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On multiparametric/explicit NMPC for Quadratically Constrained Problems ⋆

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Abstract: We present the expansion of the Basic Sensitivity Theorem to a second order Taylor approach and the implications to explicit model predictive control of quadratically constrained systems. The expansion enables the derivation of an algorithm for the analytical solution of convex multiparametric quadratically constraint programming (mpQCQP) problems and explicit quadratically constrained NMPC problems. We derive the analytical parametric expressions of the control actions for a quadratically constrained MPC problem and its corresponding critical regions. We show the piecewise non-linear form of the solution and closed-loop validation of the results.

Keywords: Multiparametric programming, quadratically constrained quadratic programming, explicit MPC

1. INTRODUCTION

1.1 The Basic Sensitivity Theorem

Consider the following general optimization problem:

$$\begin{aligned} \min_{x} \quad & f(x,\theta) \\ \text{s.t.} \quad & g_i(x,\theta) \le 0 \\ & h_j(x,\theta) = 0 \\ & x \in \mathbb{R}^n \\ & \theta \in \mathbb{R}^m \end{aligned} \tag{1}$$

where x is the optimization variables, θ the uncertain parameters and sets $i \in \mathbb{I}$, $j \in \mathbb{J}$ correspond to the inequality and equality constraint sets, respectively.

IF

(1) the functions defining the problem are twice differentiable in x and if their gradients with respect to x and the constraints are once continuously differentiable in θ in a neighborhood of (x^*, θ^*),
(2) the second-order sufficient conditions for a local minimum of the problem hold at x^* with associated Lagrange multipliers λ^* and μ^*,
(3) the gradients $\nabla g_i(x^*, \theta^*)$ (for $i \in \mathbb{I}$ such that $g_i(x^*, \theta^*) = 0$) and $\nabla h_j(x^*, \theta^*)$ are linearly independent and
(4) $\lambda_i \ge 0$ for $i \in \mathbb{I}$ such that $g_i(x^*, \theta^*) = 0$

THEN

- x^* is a local isolated minimizing point of the problem and the associated Lagrange multipliers λ_i^* and μ_j^* are unique
- for θ in the neighborhood or θ^*, there exists a unique, once continuously differentiable vector function $\eta(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]^T$ satisfying the second-order sufficient conditions for a local minimum of the problem with associated unique Lagrange multipliers $\lambda(\theta)$ and $\mu(\theta)$.
- for θ near θ^* the set of binding inequalities is unchanged, strict complementarity slackness holds and the binding constraint gradients are linearly independent at $x(\theta)$.

The above conditions are known as the Basic Sensitivity Theorem (Fiacco, 1983). If there exist λ_i^* and μ_j^* such that the first order KKT conditions hold:

$$\begin{aligned} \nabla_x L(x^*, \lambda^*, \mu^*, \theta^*) &= \nabla_x f(x^*, \theta^*) + \sum_{i=1}^{k} \lambda_i^* \nabla_x g_i(x^*, \theta^*) + \\ &\quad + \sum_{i=1}^{p} \mu_j^* \nabla_x h_j(x^*, \theta^*) = 0 \\ \lambda_i^* g_i(x^*, \theta^*) &= 0 \\ h_j(x^*, \theta^*) &= 0 \\ \lambda_i^* &\ge 0 \\ \forall i \in \mathbb{I}, \forall j \in \mathbb{J} & \end{aligned} \tag{2}$$

then the following vector of equations is defined:

$$F(x, \lambda, \mu, \theta) = \begin{bmatrix} \nabla_x L(x, \lambda, \mu, \theta) \\ \lambda_i g_i(x, \theta) \\ h_j(x, \theta) \end{bmatrix} \tag{3}$$

Adding to the above, if there exists a non-zero vector $z(x)$ such that $z(x) \nabla_{xx} L(\eta, \theta) z(x) \ge 0$, the basic sensitivity theorem is identically satisfied for θ near θ^* and can be differentiated with respect to θ to yield explicit expressions for the first partial derivatives of this vector function. The aforementioned argument can be explicitly expressed as follows:

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$$M(\eta,\theta)\nabla_\theta\eta(\theta) = N(\eta,\theta) \quad (4)$$

where M is the Jacobian matrix of the vector of equations F defined in eq. (3) with respect to the vector of variables and Lagrange multipliers η and N is the negative of the Jacobian matrix of F with respect to the vector of uncertain parameters θ:

$$\begin{aligned} M(\eta,\theta) &= \nabla_\eta F(\eta,\theta) \\ N(\eta,\theta) &= -\nabla_\theta F(\eta,\theta) \end{aligned} \quad (5)$$

