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Energy-stable Runge–Kutta schemes for gradient flow models using the energy quadratization approach

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ABSTRACT

In this letter, we present a novel class of arbitrarily high-order and unconditionally energy-stable algorithms for gradient flow models by combining the energy quadratization (EQ) technique and a specific class of Runge–Kutta (RK) methods, which is named the EQRK schemes. First of all, we introduce auxiliary variables to transform the original model into an equivalent system, with the transformed free energy a quadratic functional with respect to the new variables and the modified energy dissipative law is conserved. Then a special class of RK methods is employed for the reformulated system to arrive at structure-preserving time-discrete schemes. Along with rigorous proofs, numerical experiments are presented to demonstrate the accuracy and unconditionally energy-stability of the EQRK schemes.

1. Introduction

Many dissipative dynamics are driven by an effective energy that is decreasing with time, where the decreasing path is controlled by a certain dissipative mechanism [1–3]. To model such dynamics, a gradient flow model is usually used. In general, the gradient flow model for a single state variable ϕ reads

$$\partial_t \phi = \mathcal{G}\frac{\delta F}{\delta \phi}, \tag{1.1}$$

with proper initial and boundary conditions, where $\mathcal{G}$ is the mobility differential or integral operator that is negative semi-definite and may depend on ϕ and its spacial derivatives, and F is the effective free energy or Lyapunov function.

Consider the domain Ω with a smooth boundary. The L^2 inner product and its norm are defined as $(f,g) = \int_\Omega fg d\mathbf{x}$ and $\|f\|_2 = \sqrt{(f,f)}$, $\forall f, g \in L^2(\Omega)$, respectively. Without loss of generality, we consider a single state variable ϕ and the generic form of the effective free energy

$$F = \frac{1}{2}(\mathcal{L}\phi, \phi) + (g, 1), \tag{1.2}$$

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where $\mathcal{L}$ is a linear, self-adjoint and positive semi-definite operator, and g is a potential function density that might depend on ϕ and its low order spacial derivatives.

With the effective free energy (1.2), the gradient flow model (1.1) could be rewritten as

$$\partial_t \phi = \mathcal{G}\Big(\mathcal{L}\phi + \frac{\delta g}{\delta \phi}\Big). \tag{1.3}$$

Here we note that $\mathcal{G}$ is negative semi-definite, i.e., under proper boundary conditions, it satisfies $(u, \mathcal{G}u) \leq 0$, $\forall u \in L^2(\Omega)$. Therefore, the gradient flow system (1.3) satisfies the following energy dissipation law

$$\frac{dF}{dt} = \Big(\mathcal{L}\phi + \frac{\delta g}{\delta \phi}, \partial_t \phi\Big) = \Big(\mathcal{L}\phi + \frac{\delta g}{\delta \phi}, \mathcal{G}(\mathcal{L}\phi + \frac{\delta g}{\delta \phi})\Big) \leq 0. \tag{1.4}$$

Along with the wide applications of gradient flow models, numerical experts are dedicated to developing discrete numerical approximations such that the energy decreasing property (1.4) could be preserved in the discrete level. A numerical scheme possesses such property is usually named an energy-stable scheme. In addition, if such numerical stability does not have any restrictions on the time step, it is usually named 'unconditionally' energy stable. Such energy-stable numerical schemes are always desired as it mimics the physical structure of the original problem, and thus often performs excellent numerical stability even with large time step size.

However, among all of the existing numerical approaches so far, for instance [4–10], most of the schemes are up to second-order. There are few higher-order energy stable schemes in the literature [11–13], most of which only apply to restricted models and lack necessary theoretical proofs for unconditionally energy stability.

The goal of this letter is to shed light on this topic of high-order unconditionally energy-stable numerical approximations of dissipative systems. In particular, we propose a new class of arbitrarily high order schemes for the general gradient flow model. The proposed schemes are proven rigorously to be unconditionally energy stable. And they could be readily applied to solve any gradient flow models.

