $m(c,r)f_{x-c}(1-r)$. With similar argument, $[a(x,c,r)-a(x,c,r-\epsilon)]/\epsilon$ converges to the same limit as $\epsilon \to 0$. Then, (11) is proved.

To prove the rest of the corollary, integrate (11) over $x < c$. From the identity $\int_0^\infty f_{-x}(s)\,\mathrm{d}x = s^{1/\alpha-1}/\Gamma(1/\alpha)$ ([18], p. 270), it follows that $\overline{G}_1$ and $\overline{X}_1$ have joint p.d.f.

$$\frac{\mathbb{P}\{\overline{G}_1\in \mathrm{d}r,\overline{X}_1\in \mathrm{d}c\}}{\mathrm{d}r\,\mathrm{d}c}=\frac{m(c,r)(1-r)^{1/\alpha-1}}{\Gamma(1/\alpha)}. \tag{40}$$

The conditional independence of $\overline{X}_1$ and $\overline{X}_1 - X_1$ given $\overline{G}_1$ follows from (11). As noted earlier, $\overline{G}_1$ follows the $\mathrm{Beta}(1-1/\alpha, 1/\alpha)$ distribution. This can be directly proved by integrating the above joint p.d.f. over $c > 0$. Since by (13),

$$\int_0^\infty m(c,r)\,\mathrm{d}c=\frac{\sin(\pi/\alpha)}{\pi}\int_0^\infty s^{1/\alpha}E_{\alpha,\alpha}(-s)\left[\int_0^\infty c^{-2}e^{-sr/c^\alpha}\,\mathrm{d}c\right]\mathrm{d}s\propto r^{-1/\alpha},$$

the p.d.f. of $\overline{G}_1$ is in proportion to $r^{-1/\alpha}(1-r)^{1/\alpha-1}$, so it must be $\mathrm{Beta}(1-1/\alpha,1/\alpha)$. Then, conditionally on $\overline{G}_1$, the p.d.f. of $\overline{X}_1$ follows by dividing the joint p.d.f. of $\overline{X}_1$ and $\overline{G}_1$ by the p.d.f. of $\overline{G}_1$, and the p.d.f. of $\overline{X}_1 - X_1$ follows from integrating $f_{x-c}(1-r)$ over $x < c$ and Kendall's identity (1). □

Proof of Lemma 15 From the definition of $I(u,v)$, to show the integrability of $(u/v)I(u,v)$, it suffices to show

$$I_1 := \int_{u,v>0} uv^{-1-1/\alpha}\left[\int_0^1 g_{1-w}(u)f_{-1}(w/v)\,\mathrm{d}w\right]\mathrm{d}u\,\mathrm{d}v<\infty,$$
$$I_2 := \int_{u,v>0} uv^{-2-1/\alpha}\left[\int_0^1 wg_{1-w}(u)|f'_{-1}(w/v)|\,\mathrm{d}w\right]\mathrm{d}u\,\mathrm{d}v<\infty,$$
$$I_3 = \int_{u,v>0} uv^{-1}\left[\int_0^1 |g'_{1-w}(u)|f_{-1}(w/v)\,\mathrm{d}w\right]\mathrm{d}u\,\mathrm{d}v<\infty.$$

By (28), for $w\in(0,1)$, $\int_{u>0} ug_{1-w}(u)\,\mathrm{d}u=\mathbb{E}[X_{1-w}\vee 0]=(1-w)^{1/\alpha}/\Gamma(1/\alpha)$. Then, by Fubini's theorem and (29),

$$\begin{aligned}I_1&=\frac{1}{\Gamma(1/\alpha)}\int_0^1(1-w)^{1/\alpha}\left[\int_0^\infty v^{-1-1/\alpha}f_{-1}(w/v)\,\mathrm{d}v\right]\mathrm{d}w\\&=\frac{1}{\Gamma(1/\alpha)}\int_0^1 w^{-1/\alpha}(1-w)^{1/\alpha}\left[\int_0^\infty t^{1/\alpha-1}f_{-1}(t)\,\mathrm{d}t\right]\mathrm{d}w=\frac{\Gamma(\alpha)}{\alpha-1}.\end{aligned}$$

Similarly,

$$I_2 = \frac{1}{\Gamma(1/\alpha)} \int_0^1 w(1-w)^{1/\alpha} \left[\int_0^\infty v^{-2-1/\alpha} |f'_{-1}(w/v)| \, \mathrm{d}v \right] \mathrm{d}w$$
$$= \frac{1}{\Gamma(1/\alpha)} \int_0^1 w^{-1/\alpha}(1-w)^{1/\alpha} \left[\int_0^\infty t^{1/\alpha} |f'_{-1}(t)| \, \mathrm{d}t \right] \mathrm{d}w < \infty.$$

It follows that

$$\tilde{I}_2 := \int_{u,v>0} uv^{-2-1/\alpha} \left[\int_0^1 w g_{1-w}(u) f'_{-1}(w/v) \, \mathrm{d}w \right] \mathrm{d}u \, \mathrm{d}v$$
$$= \frac{1}{\Gamma(1/\alpha)} \int_0^1 w^{-1/\alpha}(1-w)^{1/\alpha} \left[\int_0^\infty t^{1/\alpha} f'_{-1}(t) \, \mathrm{d}t \right] \mathrm{d}w$$
$$= -\frac{1}{\alpha\Gamma(1/\alpha)} \int_0^1 w^{-1/\alpha}(1-w)^{1/\alpha} \left[\int_0^\infty t^{1/\alpha-1} f_{-1}(t) \, \mathrm{d}t \right] \mathrm{d}w = -\frac{\Gamma(\alpha)}{\alpha(\alpha-1)}.$$