The first order estimate of the variation of an isolated local solution $x(\theta)$ of the original problem and the associated unique Lagrange multipliers $\lambda(\theta)$ and $\mu(\theta)$ can be approximated, given that $\nabla_\theta\eta(\theta^*)$ is available, using the first order Taylor expansion, as follows:

Let a be the concatenation of the η and θ vectors: $a = [\eta^T, \theta^T]^T$

$$F(a) = \nabla_a F(a^*)(a-a^*) + F(a^*) \quad (6)$$

Based on the principles of the Basic Sensitivity Theorem, the solution $\eta(\theta)$ in a neighborhood of a^* does not change, therefore the value of $F(a)$ in a neighborhood of a^* remains zero.

$$\begin{aligned} &\nabla_a F(a^*)(a-a^*) = 0 \Leftrightarrow \\ &[\nabla_\eta F(a^*)|\nabla_\theta F(a^*)]\left[(\eta-\eta^*)^T|(\theta-\theta^*)^T\right]^T = 0 \Leftrightarrow \\ &\nabla_\eta F(a^*)(\eta-\eta^*) = -\nabla_\theta F(a^*)(\theta-\theta^*) \Leftrightarrow \\ &M(\eta^*,\theta^*)(\eta-\eta^*) = N(\eta^*,\theta^*)(\theta-\theta^*) \end{aligned} \quad (7)$$

The last argument of eq. (6) is the exact solution of the basic sensitivity theorem for systems that the first order Taylor expansion can describe exactly, i.e. for systems that consist of linear constraints and up to second degree polynomial objective functions in terms of the continuous variables and the uncertain parameters. Therefore, the general form of problems including only continuous variables for which eq. (6) is exact is as follows:

$$\begin{aligned} \min_x \quad & \frac{1}{2}x^TQx + x^TH^T\theta + c_x^Tx \\ \text{s.t.} \quad & A_ix \le b_i + F_i\theta \\ & A_jx = b_j + F_j\theta \\ & CR_A\theta \le CR_b \\ & x \in \mathbb{R}^n \\ & \theta \in \mathbb{R}^m \end{aligned} \quad (8)$$

Note that the terms A_i, b_i, F_i correspond to the i^{th} inequality constraint. Equivalently for j equality constraints. Furthermore, CR_A and CR_b refer to matrices of appropriate size, which define the parameter space. Also note that parameter-only dependent terms and the constant term have been omitted from the objective function as they do not affect the outcome of the optimization problem on the the $x(\theta)$, $\lambda(\theta)$ and $\mu(\theta)$ domains. Based on the findings in eq. (6) the exact solutions of *bounded* multiparametric programming problems that comply with the form presented in eq. (8) was developed. Therefore, the exact solution of multi-parametric Quadratic Programs (mpQP) and multiparametric Linear Programs (mpLP), as well as their mixed integer equivalents (mpMIQP, mpMILP respectively) were derived (for a complete literature review the reader is referred to Oberdieck et al. (2016)).

Providing exact solutions to the multiparametric nonlinear programming problem (mpNLP) is a challenging task (Fiacco, 1983; Fiacco and Kyparisis, 1986). Most efforts have focused on providing approximate solutions to the problem. In the work of Dua and Pistikopoulos (1999), an outer-approximation of the mpNLP is created through the linearization of the nonlinear terms of the objective function and the constraints. Thus, the mpNLP is transformed into a mpLP. In Johansen et al. (2002) a quadratic approximation to the objective function and linear approximations to the constraints are obtained and the mpNLP is approximated by a mpQP. Johansen (2004), proposed an approximate mp-NLP algorithm by partinioning the parameter space into hybercubes. Fotiou et al. (2006) proposed an algorithm for the solution of nonlinear parametric optimization of polynomial functions subject to polynomial constraints based on cylindrical algebraic decomposition. More recently, Domínguez and Pistikopoulos (2013) proposed the decomposition of a mpMINLP into a series of approximate mpQPs, while Dua (2015); Charitopoulos and Dua (2016) focused of multiparametric polynomial programming (mpPP). The reader is referred to Oberdieck et al. (2016) for a more detailed discussion on the topic.

In the following section, we present the second order Taylor expansion approach to the basic sensitivity theorem and how this enables the exact solution of convex multiparametric Quadratically Constrained Quadratic Programs (mpQCQP) and the application to Quadratically Constrained MPC problems.