2. Numerical approximations of gradient flow models

In this section, we introduce the general framework for developing arbitrarily high-order structure-preserving schemes for gradient flow models.

First of all, we introduce one or more auxiliary variables to reformulate the original gradient flow model (1.3) into an equivalent form, which has a quadratic energy functional and obeys the same energy dissipation law. This process is called energy quadratization (EQ) reformulation, because the free energy of the original system is transformed into a quadratic form [8]. So far, there are two ways to introduce auxiliary variables, i.e. the invariant energy quadratization (IEQ) approach [8,14] and the scalar auxiliary variable (SAV) approach [12]. The EQ reformulation for gradient flow models will provide an elegant platform for developing arbitrarily high-order unconditionally energy stable schemes, which is the major focus of this paper.

2.1. Energy quadratization reformulation

For simplicity of notations, we assume g only depends on ϕ, but not its spatial derivatives. However, we note that the proposed approaches in this paper are suitable for a more general g.

On the one hand, we utilize the IEQ approach to obtain the EQ reformulation, by introducing an auxiliary variable

$$q(\mathbf{x}, t) = \sqrt{g(\phi) + C_0}, \tag{2.1}$$

where C_0 is a positive number such that $g(\phi)+C_0>0$. The energy functional (1.2) could be rewritten as a quadratic form

$$E=\frac{1}{2}(\mathcal{L}\phi,\phi)+\|q\|^2-C_0|\Omega|. \tag{2.2}$$

Then we can rewrite the original gradient flow system (1.3) into the IEQ reformulated system

$$\begin{cases}\partial_t\phi=\mathcal{G}\Big(\mathcal{L}\phi+\frac{qg'(\phi)}{\sqrt{g(\phi)+C_0}}\Big),\\ \partial_t q=\frac{g'(\phi)}{2\sqrt{g(\phi)+C_0}}\partial_t\phi,\end{cases} \tag{2.3}$$

with the consistent initial condition $q|_{t=0}=\sqrt{g(\phi|_{t=0})+C_0}$. By a straightforward calculation, the new system (2.3) possesses the following energy dissipation law

$$\frac{dE}{dt}=\Big(\mathcal{L}\phi+\frac{qg'(\phi)}{\sqrt{g(\phi)+C_0}},\mathcal{G}\Big(\mathcal{L}\phi+\frac{qg'(\phi)}{\sqrt{g(\phi)+C_0}}\Big)\Big)\le 0. \tag{2.4}$$

On the other hand, we may apply the SAV approach to obtain another EQ reformulation, by introducing a scalar auxiliary variable

$$q(t)=\sqrt{\big(g(\phi),1\big)+C_0}, \tag{2.5}$$

where C_0 is a positive number such that $\big(g(\phi),1\big)+C_0>0$. The energy functional (1.2) could be denoted as the following quadratic form

$$E=\frac{1}{2}(\mathcal{L}\phi,\phi)+q^2-C_0. \tag{2.6}$$

Then we can transform the original system (1.3) into the SAV reformulated system

$$\begin{cases}\partial_t\phi=\mathcal{G}\Big(\mathcal{L}\phi+\frac{qg'(\phi)}{\sqrt{\big(g(\phi),1\big)+C_0}}\Big),\\ \partial_t q=\Big(\frac{g'(\phi)}{2\sqrt{\big(g(\phi),1\big)+C_0}},\partial_t\phi\Big),\end{cases} \tag{2.7}$$

with the consistent initial condition $q|_{t=0}=\sqrt{\big(g(\phi|_{t=0}),1\big)+C_0}$. It could be readily shown that the SAV reformulated system (2.7) satisfies the following energy dissipation law

$$\frac{dE}{dt}=\Big(\mathcal{L}\phi+\frac{qg'(\phi)}{\sqrt{\big(g(\phi),1\big)+C_0}},\mathcal{G}\Big(\mathcal{L}\phi+\frac{qg'(\phi)}{\sqrt{\big(g(\phi),1\big)+C_0}}\Big)\Big)\le 0. \tag{2.8}$$