Next, since $C := \int_0^\infty u|g'_1(u)| \, \mathrm{d}u < \infty$ and $g'_{1-w}(u) = (1-w)^{-2/\alpha} g'_1((1-w)^{-1/\alpha}u)$,

$$I_3 = \int_0^1 \left\{ \int_0^\infty v^{-1} f_{-1}(w/v) \left[\int_0^\infty u|g'_{1-w}(u)| \, \mathrm{d}u \right] \mathrm{d}v \right\} \mathrm{d}w$$
$$= C \int_0^1 \left\{ \int_0^\infty v^{-1} f_{-1}(w/v) \mathrm{d}v \right\} \mathrm{d}w = C\Gamma(1+\alpha) < \infty$$

and

$$\tilde{I}_3 := \int_0^1 \left\{ \int_0^\infty v^{-1} f_{-1}(w/v) \left[\int_0^\infty u g'_{1-w}(u) \, \mathrm{d}u \right] \mathrm{d}v \right\} \mathrm{d}w$$
$$= \Gamma(\alpha+1) \int_0^\infty u g'_1(u) \, \mathrm{d}u = -\Gamma(\alpha)(\alpha-1).$$

Since $\int (u/v) I(u,v) \, \mathrm{d}u \, \mathrm{d}v = \alpha(I_1 + \tilde{I}_2) - \tilde{I}_3$, the proof then follows. ◻

Proof of Lemma 16 (a) From the absolute convergence of the series (36), as $x \to 0+$, $h_c(c-x,t)/x$ converges to the limit with the expression (12), which is $m(c,t)$. To show that $m(c,t)$ has the integral expression (13), first prove it for $n(t) := m(1,t) = \lim_{x\to 0+}[h_1(1-x,t)/x]$ using the integral representation in Corollary 2 and then prove it in general using scaling. Finally, by (21) on p. 210 of [8], $E_{\alpha,\alpha}(-s) = O(s^{-2})$ as $s \to \infty$. Then, given $t > 0$,

$$|n(t)| \le \pi^{-1} \int_0^\infty s^{1/\alpha} |E_{\alpha,\alpha}(-s)| \, \mathrm{d}s < \infty,$$

so $h_1(1-x,t)/x$ is bounded for $x \in (0, x_0)$ for small enough $x_0 > 0$. On the other hand, by (22) $h_1(1-x,t) < g_t(1-x)$, so $h_1(1-x,t)/x$ is bounded on

$[x_0, \infty)$. Thus, $h_1(1-x,t)/x$ is bounded on $(0,\infty)$. Furthermore, from (12), $n(\cdot)$ has an analytic extension to $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$. As a result, $n(t) > 0$ for almost every $t > 0$ under the Lebesgue measure. Fix $r \in (0,t)$. By the Markov property, for any $x > 0$,

$$\begin{aligned} h_1(1-x,t) &= \int_{u>0} \mathbb{P}\{X_r \in 1-\mathrm{d}u, \overline{X}_r < 1\} h_u(u-x, t-r) \\ &= \int_{u>0} h_1(1-u,r) h_u(u-x,t-r)\,\mathrm{d}u. \end{aligned}$$

Divide both sides by x and let $x \to 0+$. By Fatou's lemma and $m(c,t) = n(t/c^\alpha)/c^2$,

$$n(t) \ge \int_{u>0} h_1(1-u,r) m(u,t-r)\,\mathrm{d}u = \int_{u>0} h_1(1-u,r) \frac{n((t-r)/u^\alpha)}{u^2}\,\mathrm{d}u.$$

By Corollary 3, $h_1(1-u,r) > 0$ for all $u > 0$. Then, the integral on the r.h.s. is positive, and so $n(t) > 0$.

(b) The convergence follows from (35). That $h_c(c-x,t)/c^{\alpha-1}$ is a bounded function of c on $(0,\infty)$ can be similarly proved as in (a).

□

Remark By duality, for $t = 1$, the limit in Lemma 16(a) can be written as

$$\frac{\mathbb{P}\{X_1 \in \mathrm{d}c - x \mid \underline{X}_1 > -x\}}{\mathrm{d}c} \times \frac{\mathbb{P}\{\underline{X}_1 > -x\}}{x} \to m(c,1), \quad x \to 0.$$

Since $\mathbb{P}\{\underline{X}_1 > -x\} = \mathbb{P}\{\tau_{-x} > 1\} = \mathbb{P}\{\tau_{-1} > x^{-\alpha}\} \sim x/\Gamma(1-1/\alpha)$ as $x \to 0$, then the display suggests that $\Gamma(1-1/\alpha)m(c,1)$ can be regarded as the conditional p.d.f. of X_1 at $c > 0$ given $\underline{X}_1 \ge 0$.

4 Exact Sampling for First Passage

In this section, it will be shown that it is possible to conduct exact joint sampling of T_c, X_{T_c-}, and Δ_{T_c} for a spectrally positive stable X satisfying (3). From Proposition 8, this may be done in two steps. The first step is to jointly sample X_{T_c-} and Δ_{T_c}, which is standard. The second step is to sample T_c given X_{T_c-}, which is the focus of the section. Since by scaling, $(T_c, X_{T_c-}, \Delta_{T_c}) \sim (c^\alpha T_1, cX_{T_1-}, c\Delta_{T_1})$, it suffices to consider $c = 1$.

4.1 Sampling of Pre-passage Value and Jump

Because $\mathbb{P}\{X_1 > 0\} = 1 - 1/\alpha$ ([2], p. 218), from Example 7 in [7], at every (x,z,w), the joint p.d.f. of X_{T_1-}, Δ_{T_1}, and $\overline{X}_{T_1-}$ takes value $C\mathbf{1}\{x \vee 0 \le w < 1, z > 1-x > 0\}$

$w^{\alpha-2}z^{-1-\alpha}$, where $C = \pi^{-1}\alpha(\alpha-1)\sin((\alpha-1)\pi)$. It follows that $X_{T_1-} \sim \xi$, where ξ has p.d.f.

