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Representations of the Kauffman bracket skein algebra III: closed surfaces and naturality

Francis Bonahon and Helen Wong[1]

Abstract. This is the third article in the series begun with [8, 10], devoted to finite-dimensional representations of the Kauffman bracket skein algebra of an oriented surface S. In [8] we associated a *classical shadow* to an irreducible representation ρ of the skein algebra, which is a character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ represented by a group homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. The main result of the current article is that, when the surface S is closed, every character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ occurs as the classical shadow of an irreducible representation of the Kauffman bracket skein algebra. We also prove that the construction used in our proof is natural, and associates to each group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ a representation of the skein algebra $\mathcal{S}^A(S)$ that is uniquely determined up to isomorphism.

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Contents

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This article is the third in the series begun with [8, 10], devoted to the analysis and construction of finite-dimensional representations of the skein algebra of a surface. See also [6] for a description of the corresponding program.

The Kauffman bracket skein algebra $\mathcal{S}^A(S)$ of an oriented surface S of finite topological type takes its origins in the construction of the Jones polynomial invariant [22, 23, 25, 26] of knots and links. It can be interpreted [38, 11, 12, 33] as a quantization of the character variety

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\}/\!\!/\mathrm{SL}_2(\mathbb{C})$$

with respect to its Atiyah-Bott-Goldman [2, 18, 19] Poisson structure. More accurately, the points of such a quantization are representations of the algebra $\mathcal{S}^A(S)$.

When the parameter $A = \mathrm{e}^{-\pi\mathrm{i}\hbar}$ is a root of unity, a celebrated example of finite-dimensional representation of the skein algebra $\mathcal{S}^A(S)$ arises from Witten's quantum field interpretation of the Jones polynomial [40], and more precisely from the Witten-Reshetikhin-Turaev topological quantum field theory associated to the fundamental representation of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$ [40, 34, 3, 39, 9]. In the current article, we construct a large family of new finite-dimensional representations of $\mathcal{S}^A(S)$, while providing a converse to the results of [8].

This article is mostly concerned with the case where the surface S is closed. The case where S has at least one puncture is much easier (at least assuming the results of [7] and [8]), and was treated in [10]. The current closed surface case requires many more ideas, and also involves several very surprising properties.

More specifically, when A^2 is a primitive N–root of unity with N odd, we identified in [8] certain invariants for irreducible representations $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$. It is easier to restrict attention to the case where $A^N = -1$; this is no loss of generality, as [10, §5] indicates how to deduce the case $A^N = +1$ from this one, by using spin structures on the surface. When the surface is closed, there is only one invariant, consisting of a point in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. By definition [29] of the geometric invariant theory quotient involved in the definition of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, two homomorphisms $r,\ r'\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ represent the same point of

$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if and only if they induce the same trace functions, namely if and only if $\operatorname{Tr} r(\gamma) = \operatorname{Tr} r'(\gamma)$ for every $\gamma \in \pi_1(S)$.

Theorem 1 ([8]). *Let S be a closed oriented surface, let A be a primitive N–root of -1 with N odd, and let $T_N(x)$ be the N–th normalized Chebyshev polynomial of the first kind, characterized by the trigonometric identity that $2\cos N\theta = T_N(2\cos\theta)$. For every irreducible finite-dimensional representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ of the Kauffman bracket skein algebra, there exists a unique character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that*

$$T_N\big(\rho([K])\big) = -\big(\operatorname{Tr} r_\rho(K)\big)\mathrm{Id}_E$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical. □

The character $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}$ is the *classical shadow* of the irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$. In [8] we prove a stronger version of Theorem 1, which is valid for all framed links $K \subset S \times [0,1]$ and involves the element $[K^{T_N}] \in \mathcal{S}^A(S)$ defined by threading the Chebyshev polynomial T_N along all components of K. The above version is easier to state and sufficient for our purposes.

See also [27] for a different approach to the key results underlying Theorem 1.

The main result of this article is the following converse statement.

Theorem 2 (Realization Theorem). *Let S be a closed oriented surface, and let A be a primitive N–root of -1 with N odd. Then, every character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is the classical shadow of an irreducible representation $\rho_r\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$.*

For the classical example of the Witten-Reshetikhin-Turaev representation $\rho_{\mathrm{WRT}}\colon \mathcal{S}^A(S) \to \mathrm{End}(W_{\mathrm{WRT}})$, also defined when A is a primitive $2N$–root of unity with N odd, the classical shadow of ρ_{WRT} is the trivial character [9]. We therefore construct a much broader family of representations of the skein algebra $\mathcal{S}^A(S)$ than this historic example.

As explained in Theorem 4 below, our construction is natural in the sense that it provides a representation $\rho_r\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ that depends only on the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, up to isomorphism and other symmetries of the data. In earlier versions of this article we conjectured that, when the character belongs to a Zariski dense open subset

of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, the representation ρ_r is the only irreducible representation of $\mathcal{S}^A(S)$ with classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. This conjecture is now proved [17] See [37] for explicit proof for small punctured surfaces, such as the one-puncture torus or the four-puncture sphere. The property is definitely false without the genericity hypothesis, as can for instance be proved by combining the results of [20] with the techniques of [37].

The strategy for proving Theorem 2 is somewhat unconventional. In addition to using classical hyperbolic geometry as a guide for quantum topology constructions, it relies on the fact that punctured surfaces are easier to deal with than closed surfaces, and follows the slogan "drill, baby, drill"[1]. Namely, we drill punctures from the closed surface S to obtain a punctured surface S_λ, by removing from S the vertices of a triangulation λ, and the more punctures the better. If we are given a homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ representing the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and if the triangulation λ is complicated enough, we can then choose additional data at the punctures of S_λ (called a *λ–enhancement* of the homomorphism r) and apply the results of [10] to the punctured surface S_λ. This provides a representation $\rho_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ of the skein algebra of the punctured surface S_λ, whose classical shadow is equal to the character $r_\lambda \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S_\lambda)$ induced by $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ in the sense that

$$T_N\big(\rho_\lambda([K])\big) = -\big(\mathrm{Tr}\, r_\lambda(K)\big)\mathrm{Id}_{E_\lambda} = -\big(\mathrm{Tr}\, r(K)\big)\mathrm{Id}_{E_\lambda}$$

for every framed knot $K \subset S_\lambda \times [0,1]$ whose projection to S_λ has no crossing and whose framing is vertical. This last property, proved in [10], heavily relies on the miraculous cancellations of [8].

However, there is no reason for ρ_λ to induce a representation of the skein algebra $\mathcal{S}^A(S)$ of the closed surface S. Namely, if the two framed links K, $K' \subset S_\lambda \times [0,1]$ are isotopic in $S \times [0,1]$, by an isotopy sweeping through the punctures of S_λ, the two endomorphisms $\rho_\lambda\big([K]\big)$, $\rho_\lambda\big([K']\big) \in \mathrm{End}(E_\lambda)$ will in general be different. Nevertheless, we are able to identify a subspace $F_\lambda \subset E_\lambda$ where $\rho_\lambda\big([K]\big)$ and $\rho_\lambda\big([K']\big)$ do coincide. This subspace $F_\lambda \subset E_\lambda$ is called the *total off-diagonal kernel*, for reasons that are explained in §§4.1 and 4.2.

[1] Popularized during the 2008 United States presidential campaign [32, §3], when the ideas behind this article were beginning to take shape.

Theorem 3. *Let the punctured surface S_λ be obtained from the closed surface S by removing the vertices of the triangulation λ of S, and let $F_\lambda \subset E_\lambda$ be the total off-diagonal kernel of the representation $\rho_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ introduced above. Then*

(1) *F_λ is invariant under the image $\rho_\lambda\big(\mathcal{S}^A(S_\lambda)\big) \subset \mathrm{End}(E_\lambda)$;*

(2) *if the two framed links K, $K' \subset S_\lambda \times [0,1]$ are isotopic in $S \times [0,1]$, the induced endomorphisms*

$$\rho_\lambda\big([K]\big)_{|F_\lambda} = \rho_\lambda\big([K']\big)_{|F_\lambda} \in \mathrm{End}(F_\lambda)$$

are equal.

The definition of the total off-diagonal kernel F_λ was devised by wishful thinking, as the largest subspace where the second conclusion of Theorem 3 could hold. The really unexpected properties are that this subspace is non-trivial (see Theorem 5 below) and that F_λ is invariant under the image of ρ_λ. Indeed, although $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ is defined as a composition of the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathcal{Z}^\omega(\lambda)$ of [7] with an *irreducible* representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ of the triangulation λ, the invariance of F_λ shows that the representation ρ_λ is reducible. This reducibility property for $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda$ would be false if we replaced μ_λ by an arbitrary irreducible representation of $\mathcal{Z}^\omega(\lambda)$.

Theorem 3 is proved in §4.3 and §4.4 when the triangulation λ is sufficiently complicated, and in §5.3 for general triangulations.

A consequence of Theorem 3 is that the representation $\rho_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ induces a representation $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ of the skein algebra of the closed surface S. This representation has a classical shadow equal to the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, in the sense that

$$T_N\big(\check{\rho}_\lambda([K])\big) = -\big(\mathrm{Tr}\, r(K)\big)\mathrm{Id}_{F_\lambda}$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical.