We note that the auxiliary variable $q(\mathbf{x},t)$ in (2.1) is a function of space variable $\mathbf{x}$ and time variable t in the IEQ reformulation while the scalar auxiliary variable $q(t)$ in (2.5) only depends on time variable t in the SAV reformulation. Nevertheless, their common aim is to transform the original system into a new equivalent system with a quadratic energy functional and the modified energy dissipation law. And the choice of the intermediate variable q is not unique. For some complicated free energy functionals, we may need to introduce more variables. Next, we will develop arbitrarily high-order unconditionally energy stable numerical approximations for the EQ reformulated system (2.3) or (2.7), which in turn solve (1.3).

2.2. EQRK method

In this section, we first derive the RK method for the IEQ reformulated system (2.3) and the SAV reformulated system (2.7), respectively. Then a class of RK methods with the stability condition are shown to preserve the discrete energy dissipation property, i.e. unconditionally energy stable.

On the one hand, applying an s-stage RK method to the IEQ reformulated system (2.3), we obtain the following IEQ-RK scheme.

Scheme 2.1 (*s-stage IEQ-RK Scheme*). *Let b_i, a_{ij} $(i,j=1,\dots,s)$ be real numbers and let $c_i=\sum_{j=1}^{s}a_{ij}$. For given (ϕ^n,q^n), the following intermediate values are first calculated by*

$$\begin{aligned}\Phi_i&=\phi^n+\Delta t\sum_{j=1}^{s}a_{ij}k_j,\\ Q_i&=q^n+\Delta t\sum_{j=1}^{s}a_{ij}l_j,\end{aligned}\tag{2.9}$$

where $k_i=\mathcal{G}\Big(\mathcal{L}\Phi_i+\frac{Q_ig'(\Phi_i)}{\sqrt{g(\Phi_i)+C_0}}\Big)$, and $l_i=\frac{g'(\Phi_i)}{2\sqrt{g(\Phi_i)+C_0}}k_i$. Then (ϕ^{n+1},q^{n+1}) is updated via

$$\begin{aligned}\phi^{n+1}&=\phi^n+\Delta t\sum_{i=1}^{s}b_ik_i,\\ q^{n+1}&=q^n+\Delta t\sum_{i=1}^{s}b_il_i.\end{aligned}\tag{2.10}$$

On the other hand, applying an s-stage RK method to the SAV reformulated system (2.7), we obtain the corresponding SAV-RK scheme.

Scheme 2.2 (*s-stage SAV-RK Scheme*). *Let b_i, a_{ij} $(i,j=1,\dots,s)$ be real numbers and let $c_i=\sum_{j=1}^{s}a_{ij}$. For given (ϕ^n,q^n), the following intermediate values are first calculated by*

$$\begin{aligned}\Phi_i&=\phi^n+\Delta t\sum_{j=1}^{s}a_{ij}k_j,\\ Q_i&=q^n+\Delta t\sum_{j=1}^{s}a_{ij}l_j,\end{aligned}\tag{2.11}$$

where $k_i=\mathcal{G}\Big(\mathcal{L}\Phi_i+\frac{Q_ig'(\Phi_i)}{\sqrt{\big(g(\Phi_i),1\big)+C_0}}\Big)$ and $l_i=\left(\frac{g'(\Phi_i)}{2\sqrt{\big(g(\Phi_i),1\big)+C_0}},k_i\right)$. Then (ϕ^{n+1},q^{n+1}) is updated via

$$\begin{aligned}\phi^{n+1}&=\phi^n+\Delta t\sum_{i=1}^{s}b_ik_i,\\ q^{n+1}&=q^n+\Delta t\sum_{i=1}^{s}b_il_i.\end{aligned}\tag{2.12}$$