$$p(x) = C'\mathbf{1}\{x < 1\}\,[1-(x\vee 0)^{\alpha-1}](1-x)^{-\alpha}$$

with $C' > 0$ a constant, and for every $x < 1$, conditionally on $X_{T_1-} = x$, $\Delta_{T_1} \sim (1-x)\zeta$, where ζ has p.d.f. $q(z) = \alpha\mathbf{1}\{z > 1\}\,z^{-\alpha-1}$. Thus, the joint sampling of X_{T_1-} and Δ_{T_1} boils down to that of independent $\xi \sim p$ and $\zeta \sim q$. The sampling of ζ is straightforward as $\zeta \sim U^{-1/\alpha}$, where $U \sim \text{Uniform}(0, 1)$. To sample ξ, it can be seen that $p(x) = \theta p_1(x) + (1-\theta)p_2$, where $\theta = m_1/(m_1+m_2)$ with $m_1 = (\alpha-1)^{-1}$, $m_2 = \pi/\sin((\alpha-1)\pi) - (\alpha-1)^{-1}$, and

$$p_1(x) = \mathbf{1}\{x \le 0\}\,(1-x)^{-\alpha}/m_1, \quad p_2(x) = \mathbf{1}\{0 < x < 1\}\,(1-x^{\alpha-1})(1-x)^{-\alpha}/m_2$$

are two p.d.f.'s. On one hand, $p_1(x)$ is the p.d.f. of $1 - U^{-1/(\alpha-1)}$. On the other, $p_2(x) \propto \mathbf{1}\{0 < x < 1\}\,(1-x^{\alpha-1})(1-x)^{-\alpha} < \rho(x) := \mathbf{1}\{0 < x < 1\}\,(1-x)^{-\alpha+1}$. Using the fact that $\rho(x)$ is proportional to the p.d.f. of $1-U^{1/(2-\alpha)}$, p_2 can be sampled by the rejection sampling method ([5], Chapter II). In summary, $p(x)$ can be sampled as follows.

(a) Sample I from $\{1, 2\}$ such that $\mathbb{P}\{I = 1\} = m_1/(m_1 + m_2)$
(b) If $I = 1$, then sample $U \sim \text{Uniform}(0, 1)$ and output $1 - U^{-1/(\alpha-1)}$, otherwise, do the following iteration until an output is made.

- Sample U, V i.i.d. $\sim$ Uniform$(0, 1)$, and set $x = 1 - U^{1/(2-\alpha)}$. If $V \le (1-x^{\alpha-1})/(1-x)$, then output x, otherwise repeat.

4.2 Sampling of Time of First Passage

We now consider the sampling of T_1 conditionally on $X_{T_1-} = x \in (-\infty, 1)$. By Proposition 9, if $x < 0$, then $h_1(x,\cdot)/v_1(x)$ is the p.d.f. of $\tau' + \xi$, with $\tau' \sim h_1(0,\cdot)/v_1(0)$ and $\xi \sim f_x$ being independent. Since the sampling of ξ is well known [4], the sampling of $h_1(x,\cdot)/v_1(x)$ can be reduced to that of $h_1(0,\cdot)/v_1(0)$. As a result, it only remains to consider the case $0 \le x < 1$.

We again will use the rejection sampling method. For this method, the normalizing constant $v_c(x)$ is not important and one can just focus on $h_1(x,\cdot)$. We will use the power series representation (5) of $h_1(x,\cdot)$. In order to handle the infinite number of positive and negative terms in the series, we first describe the general approach to use.

Let p and q be two p.d.f.'s that are proportional to some explicit functions f and g, respectively, whose normalizing constants may be intractable; g is known as an envelope function. For the rejection sampling method, q must be easy to sample. Suppose f can be decomposed as

$$\begin{aligned} f(t) = \sum_{l=1}^{\infty} \phi_l(t) \quad &\text{such that for some explicit constants } c_1, c_2, \ldots \\ 0 \le \phi_l(t) \le c_l g(t) \quad &\text{with } C := \sum c_l < \infty. \end{aligned} \tag{41}$$

Then, p can be sampled as follows.

- Independently sample $T \sim q$, $U \sim \text{Uniform}(0, 1)$, and ℓ from the probability mass function $\mathbb{P}\{\ell = l\} = c_l/C$. If $U \le \phi_\ell(T)/(c_\ell g(T))$, then output T and stop, otherwise repeat.

Indeed, by standard argument of the rejection sampling method, the p.d.f. of the output of the procedure is proportional to

$$g(t)\sum_l \left[\frac{c_l}{C} \times \frac{\phi_l(t)}{c_l g(t)}\right] = \sum_l \phi_l(t)/C = f(t)/C,$$

so it must be p. The point is that when $f(t)$ is an infinite series that cannot be evaluated in closed form, say $f(t) = \sum_{a\in A} f_a(t)$, it is possible to have each $\phi_l(t)$ equal to the sum of a finite set of $f_a(t)$. More precisely, $\phi_l(t) = \sum_{a\in A_l(t)} f_a(t)$, where $A_l(t)$ is a finite subset of A that may depend on t, and given t, $A_1(t)$, $A_2(t)$, …, form a partition of A. It is also critical the $A_l(t)$'s are such that $\phi_l(t) \ge 0$ for all l and t. In each iteration, once T and ℓ are sampled, only $\phi_l(T)$ with l equal to the value of ℓ needs to be evaluated. As long as for any t, each $f_a(t)$ is easy to evaluate, and the set $A_l(t)$ can be enumerated in a finite number of steps, $\phi_l(t)$ can be evaluated exactly.

To apply the above approach to $h_1(x, t)$, where $x < 1$ is fixed, the main issue is the construction of the envelop function and the $\phi_l(t)$'s. The next lemma gives an option for the envelope function.

Lemma 17 *Fixing any* $D \ge \sup_{n\ge1} 2^{n-1}\Gamma(n)/\Gamma(\alpha n)$, *define*

$$\theta = 4^{1/(\alpha-1)}, \quad C_\alpha = (\alpha\Gamma(1 - 1/\alpha))^{-1} \vee [D(\theta^\alpha e^\theta + 4)],$$
$$H_\alpha(t) = C_\alpha t^{-1/\alpha} \wedge t^{-1-\alpha}, \quad t > 0.$$

Then, for every $0 \le x < 1$ *and* $t > 0$, $h_1(x, t) \le H_\alpha(t)$.