The following property shows that our construction is very natural. For most homomorphisms $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, this result states that the representation $\check{\rho}_\lambda$ is unique up to isomorphism. However, an additional ambiguity can arise for special characters that admit internal symmetries called sign-reversal symmetries. The cohomology group $H^1(S;\mathbb{Z}_2)$ acts on the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and on the skein algebra $\mathcal{S}^A(S)$;

see §2. A *sign-reversal symmetry* for the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is a class $\varepsilon \in H^1(S;\mathbb{Z}_2)$ that fixes r; characters that admit non-trivial sign-reversal symmetries are rare, and form a high codimension subset of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. If $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ is a representation with classical shadow r, composing it with the action of a sign-reversal symmetry $\varepsilon \in H^1(S;\mathbb{Z}_2)$ on $\mathcal{S}^A(S)$ gives another representation $\rho \circ \varepsilon$ with classical shadow $\varepsilon r = r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. Therefore, sign-reversal symmetries of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ are intrinsic symmetries of the problem of finding representations of $\mathcal{S}^A(S)$ with classical shadow r.

Theorem 4 (Naturality Theorem). *Up to isomorphism and up to sign-reversal symmetry of the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ (if any exists), the representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ depends only on the group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, not on the choice of the triangulation λ or of the λ–enhancement ξ used in the construction.*

In particular, although the dimension of E_λ grows exponentially with the number of punctures of the drilled surface S_λ, the dimension of the off-diagonal kernel F_λ is independent of the topology of λ. A consequence is that the construction is natural with respect to the action of the mapping class group of S.

The proof of Theorem 4, given in §5.4, relies on invariance under Pachner moves to go from one triangulation to another. It is a good illustration of the "drill, baby, drill" philosophy, as showing that two triangulations λ and λ' induce the isomorphic representations of $\mathcal{S}^A(S)$ usually involves surfaces with many more punctures than S_λ and $S_{\lambda'}$. Here, the invariance under the face subdivision move considered in §5.1, which adds one vertex to the triangulation but does not change the representation, is probably the most surprising.

Conjugating r by an element of $\mathrm{SL}_2(\mathbb{C})$ also leaves $\check{\rho}_\lambda$ unchanged, up to isomorphism. For a generic character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, two homomorphisms $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ representing r are always conjugate by an element of $\mathrm{SL}_2(\mathbb{C})$ and therefore determine the same representation of $\mathcal{S}^A(S)$. However, for those special characters for which the property fails (namely reducible characters), we do not know if the representation $\check{\rho}_\lambda$ depends only on the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, or on subtler properties of the specific homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ representing r that we used in the construction.

It is also quite possible that the need to consider sign-reversal symmetries is an artifact of our proof, and of its reliance on insights from the character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. Indeed, the characters that admit non-trivial sign-reversal symmetries are precisely the branch points of the projection $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \to \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. It appears that composing our representation $\check{\rho}_\lambda$ with a sign-reversal symmetry of its classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ often produces a representation $\check{\rho}_\lambda \circ \varepsilon$ that is isomorphic to $\check{\rho}_\lambda$, but we have not been able to confirm this fact in full generality.

At this point, we still have a major problem, which is that we do not know that the off-diagonal kernel F_λ is different from 0. This property may even seem unlikely at first, as the off-diagonal kernel F_λ is defined as an intersection of kernels of endomorphisms of the vector space E_λ. This question is addressed in §6, and provides another one of the surprising twists in this article.

Theorem 5. *If the closed oriented surface S has genus g, the representation $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ with classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ provided by Theorem 3 has dimension*

$$\dim F_\lambda \geqslant \begin{cases} N^{3(g-1)} & \text{if } g \geqslant 2 \\ N & \text{if } g = 1 \\ 1 & \text{if } g = 0. \end{cases}$$

The above inequality is an equality for $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ generic, namely for r in an explicit Zariski dense open subset of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

In particular, the representation $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ is non-trivial. It may be reducible. In fact, although we conjecture that $\check{\rho}_\lambda$ is irreducible for generic $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, it is definitely reducible for highly non-generic homomorphisms $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ such as the trivial homomorphism. However, restricting $\check{\rho}_\lambda$ to an irreducible component proves our main Theorem 2.

We suspect that the inequalities of Theorem 5 are always equalities. Our proof of Theorem 5 departs from the "drill, baby, drill" and "more punctures is better" philosophy, and is based on a careful analysis of explicit triangulations λ with a very small number of vertices.

The results and methodology of this article were announced in [6]. The more recent articles and preprints [1, 15, 16] develop another construction of representations of $\mathcal{S}^A(S)$ with a given classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$,

valid for r in a Zariski dense open subset of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. The construction of [1, 15] is simpler, but ours is more explicit. We also believe that many of the ideas introduced in this paper are susceptible to have further applications in other contexts.

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1. The Kauffman bracket skein algebra

Let S be an oriented surface of finite topological type without boundary. The *Kauffman bracket skein algebra* $\mathcal{S}^A(S)$ depends on a parameter $A = \mathrm{e}^{-\pi\mathrm{i}\hbar} \in \mathbb{C} - \{0\}$, and is defined as follows: One first considers the vector space freely generated by all isotopy classes of framed links in the thickened surface $S \times [0,1]$, and then one takes the quotient of this space by two relations. The first and main relation is the *skein relation*, which states that

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

whenever the three links K_1, K_0 and $K_\infty \subset S \times [0,1]$ differ only in a little ball where they are as represented in Figure 1, and where $[K]$ denotes the class of $\mathcal{S}^A(S)$ represented by the framed link K. The second relation is the *trivial knot relation*, which asserts that

$$[K \cup O] = -(A^2 + A^{-2})[K]$$

whenever O is the boundary of a disk $D \subset K \times [0,1]$ disjoint from K, and is endowed with a framing transverse to D.

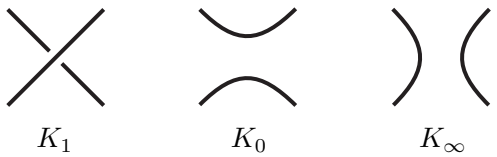


Figure 1. A Kauffman triple

The algebra multiplication is provided by the operation of superposition, where the product $[K] \cdot [L]$ is represented by the union $[K' \cup L']$ where $K' \subset S \times [0, \frac{1}{2}]$ and $L' \subset S \times [\frac{1}{2}, 1]$ are respectively obtained by rescaling the framed links $K \subset S \times [0,1]$ and $L' \subset S \times [0,1]$.

2. Sign-reversal symmetries

The character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and the skein algebra $\mathcal{S}^A(S)$ both admit natural actions of the cohomology group $H^1(S;\mathbb{Z}_2)$. Indeed, for a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ represented by a homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and a cohomology class $\varepsilon \in H^1(S;\mathbb{Z}_2)$, its image $\varepsilon r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is represented by the homomorphism εr defined by

$$\varepsilon r(\gamma) = (-1)^{\varepsilon(\gamma)} r(\gamma) \in \mathrm{SL}_2(\mathbb{C})$$

for every $\gamma \in \pi_1(S)$. The action of $H^1(S;\mathbb{Z}_2)$ on $\mathcal{S}^A(S)$ is similarly defined by the property that

$$\varepsilon[K] = [(-1)^{\varepsilon(K)} K] \in \mathcal{S}^A(S)$$

for every framed link $K \subset S \times [0,1]$ and $\varepsilon \in H^1(S;\mathbb{Z}_2)$.

If the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is fixed under the action of some $\varepsilon \in H^1(S;\mathbb{Z}_2)$, we say that $\varepsilon \in H^1(S;\mathbb{Z}_2)$ is a *sign-reversal symmetry* for the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. This is equivalent to the property that the trace $\operatorname{Tr} r(\gamma)$ is equal to 0 for every $\gamma \in \pi_1(S)$ with $\varepsilon(\gamma) \neq 0$.

Because of our assumption that N is odd, the Chebyshev polynomial $T_N(x)$ is a sum of monomials of odd degree. It follows that, if the representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ has classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, its composition $\rho \circ \varepsilon$ with the action of $\varepsilon \in H^1(S;\mathbb{Z}_2)$ on $\mathcal{S}^A(S)$ has classical shadow $\varepsilon r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. In particular, if the classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ of the representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ has a sign-reversal symmetry $\varepsilon \in H^1(S;\mathbb{Z}_2)$, the representation $\rho \circ \varepsilon$ also has classical shadow $\varepsilon r = r$. Sign-reversal symmetries of a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ are therefore intrinsic symmetries of the problem of finding representations of $\mathcal{S}^A(S)$ with classical shadow r, which explains why they will occur in many statements of our article.

Characters with non-trivial sign-reversal symmetries exist, but are rare. For instance, they form an algebraic subset of complex dimension $2g-2$ in the $(6g-6)$–dimensional character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, where g is the genus of the surface S; see [10, §5.1].

3. Constructing representations for punctured surfaces

Throughout the article, A will be a primitive N–root of -1 with N odd. Namely, $A^N = -1$ and N is the smallest positive integer with this property (and N is odd). We also use a choice of square root $\omega = \sqrt{A^{-1}}$.

3.1. The balanced Chekhov-Fock algebra of a triangulation. Let λ be a triangulation of the closed oriented surface S. For most of the article, we are allowing an edge to go from one vertex to itself, as well as two edges to have the same endpoints. However, we will always require that the sides of a face of λ correspond to three distinct edges, for reasons that will become apparent in Remark 6.