The RK coefficients are usually displayed by a Butcher table [15,16]

$$\begin{array}{c|c}\mathbf{c}&\mathbf{A}\\\hline&\mathbf{b}^T\end{array}=\begin{array}{c|ccc}c_1&a_{11}&\cdots&a_{1s}\\\vdots&\vdots&&\vdots\\c_s&a_{s1}&\cdots&a_{ss}\\\hline&b_1&\cdots&b_s\end{array},$$

where $\mathbf{A}\in\mathbb{R}^{s,s}$, $\mathbf{b}\in\mathbb{R}^s$, and $\mathbf{c}=\mathbf{A}\mathbf{l}$ with $\mathbf{l}=(1,1,\dots,1)^T\in\mathbb{R}^s$.

Definition 2.1 (*Stability Condition*). Define a symmetric matrix $\mathbf{M}$ given by

$$\mathbf{M}=\mathrm{diag}(\mathbf{b})\mathbf{A}+\mathbf{A}^T\mathrm{diag}(\mathbf{b})-\mathbf{b}\mathbf{b}^T.$$

The stability condition is defined as

$$b_i\geq 0,\ \forall i=1,2,\dots,s,\quad \text{and } \mathbf{M} \text{ is positive semi-definite.}\tag{2.13}$$

Next, we show that both the IEQ-RK scheme and the SAV-RK scheme with their RK coefficients satisfying the stability condition are unconditionally energy stable.

Theorem 2.1. *Both the IEQ-RK scheme and the SAV-RK scheme with their RK coefficients satisfying the stability condition (2.13) are unconditionally energy stable, i.e., they satisfy*

$$E^{n+1} \leq E^n, \tag{2.14}$$

where

$$E^n = \frac{1}{2}(\mathcal{L}\phi^n, \phi^n) + \|q^n\|^2 - C_0|\Omega|$$

in the IEQ-RK scheme and

$$E^n = \frac{1}{2}(\mathcal{L}\phi^n, \phi^n) + (q^n)^2 - C_0$$

in the SAV-RK scheme, respectively.

Proof. Here we give the rigorous proof for the IEQ-RK scheme. The SAV-RK scheme can be discussed analogously. Denoting $\phi^{n+1} = \phi^n + \Delta t \sum_{i=1}^s b_i k_i$ and noticing that the operator $\mathcal{L}$ is linear and self-adjoint, we have

$$\frac{1}{2}(\mathcal{L}\phi^{n+1}, \phi^{n+1}) - \frac{1}{2}(\mathcal{L}\phi^n, \phi^n) = \Delta t \sum_{i=1}^s b_i(k_i, \mathcal{L}\phi^n) + \frac{\Delta t^2}{2} \sum_{i,j=1}^s b_i b_j (k_i, \mathcal{L}k_j). \tag{2.15}$$

Applying $\phi^n = \Phi_i - \Delta t \sum_{j=1}^s a_{ij} k_j$ to the right of (2.15), we can deduce

$$\frac{1}{2}(\mathcal{L}\phi^{n+1}, \phi^{n+1}) - \frac{1}{2}(\mathcal{L}\phi^n, \phi^n) = \Delta t \sum_{i=1}^s b_i(k_i, \mathcal{L}\Phi_i) - \frac{\Delta t^2}{2} \sum_{i,j=1}^s \mathbf{M}_{ij}(k_i, \mathcal{L}k_j), \tag{2.16}$$

where $\sum_{i,j=1}^s b_i a_{ij}(k_i, \mathcal{L}k_j) = \sum_{i,j=1}^s b_j a_{ji}(k_i, \mathcal{L}k_j)$ and $\mathbf{M}_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j$ were used. Note that $\mathcal{L}$ can be denoted as $\mathcal{L} = \sum_m \mathcal{L}_m^* \mathcal{L}_m$, where $\mathcal{L}_m$ is a linear operator and $\mathcal{L}_m^*$ is the adjoint operator of $\mathcal{L}_m$. Since $\mathbf{M}$ is positive semi-definite, we have

$$\sum_{i,j=1}^s \mathbf{M}_{ij}(k_i, \mathcal{L}k_j) = \sum_m \sum_{i,j=1}^s \mathbf{M}_{ij}(\mathcal{L}_m k_i, \mathcal{L}_m k_j) \geq 0. \tag{2.17}$$