The normalized $H_\alpha(t)$ is $\theta p_1(t) + (1 - \theta)p_2(t)$, where $p_1(t) = (1 - 1/\alpha)\mathbf{1}\{0 < t < 1\}t^{-1/\alpha}$ and $p_2(t) = \alpha\mathbf{1}\{t > 1\}t^{-\alpha-1}$ are p.d.f.'s and $\theta = \alpha^2/(\alpha^2 + \alpha - 1)$. Thus, the normalized H_α can be sampled as follows.

- Sample U, V i.i.d. $\sim \text{Uniform}(0, 1)$. If $U \le \theta$, return $V^{\alpha/(\alpha-1)}$, otherwise return $V^{-1/\alpha}$.

As a result, H_α can be used as an envelope function.

Now, consider the construction of $\phi_l(t)$. Let $c_l = 2^{-l+1}$. Then, from (41), we wish to construct $0 < \phi_l(t) < 2^{-l+1}H_\alpha(t)$ such that $h_1(x, t) = \sum_{l=1}^\infty \phi_l(t)$. Write

$$m_{k,n}(s, u) = \frac{\Gamma(k/\alpha + n)s^k u^n}{\pi k!\Gamma(\alpha n)},$$

so that

$$h_1(x, t) = \sum_{k,n=1}^\infty \sin(\pi k/\alpha) m_{k,n}(-(1 - x)t^{-1/\alpha}, -t^{-1}).$$

We shall construct for each $t > 0$ a sequence of finite sets $\Lambda_l(t) \subset \mathbb{N} \times \mathbb{N}$, $l \geq 0$, such that $\Lambda_l(t) \subset \Lambda_{l+1}(t)$, $\bigcup_{l=1}^{\infty} \Lambda_l(t) = \mathbb{N} \times \mathbb{N}$ and

$$F_l(t) := \sum_{(k,n)\in\Lambda_l(t)} (-1)^{k+n} \sin(\pi k/\alpha) M_{k,n}$$

is strictly increasing in l such that $0 < h_1(x,t) - F_l(t) \leq 2^{-l} H_\alpha(t)$, where $M_{k,n} = m_{k,n}(s,u)$ with $s = (1-x)t^{-1/\alpha}$ and $u = t^{-1}$. Once this is done, let $\phi_l(t) = F_l(t) - F_{l-1}(t)$. Then, $\sum_{l=0}^{\infty} \phi_l(t) = \lim_l F_l(t) = h_1(x,t)$ and $0 < \phi_l(t) < h_1(x,t) - F_{l-1}(t) \leq 2^{-l+1} H_\alpha(t)$, as desired. The construction is based on the following two lemmas.

Lemma 18 *Fix $\epsilon \in (0, 1/2)$ and $s, u > 0$. Let k and $n \in \mathbb{N}$ such that $n > (2u/\epsilon)^{1/(\alpha-1)}$, $k > (2s/\epsilon)^{\alpha/(\alpha-1)}$, and $k/\alpha \leq n \leq (2-1/\alpha)k$, then*

$$\sum_{i,j=0,i+j\geq 1}^{\infty} m_{k+i,n+j}(s,u) \leq 24\epsilon m_{k,n}(s,u), \tag{42}$$

$$\sum_{j=1}^{\infty} m_{k',n+j}(s,u) \leq 2\epsilon m_{k',n}(s,u) \quad \forall k' \leq k, \tag{43}$$

$$\sum_{i=1}^{\infty} m_{k+i,n'}(s,u) \leq 2\epsilon m_{k,n'}(s,u) \quad \forall n' \leq n. \tag{44}$$

Lemma 19 *Let $d_\alpha = (1/\alpha - 1/2) \wedge [1/2 - 1/(2\alpha)]$ and $L_\alpha = \lfloor (\alpha - 1/2)/(\alpha-1) \rfloor + 1 \geq 2$. Then, among any $2L_\alpha$ consecutive integers, there exist an even number and an odd number both belonging to $A_\alpha := \cup_{j\in\mathbb{Z}} I_j$, where $I_j = [(2j+d_\alpha)\alpha, (2j+1-d_\alpha)\alpha]$.*

Assume the two lemmas are true for now. Let $\Lambda_0(t) = \emptyset$ and $F_0(t) = 0$. By Corollary 3 and Lemma 17, $0 < h_1(x,t) - F_0(t) = h_1(x,t) \leq H_\alpha(t)$. Suppose $\Lambda_l(t)$ has been constructed, such that $F_l(t) \geq 0$ and $0 < h_1(x,t) - F_l(t) \leq 2^{-l+1} H_\alpha(t)$. We need to construct $\Lambda_{l+1}(t) \supset \Lambda_l(t)$, such that $F_{l+1}(t) > F_l(t)$ and $0 < h_1(x,t) - F_{l+1}(t) \leq 2^{-l} H_\alpha(t)$.

For $r \in \mathbb{N}$, denote $S_r = \{(k,n) : k, n = 1, \ldots, r\}$ and $\partial S_r = \{(k,n) \in S_r : k \vee n = r\}$ its "boundary." Let d_α and A_α be as in Lemma 19. Let $\delta_\alpha = \sin(d_\alpha \pi)$ and $K_\alpha = \mathbb{Z} \cap A_\alpha$. Then, $\delta_\alpha > 0$ and for $k \in K_\alpha$, $\pi k/\alpha \in [(2j+d_\alpha)\pi, (2j+1-d_\alpha)\pi]$ for some $j \in \mathbb{Z}$, so for n of the same parity as k,

$$(-1)^{k+n} \sin(k\pi/\alpha) = \sin(k\pi/\alpha) \geq \delta_\alpha > 0.$$

Put $\epsilon = \delta_\alpha/24$. Let R be the smallest integer such that

$$R > \frac{2L_\alpha}{\alpha-1} \vee \left(\frac{2u}{\epsilon}\right)^{1/(\alpha-1)} \vee \left(\frac{2s}{\epsilon}\right)^{\alpha/(\alpha-1)}, \quad \Lambda_l(t) \subset S_R, \quad \sum_{(k,n)\in\partial S_R} M_{k,n} \leq \frac{2^{-l} H_\alpha(t)}{24\epsilon}.$$

Starting with $r = R$, do the following iteration.