We will sometimes restrict attention to triangulations where each edge has distinct endpoints, and where distinct edges have distinct pairs of endpoints. In this case, we will say that the triangulation is *combinatorial*, since this corresponds to the usual convention of combinatorial piecewise linear topology.

Let e_1, e_2, …, e_n be the edges of λ. After choosing an auxiliary number ω such that $\omega^2 = A^{-1}$, the *Chekhov-Fock algebra* of λ is the algebra $\mathcal{T}^\omega(\lambda)$ defined by generators $Z_1^{\pm 1}$, $Z_2^{\pm 1}$, …, $Z_n^{\pm 1}$ respectively associated to the edges e_1, e_2, …, e_n of λ, and by the relations

$$Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i.$$

where $\sigma_{ij} = a_{ij} - a_{ji} \in \{-2, -1, 0, 1, 2\}$ and where $a_i \in \{0, 1, 2\}$ is the number of times an end of the edge e_j immediately succeeds an end of e_i when going counterclockwise around a vertex of λ.

An element of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ is a linear combination of monomials $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}$ in the generators Z_i, with k_1, k_2, …, $k_n \in \mathbb{Z}$. Such a monomial $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}$ is *balanced* if its exponents k_i satisfy the following parity condition: for every triangle T_j of the ideal triangulation λ, the sum $k_{i_1} + k_{i_2} + k_{i_3}$ of the exponents of the generators Z_{i_1}, Z_{i_2}, Z_{i_3} associated to the sides of T_j is even.

The *balanced Chekhov-Fock algebra* $\mathcal{Z}^\omega(\lambda)$ of the triangulation λ is the subalgebra of $\mathcal{T}^\omega(\lambda)$ generated by all monomials satisfying this parity condition.

There are two reasons to be interested in the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$, whose combination is particularly useful for our purposes. The first one is the existence of an injective algebra homomorphism

$$\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}^A(S_\lambda) \to \mathcal{Z}^\omega(\lambda)$$

from the skein algebra of the punctured surface $S_\lambda = S - V_\lambda$, obtained by removing from S the set V_λ of vertices of λ, to the algebra $\mathcal{Z}^\omega(\lambda)$; this *quantum trace homomorphism* $\mathrm{Tr}_\lambda^\omega$ is constructed in [7]. The second reason is that the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ is fairly simple, so that its

representations are easily classified (see [10, §2], and the next section). This enables us to obtain representations of $\mathcal{S}^A(S_\lambda)$ by composing the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda$ with suitable representations of $\mathcal{Z}^\omega(\lambda)$. We will then show that these representations of the skein algebra $\mathcal{S}^A(S_\lambda)$ of the punctured surface S_λ induce representations of the skein algebra $\mathcal{S}^A(S)$ of the closed surface S, which is the object of interest to us.

Because of the skew-commutativity relations $Z_iZ_j = \omega^{2\sigma_{ij}}Z_jZ_i$, the order of the variables in a monomial $Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}$ is quite important. We will make heavy use of the following symmetrization trick. The *Weyl quantum ordering* for $Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}$ is the monomial

$$[Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}] = \omega^{-\sum_{u<v} n_u n_v \sigma_{i_u i_v}} Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}.$$

The formula is specially designed that $[Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}] \in \mathcal{T}^\omega(\lambda)$ is invariant under any permutation of the $Z_{i_u}^{n_u}$.

3.2. Enhanced homomorphisms from $\pi_1(S)$ to $\mathrm{SL}_2(\mathbb{C})$. We are given a character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, represented by a homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, and a triangulation λ of the closed surface S. Let $\widetilde{S}$ be the universal covering of S, and let $\widetilde{\lambda}$ be the triangulation of $\widetilde{S}$ induced by λ. Let $V_\lambda \subset S$ and $\widetilde{V}_\lambda \subset \widetilde{S}$ be the respective vertex sets of λ and $\widetilde{\lambda}$.

A λ–*enhancement* for the group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ is a map $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ such that:

(1) ξ is r–equivariant, in the sense that $\xi(\gamma\widetilde{v}) = r(\gamma)\xi(\widetilde{v})$ for every $\widetilde{v} \in \widetilde{V}_\lambda$ and every $\gamma \in \pi_1(S)$ (for the standard action of $\mathrm{SL}_2(\mathbb{C})$ on the projective line $\mathbb{CP}^1$);

(2) for every edge $\widetilde{e}$ of $\widetilde{\lambda}$, the elements $\xi(\widetilde{v})$ and $\xi(\widetilde{v}') \in \mathbb{CP}^1$ respectively associated to the end points $\widetilde{v}$ and $\widetilde{v}'$ of $\widetilde{e}$ are distinct.

Remark 6. Note an easy consequence of our assumption that the sides of each face of a triangulation λ correspond to three distinct edges. If e is an edge of λ whose endpoints are equal to the same vertex v, an Euler characteristic argument shows that the closed loop formed by e cannot bound a disk in S. As a consequence every edge $\widetilde{e}$ of $\widetilde{\lambda}$ has distinct endpoints, which makes Condition (2) above more likely. Also, for the same reason, every edge e of λ whose endpoints are equal determines a non-trivial element of $\pi_1(S)$, well-defined up to conjugation.

Lemma 7. *Consider a triangulation λ of the surface S and a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ satisfying the following property: for every edge e of λ whose endpoints are equal to the same vertex v, the element $r(e)$ is different from $\pm\mathrm{Id}$ in $\mathrm{SL}_2(\mathbb{C})$. Then the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ admits a λ–enhancement.*

Note that, in particular, the hypotheses of Lemma 7 are automatically satisfied if every edge of λ has distinct endpoints, or if r is injective by Remark 6.

Proof. To construct an r–equivariant map $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$, we proceed orbit by orbit for the action of $\pi_1(S)$ on the vertex set $\widetilde{V}_\lambda$.

For a vertex $v \in V_\lambda$, pick a point $\widetilde{v} \in \widetilde{V}_\lambda$ in its preimage. As a first approximation, define $\xi(\widetilde{v})$ to be an arbitrary point of the projective line $\mathbb{CP}^1$. Then there is a unique way to r–equivariantly extend ξ to the whole preimage of v, namely to the orbit $\pi_1(S)\widetilde{v}$ of $\widetilde{v}$ under the action of $\pi_1(S)$: define ξ on this orbit by the property that $\xi(\gamma\widetilde{v}) = r(\gamma)\xi(\widetilde{v})$ for every $\gamma \in \pi_1(S)$.

Performing this operation for each vertex v of λ defines an r–equivariant map $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$.

In addition, we can require that, at each step, the initial point $\xi(\widetilde{v}) \in \mathbb{CP}^1$ is chosen to satisfy the following two conditions: $\xi(\widetilde{v})$ is not in the image under ξ of the orbits considered in earlier steps; for every edge e of λ whose endpoints are both equal to v, the point $\xi(\widetilde{v})$ is not fixed by the image under r of any conjugate of $e \in \pi_1(S)$. Because of our hypothesis that $r(e) \neq \pm\mathrm{Id}$ in the second case, these two conditions are easily satisfied by suitably choosing $\xi(\widetilde{v}) \in \mathbb{CP}^1$ outside of a countable number of forbidden values.

It is then immediate that the map $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ so constructed in a λ–enhancement. □

A λ–enhancement $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ for the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ assigns a non-zero complex weight $x_i \in \mathbb{C}^* = \mathbb{C} - \{0\}$ to the i–th edge e_i of λ as follows. Lift e_i to an edge $\widetilde{e}_i$ of the triangulation $\widetilde{\lambda}$ of the universal covering $\widetilde{S}$. Arbitrarily orient $\widetilde{e}_i$, and let $\widetilde{v}_i^+$ and $\widetilde{v}_i^-$ be the positive and negative endpoints of $\widetilde{e}_i$. Consider the two faces of $\widetilde{\lambda}$ that are adjacent to $\widetilde{e}_i$, let $\widetilde{v}_i^{\mathrm{left}}$ be the third vertex of the face to the left, and let $\widetilde{v}_i^{\mathrm{right}}$ be the third vertex of the face to the right. Then, x_i is defined as minus the crossratio of the four points $\xi(\widetilde{v}_i^+)$, $\xi(\widetilde{v}_i^-)$, $\xi(\widetilde{v}_i^{\mathrm{left}})$, $\xi(\widetilde{v}_i^{\mathrm{right}}) \in \mathbb{CP}^1$. More

precisely, for the standard identification $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$,

$$x_i = -\frac{\big(\xi(\widetilde{v}_i^{\,\text{left}}) - \xi(\widetilde{v}_i^{+})\big)\big(\xi(\widetilde{v}_i^{\,\text{right}}) - \xi(\widetilde{v}_i^{-})\big)}{\big(\xi(\widetilde{v}_i^{\,\text{left}}) - \xi(\widetilde{v}_i^{-})\big)\big(\xi(\widetilde{v}_i^{\,\text{right}}) - \xi(\widetilde{v}_i^{+})\big)}.$$

Note that reversing the orientation of $\widetilde{e}_i$ leaves x_i unchanged. Also, the two conditions in the definition of λ–enhancements guarantee that x_i is a well-defined element of $\mathbb{C}^*$ and is independent of the choice of the lift $\widetilde{e}_i$ of e_i, by invariance of crossratios under the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{CP}^1$.