Combining Eqs. (2.16) and (2.17) leads to

$$\frac{1}{2}(\mathcal{L}\phi^{n+1}, \phi^{n+1}) - \frac{1}{2}(\mathcal{L}\phi^n, \phi^n) \leq \Delta t \sum_{i=1}^s b_i(k_i, \mathcal{L}\Phi_i). \tag{2.18}$$

Similarly, we have

$$\|q^{n+1}\|^2 - \|q^n\|^2 \leq 2\Delta t \sum_{i=1}^s b_i(l_i, Q_i) = \Delta t \sum_{i=1}^s b_i \Big(\frac{Q_i g'(\Phi_i)}{\sqrt{g(\Phi_i) + C_0}}, k_i \Big). \tag{2.19}$$

Adding (2.18) and (2.19) leads to

$$E^{n+1} - E^n \leq \Delta t \sum_{i=1}^s b_i \Big(\mathcal{L}\Phi_i + \frac{Q_i g'(\Phi_i)}{\sqrt{g(\Phi_i) + C_0}}, k_i \Big). \tag{2.20}$$

Replacing $k_i = \mathcal{G}\left(\mathcal{L}\Phi_i + \frac{Q_i g'(\Phi_i)}{\sqrt{g(\Phi_i)+C_0}} \right)$ to (2.20) and noticing the negative semi-definite property of $\mathcal{G}$ and $b_i \geq 0, \forall i$, we can arrive at $E^{n+1} - E^n \leq 0$. This completes the proof. □

Table 2.1
Butcher tableaus of SDIRK(2,1), SDIRK(3,2) and SDIRK(4,3).

$\frac{1}{2}$	$\frac{1}{2}$	σ	σ	0	σ	σ	0	0
	1	$1-\sigma$	$1-2\sigma$	σ	$\frac{1}{2}$	$\frac{1}{2}-\sigma$	σ	0
			$\frac{1}{2}$	$\frac{1}{2}$	$1-\sigma$	2σ	$1-4\sigma$	σ
						μ	$1-2\mu$	μ

Examples of the RK coefficients, satisfying the stability condition (2.13), include the singly diagonal implicit Runge–Kutta (SDIRK) method and the Gauss–Legendre (GL) method. The SDIRK schemes from the second order to fourth order are presented in Table 2.1. In Table 2.1, we set $\sigma = (3+\sqrt{3})/6$ for SDIRK(3,2), and $\sigma = \cos(\pi/18)/\sqrt{3}+1/2$, $\mu = 1/\big(6(2\sigma-1)^2\big)$ for SDIRK(4,3). For more discussions, please refer to [16,17].

Remark 2.1. Though the energy (1.2), (2.2) and (2.6) are equivalent in the continuum form, the proposed schemes only satisfy an energy dissipation law in the discrete form of (2.2) (or (2.6)), instead of the original energy in (1.2). And unlike the result in [6], the error between the discrete versions of (2.2) (or (2.6)) and (1.2) is not analytically quantified yet. However, we point out the discrete version of (2.2) (or (2.6)) is a high-order approximation of (2.2) (or (2.6)), i.e. (1.2).

3. Numerical examples

In this section, we briefly present several numerical examples to demonstrate the practicability, accuracy, as well as unconditional energy stability of our proposed IEQ-RK and SAV-RK schemes.

