Mathematika 65 (2019) 702–707
doi:10.1112/S0025579319000135

THE ERDŐS–SZEKERES PROBLEM AND AN INDUCED RAMSEY QUESTION

DHRUV MUBAYI AND ANDREW SUK

Abstract. Motivated by the Erdős–Szekeres convex polytope conjecture in $\mathbb{R}^d$, we initiate the study of the following induced Ramsey problem for hypergraphs. Given integers $n > k \geqslant 5$, what is the minimum integer $g_k(n)$ such that any k-uniform hypergraph on $g_k(n)$ vertices with the property that any set of $k+1$ vertices induces 0, 2, or 4 edges, contains an independent set of size n. Our main result shows that $g_k(n) > 2^{cn^{k-4}}$, where $c = c(k)$.

§1. *Introduction.* Given a finite point set P in d-dimensional Euclidean space $\mathbb{R}^d$, we say that P is in *general position* if no $d+1$ members lie on a common hyperplane. Let $ES_d(n)$ denote the minimum integer N, such that any set of N points in $\mathbb{R}^d$ in general position contains n members in *convex position*, that is, n points that form the vertex set of a convex polytope. In their classic 1935 paper, Erdős and Szekeres [1] proved that in the plane, $ES_2(n) \leqslant 4^n$. In 1960 [2], they showed that $ES_2(n) \geqslant 2^{n-2}+1$ and conjectured this to be sharp for every integer $n \geqslant 3$. Their conjecture has been verified for $n \leqslant 6$ [1, 7], and determining the exact value of $ES_2(n)$ for $n \geqslant 7$ is one of the longest-standing open problems in Ramsey theory/discrete geometry. Recently [8], the second author asymptotically verified the Erdős–Szekeres conjecture by showing that $ES_2(n) = 2^{n+o(n)}$.

In higher dimensions, $d \geqslant 3$, much less is known about $ES_d(n)$. In [3], Károlyi showed that projections into lower-dimensional spaces can be used to bound these functions, since most generic projections preserve general position, and the preimage of a set in convex position must itself be in convex position. Hence, $ES_d(n) \leqslant ES_2(n) = 2^{n+o(n)}$. However, the best known lower bound for $ES_d(n)$ is only of the order of $2^{cn^{1/(d-1)}}$, due to Károlyi and Valtr [4]. An old conjecture of Füredi (see [5, Ch. 3]) says that this lower bound is essentially the truth.

CONJECTURE 1.1. *For $d \geqslant 3$, $ES_d(n) = 2^{\Theta(n^{1/(d-1)})}$.*

It was observed by Motzkin [6] that any set of $d+3$ points in $\mathbb{R}^d$ in general position contains either 0, 2, or 4 $(d+2)$-tuples not in convex position. By defining a hypergraph H whose vertices are N points in $\mathbb{R}^d$ in general position,

Received 31 July 2018.
MSC (2010): 05D10 (primary).
The first author's research was partially supported by NSF grant DMS-1763317. The second author was supported by an NSF CAREER award and an Alfred Sloan Fellowship.

and edges are $(d+2)$-tuples not in convex position, then every set of $d+3$ vertices induces 0, 2, or 4 edges. Moreover, by Carathéodory's theorem (see [5, Theorem 1.2.3]), an independent set in H would correspond to a set of points in convex position. This leads us to the following combinatorial parameter.

Let $g_k(n)$ be the minimum integer N such that any k-uniform hypergraph on N vertices with the property that every set of $k+1$ vertices induces 0, 2, or 4 edges, contains an independent set of size n. For $k \geqslant 5$, the geometric construction of Károlyi and Valtr [4] mentioned earlier implies that

$$g_k(n) \geqslant ES_{k-2}(n) \geqslant 2^{cn^{1/(k-3)}},$$

where $c = c(k)$. One might be tempted to prove Conjecture 1.1 by establishing a similar upper bound for $g_k(n)$. However, our main result shows that this is not possible.