1.2 Basic Sensitivity Theorem: The quadratic case

"If the conditions of the Basic Sensitivity Theorem hold, with the respective assumed orders of differentiability being $p-1$ more than that assumed, with $p \ge 1$, then $\eta(\theta) \equiv [x(\theta), \lambda(\theta), \mu(\theta)]^T \in C^p$ in a neighborhood of θ^*. If the problem functions are analytic in (x,θ) in a neighborhood of (x^*,θ^*), then $\eta(\theta)$ is analytic in a neighborhood of θ^*" (Fiacco, 1983). Following that, given that the conditions for the Basic Sensitivity Theorem are fulfilled and both $\nabla_\theta\eta(\theta^*)$ and $\nabla_{\theta\theta}\eta(\theta^*)$ exist, the formulation of eq. (6) can be expanded to a quadratic approach, thus yielding:

$$\begin{aligned} F(a) = &\frac{1}{2}(a-a^*)^T\nabla_{aa}F(a^*)(a-a^*)+ \\ &+\nabla_a F(a^*)(a-a^*) + F(a^*) \end{aligned} \quad (9)$$

Based on the principles of the Basic Sensitivity Theorem, the solution $\eta(\theta)$ in a neighborhood of a^* does not change, therefore the value of $F(a)$ in a neighborhood of a^* remains zero.

$$[\frac{1}{2}(a-a^*)^T\nabla_{aa}F(a^*) + \nabla_a F(a^*)](a-a^*) = 0 \quad (10)$$

The Taylor expansion of eq. (10) is exact for convex problems in x with:

- Cubic or quadratic objective function. Bilinear and trilinear terms can be included as long as the convexity of the problem is preserved.
- Quadratic, linear and left-hand-side uncertainty constraints.

The aforementioned problems have in common that the function $F(a)$ will consist of equations of up to a quadratic polynomial order.

2. MULTIPARAMETRIC QUADRATICALLY CONSTRAINED QUADRATIC PROBLEMS

Consider the convex multiparametric Quadratically Constrained Quadratic Programming (mpQCQP) problem (11):

$$\begin{aligned}\min_x \ & \frac{1}{2}x^TQx + x^TH^T\theta + c_x^Tx\\ \text{s.t.} \ & g_i(x,\theta) = x^TQ_ix + x^TH_i^T\theta + A_ix \le b_i + F_i\theta + \theta^TQ_{\theta,i}\theta\\ & h_j(x,\theta) = x^TQ_jx + x^TH_j^T\theta + A_jx = b_j + F_j\theta + \theta^TQ_{\theta,j}\theta\\ & CR_A\theta \le CR_b\\ & x\in\mathbb{R}^n, \theta\in\mathbb{R}^m\\ & i\in\mathbb{I}, j\in\mathbb{J}\end{aligned} \tag{11}$$

Similarly to problem (8) the subscript i and j corresponds to the i^{th} inequality and j^{th} equality respectively. The quadratic matrices Q, Q_i and Q_j are positive definite and Q is symmetric. Therefore, the dimensions of the matrices in problem (11) are as follows:

$$\begin{array}{lll} Q, Q_i, Q_j : [n\times n] & H, H_i, H_j : [m\times n] & c_x, A_i, A_j : [1\times n]\\ F_i, F_j : [1\times m] & Q_{\theta,i}, Q_{\theta,j} : [m\times m] & \\ CR_A : [r\times m] & CR_b : [r\times 1] & \end{array} \tag{12}$$

Let there be a solution x^* for a nominal parametric value denoted by θ^*. Furthermore, let the Lagrangian function of the above system for any active set be convex, a condition that holds for every problem with a convex objective function and convex quadratic constraints, then the KKT conditions are satisfied and hold true:

$$\begin{array}{ll} \nabla_x L(x^*,\lambda^*,\mu^*,\theta^*) & = 0\\ \lambda_i^* g_i(x^*,\theta^*) & = 0\\ h_j(x^*,\theta^*) & = 0\\ \lambda_i^* & \ge 0\\ \forall i\in\mathbb{I}, \forall j\in\mathbb{J} & \end{array} \tag{13}$$

In the case of the general form of eq. (11), the set of relations in eq. (13) become for an active set of $\mathbb{A}$ of k active inequality constraints, without loss of generality:

$$\begin{aligned} & F = \\ & \begin{bmatrix} \left[Q + \sum_{i=1}^{k}\lambda_iQ_i\right]x + \left[H^T + \sum_{i=1}^{k}\lambda_iH_i^T\right]\theta + c + \sum_{i=1}^{k}\lambda_iA_i \\ x^TQ_ix + x^TH_i^T\theta + A_ix - b_i - F_i\theta - \theta^TQ_{\theta,i}\theta \\ x^TQ_jx + x^TH_j^T\theta + A_jx - b_j - F_j\theta - \theta^TQ_{\theta,j}\theta \end{bmatrix} \\ & = 0 \\ & \lambda_i \ge 0, \forall i\in\mathbb{A}, \forall j\in\mathbb{J} \end{aligned} \tag{14}$$