For simplicity of explanation, we will consider periodic boundary conditions for the examples below. To make the order of accuracy in space compatible with the arbitrarily high-order in time, we will employ the Fourier pseudo-spectral method in space for Scheme 2.1 and Scheme 2.2. For more details of the pseudo-spectral spacial discretization, please refer to our previous work [18].

Example 1. [Cahn–Hilliard Type Equations] One broad class of gradient flow models is the Cahn–Hilliard type equations. The proposed schemes above are rather general to be readily applied to solve all Cahn–Hilliard type equations. For illustration purpose, we study the widely-used Cahn–Hilliard equation as follows [19]

$$\partial_t \phi = \lambda \Delta(-\varepsilon^2 \Delta\phi + \phi^3 - \phi), \tag{3.1}$$

with the Ginzburg–Landau free energy $F = \frac{\varepsilon^2}{2}\|\nabla\phi\|^2 + \frac{1}{4}\|\phi^2-1\|^2$. By introducing a new variable $q = \frac{1}{2}(\phi^2 - 1 - \gamma_0)$ or a new scalar variable $q = \frac{1}{2}\sqrt{\|\phi^2-1-\gamma_0\|^2 + C_0}$, we can deduce the model into its EQ reformulated form, such that the proposed schemes could be readily applied. Here we note that the new energy is transformed into $E = \frac{\varepsilon^2}{2}\|\nabla\phi\|^2 + \frac{\gamma_0}{2}\|\phi\|^2 + \|q\|^2 - \frac{2\gamma_0+\gamma_0^2}{4}|\Omega|$ in the reformulated IEQ model and $E = \frac{\varepsilon^2}{2}\|\nabla\phi\|^2 + \frac{\gamma_0}{2}\|\phi\|^2 + q^2 - \frac{C_0}{4} - \frac{2\gamma_0+\gamma_0^2}{4}|\Omega|$ in the reformulated SAV model, which imply that L^2 boundedness of ϕ is still inherited in the EQ reformulated models. A detailed discussion on γ_0 and C_0, readers is referred to [12,20].

Consider the domain $\Omega = [0,1]^2$ and the parameters $\varepsilon = 10^{-2}$, $\lambda = 10^{-3}$. We set the initial condition as $\phi = 0.25\sin(2\pi x)\cos(2\pi y)$, and use 256×256 meshes, $\gamma_0 = 1$, $C_0 = 1$ for calculation. As the analytical solution is unknown, we use the numerical solution from the IEQ-RK scheme with SDIRK(4,3) and $\delta t = 10^{-5}$ as the 'real' solution. The L^2 errors obtained using IEQ-RK schemes and SAV-RK schemes with SDIRK(3,2), SDIRK(4,3) at $t = 0.4$ are summarized in Table 3.1. We observe that the expected order of accuracy is reached when the time step is relatively small enough. In addition, the IEQ-RK schemes and their corresponding SAV-RK schemes with the same RK coefficients have similar accuracy.

Table 3.1
The L^2 errors calculated using IEQ-SDIRK(3,2), IEQ-SDIRK(4,3), SAV-SDIRK(3,2) and SAV-SDIRK(4,3), respectively.

δt	L^2 error	Order	L^2 error	Order	L^2 error	Order	L^2 error	Order
0.2	1.918e−5		1.305e−5		1.865e−5		1.245e−5	
0.1	2.124e−6	3.17	4.747e−7	4.78	2.069e−6	3.17	4.557e−7	4.77
0.05	2.519e−7	3.07	2.438e−8	4.28	2.456e−7	3.07	2.345e−8	4.28
0.025	3.071e−8	3.03	1.396e−9	4.12	2.996e−8	3.03	1.344e−9	4.13
0.0125	3.793e−9	3.02	8.309e−11	4.07	3.701e−9	3.02	8.004e−11	4.07
0.00625	4.714e−10	3.00	4.902e−12	4.08	4.600e−10	3.01	4.715e−12	4.08