THEOREM 1.2. *For each $k \geqslant 5$ there exists $c = c(k) > 0$ such that for any $n \geqslant k$ we have $g_k(n) > 2^{cn^{k-4}}$.*

In the other direction, we can bound $g_k(n)$ from above as follows. For $n \geqslant k \geqslant 5$ and $t < k$, let $h_k(t, n)$ be the minimum integer N such that any k-uniform hypergraph on N vertices with the property that any set of $k+1$ vertices induces at most t edges, contains an independent set of size n. In a forthcoming paper, the authors prove the following.

THEOREM 1.3. *For $k \geqslant 5$ and $t < k$, there is a positive constant $c' = c'(k, t)$ such that*

$$h_k(t, n) \leqslant \mathrm{twr}_t(c' n^{k-t} \log n),$$

where twr *is defined recursively as* $\mathrm{twr}_1(x) = x$ *and* $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Hence, we have the following corollary.

COROLLARY 1.4. *For $k \geqslant 5$, there is a constant $c' = c'(k)$ such that*

$$g_k(n) \leqslant h_k(4, n) \leqslant 2^{2^{2^{c' n^{k-4} \log n}}}.$$

It is an interesting open problem to improve either the upper or lower bounds for $g_k(n)$.

Problem 1.5. Determine the tower growth rate for $g_k(n)$.

Actually, this Ramsey function can be generalized further as follows: for every $S \subset \{0, 1, \ldots, k\}$, define $g_k(n, S)$ to be the minimum integer N such that any N-vertex k-uniform hypergraph with the property that every set of $k+1$ vertices induces s edges for some $s \in S$, contains an independent set of size n.

General results for $g_k(n, S)$ may shed light on classical Ramsey problems, but it appears difficult to determine even the tower height for any nontrivial cases.

§2. *Proof of Theorem 1.2.* Let $k \geqslant 5$ and $N = 2^{cn^{k-4}}$ where $c = c_k > 0$ is sufficiently small to be chosen later. We are to produce a k-uniform hypergraph H on N vertices with $\alpha(H) < n$ and every $k+1$ vertices of H span 0, 2, or 4 edges. Let $\phi : \binom{[N]}{k-3} \to \binom{[k-1]}{2}$ be a random $\binom{k-1}{2}$-coloring, where each color appears on each $(k-3)$-tuple independently with probability $1/\binom{k-1}{2}$. For $f = (v_1, \ldots, v_{k-1}) \in \binom{[N]}{k-1}$, where $v_1 < v_2 < \cdots < v_{k-1}$, define the function $\chi_f : \binom{f}{k-3} \to \binom{[k-1]}{2}$ as follows: for all $\{i, j\} \in \binom{[k-1]}{2}$, let

$$\chi_f(f \setminus \{v_i, v_j\}) = \{i, j\}.$$

We define the $(k-1)$-uniform hypergraph G, whose vertex set is $[N]$, such that

$$G = G_\phi := \left\{ f \in \binom{[N]}{k-1} : \phi(f \setminus \{u, v\}) = \chi_f(f \setminus \{u, v\}) \text{ for all } \{u, v\} \in \binom{f}{2} \right\}.$$

For example, if $k = 4$ (which is excluded for the theorem but we allow it to illustrate this construction) then $\phi : [N] \to \{12, 13, 23\}$ and for $f = (v_1, v_2, v_3)$, where $v_1 < v_2 < v_3$, we have $f \in G$ if and only if $\phi(v_1) = 23$, $\phi(v_2) = 13$, and $\phi(v_3) = 12$.

Given a subset $S \subset [N]$, let $G[S]$ be the subhypergraph of G induced by the vertex set S. Finally, we define the k-uniform hypergraph H, whose vertex set is $[N]$, such that

$$H = H_\phi := \left\{ e \in \binom{[N]}{k} : |G[e]| \text{ is odd} \right\}.$$

CLAIM 2.1. *$|H[S]|$ is even for every $S \in \binom{[N]}{k+1}$.*

Proof. Let $S \in \binom{[N]}{k+1}$ and suppose for contradiction that $|H[S]|$ is odd. Then

$$2|G[S]| = \sum_{f \in G[S]} 2 = \sum_{f \in G[S]} \sum_{\substack{e \in \binom{S}{k} \\ e \supset f}} 1 = \sum_{e \in \binom{S}{k}} |G[e]| = \sum_{e \notin H[S]} |G[e]| + \sum_{e \in H[S]} |G[e]|.$$