Therefore, given that $\eta = [x^T, \lambda_i^T, \mu_j^T]^T, \forall i \in \mathbb{A}, \forall j$ and $a = [\eta^T, \theta^T]^T$ the first order partial derivative of F with respect to a is defined as:

$$\nabla_a F = \begin{bmatrix} \nabla_{xx}L & \nabla_{x\lambda_i}L & \nabla_{x\mu_j}L & \nabla_{x\theta}L \\ \nabla_x g_i^T & & & \nabla_\theta g_i^T \\ \nabla_x h_j^T & & 0 & \nabla_\theta h_j^T \end{bmatrix} \quad \forall i\in\mathbb{A}, \forall j\in\mathbb{J} \tag{15}$$

where the analytical form of the terms in eq. (15) are given in eq. (16). Note that the partial derivatives of the equality constraints are omitted for brevity.

$$\begin{array}{ll} \nabla_{xx}L & = Q + \sum_{i=1}^{k}\lambda Q_i^T + \sum_{j\in\mathbb{J}}\mu Q_j^T \\ \nabla_{x\lambda_i}L & = Q_ix + H_i^T\theta + A_i \\ \nabla_{x\mu_j}L & = Q_jx + H_j^T\theta + A_j \\ \nabla_{x\theta}L & = H^T + \sum_{i=1}^{k}\lambda H_i^T + \sum_{j\in\mathbb{J}}\mu H_j^T \\ \nabla_x g_i^T & = 2x^TQ_i^T + \theta^TH_i + A_i \\ \nabla_\theta g_i^T & = x^TH_i^T - 2\theta^TQ_{\theta,i}^T - F_i \end{array} \tag{16}$$

Note that on the contrary to the case where the optimization problem consists only of linear constraints, here the $\nabla_a F$ matrix remains a function of (η,θ), i.e. a first order Taylor expansion for the classes of problem with quadratic constraints would only yield valid (approximate) linear parametric solutions around a $a^* = (\eta^*,\theta^*)$. The quadratic approach presented here requires the derivation of the matrix $\nabla_{aa}F$. Given that the, as shown in eq. (15) the first order partial derivative is a 2–dimensional matrix, it follows that the 2^{nd} order partial derivative is augmented by one dimension – a 3–dimensional tensor. Therefore the matrix $\nabla_{ax_1}F$ [1] will be derived here to show that the matrix remains invariable with respect to all elements of the vector a (eq. (17)).

$$\nabla_{ax_1}F = \begin{bmatrix} 0 & Q_i(:,1)^T & Q_j(:,1)^T & 0 \\ 2Q_i(:,1)^T & & & H_i(:,1)^T \\ 2Q_j(:,1)^T & & 0 & H_j(:,1)^T \end{bmatrix} \quad \forall i\in\mathbb{A}, \forall j\in\mathbb{J} \tag{17}$$

where $Q_i(:,1)$ corresponds to the vector of the elements of the first column of matrix Q_i and equivalently for Q_j. The rest of the elements of the tensor $\nabla_{aa}F$ are similarly derived. Note that $\nabla_{aa}F(a) = \nabla_{aa}F(a^*), \forall a \in \mathbb{R}^{n+k+j+m}$. Pseudo-algorithm 1 briefly describes the derivation of the parametric solution based on the analytical solution of eq. (10). Alternatively, a space exploration that considers all possible combinations of active sets can be followed. Such an approach was suggested for mpQPs by Gupta et al. (2011) and can be applicable on the convex mpQCQP case as well. Note that due to the non-linearity of the constraints, a single active set may have multiple non-linear solutions on $\eta(\theta)$. Therefore, every $\eta(\theta)$ solution may yield different critical regions for the same active set. Furthermore, the critical regions defined here are non-convex as the result of the solution of a quadratic system of equations, hence the need for a global optimizer after the first iteration of the algorithm even for a convex QCQP problem definition. Primal degeneracy can be handled by identifying strongly and weakly active constraints per active set (Gupta et al., 2011). Dual degeneracy cannot occur since the matrices Q, Q_i and Q_j in problem (11) are positive definite, Oberdieck et al. (2017).

3. ILLUSTRATIVE MULTIPARAMETRIC/EXPLICIT NMPC EXAMPLE

Consider problem (18), (Bemporad et al., 2002):

[1] In this context, x_1 corresponds to the first element of the optimization variable vector x.