- For each n, let k_n be the smallest number in $K_\alpha \cap [r+1, \infty)$ that has the same parity as n; k_n exists because by Lemma 19, $K_\alpha \cap \{r+1, \ldots, r+2L_\alpha\}$ contains an even number and an odd number. In particular, $1 \le k_n - r \le 2L_\alpha$. Define

$$S''_r = S_r \cup \bigcup_{n=1}^{r} \{(k,n) : r < k < k_n\}$$

and $S'_r = S''_r \cup \{(k, r+1) : (k,r) \in S''_r, (-1)^{k+r} \sin(\pi k/\alpha) > 0\}$. If

$$\sum_{(k,n)\in S'_r} (-1)^{k+n} \sin(\pi k/\alpha) M_{k,n} > F_l(t),$$

then let $\Lambda_{l+1}(t) = S'_r$ and stop. Otherwise increase r by 1 and repeat.

Since $h_1(x,t) - F_l(t) > 0$ and $\sum_{(k,n)\in S'_r} (-1)^{k+n} \sin(\pi k/\alpha) M_{k,n} \to h_1(x,t)$ as $r \to \infty$, the iteration eventually will stop. It is clear that $\Lambda_{l+1}(t) = S_r \supset S_R \supset \Lambda_l(t)$ and $F_{l+1}(t) > F_l(t)$. Next,

$$h_1(x,t) - F_{l+1}(t) = \sum_{(k,n)\notin S'_r} (-1)^{k+n} \sin(\pi k/\alpha) M_{k,n} \le \sum_{(k,n)\notin S_r} M_{k,n} \le \sum_{(k,n)\notin S_R} M_{k,n}.$$

Since $R > (2u/\epsilon)^{1/(\alpha-1)} \vee (2s/\epsilon)^{\alpha/(\alpha-1)}$, by Lemma 18,

$$\begin{aligned}\sum_{(k,n)\notin S_R} M_{k,n} &= \sum_{i,j=0, i+j\ge 1}^{\infty} M_{R+i,R+j} + \sum_{k=1}^{R-1}\sum_{j=1}^{\infty} M_{k,R+j} + \sum_{n=1}^{R-1}\sum_{i=1}^{\infty} M_{R+i,n} \\ &\le 24\epsilon M_{R,R} + 2\epsilon \sum_{k=1}^{R-1} M_{k,R} + 2\epsilon \sum_{n=1}^{R-1} M_{R,n} \le 24\epsilon \sum_{(k,n)\in \partial S_R} M_{k,n}.\end{aligned}$$

By the choice of R, the above two displays give $h_1(x,t) - F_{l+1}(t) < 2^{-l} H_\alpha(t)$. It only remains to show $h_1(x,t) - F_{l+1}(t) > 0$, i.e., $\sum_{(k,n)\notin S'_r} (-1)^{k+n} \sin(\pi k/\alpha) M_{k,n} > 0$. It can be seen that $(\mathbb{N} \times \mathbb{N}) \setminus S'_r$ can be partitioned into the following sets:

$$\begin{aligned} E_1 &= \{(k,n) : k < k_r, (-1)^{k+r} \sin(k\pi/\alpha) \le 0, n \ge r+1\}, \\ E_2 &= \{(k,n) : k < k_r, (-1)^{k+r} \sin(k\pi/\alpha) > 0, n \ge r+2\}, \\ E_3 &= \{(k,n) : k \ge k_n, n \le r-1\}, \\ E_4 &= \{(k,n) : k \ge k_r, n \ge r\}. \end{aligned}$$

As already seen, $1 \le k_r - r \le 2L_\alpha$. Then, $k_r/\alpha \le r+1$ and $r+2 \le (2-1/\alpha)k_r$, the first one due to $r - k_r/\alpha \ge (1-1/\alpha)r - 2L_\alpha/\alpha \ge (1-1/\alpha)R - 2L_\alpha/\alpha > 0$ and the second one $(2-1/\alpha)k_r - r - 2 \ge (2-1/\alpha)(r+1) - r - 2 \ge (1-1/\alpha)R - 1/\alpha \ge 0$.

Also, $r > (2u/\epsilon)^{1/(\alpha-1)}$ and $k_r > (2s/\epsilon)^{\alpha/(\alpha-1)}$. Then, by (43) in Lemma 18, for every $k < k_r$, $\sum_{j=1}^{\infty} M_{k,r+1+j} \le 2\epsilon M_{k,r+1}$, giving

$$\sum_{n:(k,n)\in E_1} (-1)^{k+n} \sin(k\pi/\alpha) M_{k,n} = (-1)^{k+r+1} \sin(k\pi/\alpha) \left[M_{k,r+1} - \sum_{n\ge r+2} (-1)^{n-r-1} M_{k,n} \right]$$
$$\ge |\sin(k\pi/\alpha)|(1-2\epsilon) M_{k,r+1} \ge 0.$$

Since $k_r \ge 2L_\alpha + 1$, by Lemma 19, $(-1)^{k+r+1}\sin(k\pi/\alpha) > 0$ for at least one $k < k_r$. Thus the sum over E_1 is strictly positive. Likewise, the sum over E_2 is strictly positive. Next, the sum over E_3 is at least

$$\sum_{n=1}^{r-1} \left((-1)^{k_n+n} \sin(k_n\pi/\alpha) M_{k_n,n} - \sum_{j=1}^{\infty} M_{k_n,n+j} \right) \ge \sum_{n=1}^{r-1} \left(\delta_0 M_{k_n,n} - \sum_{j=1}^{\infty} M_{k_n,n+j} \right).$$

By (43) in Lemma 18, the last sum is strictly positive. Similar, using (42) in Lemma 18, the sum over E_4 is strictly positive. Thus $h_1(x,t) - F_{l+1}(t) > 0$, as desired.