The first sum on the right-hand side above is even by definition of H and the second sum is odd by definition of H and the assumption that $|H[S]|$ is odd. This contradiction completes the proof. □

CLAIM 2.2. *$|G[e]| \leqslant 2$ for every $e \in \binom{[N]}{k}$.*

Proof. For sake of contradiction, suppose that for $e = (v_1, \ldots, v_k)$, where $v_1 < \cdots < v_k$, we have $|G[e]| \geqslant 3$. Let $e_p = e \setminus \{v_p\}$ for $p \in [k]$ and suppose that $e_i, e_j, e_l \in G$ with $i < j < l$. In what follows, we will find a set S of size $k-3$, where $S \subset e_i$ and $S \subset e_l$, such that $\chi_{e_i}(S) \neq \chi_{e_l}(S)$. This will give us our contradiction since $e_i, e_l \in G$ implies that $\chi_{e_i}(S) = \phi(S) = \chi_{e_l}(S)$.

Let $Y = e \setminus \{v_i, v_j, v_l\}$ and $Y' = Y \setminus \{\min Y\}$. Let us first assume that $i > 1$ so that $\min Y = v_1$. In this case,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{1, l-1\},$$

since we obtain $Y' \cup \{v_j\}$ from e_i by removing $\min Y$ and v_l which are the first and $(l-1)$st elements of e_i. Similarly,

$$\chi_{e_l}(Y' \cup \{v_j\}) = \{1, i\},$$

since we obtain $Y' \cup \{v_j\}$ from e_l by removing $\min Y$ and v_i which are the first and ith elements of e_l. Because $l > i+1$, we conclude that $\chi_{e_i}(Y' \cup \{v_j\}) \neq \chi_{e_l}(Y' \cup \{v_j\})$ as desired.

Next, we assume that $i = 1$ and $\min Y = v_q$ where $q > 1$. In this case,

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{q-1, l-1\},$$

since we obtain $Y' \cup \{v_j\}$ from e_i by removing v_q and v_l which are the $(q-1)$st and $(l-1)$st elements of e_i. Similarly,

$$\chi_{e_l}(Y' \cup \{v_j\}) = \{1, q'\} \quad \text{where } q' = q \text{ if } q < l \text{ and } q' = q-1 \text{ if } q > l,$$

since we obtain $Y' \cup \{v_j\}$ from e_l by removing $v_i = v_1$ and v_q which are the first and q'th elements of e_l. If $q \neq 2$, then we immediately obtain $\chi_{e_i}(Y' \cup \{v_j\}) \neq \chi_{e_l}(Y' \cup \{v_j\})$ as desired. On the other hand, if $q = 2$, then $q' = q = 2$ as well and $l \geqslant 4$, so $l - 1 \neq q'$ and again

$$\chi_{e_i}(Y' \cup \{v_j\}) = \{q-1, l-1\} \neq \{1, q'\} = \chi_{e_l}(Y' \cup \{v_j\}).$$

This completes the proof of the claim. □

Let T_3 be the $(k-1)$-uniform hypergraph with vertex set S with $|S| = k+1$ and three edges e_1, e_2, e_3 such that there are three pairwise disjoint pairs $p_1, p_2, p_3 \in \binom{S}{2}$ with $p_i = \{v_i, v_i'\}$ and $e_i = S \setminus p_i$ for $i \in \{1, 2, 3\}$.