Algorithm 1 An mpQCQP pseudo-algorithm

1: **procedure** MPQCQP
2: Reformulate the mpQCQP problem to the form of eq. (11).
3: *start*:
4: Treat θ as optimization variables.
5: Solve the deterministic problem via Global Optimization algorithms.
6: **if** feasible **then**
7: Acquire an active set $\mathbb{A}$ and the vector a^*.
8: Solve analytically eq. (10), $(15-17)$ for $\eta(\theta)$.
9: Keep all real solutions.
10: *loop*:
11: Derive the Critical Region (CR) by $g_i(x(\theta),\theta) \le 0, \forall i \notin \mathbb{A}$ and $\lambda_i(\theta) \ge 0 \forall i \in \mathbb{A}$.
12: **if** Empty Critical Region **then**
13: Discard solution
14: **goto** *loop*.
15: Reverse one-by-one the constraints that define the existing CR and add to the original mpQCQP.
16: **goto** *start*
17: **else**
18: Terminate.

$$\begin{aligned} \min_{u_i} \;& x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t.} \;& x_{i+1} = Ax_i + Bu_i, \forall i \in [0,...,N-1] \\ & -2 \le u_i \le 2, \forall i \in [0,...,N-1] \\ & [-1.5\ \ -1.5]^T \le x_i \le [1.5\ \ 1.5]^T, \forall i \in [0,\dots,N] \\ & u_0^2 \le x_0^T x_0 \\ & A = \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix}, B = \begin{bmatrix} 0.0861 \\ 0.0045 \end{bmatrix} \end{aligned} \tag{18}$$

where, Q is the identity matrix, $R = 0.01$, P is the terminal weight matrix derived via the discrete time Riccati equation and $N = 2$ [2]. Note that on the contrary to a standard LQR, the first input action u_0 is quadratically constrained with respect to the initial values of the states at each step of the receding horizon. Via forward substitution problem (18) is reformulated to the equivalent mpQCQP form of problem (19):

$$\begin{aligned} \min_{u_0,u_1} \;& 0.0261u_0^2 + 0.0194u_0u_1 + 0.0218u_1^2 + 0.2908u_0x_{0,1} \\ & +0.2917u_0x_{0,2} + 0.1747u_1x_{0,1} + 0.1753u_1x_{0,2} + \\ & 3.1284x_{0,1}^2 + 7.2476x_{0,1}x_{02} + 3.1284x_{0,2}^2 \\ \text{s.t.} \;& u_0^2 - x_{01}^2 - x_{0,2}^2 \le 0 \\ & 0.0623u_0 + 0.0861u_1 \le 1.5 - 0.5219x_{0,1} + 0.2968x_{0,2} \\ & 0.0119u_0 + 0.0045u_1 \le 1.5 - 0.1484x_{0,1} + 0.9671x_{0,2} \\ & -0.0623u_0 - 0.0861u_1 \le 1.5 + 0.5219x_{0,1} - 0.2968x_{0,2} \\ & -0.0119u_0 - 0.0045u_1 \le 1.5 + 0.1484x_{0,1} - 0.9671x_{0,2} \\ & 0.0861u_0 \le 1.5 - 0.7236x_{0,1} + 0.1722x_{0,2} \\ & 0.0045u_0 \le 1.5 - 0.0861x_{0,1} - 0.9909x_{0,2} \\ & -0.0861u_0 \le 1.5 + 0.7236x_{0,1} - 0.1722x_{0,2} \\ & -0.0045u_0 \le 1.5 + 0.0861x_{0,1} + 0.9909x_{0,2} \\ & u_0 \le 2, -u_0 \le 2, u_1 \le 2, -u_1 \le 2 \\ & x_{0,1} \le 1.5, -x_{0,1} \le 1.5, x_{0,2} \le 1.5, -x_{0,2} \le 1.5 \end{aligned} \tag{19}$$

[2] Note that in this particular case, the output/control horizon of the MPC does not affect the computational complexity of the problem as the quadratic constraint involves only elements of the first step of the receding horizon.

In the above formulation, the optimization variables are the control actions u_0 and u_1, while the initial states x_{01} and x_{02} are treated as uncertain parameters. Note that the objective function and the quadratic constraint are convex with respect to the optimization variables. To illustrate the solution procedure we present analytically the derivation of the solution of a single critical region corresponding to initial parametric values of $(x_{0,1}, x_{0,2}) = (1, 0)$, and following Algorithm 1 from step 5 onwards. The deterministic possibly non-convex, nonlinear problem is solved in GAMS (Tawarmalani and Sahinidis, 2005; Misener and Floudas, 2014) and the global solution along with the corresponding Lagrange multipliers are obtained (eq. (20)).

$$\begin{aligned} u_0 = -1 \;\; u_1 = -2 \\ \lambda_1 = 0.1 \;\; \lambda_2 = 0.068 \end{aligned} \tag{20}$$

The Lagrange function is constructed based on the active constraints – in this case the quadratic and upper bound for $u_1 = 2$ – and the first derivatives with respect to the optimization variables are calculated. Therefore, the function F is defined as follows (eq. (21)).