3.3. Representations of the balanced Chekhov-Fock algebra. We will use the results of [10] to associate to each group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, endowed with a λ–enhancement $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$, a representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E)$ of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$.

This representation μ_λ will be uniquely determined up to isomorphism, but also up to sign-reversal symmetry of the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. To make sense of this property, note that a monomial $Z_{\mathbf{k}} = Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n} \in \mathcal{Z}^\omega(\lambda)$ uniquely determines a homology class $[\mathbf{k}] \in H_1(S_\lambda; \mathbb{Z}_2)$ in the punctured surface S_λ, by the property that the algebraic intersection number of $[\mathbf{k}]$ with each edge e_i has the same parity as the exponent k_i of the corresponding generator Z_i; see [10, Lemma 9]. A cohomology class $\varepsilon \in H^1(S_\lambda; \mathbb{Z}_2)$ then acts on $\mathcal{Z}^\omega(\lambda)$ by sending each $Z_{\mathbf{k}}$ to $(-1)^{\varepsilon([\mathbf{k}])} Z_{\mathbf{k}}$. By restriction, this defines an action of $H^1(S; \mathbb{Z}_2)$ on $\mathcal{Z}^\omega(\lambda)$.

Also, a vertex v of λ determines an element

$$H_v = [Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}] = \omega^{-\sum_{i<j} k_i k_j \sigma_{ij}} Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n} \in \mathcal{Z}^\omega(\lambda)$$

where $k_i \in \{0, 1, 2\}$ is the number of endpoints of the edge e_i that are equal to v, and where $[\ \ ]$ denotes the Weyl quantum ordering defined in §3.1. This element H_v is central in $\mathcal{Z}^\omega(\lambda)$, as proved in [5, §3] or [10, §2.2].

A final observation is that the generator $Z_i \in \mathcal{T}^\omega(\lambda)$ associated to the edge e_i of λ does not belong to the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$, as it does not satisfy the required exponent parity condition. However, its square $Z_i^2 \in \mathcal{Z}^\omega(\lambda)$ does.

We will make repeated use of the following result, borrowed from [10].

Proposition 8. *For a triangulation λ of the surface S, consider a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ endowed with a λ–enhancement $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$. Then, up to isomorphism and up to the action of a sign-reversal symmetry of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ (if r admits any), there exists a unique representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ with the following properties.*

(1) *The dimension of E_λ is equal to $N^{3g+p_\lambda-3}$, where g is the genus of the surface S and where p_λ is the number of vertices of the triangulation λ.*

(2) *For every edge e_i of λ, let $x_i \in \mathbb{C}^*$ be the crossratio weight associated to e_i by the enhancement ξ as above, and let Z_i be the corresponding generator of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$. Then,*

$$\mu_\lambda(Z_i^{2N}) = x_i \,\mathrm{Id}_{E_\lambda}.$$

(3) *For every vertex v of λ with associated central element $H_v \in \mathcal{Z}^\omega(\lambda)$,*

$$\mu_\lambda(H_v) = -\omega^4 \,\mathrm{Id}_{E_\lambda}.$$

(4) *The representation $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ has classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, in the sense that*

$$T_N\big(\rho_\lambda([K])\big) = -\mathrm{Tr}\, r(K)\,\mathrm{Id}_{E_\lambda}$$

for every framed knot $K \subset S_\lambda \times [0,1]$ whose projection to S_λ has no crossing and whose framing is vertical.

In addition, μ_λ is irreducible.

Proof. This is a special case of the combination of Propositions 22 and 23 of [10]. The only minor difference is that these results are expressed in terms of pleated surfaces instead of λ–enhancements.

To connect the two viewpoints, note that the triangulation λ can also be interpreted as an ideal triangulation of the punctured surface $S_\lambda = S - V_\lambda$, obtained by removing from S the vertex set V_λ of λ. Similarly, the lift $\widetilde{\lambda}$ of λ to the universal covering $\widetilde{S}$ of S gives an ideal triangulation of the preimage $\widetilde{S}_\lambda = \widetilde{S} - \widetilde{V}_\lambda$ of S_λ in $\widetilde{S}$. The λ–enhancement $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ then determines an r–equivariant pleated surface $\widetilde{f}_\lambda\colon \widetilde{S}_\lambda \to \mathbb{H}^3$ that sends each face $\widetilde{T}$ of $\widetilde{\lambda}$ with vertices $\widetilde{v}_1$, $\widetilde{v}_2$, $\widetilde{v}_3 \in \widetilde{V}_\lambda$ to the ideal triangle $\widetilde{f}_\lambda(\widetilde{T}) \subset \mathbb{H}^3$ with vertices $\xi(\widetilde{v}_1)$, $\xi(\widetilde{v}_2)$, $\xi(\widetilde{v}_3) \in \mathbb{CP}^1 = \partial_\infty \mathbb{H}^3$. We can then lift $\widetilde{f}_\lambda$ to an r_λ–equivariant pleated surface $\widehat{f}_\lambda\colon \widehat{S}_\lambda \to \mathbb{H}^3$, where $\widehat{S}_\lambda$ is the universal

cover of the punctured surface S_λ and where $r_\lambda\colon \pi_1(S_\lambda) \to \mathrm{SL}_2(\mathbb{C})$ is the composition of $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ with the homomorphism $\pi_1(S_\lambda) \to \pi_1(S)$ induced by the inclusion map.

The pleated surface $(\widehat{f}_\lambda, r_\lambda)$ is exactly the setup needed to apply Proposition 23 of [10] to the punctured surface S_λ. By construction, the shearbend parameter associated by this pleated surface to the edge e_i of λ is exactly the crossratio weight x_i defined as above by the λ–enhancement ξ.

Proposition 23 of [10] has an additional degree of freedom for each puncture v of S_λ. Specifically, the hypotheses of that statement require that we choose an N–root $h_v = \mu_\zeta(H_v)^{\frac{1}{N}}$ for a certain number $\mu_\zeta(H_v) \in \mathbb{C}^*$ provided by [10, Proposition 22] (using the notation of [10]). In addition, this number $\mu_\zeta(H_v)$ is such that $\mu_\zeta(H_v) + \mu_\zeta(H_v)^{-1} = -\mathrm{Tr}\, r(P_v)$, where P_v is a small loop going around the puncture v of S_λ. In our case $r_\lambda(P_v)$ is the identity and consequently has trace equal to 2, so that $\mu_\zeta(H_v) = -1$.

We can therefore apply [10, Proposition 23] to the N–root $h_v = -\omega^4$ of $\mu_\zeta(H_v) = -1$, since N is odd and $\omega^{4N} = A^{-2N} = 1$. This provides a representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ satisfying the conclusions of Proposition 8.

The uniqueness parts of Propositions 22 and 23 of [10] show that μ_λ is unique up to isomorphism and up to the action of a sign-reversal symmetry $\varepsilon_\lambda \in H^1(S_\lambda; \mathbb{Z}_2)$ of the restriction $r_\lambda \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S_\lambda)$ of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. For every puncture v of S_λ, $\mathrm{Tr}\, r_\lambda(P_v) = 2 \neq 0$ and the sign-reversal symmetry ε_λ is consequently trivial on the loop P_v going around v. It follows that ε_λ is the restriction of a sign-reversal symmetry $\varepsilon \in H^1(S; \mathbb{Z}_2)$ of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. This proves the uniqueness statement for the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$. □

Remark 9. As indicated in the above discussion, we could have replaced Conclusion (3) of Proposition 8 by the property that $\mu_\lambda(H_v) = h_v\, \mathrm{Id}_{E_\lambda}$ for an arbitrary N–root h_v of -1. However our subsequent applications of Proposition 8 will require that $h_v = -\omega^4$ in a crucial way.

Complement 10. *The representation μ_λ of Proposition 8 continuously depends on the enhanced homomorphism (r, ξ) as follows. For each edge e_i of λ, consider the corresponding crossratio weight $x_i \in \mathbb{C} - \{0\}$ as a function of the pair (r, ξ), and let $u_i = \sqrt[2N]{x_i}$ be a local determination of the $2N$–root of x_i defined for (r, ξ) in an open subset $\mathcal{U}$ of the space of all such pairs. Then the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ can be chosen so that, for every monomial $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n} \in \mathcal{Z}^\omega(\lambda)$,*

$$\mu_\lambda(Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}) = u_1^{k_1} u_2^{k_2} \dots u_n^{k_n}\, A_{k_1 k_2 \dots k_n}$$

for some linear isomorphism $A_{k_1 k_2 \dots k_n} \in \mathrm{End}(E_\lambda)$ *independent of* $(r, \xi) \in \mathcal{U}$.

Proof. This is an immediate consequence of the proofs of Propositions 15, 22 and 23 in [10] (where Proposition 15 is a key step in the proof of Proposition 23). □

3.4. Representing the skein algebra of the punctured surface S_λ. We now begin our construction of an irreducible representation of the skein algebra $\mathcal{S}^A(S)$ whose classical shadow is equal to the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

Represent the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ by a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Let λ be a triangulation of S for which this homomorphism r admits a λ–enhancement ξ. For instance, any combinatorial triangulation has this property by Lemma 7. Let $S_\lambda = S - V_\lambda$ be the punctured surface obtained by removing the vertex set of λ from S.