CLAIM 2.3. *$T_3 \not\subset G$.*

Proof. Suppose for a contradiction that there is a subset $S \subset [N]$ of size $k+1$ such that $T_3 \subset G[S]$. Using the notation above, assume without loss of generality that $v_1 = \min \cup_i p_i$ and $v_2 = \min(p_2 \cup p_3)$. Let $Y = S \setminus (p_1 \cup p_3)$ and note that

$Y \in \binom{e_1 \cap e_3}{k-3}$. Let $Y_1 \subset Y$ be the set of elements in Y that are smaller than v_1, so we have the ordering

$$Y_1 < v_1 < v_2 < \{v_3, v_3'\}.$$

Now, $\chi_{e_1}(Y)$ is the pair of positions of v_3 and v_3' in e_1. Both of these positions are at least $|Y_1| + 2$ as $Y_1 \cup \{v_2\}$ lies before p_3. On the other hand, the smallest element of $\chi_{e_3}(Y)$ is $|Y_1| + 1$ which is the position of v_1 in e_3. This shows that $\chi_{e_1}(Y) \neq \chi_{e_3}(Y)$, which is a contradiction as both must be equal to $\phi(Y)$ as $e_1, e_3 \subset G$. □

We now show that every $(k+1)$-set $S \subset [N]$ spans $0, 2$ or 4 edges of H. By Claim 2.1, $|H[S]|$ is even. Let G' be the graph with vertex set S and edge set $\{S \setminus f : f \in G[S]\}$. So there is a one-to-one correspondence between $G[S]$ and G' via the map $f \to S \setminus f$. If G' has a vertex x of degree at least three, then $|G[S \setminus \{x\}]| \geqslant 3$ which contradicts Claim 2.2. Therefore G' consists of disjoint paths, cycles, and isolated vertices. This implies that a k-set $A \subset S$ is an edge in H exactly when $S \setminus A$ is a vertex of degree one in G'. Next, observe that Claim 2.3 implies that G' does not contain a matching of size three, for the complementary sets of this matching yield a copy of $T_3 \subset G$. Hence, the number of degree-one vertices in G' is 0, 2, or 4, and therefore $|H[S]| \in \{0, 2, 4\}$ for all $S \in \binom{[N]}{k+1}$.

Let us now argue that $\alpha(H) < n$, which is a straightforward application of the probabilistic method. Indeed, we will show that this happens with positive probability and conclude that an H with this property exists. For a given k-set, the probability that it is an edge of H is $p > 0$, where p depends only on k. Consequently, the probability that H has an independent set of size n is at most

$$\binom{N}{n}(1-p)^{c'n^{k-3}}$$

for some $c' > 0$. Note that the exponent $k-3$ above is obtained by taking a partial Steiner $(n, k, k-3)$ system S within a potential independent set of size n and observing that we have independence within the edges of S. A short calculation shows that this probability is less than 1 as long as c is sufficiently small. This completes the proof of Theorem 1.2. □

Acknowledgement. We thank the referee for helpful comments.

References

1. P. Erdős and G. Szekeres, A combinatorial problem in geometry. *Compos. Math.* **2** (1935), 463–470.
2. P. Erdős and G. Szekeres, On some extremum problems in elementary geometry. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **3–4** (1960–61), 53–62.
3. G. Károlyi, Ramsey-remainder for convex sets and the Erdős–Szekeres theorem. *Discrete Appl. Math.* **109** (2001), 163–175.
4. G. Károlyi and P. Valtr, Point configurations in d-space without large subsets in convex position. *Discrete Comput. Geom.* **30** (2003), 277–286.
5. J. Matoušek, *Lectures in Discrete Geometry*, Springer (2002).
6. T. Motzkin, Cooperative classes of finite sets in one and more dimensions. *J. Combin. Theory* **3** (1967), 244–251.

7. G. Szekeres and L. Peters, Computer solution to the 17-point Erdős–Szekeres problem. *ANZIAM J.* **48** (2006), 151–164.
8. A. Suk, On the Erdős–Szekeres convex polygon problem. *J. Amer. Math. Soc.* **30** (2017), 1047–1053.

Dhruv Mubayi,
Department of Mathematics, Statistics, and Computer Science,
University of Illinois,
Chicago,
IL 60607,
U.S.A.
E-mail: mubayi@uic.edu

Andrew Suk,
Department of Mathematics,
University of California at San Diego,
La Jolla,
CA 92093,
U.S.A.
E-mail: asuk@ucsd.edu