$$\begin{aligned} & F = \\ & \begin{bmatrix} 0.0522u_0 + 0.0194u_1 + 0.2908x_{01} + 0.2917x_{02} + 2\lambda_1 u_0 \\ 0.0194u_0 + 0.0436u_1 + 0.1747x_{01} + 0.1753x_{02} - \lambda_2 \\ u_0^2 - x_{01}^2 - x_{02}^2 \\ -u_1 - 2 \end{bmatrix} \\ & = 0 \end{aligned} \tag{21}$$

A second-order Taylor expansion around the nominal point $a^* = [u_0^*, u_1^*, \lambda_i^*, x_{0,1}^*, x_{0,2}^*] = \{-1, -2, 0.1, 0.068, 1, 0\}$ is constructed as follows

$$\nabla_a F = \begin{bmatrix} 0.0522 + 2\lambda_1 & 0.0194 & 2u_0 & 0 & 0.2908 & 0.2917 \\ 0.0194 & 0.0436 & 0 & -1 & 0.1747 & 0.1753 \\ 2u_0 & 0 & 0 & 0 & -2x_{01} & -2x_{02} \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$\nabla_a F\left(a^*\right) = \begin{bmatrix} 0.2522 & 0.0194 & -2 & 0 & 0.2908 & 0.2917 \\ 0.0194 & 0.0436 & 0 & -1 & 0.1747 & 0.1753 \\ -2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

For the $\nabla_{aa}F$ calculation only nonzero *elements* of matrices are reported

$$\begin{aligned} \nabla_{au_0} F\{1,3\} = 2 \\ \nabla_{au_0} F\{3,1\} = 2 \\ \nabla_{au_1} F = 0 \\ \nabla_{a\lambda_1} F\{1,1\} = 2 \\ \nabla_{a\lambda_2} F = 0 \\ \nabla_{ax_{0,1}} F\{3,5\} = -2 \\ \nabla_{ax_{0,2}} F\{3,6\} = -2 \end{aligned} \tag{24}$$

The analytical solution of the system of eq. (10) for the problem at hand yields the parametric expressions for the control actions and Lagrange multipliers of eq. (25).

$$\begin{aligned} & u_0 = -\sqrt{x_{0,1}^2 + x_{0,2}^2} \\ & u_1 = -2 \\ & \lambda_1 = \frac{1}{u_0}\left(-0.0261u_0 - 0.1454x_{0,1} - 0.1459x_{0,2} + 0.0193\right) \\ & \lambda_2 = 0.0194u_0 + 0.1747x_{0,1} + 0.1753x_{0,2} - 0.0873 \end{aligned} \tag{25}$$

Table 1. The full solution of the mpNMPC problem per CR: Optimal Control Action Definition

	$OptimalControlActionDefinition$
CR_1	$u_0 = -\sqrt{x_{0,1}^2 + x_{0,2}^2}$ $u_1 = -2$ $\lambda_1 = \frac{1}{u_0}(-0.0261u_0 - 0.1454x_{0,1} - 0.1459x_{0,2} + 0.0193)$ $\lambda_2 = 0.0194u_0 + 0.1747x_{0,1} + 0.1753x_{0,2} - 0.0873$
CR_2	$u_0 = \sqrt{x_{0,1}^2 + x_{0,2}^2}$ $u_1 = 2$ $\lambda_1 = \frac{1}{u_0}(-0.0261u_0 - 0.1454x_{0,1} - 0.1458x_{0,2} - 0.0193)$ $\lambda_2 = -0.0194u_0 - 0.1747x_{0,1} - 0.1753x_{0,2} - 0.0873$
CR_3	$u_0 = \sqrt{x_{0,1}^2 + x_{0,2}^2}$ $u_1 = -0.4450u_0 - 4.0069x_{0,1} - 4.0206x_{0,2}$ $\lambda_1 = \frac{1}{u_0}(-0.0218u_0 - 0.1065x_{0,1} - 0.1068x_{0,2})$
CR_4	$u_0 = -\sqrt{x_{0,1}^2 + x_{0,2}^2}$ $u_1 = -0.4450u_0 - 4.0069x_{0,1} - 4.0206x_{0,2}$ $\lambda_1 = \frac{1}{u_0}(-0.0218u_0 - 0.1065x_{0,1} - 0.1068x_{0,2})$
CR_5	$u_0 = -4.8904x_{0,1} - 4.9050x_{0,2}$ $u_1 = -1.8309x_{0,1} - 1.8381x_{0,2}$
CR_6	$u_0 = -2$ $u_1 = -2$ $\lambda_1 = 0.2908x_{0,1} + 0.2917x_{0,2} - 0.1432$ $\lambda_2 = 0.1747x_{0,1} + 0.1753x_{0,2} - 0.1260$
CR_7	$u_0 = 2$ $u_1 = 2$ $\lambda_1 = -0.2908x_{0,1} - 0.2917x_{0,2} - 0.1432$ $\lambda_2 = -0.1747x_{0,1} - 0.1753x_{0,2} - 0.1260$