We can then consider the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ associated to the enhanced homomorphism (r, ξ) by Proposition 8. Composing μ_λ with the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathcal{Z}^\omega(\lambda)$ of [7] now defines a representation

$$\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda).$$

This is only a representation of the skein algebra $\mathcal{S}^A(S_\lambda)$ of the punctured surface S_λ, whereas we want to represent the skein algebra $\mathcal{S}^A(S)$ of the closed surface S. The rest of the article is devoted to showing how ρ_λ induces a non-trivial representation of $\mathcal{S}^A(S)$.

4. The off-diagonal kernels

4.1. The classical off-diagonal term of a vertex. This section is intended to motivate the definition of the next section.

Consider a vertex v of the triangulation λ. Let e_{i_1}, e_{i_2}, …, e_{i_u} be the edges of λ that emanate from v, indexed in counterclockwise order around v, and with possible repetitions when the two endpoints of an edge are equal to v. As in §3.2, let $x_i \in \mathbb{C}^*$ be the crossratio weight associated to the edge e_i of λ by the enhancement ξ.

Lemma 11.

$$1 + x_{i_1} + x_{i_1}x_{i_2} + \cdots + x_{i_1}x_{i_2}\dots x_{i_{u-1}} = 0$$

Proof. Let P_v be a small loop going around the vertex v, oriented counterclockwise. A standard computation (see for instance Exercises 8.5–8.7 and 10.14 in [4]) enables us to compute the image of any element of $\pi_1(S_\lambda)$ under the homomorphism $r_\lambda\colon \pi_1(S_\lambda) \to \mathrm{SL}_2(\mathbb{C})$ induced by r, namely the homomorphism r_λ obtained by composing $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ with the homomorphism $\pi_1(S_\lambda) \to \pi_1(S)$ induced by the inclusion map. For P_v, this gives that, up to conjugation,

$$\begin{aligned} r_\lambda(P_k) &= \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_{i_1} & 0 \\ 0 & z_{i_1}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_{i_2} & 0 \\ 0 & z_{i_2}^{-1} \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_{i_u} & 0 \\ 0 & z_{i_u}^{-1} \end{pmatrix} \\ &= \pm \begin{pmatrix} z_{i_1}z_{i_2}\dots z_{i_u} & \sum_{j=1}^{u} z_{i_1}z_{i_2}\dots z_{i_{j-1}}z_{i_j}^{-1}z_{ij+1}^{-1}\dots z_{i_u}^{-1} \\ 0 & z_{i_1}^{-1}z_{i_2}^{-1}\dots z_{i_u}^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \end{aligned}$$

for arbitrary choices of square roots $z_i = \sqrt{x_i}$. The $\pm$ sign depends on these choices of square roots.

Since P_v is homotopic to 0 in S, $r_\lambda(P_v) = r(P_v) = \mathrm{Id} \in \mathrm{SL}_2(\mathbb{C})$. Therefore, $z_{i_1}z_{i_2}\dots z_{i_u} = \pm 1$ and the off-diagonal term

$$\begin{aligned} &\sum_{j=1}^{u} z_{i_1}z_{i_2}\dots z_{i_{j-1}}z_{i_j}^{-1}z_{ij+1}^{-1}\dots z_{i_u}^{-1} \\ &\qquad = z_{i_1}^{-1}z_{i_2}^{-1}\dots z_{i_u}^{-1}\left(\sum_{j=1}^{u} z_{i_1}^2 z_{i_2}^2 \dots z_{i_{j-1}}^2\right) \\ &\qquad = \pm\left(1 + z_{i_1}^2 + z_{i_1}^2 z_{i_2}^2 + \cdots + z_{i_1}^2 z_{i_2}^2 \dots z_{i_{u-1}}^2\right) \\ &\qquad = \pm\left(1 + x_{i_1} + x_{i_1}x_{i_2} + \cdots + x_{i_1}x_{i_2}\dots x_{i_{u-1}}\right) \end{aligned}$$

is equal to 0. □

We will consider a quantum analogue of the equation

$$1 + x_{i_1} + x_{i_1}x_{i_2} + \cdots + x_{i_1}x_{i_2}\dots x_{i_{u-1}} = 0$$

or, equivalently,

$$1 + z_{i_1}^2 + z_{i_1}^2 z_{i_2}^2 + \cdots + z_{i_1}^2 z_{i_2}^2 \dots z_{i_{u-1}}^2 = 0$$

for the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ of Proposition 8. The major difference is that this equation will not be realized everywhere, but only on a subspace F_v of E_λ.

4.2. The off-diagonal term and kernel of a vertex. As in the previous section, we consider a vertex v of the triangulation λ, and we index the edges of λ emerging from v as e_{i_1}, e_{i_2}, …, e_{i_u} in counterclockwise order around v.

Note that the indexing of the e_{i_j} depends on our choice of the first edge e_{i_1}. For this choice of indexing, the *off-diagonal term* of the puncture v is the element

$$\begin{aligned} Q_v &= \sum_{j=0}^{u-1} \omega^{-4j} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_j}^2 \\ &= 1 + \omega^{-4} Z_{i_1}^2 + \omega^{-8} Z_{i_1}^2 Z_{i_2}^2 + \dots + \omega^{-4(u-1)} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_{u-1}}^2 \end{aligned}$$

of $\mathcal{Z}^\omega(\lambda)$. The term corresponding to $j = 0$ is here 1 by convention.

For the representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ of Proposition 8, the *off-diagonal kernel* of the vertex v for the representation μ_λ is the subspace $F_v = \ker \mu_\lambda(Q_v)$ of E_λ. To relate this definition to the relation of Lemma 11, observe that the off-diagonal kernel F_v is the set of vectors $w \in E_\lambda$ such that

$$\mu_\lambda\big(1 + \omega^{-4} Z_{i_1}^2 + \omega^{-8} Z_{i_1}^2 Z_{i_2}^2 + \dots + \omega^{-4(u-1)} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_{u-1}}^2\big)(w) = 0.$$

Note the analogy with the last displayed equation of §4.1.

The *total off-diagonal kernel* of μ_λ is the intersection $F_\lambda = \bigcap_{v \in V_\lambda} F_v$ of the off-diagonal kernels of all vertices of λ.

The off-diagonal term $Q_v \in \mathcal{Z}^\omega(\lambda)$ clearly depends of the indexing of the edges of λ around v. We will show in Lemma 14 below that, on the contrary, the off-diagonal kernel $F_v \subset E_\lambda$ depends only on the vertex v. As a first step towards the proof of that statement, we begin with a preliminary lemma.

By invariance of the Weyl quantum ordering under permutation, the central element $H_v \in \mathcal{Z}^\omega(\lambda)$ associated to the vertex v can be written as $H_v = [Z_{i_1} Z_{i_2} \dots Z_{i_u}]$. We want to compute the precise quantum ordering coefficient in this expression.

Lemma 12. *Let the edges of λ emerging from the vertex v be indexed as e_{i_1}, e_{i_2}, …, e_{i_u} in counterclockwise order around v. Then, the central element $H_v \in \mathcal{Z}^\omega(\lambda)$ associated to v is equal to*

$$H_v = \omega^{-u+2} Z_{i_1} Z_{i_2} \dots Z_{i_u}.$$

Proof. The proof is straightforward when the edges e_{i_k} are all distinct, and in particular when the triangulation λ is combinatorial. Indeed, in this case, $Z_{i_k}Z_{i_{k+1}} = \omega^2 Z_{i_{k+1}}Z_{i_k}$ whenever $1 \leqslant k < u$, $Z_{i_1}Z_{i_u} = \omega^2 Z_{i_u}Z_{i_1}$, and all other pairs of generators Z_{i_k}, Z_{i_l} commute. The general case could be deduced from this one with the change of triangulation techniques developed in §§5.1 and 5.2, but we prefer to give a combinatorial proof right away. See also the very indirect argument that we used in the proof of [10, Lemma 18].

By definition of the Weyl quantum ordering,

$$H_v = [Z_{i_1}Z_{i_2}\dots Z_{i_u}] = \omega^{-\sum_{1\leqslant k<l\leqslant u}\sigma_{i_k i_l}} Z_{i_1}Z_{i_2}\dots Z_{i_u}$$

where the skew commutativity coefficient $\sigma_{ij} \in \{0, \pm 1, \pm 2\}$ is defined as in §3.1, and in particular is such that $Z_iZ_j = \omega^{2\sigma_{ij}} Z_jZ_i$.

By definition of σ_{ij}, we can write

$$\sum_{1\leqslant k<l\leqslant u} \sigma_{i_k i_l} = \sum_{l=2}^{u} \sum_{\substack{1\leqslant k<l \text{ and} \\ a \text{ angular sector} \\ \text{from } e_{i_k} \text{ to } e_{i_l}}} \varepsilon(k,l,a)$$

where, for every angular sector a of a triangle T_j that is locally bounded by the edges e_{i_k} and e_{i_l} near the vertex of a, $\varepsilon(k,l,a)$ is equal to $+1$ if one goes from e_{i_k} to e_{i_l} counterclockwise at a, and is equal to -1 otherwise. The angular sectors a contributing to this sum include the angular sectors a_1, a_2, …, a_u that are adjacent to v, indexed in such a way that a_k is locally bounded by the edges e_{i_k} and $e_{i_{k+1}}$ near v. There may be contributions from additional angular sectors when the edges e_{i_k} are not distinct.