4.3 Proof of Lemmas

Proof of Lemma 17 Given $x \in [0, 1)$, by (21), (25), and g_t being decreasing on $[0, \infty)$ ([18], p. 416),

$$h_1(x,t) \le g_t(x) \le g_t(0) = t^{-1/\alpha}/(\alpha\Gamma(1-1/\alpha)), \quad t > 0.$$

On the other hand, for $t \ge 1$, from (33),

$$h_1(x,t) \le t^{-1/\alpha-1} \sum_{k,n=1}^{\infty} \frac{\Gamma(k/\alpha+n)}{k!\Gamma(\alpha n)} \le B t^{-1/\alpha-1},$$

where $B = \int_0^\infty E_{\alpha,\alpha}(s) e^{s^{1/\alpha}-s}\,ds$. By $E_{\alpha,\alpha}(s) = \sum_{n=1}^\infty s^{n-1}/\Gamma(\alpha n) \le D \sum_{n=1}^\infty (s/2)^{n-1}/\Gamma(n) = De^{s/2}$, $B \le D\int_0^\infty e^{s^{1/\alpha}-s/2}\,ds \le D(\int_0^{\theta^\alpha} e^{s^{1/\alpha}}\,ds + \int_{\theta^\alpha}^\infty e^{-s/4}\,ds) \le D(\theta^\alpha e^\theta + 4)$, which together with the displays yields the proof. □

To prove Lemma 18, we need the following.

Lemma 20 *Let k and $n \in \mathbb{N}$, and $s, u > 0$.*

(a) *If $n \ge k/\alpha$, then $2um_{k,n}(s,u)/(n-1)^{\alpha-1} > m_{k,n+1}(s,u)$.*
(b) *If $n \le (2-1/\alpha)(k+1)$, then $2sm_{k,n}(s,u)/(k+1)^{1-1/\alpha} > m_{k+1,n}(s,u)$.*
(c) *If $k/\alpha \le n \le (2-1/\alpha)(k+1)$, then $6sum_{k,n}(s,u)/(n-1)^{\alpha-1}(k+1)^{1-1/\alpha} > m_{k+1,n+1}(s,u)$.*

Proof (a) For $k \ge 1$ and $n \ge k/\alpha$,

$$\frac{m_{k,n}(s,u)}{m_{k,n+1}(s,u)} = \frac{u^{-1}\Gamma(\alpha n+\alpha)}{\Gamma(\alpha n)(k/\alpha+n)} \ge \frac{u^{-1}}{2n}\frac{\Gamma(\alpha n+\alpha)}{\Gamma(\alpha n)}.$$

By Gautschi's inequality ([15], p. 138), $\Gamma(\alpha n+\alpha)/\Gamma(\alpha n) > (\alpha n+\alpha-1)(\alpha n+\alpha-2)^{\alpha-1} > n(n-1)^{\alpha-1}$, which together with the display yields the proof.

(b) By Gautschi's inequality, $\Gamma((k+1)/\alpha+n) < \Gamma(k/\alpha+n)((k+1)/\alpha+n)^{1/\alpha}$. Then,

$$\frac{m_{k,n}(s,u)}{m_{k+1,n}(s,u)} = \frac{s^{-1}\Gamma(k/\alpha+n)(k+1)}{\Gamma((k+1)/\alpha+n)} > \frac{s^{-1}(k+1)}{((k+1)/\alpha+n)^{1/\alpha}}.$$

If $n \le (2-1/\alpha)(k+1)$, then $((k+1)/\alpha+n)^{1/\alpha} \le [2(k+1)]^{1/\alpha} < 2(k+1)^{1/\alpha}$, leading to the proof.

(c) From the above argument, if $n \ge k/\alpha$, then

$$\frac{m_{k,n}(s,u)}{m_{k+1,n+1}(s,u)} = \frac{m_{k,n}(s,u)}{m_{k,n+1}(s,u)}\frac{m_{k,n+1}(s,u)}{m_{k+1,n+1}(s,u)} \ge \frac{(su)^{-1}(n-1)^{\alpha-1}}{2}\frac{(k+1)}{((k+1)/\alpha+n+1)^{1/\alpha}}.$$

Then, for $n \le (2-1/\alpha)(k+1)$, $(k+1)/\alpha+n+1 \le 2(k+1)+1 \le 3(k+1)$, so

$$\frac{(k+1)}{((k+1)/\alpha+n+1)^{1/\alpha}} \ge \frac{(k+1)}{(3(k+1))^{1/\alpha}},$$

which together with the previous display yields the proof.

□

Proof of Lemma 18 Write $m_{k,n} = m_{k,n}(s,u)$ and $S_{k,n} = \sum_{i,j=0}^{\infty} m_{k+i,n+j}$. Then, (42) is equivalent to $S_{k,n} \le (1+24\epsilon)m_{k,n}$ for k, n satisfying the conditions in the lemma. Let $k_0 = k$ and for $l \ge 1$, $k_l = \lfloor \alpha(n+l-1)+1 \rfloor$. Then, by $\alpha \in (1,2)$, $k_0 < k_1 < k_2 < \ldots$ and $k_l/\alpha \le (k_{l+1}-1)/\alpha \le n+l \le (2-1/\alpha)k_l$ for $l \ge 0$. Put $d_l = k_{l+1} - k_l$. Then,

$$S_{k,n} = \sum_{l=0}^{\infty}\left(\sum_{i=0}^{d_l-1}\sum_{j=1}^{\infty} m_{k_l+i,n+l+j} + \sum_{i=0}^{\infty} m_{k_l+i,n+l}\right).$$

For $0 \le i < d_l$, and $j \ge 1$, since $n+l+j-1 \ge n+l \ge (k_{l+1}-1)/\alpha \ge (k_l+i)/\alpha$, by Lemma 20(a), $m_{k_l+i,n+l+j}/m_{k_l+i,n+l+j-1} \le 2u(n+l+j-2)^{1-\alpha} \le 2u(n-1)^{1-\alpha} < \epsilon$. Then, by induction,

$$\sum_{i=0}^{d_l-1}\sum_{j=1}^{\infty} m_{k_l+i,n+l+j} < \sum_{i=0}^{d_l-1}\sum_{j=1}^{\infty} \epsilon^j m_{k_l+i,n+l} \le \frac{\epsilon}{1-\epsilon}\sum_{i=0}^{\infty} m_{k_l+i,n+l}$$

and hence

$$S_{k,n} < \frac{1}{1-\epsilon} \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} m_{k_l+i,n+l}.$$