Table 2. The full solution of the mpNMPC problem per CR: Analytical Critical Region Definition

	$CriticalRegionDefinition$
CR_1	$-u_0 - 2 \leq 0$ $0.0045u_0 + 0.0861x_{0,1} + 0.9909x_{0,2} - 1.5 \leq 0$ $0.0119u_0 + 0.1484x_{0,1} + 0.9671x_{0,2} - 1.5090 \leq 0$ $-0.0194u_0 - 0.1747x_{0,1} - 0.1753x_{0,2} + 0.0873 \leq 0$ $x_{0,1} - 1.5 \leq 0$ $x_{0,2} - 1.5 \leq 0$
CR_2	$u_0 - 2 \leq 0$ $-0.0045u_0 - 0.0861x_{0,1} - 0.9909x_{0,2} - 1.5 \leq 0$ $-0.0119u_0 - 0.1484x_{0,1} - 0.9671x_{0,2} - 1.5090 \leq 0$ $0.0194u_0 + 0.1747x_{0,1} + 0.1753x_{0,2} + 0.0873 \leq 0$ $-x_{0,1} - 1.5 \leq 0$ $-x_{0,2} - 1.5 \leq 0$
CR_3	$-0.4450u_0 - 4.0069x_{0,1} - 4.0206x_{0,2} - 2 \leq 0$ $\frac{1}{u_0}(0.0218u_0 + 0.1065x_{0,1} + 0.1068x_{0,2}) \leq 0$ $-x_{0,1} - 1.5 \leq 0$ $-x_{0,2} - 1.5 \leq 0$
CR_4	$0.4450u_0 + 4.0069x_{0,1} + 4.0206x_{0,2} - 2 \leq 0$ $\frac{1}{u_0}(0.0218u_0 + 0.1065x_{0,1} + 0.1068x_{0,2}) \leq 0$ $x_{0,1} - 1.5 \leq 0$ $x_{0,2} - 1.5 \leq 0$
CR_5	$u_0^2 - x_{0,1}^2 - x_{0,2}^2 \leq 0$ $x_{0,1} - 1.5 \leq 0$ $x_{0,2} - 1.5 \leq 0$ $-x_{0,1} - 1.5 \leq 0$ $-x_{0,2} - 1.5 \leq 0$
CR_6	$0.1484x_{0,1} + 0.9671x_{0,2} - 1.5328 \leq 0$ $0.0861x_{0,1} + 0.9909x_{0,2} - 1.5090 \leq 0$ $-x_{0,1}^2 - x_{0,2}^2 + 4 \leq 0$ $x_{0,1} - 1.5 \leq 0$
CR_7	$-0.1484x_{0,1} - 0.9671x_{0,2} - 1.5328 \leq 0$ $-0.0861x_{0,1} - 0.9909x_{0,2} - 1.5090 \leq 0$ $-x_{0,1}^2 - x_{0,2}^2 + 4 \leq 0$ $-x_{0,1} - 1.5 \leq 0$

By substituting the parametric solution to the inactive constraints and imposing positive Lagrange multipliers the resulting critical region is defined (eq. (26))

$$\begin{aligned}
&\sqrt{x_{0,1}^2 + x_{0,2}^2} - 2 \leq 0\\
&-0.0045\sqrt{x_{0,1}^2 + x_{0,2}^2} + 0.0861x_{0,1} + 0.9909x_{0,2} - 1.5 \leq 0\\
&-0.0119\sqrt{x_{0,1}^2 + x_{0,2}^2} + 0.1484x_{0,1} + 0.9671x_{0,2} - 1.5090 \leq 0\\
&0.0194\sqrt{x_{0,1}^2 + x_{0,2}^2} - 0.1747x_{0,1} - 0.1753x_{0,2} + 0.0873 \leq 0\\
&x_{0,1} - 1.5 \leq 0\\
&x_{0,2} - 1.5 \leq 0
\end{aligned} \quad (26)$$

The procedure is repeated until the termination of algorithm 1 and the results are summarized in Tables 1 and 2. Note that in $CR_1 - CR_4$ the parametric results of the Lagrange multipliers and critical region definitions are a function of u_0, i.e. the term $\pm\sqrt{x_{0,1}^2 + x_{0,2}^2}$ is replaced with u_0. Observe that:

- The definition of critical regions in CR_1 and CR_2 are linear with respect to u_0, $x_{0,1}$ and $x_{0,2}$ because (i) the quadratic constraint involves only u_0 (not u_1) and (ii) the active set of the aforementioned CRs includes the quadratic constraint.
- The term $\pm\sqrt{x_{0,1}^2 + x_{0,2}^2}$ appears in the denominators of fractions only in critical regions were the point $(x_{0,1}, x_{0,2}) = (0, 0)$ is not included.