Fixing an index l, we want to analyze its contribution $\sigma(l) = \sum_{k,a} \varepsilon(k,l,a)$ of the edge e_{i_l} to the above sum. If an index k contributes to $\sigma(l)$, then the edge e_{i_k} is contained in a face of λ that also contains e_{i_l}, and one of the two edges $e_{i_{k\pm1}}$ is also contained in the same face. Analyzing the possible configurations in the union of the two faces of λ containing e_{i_l}, we see that most of the couples (k,a) contributing to the sum can be grouped into pairs

(1) $\{(k,a'_k),(k+1,a''_k)\}$ when the angular sector a_k is opposite e_{i_l} in a face of λ, and where a'_k and a''_k are the other two angular sectors of this face;

(2) $\{(k-1,a_{k-1}),(k+1,a_k)\}$ when $e_{i_k} = e_{i_l}$.

The first type of pair $\{(k,a'_k),(k+1,a''_k)\}$ contributes $\varepsilon(k,l,a'_k) + \varepsilon(k+1,l,a''_k) = +1-1 = 0$. The second type $\{(k-1,a_{k-1}),(k+1,a_k)\}$ contributes

$\varepsilon(k, l-1, a_{l-1}) + \varepsilon(k, l+1, a_l) = 1 - 1 = 0$. In particular, the corresponding terms cancel out.

The only terms that do not cancel out in such a pair are those where the potential pair would involve an index that is not in the interval $[1, l-1]$. This always occurs for $\varepsilon(l-1, l, a_{l-1}) = +1$, and for $\varepsilon(1, u, a_u) = -1$ when $l = u$. A more special instance arises when the angular sector a_u is opposite e_{i_l} in a face of λ, in which case $\varepsilon(1, l, a_u'') = 1$ cannot be cancelled by a term $\varepsilon(u, l, a_u') = -1$. Similarly, when $e_{i_l} = e_{i_1}$ or $e_{i_l} = e_{i_u}$ with $l < u$, the terms $\varepsilon(2, l, a_1) = -1$ or $\varepsilon(1, l, a_u) = -1$, respectively, are not cancelled by another term.

Using our convention that the three sides of each face of the triangulation λ are all distinct, one easily sees that these are the only terms that do not cancel out. Note that, outside of $\varepsilon(l-1, l, a_{l-1}) = +1$, all the other exceptions occur precisely when the face of λ containing the angular sector a_u also contains the edge e_{i_l}. Summing over l and combining the above observations, it follows that

$$\sum_{1 \leqslant k < l \leqslant u} \sigma_{i_k i_l} = u - 1 + n_1 - n_2$$

where $n_1 \in \{0, 1, 2\}$ is the number of indices $l \in [2, u]$ for which the edge e_{i_l} is opposite the angular sector a_u in the face that contains it, and $n_2 \in \{1, 2, 3\}$ is the number of indices $l \in [2, u]$ for which the edge e_{i_l} is adjacent to a_u.

Now, consider the face of λ that contains the angular sector a_u. There are three cases to consider, according to whether 1, 2 or 3 of the vertices of this face are equal to v. An immediate count gives that $n_2 = n_1 + 1$ in all three cases. This proves that

$$\sum_{1 \leqslant k < l \leqslant u} \sigma_{i_k i_l} = u - 2$$

and completes the proof of Lemma 12. □

This combinatorial proof of Lemma 12 also enables us to compute the Weyl quantum ordering of monomials similar to the central element H_v. These computations will be used in §6.

Lemma 13. *Let the edges of λ emerging from the vertex v be indexed as e_{i_1}, e_{i_2}, …, e_{i_u} in counterclockwise order around v. Then, for every k_0 with $1 < k_0 < u$,*

$$[Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}}] = \begin{cases} \omega^{-k_0+1} Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}} & \text{if } e_{i_{k_0}} \neq e_{i_1} \text{ and } e_{i_{k_0+1}} \neq e_{i_u} \\ \omega^{-k_0+2} Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}} & \text{if } e_{i_{k_0}} = e_{i_1} \\ \omega^{-k_0} Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}} & \text{if } e_{i_{k_0+1}} = e_{i_u}. \end{cases}$$

In particular, $[Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}}] = \omega^{-k_0+1} Z_{i_1} Z_{i_2} \dots, Z_{i_{k_0}}$ if the triangulation λ is combinatorial.

Proof. By definition of the Weyl quantum ordering,

$$[Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}}] = \omega^{-\sum_{1 \leqslant k < l \leqslant k_0} \sigma_{i_k i_l}} Z_{i_1} Z_{i_2} \dots Z_{i_{k_0}}.$$

The same arguments as in the proof of Lemma 12 then give that

$$\sum_{1 \leqslant k < l \leqslant k_0} \sigma_{i_k i_l} = k_0 - 1 + n_1 - n_2$$

where $n_1 \in \{0, 1, 2\}$ is the number of indices $l \in [2, k_0]$ for which the edge e_{i_l} is opposite the angular sector a_u in a face of λ, and $n_2 \in \{0, 1, 2\}$ is the number of indices $l \in [2, k_0]$ for which the edge e_{i_l} is adjacent to a_u.

The fact that indices are truncated at k_0 introduces minor differences with the case of Lemma 12. More precisely, the case-by-case analysis now gives that

$$n_2 = \begin{cases} n_1 & \text{if } e_{i_{k_0}} \neq e_{i_1} \text{ and } e_{i_{k_0+1}} \neq e_{i_u} \\ n_1 + 1 & \text{if } e_{i_{k_0}} = e_{i_1} \\ n_1 - 1 & \text{if } e_{i_{k_0+1}} = e_{i_u}. \end{cases}$$

The stated computation immediately follows. □

We are now ready to prove the promised result, that the off-diagonal kernel $F_v \subset E_\lambda$ depends only on the vertex v.

Lemma 14. *The off-diagonal kernel $F_v = \ker \mu_\lambda(Q_v)$ of v is independent of the counterclockwise indexing of the edges e_{i_1}, e_{i_2}, …, e_{i_u}, $e_{i_{u+1}} = e_{i_1}$ of λ around the vertex v.*

Proof. We can clearly restrict attention to the case where we shift the indexing by 1, and start at the last edge e_{i_u} instead. Then the off-diagonal term

$$Q_v = 1 + \omega^{-4} Z_{i_1}^2 + \omega^{-8} Z_{i_1}^2 Z_{i_2}^2 + \dots + \omega^{-4(u-1)} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_{u-1}}^2$$

gets replaced by

$$\begin{aligned} Q'_v &= 1 + \omega^{-4} Z_{i_u}^2 + \omega^{-8} Z_{i_u}^2 Z_{i_1}^2 + \omega^{-12} Z_{i_u}^2 Z_{i_1}^2 Z_{i_2}^2 + \dots \\ &\qquad\qquad \dots + \omega^{-4(u-1)} Z_{i_u}^2 Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_{u-2}}^2 \\ &= 1 + \omega^{-4} Z_{i_u}^2 Q_v - \omega^{-4u} Z_{i_u}^2 Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_{u-2}}^2 Z_{i_{u-1}}^2. \end{aligned}$$

We now have to remember that $\mu_\lambda(H_v) = -\omega^4 \operatorname{Id}_{E_\lambda}$ by choice of the representation μ_λ in Proposition 8, so that

$$\mu_\lambda(\omega^{-4u} Z_{i_u}^2 Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_{u-2}}^2 Z_{i_{u-1}}^2) = \mu_\lambda(\omega^{-8} H_v^2) = \operatorname{Id}_{E_\lambda}$$

by Lemma 12. It follows that

$$\mu_\lambda(Q'_v) = \operatorname{Id}_{E_\lambda} + \mu_\lambda(\omega^{-4} Z_{i_u}^2 Q_v) - \operatorname{Id}_{E_\lambda} = \omega^{-4} \mu_\lambda(Z_{i_u}^2) \circ \mu_\lambda(Q_v).$$

The element $Z_{i_u}^2$ is invertible in $\mathcal{Z}^\omega(\lambda)$. Therefore $\mu_\lambda(Z_{i_u}^2)$ is invertible in $\operatorname{End}(E_\lambda)$ and it follows that $\ker \mu_\lambda(Q'_v) = \ker \mu_\lambda(Q_v)$, as desired. □

4.3. Invariance under the action of the skein algebra. The off-diagonal kernel $F_v \subset E_\lambda$ cannot be invariant under $\mu_\lambda\big(\mathcal{Z}^\omega(\lambda)\big)$, since the representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \operatorname{End}(E_\lambda)$ is irreducible. However, it is invariant under the image of the representation $\rho_\lambda = \mu_\lambda \circ \operatorname{Tr}_\lambda^\omega \colon \mathcal{S}^A(S_\lambda) \to \operatorname{End}(E_\lambda)$.

In this section, we restrict attention to the case where the triangulation λ is combinatorial. We will later see, in §5.3, that the property holds without this hypothesis.