For each $i \ge 1$, since $n+l \le (2-1/\alpha)(k_l+i)$, by Lemma 20(b), $m_{k_l+i,n+l}/m_{k_l+i-1,n+l} \le 2s(k_l+i)^{1/\alpha-1} < 2sk^{1/\alpha-1} < \epsilon$. Then, by induction, $m_{k_l+i,n+l} \le \epsilon^i m_{k_l,n+l}$, resulting in

$$S_{k,n} < \frac{1}{(1-\epsilon)^2} \sum_{l=0}^{\infty} m_{k_l,n+l}.$$

For each $l \ge 1$, since $(k_l-1)/\alpha \le n+l-1 \le (2-1/\alpha)(k_l-1)$, by Lemma 20(c) $m_{k_l,n+l}/m_{k_{l-1},n+l-1} < 6su(n-1)^{1-\alpha}k^{1/\alpha-1} < \epsilon$. Then, by induction, $S_{k,n} < (1-\epsilon)^{-3}m_{k,n} < (1+24\epsilon)m_{k,n}$, as desired. The proof for (43) and (44) is very similar to that for (42) and hence is omitted. □

Proof of Lemma 19 Recall that I_j is defined to be $[(2j+d_\alpha)\alpha, (2j+1-d_\alpha)\alpha]$. Then, $|I_j| \ge 1$. Let $B_j = ((2j+1-d_\alpha)\alpha, (2j+2+d_\alpha)\alpha)$. Then, $|B_j| \le 2$ and $A_\alpha^c = \cup_{j\in\mathbb{Z}} B_j$. If two consecutive integers both belong to A_α^c, they must belong to the same B_j, for otherwise there would be an I_i strictly between the two, implying $|I_i| < 1$. Moreover, no three consecutive integers can all belong to A_α^c, for otherwise they had to be in the same B_j, implying $|B_j| > 2$. Assume that for some i, none of the even numbers in $S = \{i+1, i+2, \dots, i+2L_\alpha\}$ is in K_α. Then, all the odd numbers in S are in K_α. Consequently, the even numbers belong to L_α different B_j's, and the odd ones to L_α different I_j's. The union of these intervals has Lebesgue measure $2\alpha L_\alpha$. Since the union lies between $i+1-|C|$ and $i+2L_\alpha+|D|$, where C is the interval containing $i+1$ and D the one containing $i+2L_\alpha$, then $2\alpha L_\alpha \le L_\alpha - 1 + |C| + |D|$. Observe that either C is an I_j and D is a B_l, or vice versa. Then, $|C|+|D| = 2\alpha$, so $2\alpha L_\alpha \le 2L_\alpha - 1 + 2\alpha$, contradicting the choice for L_α. This shows there is at least one even number in S belonging to K_α. Likewise, there is at least one odd number in S belonging to K_α. □

Acknowledgements The research is partially supported by NSF Grant DMS 1720218. The author would like to thank two referees for their careful reading of the manuscript and useful suggestions, in particular, one that significantly simplifies the proof of the main theorem.

Appendix

On the connection between (9) *and* (10) When $\alpha = 2$, $\sin(\pi k/\alpha)$ is 0 if k is even and is $(-1)^j$ is $k = 2j+1$ for integer $j \ge 0$. Then, the series in (9) can be written as

$$\frac{1}{\pi} \sum_{j=0,n=1}^{\infty} \frac{\Gamma(j+1/2+n)}{(2j+1)!(2n-1)!} (-1)^{j+n+1}[(2j+1)c + (2n-1)(c-x)](c-x)^{2j}c^{2n-2}.$$

Write $n = l + 1$ and $m = j + l$. Then, the series becomes

$$\frac{1}{\pi}\sum_{j,l=0}^{\infty}\frac{\Gamma(j+l+3/2)}{(2j+1)!(2l+1)!}(-1)^{j+l}[(2j+1)(c-x)^{2j}c^{2l+1}+(2l+1)(c-x)^{2j+1}c^{2l}]$$
$$=\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+3/2)}{(2m+1)!}(-1)^m\sum_{j=0}^{m}\left[\frac{(c-x)^{2j}c^{2m-2j+1}}{(2j)!(2m-2j+1)!}+\frac{(c-x)^{2j+1}c^{2m-2j}}{(2j+1)!(2m-2j)!}\right]$$
$$=\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+3/2)}{(2m+1)!}(-1)^m\sum_{s=0}^{2m+1}\frac{(2m+1)!}{s!(2m+1-s)!}(c-x)^s c^{2m+1-s}$$
$$=\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\sqrt{\pi}}{2^{2m+1}m!}(-1)^m(2c-x)^{2m+1}$$
$$=\frac{2c-x}{2\sqrt{\pi}}\exp\left\{-\frac{(2c-x)^2}{4}\right\}.$$

Since $(X_t)_{t\geq 0} \sim (W_{2t})_{t\geq 0}$, this is essentially the same result as (10). □

Proof of Eq. (16) We need the following refined version of Lemma 16(a).