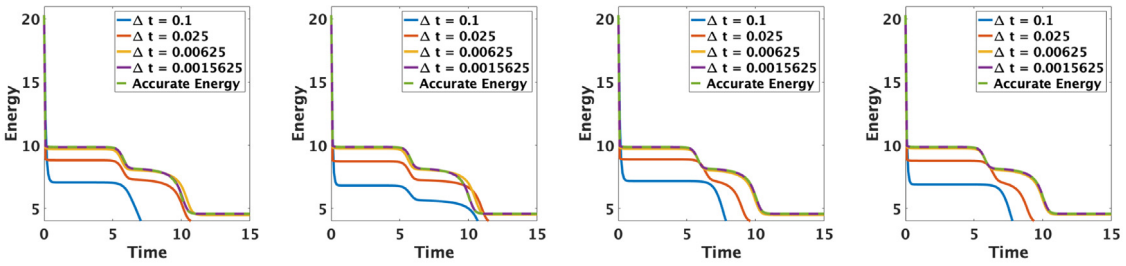

Fig. 3.1. MBE energy evolutions calculated using different numerical schemes with various time steps, namely using (a) IEQ-SDIRK(3,2); (b) IEQ-SDIRK(4,3); (c) SAV-SDIRK(3,2); (d) SAV-SDIRK(4,3) respectively.

Example 2. [Allen–Cahn Type Equations] Another type of gradient flow models is the Allen–Cahn type equation. Without lose of generality, here we consider one case of Allen–Cahn type equations, the molecular beam epitaxy (MBE) growth model without slop selection

$$\partial_t \phi = -M\Big(\varepsilon^2 \Delta^2 \phi + \nabla \cdot \big((1 - |\nabla\phi|^2)\nabla\phi\big)\Big), \tag{3.2}$$

with the free energy functional $F = \frac{\varepsilon^2}{2}\|\Delta\phi\|^2 + \frac{1}{4}\big\||\nabla\phi|^2 - 1\big\|^2$. Letting the new variable $q = \frac{1}{2}(|\nabla\phi|^2 - 1 - \gamma_0)$ or the new scalar variable $q = \frac{1}{2}\sqrt{\big\||\nabla\phi|^2 - 1 - \gamma_0\big\|^2 + C_0}$, we can obtain the corresponding EQ reformulated models, which could be solved with the proposed schemes. Here γ_0 and C_0 are artificial parameters to regularize the scheme [12,20].

In this example, we choose $\Omega = [0, 2\pi]^2$ and $\phi(t = 0) = 0.1(\sin(3x)\sin(2y) + \sin(5x)\sin(5y))$ [21], and set $\lambda = 1$, $\epsilon^2 = 0.1$, $\gamma_0 = 1$, $C_0 = 1$ and 256×256 meshes. The MBE effective energy evolution using various time step sizes are summarized in Fig. 3.1, demonstrating the energy stability of our proposed schemes. We observe the calculated energies are all decreasing with time, which agree with Theorem 2.1 ($E^{n+1} \leq E^n$ for an arbitrary time step), i.e., unconditionally energy stable. However, when the time step is large, truncation error would contaminate the numerical solution, which produces deviated energy evolution. Nevertheless, the newly proposed high order schemes are superior than the traditional second-order IEQ scheme and SAV scheme, as they provide an accurate numerical solution even with much larger time step sizes.

4. Conclusion

In this letter, we propose a novel class of arbitrarily high-order-accurate in time and energy-stable numerical schemes for solving the gradient flow models. The energy quadratization technique is first applied to reformulate the gradient flow models into an equivalent model, where the effective free energy is transformed into a quadratic form. Then a special class of RK methods is utilized to arrive at the structure-preserving schemes for the equivalent model. This numerical strategy is rather general that it applies to all gradient flow models and can reach arbitrarily high order while preserving the energy stability.

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