Proposition 15. *Suppose that the triangulation λ is combinatorial, in the sense that every edge has distinct endpoints and that no two distinct edges have the same endpoints. Then, the off-diagonal kernel F_v of each vertex v of λ is invariant under $\rho_\lambda\big(\mathcal{S}^A(S_\lambda)\big) \subset \operatorname{End}(E_\lambda)$.*

Proof. Let $\mathcal{N}(v) \subset S$ be the neighborhood of the vertex v that is the union of the faces of λ containing v. Because of our hypothesis that λ is combinatorial, there are no identifications on the boundary of $\mathcal{N}(v)$, and $\mathcal{N}(v)$ is homeomorphic to a disk. We already indexed the edges of λ emanating from v as e_{i_1}, e_{i_2}, …, e_{i_u}, going counterclockwise around v. Let e_{k_1}, e_{k_2}, …, e_{k_u} denote the edges forming the boundary of the star neighborhood $\mathcal{N}(v)$, in such a way that e_{i_j}, $e_{i_{j-1}}$ and e_{k_j} cobound a face of λ. See Figure 2.

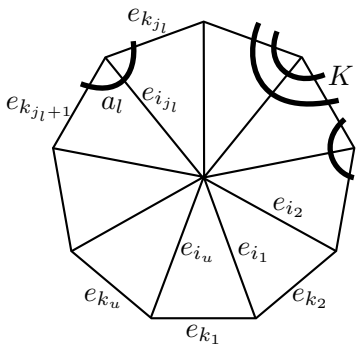


Figure 2. The star neighborhood $\mathcal{N}(v)$ of the vertex v

Let K be a framed link in $S_\lambda \times [0,1]$. We want to show that $\rho_\lambda\big([K]\big) = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\big([K]\big)$ respects F_v and, for this, we first need to understand the quantum trace $\mathrm{Tr}^\omega_\lambda\big([K]\big) \in \mathcal{Z}^\omega(\lambda)$. The precise construction of $\mathrm{Tr}^\omega_\lambda\big([K]\big)$ in [7] can be somewhat elaborate, and we first isotop K to a position where the quantum trace is easier to analyze.

Because there are no identifications on the boundary of the neighborhood $\mathcal{N}(v)$, we can arrange by an isotopy that the intersection of K with $\mathcal{N}(v) \times [0,1]$ consists of finitely many horizontal arcs $a_1 \times \{*\}$, $a_2 \times \{*\}$, $\dots$, $a_t \times \{*\}$, where each $a_l \subset \mathcal{N}(v)$ is an embedded arc that turns around one of the boundary vertices of $\mathcal{N}(v)$, in the sense that a_l goes from some boundary edge $e_{k_{j_l}}$ of $\mathcal{N}(v)$ to the next $e_{k_{j_l+1}}$, crosses the internal edge $e_{i_{j_l}}$ in one point, and meets no other edge of λ (with the convention that $k_{u+1} = k_1$). See Figure 2. In addition, we can arrange that these horizontal arcs $a_l \times \{*\}$ sit at increasing elevations $* \in [0,1]$ as l goes from 1 to t.

Then the State Sum Formula of [7] expresses $\mathrm{Tr}^\omega_\lambda\big([K]\big)$ as a sum of terms of the form

$$A_1 A_2 \dots A_t B \in \mathcal{Z}^\omega(\lambda)$$

where each term corresponds to a state for the boundary of $K \cap \big(\mathcal{N}(v) \times [0,1]\big)$, where A_l is the contribution of the arc a_l, and where B is the contribution of the complement of $K \cap \big(\mathcal{N}(v) \times [0,1]\big)$ in K. In addition, each non-zero A_l can be of only three types

$$\begin{aligned} A_l &= \omega^2 Z_{k_{j_l}} Z_{i_{j_l}} Z_{k_{j_l+1}} \\ \text{or } A_l &= Z^{-1}_{k_{j_l}} Z_{i_{j_l}} Z_{k_{j_l+1}} + Z^{-1}_{k_{j_l}} Z^{-1}_{i_{j_l}} Z_{k_{j_l+1}} \\ \text{or } A_l &= \omega^2 Z^{-1}_{k_{j_l}} Z^{-1}_{i_{j_l}} Z^{-1}_{k_{j_l+1}} \end{aligned}$$

according to the state considered, and B involves only generators Z_i corresponding to edges e_i of λ that are *not* in $\mathcal{N}(v)$.

This expression has the unfortunate feature that, although the terms A_l and B are elements of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$, they do not satisfy the exponent parity condition necessary to make them elements of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$. In particular, we cannot directly apply the representation μ_λ, since terms like $\mu_\lambda(A_l)$ and $\mu_\lambda(B)$ are not defined.

To circumvent this problem, we factor out of $A_1A_2\dots A_vB$ the product $C = Z_{j_1}^{-1}Z_{j_2}^{-1}\dots Z_{j_w}^{-1}$, where e_{j_1}, e_{j_2}, …, e_{i_w} are the edges of λ crossed by K (with possible repetitions). Note that the arc a_l contributes, up to permutation, a term $Z_{k_{j_l}}^{-1}Z_{i_{j_l}}^{-1}Z_{k_{j_l}+1}^{-1}$ to this product $C = Z_{j_1}^{-1}Z_{j_2}^{-1}\dots Z_{j_w}^{-1}$. Set

$$\begin{aligned} A'_l &= Z^2_{k_{j_l}}Z^2_{i_{j_l}}Z^2_{k_{j_l}+1} \\ \text{or } A'_l &= \omega^4 Z^2_{i_{j_l}}Z^2_{k_{j_l}+1} + Z^2_{k_{j_l}+1} \\ \text{or } A'_l &= 1 \end{aligned}$$

according to which of the above three types A_l belongs to. This is specially designed so that $A_l = \omega^\alpha A'_l Z^{-1}_{k_{j_l}}Z^{-1}_{i_{j_l}}Z^{-1}_{k_{j_l}+1}$ for some $\alpha \in \mathbb{Z}$ depending on whether the edges $e_{k_{j_l}}$ and $e_{k_{j_l}+1}$ are contained in the same face of λ (outside of $\mathcal{N}(v)$) or not.

In addition, the term $Z^{-1}_{k_{j_l}}Z^{-1}_{i_{j_l}}Z^{-1}_{k_{j_l}+1}$ ω–commutes with each $A'_{l'}$, in the sense that $Z^{-1}_{k_{j_l}}Z^{-1}_{i_{j_l}}Z^{-1}_{k_{j_l}+1}A'_{l'} = \omega^\beta A'_{l'}Z^{-1}_{k_{j_l}}Z^{-1}_{i_{j_l}}Z^{-1}_{k_{j_l}+1}$ for some $\beta \in \mathbb{Z}$. (The only case that requires some thought is when $A'_{l'} = \omega^4 Z^2_{i_{j_{l'}}}Z^2_{k_{j_{l'}+1}} + Z^2_{k_{j_{l'}+1}}$, in which case it suffices to note that $Z^{-1}_{k_{j_l}}Z^{-1}_{i_{j_l}}Z^{-1}_{k_{j_l}+1}$ commutes with $Z^2_{i_{j_{l'}}}$.) This enables us to rewrite

$$A_1A_2\dots A_tB = A'_1A'_2\dots A'_tB'C.$$

where B' involves only generators Z_i corresponding to edges e_i of λ that are outside of $\mathcal{N}(v)$.

By construction, C belongs to the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$. Each A'_l is also an element of $\mathcal{Z}^\omega(\lambda)$ since all its exponents are even. Since $A_1A_2\dots A_tB$ belongs to $\mathcal{Z}^\omega(\lambda)$ by construction of the quantum trace, it follows that B' is also in $\mathcal{Z}^\omega(\lambda)$. In particular, we can now consider the endomorphisms $\mu_\lambda(C)$, $\mu_\lambda(A'_l)$, $\mu_\lambda(B') \in \mathrm{End}(E_\lambda)$. We want to show that all these endomorphisms respect the off-diagonal kernel F_v.

Since the expression of B' involves only generators Z_i corresponding to edges e_i of λ that are not in the neiborhood $\mathcal{N}(v)$ of v, it commutes with each of the generators Z_{i_j} corresponding to edges emanating from

the vertex v. As a consequence, B' commutes with the off-diagonal term $Q_v = \sum_{j=0}^{u-1} \omega^{-4j} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_j}^2$. It follows that $\mu_\lambda(B') \in \mathrm{End}(E_\lambda)$ respects the off-diagonal kernel $F_v = \ker \mu_\lambda(Q_v)$.

In $C = Z_{j_1}^{-1} Z_{j_2}^{-1} \dots Z_{j_w}^{-1}$, the contribution $Z_{k_{j_l}}^{-1} Z_{i_{j_l}}^{-1} Z_{k_{j_l+1}}^{-1}$ of each arc a_l commutes with each of the generators Z_{i_j} corresponding to edges emanating from the vertex v. The remaining terms of C involve only generators Z_i associated to edges of λ that are not in $\mathcal{N}(v)$, and therefore also commute with the Z_{i_j}. It follows that C also commutes with the off-diagonal term Q_v. This again implies that $\mu_\lambda(C) \in \mathrm{End}(E_\lambda)$ respects the off-diagonal kernel F_v.

For $\mu_\lambda(A'_l)$, we need to distinguish cases. There is nothing to prove when $A'_l = 1$, since $\mu(A'_l) = \mathrm{Id}_{E_\lambda}$ clearly respects F_v. Also, when $A'_l = Z_{k_{j_l}}^2 Z_{i_{j_l}}^2 Z_{k_{j_l+1}}^2$, it commutes with all generators Z_{i_j} corresponding to edges e_{i_j} emanating from the vertex v, and therefore commutes with the off-diagonal term $Q_v = \sum_{j=0}^{u-1} \omega^{-4j} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_j}^2$; therefore $\mu_\lambda(A'_l)$ respects the off-diagonal kernel F_v in this case as well.