The resulting critical regions are presented in Fig. 1. The linear state-space system is simulated starting from

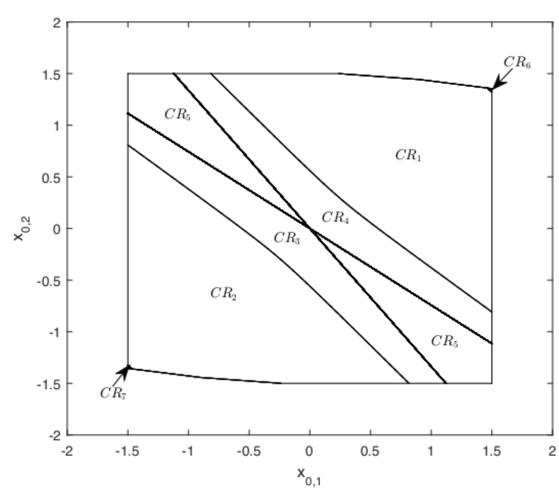

Fig. 1. The Critical Regions of the mpNMPC

$x_{0,1} = x_{0,2} = 1$ (see Fig. 2). Furthermore, the first optimal action u_0 as a function of x_0 is shown in Fig. 3. Observe that:

- The control action is continuous and piecewise nonlinear.
- Non-smooth transitions occur between critical regions

The above are a result of (i) the problem consisting only of continuous variables and (ii) the convexity of the problem with respect to the optimization variables on both the objective function (Fig. 4) and the feasible space.

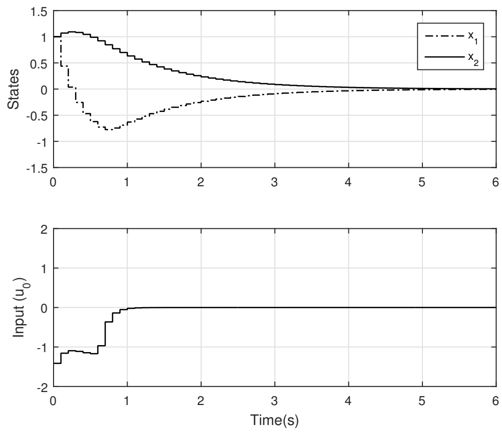

Fig. 2. mpNMPC closed-loop response ($x_{0,1} = x_{0,2} = 1$)

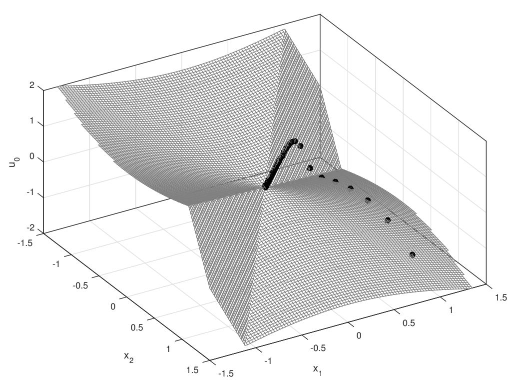

Fig. 3. Optimal input profile and closed-loop response

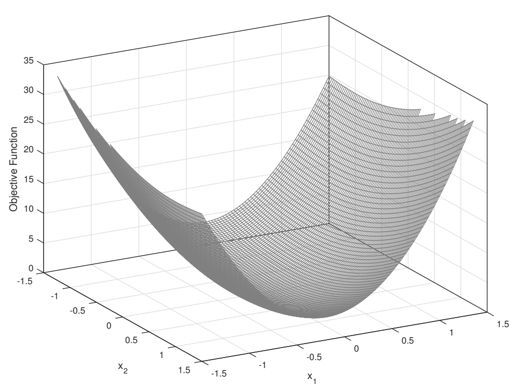

Fig. 4. Objective function value over the feasible space

4. CONCLUDING REMARKS

We presented a quadratic approach to the Basic Sensitivity Theorem and its applicability on multiparametric Quadratically Constrained Quadratic Programming problems. An analytical pseudo-algorithm for the solution of such problems was applied on a quadratically constrained MPC formulation and the results were evaluated. We showed the continuity of the critical regions and corresponding optimal actions for convex mpQCQP and discussed elements of the solution properties. Further steps include the efficient solution of the resulting quadratic systems of equations with respect to the parameters and the expansion of the procedure to different classes of QCQP and NMPC problems.

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