Lemma 21 *There is a constant* $M > 0$*, such that for all* $0 < x < 1/2$ *and all* $t > 0$,

$$h_1(1-x,t) \leq Mx(t+t^{1-1/\alpha}).$$

Assume the lemma is true for now. Then, given $c > 0$, by scaling, for all $0 < x < c/2$, $h_c(c-x,t) \leq Mx(t+t^{1-1/\alpha})$ for some $M = M(c) > 0$. Then, by Lemma 16(a) and dominated convergence, for each $q > 0$,

$$\int_0^{\infty} m(c,t)e^{-qt}\,\mathrm{d}t = \lim_{x\to 0}\frac{1}{x}\int_0^{\infty} h_c(c-x,t)e^{-qt}\,\mathrm{d}t.$$

However, by scaling (6) and Proposition 10, for $0 < x < c/2$,

$$\frac{1}{x}\int_0^{\infty} h_c(c-x,t)e^{-qt}\,\mathrm{d}t = \frac{e^{-q^{1/\alpha}x}-1}{x}\sum_{n=1}^{\infty}\frac{q^{n-1}c^{\alpha n-1}}{\Gamma(\alpha n)} + \sum_{n=1}^{\infty}\frac{q^{n-1}[c^{\alpha n-1}-(c-x)^{\alpha n-1}]}{x\Gamma(\alpha n)}.$$

As a result,

$$\int_0^{\infty} m(c,t)e^{-qt}\,\mathrm{d}t = \sum_{n=1}^{\infty}\frac{q^{n-1}c^{\alpha n-2}}{\Gamma(\alpha n-1)} - q^{1/\alpha}\sum_{n=1}^{\infty}\frac{q^{n-1}c^{\alpha n-1}}{\Gamma(\alpha n)}.$$

Provided that $\beta > q^{1/\alpha}$, integration term by term of the r.h.s. yields

$$\int_0^\infty \left(\int_0^\infty m(c,t) e^{-qt} \, \mathrm{d}t \right) e^{-\beta c} \, \mathrm{d}c = \sum_{n=1}^\infty \frac{q^{n-1}}{\beta^{\alpha n - 1}} - q^{1/\alpha} \sum_{n=1}^\infty \frac{q^{n-1}}{\beta^{\alpha n}} = \frac{\beta - q^{1/\alpha}}{\beta^\alpha - q}.$$

By analytic extension, the equality still holds for $0 \le \beta \le q^{1/\alpha}$. Then, by (15), the proof is complete. □

Proof of Lemma 21 By (22) and integral by parts,

$$h_1(1-x,t) = g_t(1-x) - g_t(1) + \int_0^t \overline{F}_{-x}(t-s) \frac{\partial g_s(1)}{\partial s} \, \mathrm{d}s, \tag{45}$$

where $\overline{F}_{-x}(t) = \int_t^\infty f_{-x}(s) \, \mathrm{d}s = \mathbb{P}\{\tau_{-x} > t\}$. For $0 < x < 1/2$, $g_t(1-x) - g_t(1) = -g_t'(z)x$ for some $z \in (1-x, 1)$. Clearly, $z > 1/2$. It is not hard to show that $M_1 := \sup_{y>0}[y^{\alpha+2}|g_1'(y)|] < \infty$ ([18], p. 88). On the other hand, by $g_t(z) = t^{-1/\alpha} g_1(t^{-1/\alpha} z)$, $g_t'(z) = t^{-2/\alpha} g_1'(t^{-1/\alpha} z)$. Then,

$$|g_t(1-x) - g_t(1)| = x t^{-2/\alpha} |g_1'(t^{-1/\alpha} z)| \le x t^{-2/\alpha} M_1 (t^{-1/\alpha} z)^{-\alpha-2} \le M_1 2^{\alpha+2} x t. \tag{46}$$

Next, by $g_s(1) = s^{-1/\alpha} g_1(s^{-1/\alpha})$, $|\partial g_s(1)/\partial s| \le (1/\alpha)[s^{-1/\alpha-1} g_1(s^{-1/\alpha}) + s^{-2/\alpha-1}|g_1'(s^{-1/\alpha})|]$ is bounded. Then, for some $M_2 > 0$,

$$\left| \int_0^t \overline{F}_{-x}(t-s) \frac{\partial g_s(1)}{\partial s} \, \mathrm{d}s \right| \le M_2 \int_0^t \overline{F}_{-x}(s) \, \mathrm{d}s = M_2 x^\alpha \int_0^{x^{-\alpha} t} \overline{F}_{-1}(s) \, \mathrm{d}s,$$

where the equality is due to $\overline{F}_{-x}(s) = \overline{F}_{-1}(x^{-\alpha} s)$ and change of variable. Because $\overline{F}_{-1}(s)$ is decreasing with $\overline{F}_{-1}(0) = 1$ and is slowly varying at ∞ with index $-1/\alpha$, there is a constant $M_3 > 0$ such that $\int_0^y \overline{F}_{-1}(s) \, \mathrm{d}s \le M_3 y^{1-1/\alpha}$ for all $y > 0$. It follows that

$$\left| \int_0^t \overline{F}_{-x}(t-s) \frac{\partial g_s(1)}{\partial s} \, \mathrm{d}s \right| \le M_2 M_3 x t^{1-1/\alpha}. \tag{47}$$

Then, the proof is complete by combining (45)–(47). □

Proof of Eq. (20) Denote the r.h.s. of (20) by $v^q(x)$. The task is to show $\widehat{v^q} = \widehat{u^q}$, where, for example, $\widehat{v^q}(x) = v^q(-x)$. Since v^q is a version of the q-resolvent density, according to the proof of Proposition I.13 of [2], $(r-q)U^r \widehat{v^q} \uparrow \widehat{u^q}$ as $r \to \infty$, where U^r is the r-resolvent operator. For $r > q$,

$$
\begin{aligned}
U^r \widehat{v^q}(x) &= \int_0^\infty e^{-rt}\mathbb{E}^x[\widehat{v^q}(X_t)]\,\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\left[\int\left(\int_0^\infty e^{-qs} g_s(-y)\,\mathrm{d}s\right) g_t(y-x)\,\mathrm{d}y\right]\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty e^{-rt-qs} g_{s+t}(-x)\,\mathrm{d}s\,\mathrm{d}t \\
&= (r-q)^{-1}\int_0^\infty (1-e^{(q-r)s})e^{-qs} g_s(-x)\,\mathrm{d}s.
\end{aligned}
$$

Then, by monotone convergence, $(r-q)U^r\widehat{v^q}(x) \to \widehat{v^q}(x)$, giving $\widehat{v^q}(x) = \widehat{u^q}(x)$.
□

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