The case $A'_l = \omega^4 Z_{i_{j_l}}^2 Z_{k_{j_l+1}}^2 + Z_{k_{j_l+1}}^2$ requires more work. Remember from Lemma 14 that we have some flexibility in the choice of Q_v. In particular, F_v is also the kernel of

$$Q'_v = 1 + \omega^{-4} Z_{i_{j_l+1}}^2 + \omega^{-8} Z_{i_{j_l+1}}^2 Z_{i_{j_l+2}}^2 + \dots + \omega^{-4(u-2)} Z_{i_{j_l+1}}^2 Z_{i_{j_l+2}}^2 \dots Z_{i_{j_l-2}}^2 + \omega^{-4(u-1)} Z_{i_{j_l+1}}^2 Z_{i_{j_l+2}}^2 \dots Z_{i_{j_l-2}}^2 Z_{i_{j_l-1}}^2$$

Because there are no identifications between the edges on the boundary of $\mathcal{N}(v)$, we observe that $Z_{i_{j_l+1}}^2 Z_{k_{j_l+1}}^2 = \omega^8 Z_{k_{j_l+1}}^2 Z_{i_{j_l+1}}^2$, and that $Z_{k_{j_l+1}}^2$ commutes with all Z_{i_j} with $j \neq j_l$, $j_l + 1$. Therefore

$$(Q'_v - 1) Z_{k_{j_l+1}}^2 = \omega^8 Z_{k_{j_l+1}}^2 (Q'_v - 1)$$

and

$$Q'_v Z_{k_{j_l+1}}^2 = \omega^8 Z_{k_{j_l+1}}^2 Q'_v + (1 - \omega^8) Z_{k_{j_l+1}}^2$$

Similarly, $Z_{i_{j_l}}^2 Z_{k_{j_l+1}}^2$ commutes with all but the last term of Q'_v, and

$$(Q'_v - \omega^{-4(u-1)} Z_{i_{j_l+1}}^2 Z_{i_{j_l+2}}^2 \dots Z_{i_{j_l-1}}^2) Z_{i_{j_l}}^2 Z_{k_{j_l+1}}^2 = Z_{i_{j_l}}^2 Z_{k_{j_l+1}}^2 (Q'_v - \omega^{-4(u-1)} Z_{i_{j_l+1}}^2 Z_{i_{j_l+2}}^2 \dots Z_{i_{j_l-1}}^2)$$

Reordering terms, we conclude that

$$\begin{aligned} Q'_v Z^2_{i_{j_l}} Z^2_{k_{j_l+1}} &= Z^2_{i_{j_l}} Z^2_{k_{j_l+1}} Q'_v \\ &\qquad + (1-\omega^{-8})\omega^{-4(u-1)} Z^2_{k_{j_l+1}} Z^2_{i_{j_l}} Z^2_{i_{j_l+1}} Z^2_{i_{j_l+2}} \dots Z^2_{i_{j_l-1}} \\ &= Z^2_{i_{j_l}} Z^2_{k_{j_l+1}} Q'_v + (1-\omega^{-8})\omega^{-4} Z^2_{k_{j_l+1}} H_v^2 \end{aligned}$$

where $H_v = [Z_{i_{j_l}} Z_{i_{j_l+1}} Z_{i_{j_l+2}} \dots Z_{i_{j_l-1}}] = \omega^{-u+2} Z_{i_{j_l}} Z_{i_{j_l+1}} Z_{i_{j_l+2}} \dots Z_{i_{j_l-1}}$ is the central element of $\mathcal{Z}^\omega(\lambda)$ associated to the vertex v.

Therefore, for every vector $w \in F_v = \ker \mu_\lambda(Q'_v)$,

$$\begin{aligned} &\mu_\lambda(Q'_v) \circ \mu_\lambda\big(A'_l\big)(w) \\ &\qquad = \omega^4 \mu_\lambda(Q'_v) \circ \mu_\lambda\big(Z^2_{i_{j_l}} Z^2_{k_{j_l+1}}\big)(w) + \mu_\lambda(Q'_v) \circ \mu_\lambda\big(Z^2_{k_{j_l+1}}\big)(w) \\ &\qquad = \omega^4 \mu_\lambda\big(Z^2_{i_{j_l}} Z^2_{k_{j_l+1}}\big) \circ \mu_\lambda(Q'_v)(w) + (1-\omega^{-8})\mu_\lambda(Z^2_{k_{j_l+1}}) \circ \mu_\lambda(H_v^2)(w) \\ &\qquad\qquad + \omega^8 \mu_\lambda(Z^2_{k_{j_l+1}}) \circ \mu_\lambda(Q'_v)(w) + (1-\omega^8)\mu_\lambda(Z^2_{k_{j_l+1}})(w) \\ &\qquad = (1-\omega^{-8})\mu_\lambda(Z^2_{k_{j_l+1}})(\omega^8 w) + (1-\omega^8)\mu_\lambda(Z^2_{k_{j_l+1}})(w) = 0 \end{aligned}$$

since $\mu_\lambda(Q'_v)(w) = 0$ and $\mu_\lambda(H_v) = -\omega^4 \mathrm{Id}_V$. As a consequence, the image of $w \in F_v = \ker \mu_\lambda(Q'_v)$ under $\mu_\lambda(A'_l)$ is also in F_v.

This proves that $\mu_\lambda(A'_l)$ respects the off-diagonal kernel F_v in all three cases.

As a summary, for every skein $[K] \in \mathcal{S}^A(S_\lambda)$, we showed that the linear map $\rho_\lambda\big([K]\big) \in \mathrm{End}(E_\lambda)$ is a sum of terms

$$\mu_\lambda(A'_1) \circ \mu_\lambda(A'_2) \circ \dots \circ \mu_\lambda(A'_v) \circ \mu_\lambda(B') \circ \mu_\lambda(C)$$

such that each factor in this composition respects the off-diagonal kernel F_v. This proves that the image $\rho_\lambda\big(\mathcal{S}^A(S_\lambda)\big) \subset \mathrm{End}(E_\lambda)$ respects F_v, and completes the proof of Proposition 15. □

Remark 16. Although the hypotheses of Proposition 15 require that the triangulation λ be combinatorial, the proof shows that this statement is valid under the weaker hypothesis that no edge of λ connects the vertex v to itself, and that no two distinct edges connect v to the same vertex of λ. We will use this observation in §5.1.

4.4. Constructing a representation of the skein algebra $\mathcal{S}^A(S)$. We now consider the total off-diagonal kernel $F_\lambda = \bigcap_{v \in V_\lambda} F_v \subset E_\lambda$.

Let $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ be the representation associated in §3.4 to a homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ endowed with a

λ–enhancement ξ. Assuming that the triangulation λ is combinatorial, Proposition 15 shows that the total off-diagonal kernel F_λ is invariant under the image $\rho_\lambda\big(\mathcal{S}^A(S_\lambda)\big) \subset \mathrm{End}(E_\lambda)$. For every framed link $K \subset S_\lambda \times [0,1]$, we can therefore consider the restriction $\rho_\lambda\big([K]\big)_{|F_\lambda} \in \mathrm{End}(F_\lambda)$.

We now show that $\rho_\lambda\big([K]\big)_{|F_\lambda} \in \mathrm{End}(F_\lambda)$ remains invariant if we modify K by an isotopy in $S \times [0,1]$, not just in $S_\lambda \times [0,1]$.

Proposition 17. *Suppose that the triangulation λ is combinatorial, in the sense that each edge has distinct endpoints and that no two distinct edges have the same endpoints. Let the two framed links K, $K' \subset S_\lambda \times [0,1]$ be isotopic in $S \times [0,1]$. Then $\rho_\lambda\big([K]\big)_{|F_\lambda} = \rho_\lambda\big([K']\big)_{|F_\lambda}$ in $\mathrm{End}(F_\lambda)$.*

Again, the hypothesis that λ is combinatorial is here only to simplify the proof, and the property holds without this condition; see Theorem 35 in §5.3.

Proof. We can choose the isotopy from K to K' so that it sweeps through punctures of S_λ at only finitely many times. This reduces the problem to the case where the isotopy sweeps only once through a puncture. Let v be the vertex of the triangulation λ corresponding to this puncture.

We will be using the same labeling conventions as in the proof of Proposition 15. In particular, $\mathcal{N}(v)$ denotes the union of the faces of λ that contain v. The edges emanating from v are indexed as e_{i_1}, e_{i_2}, …, e_{i_u} in counterclockwise order around v. The edges of the boundary of $\mathcal{N}(v)$ are e_{k_1}, e_{k_2}, …, e_{k_u}, where e_{k_j} is the third side of the face containing $e_{i_{j-1}}$ and e_{i_j}. See Figure 3.

Since the skeins $[K]$, $[K'] \in \mathcal{S}^A(S_\lambda)$ are invariant under isotopy in $S_\lambda \times [0,1]$, we can restrict attention to the case of Figure 3, where the two pieces of K and K' represented are endowed with the vertical framing, and where the remaining portions of K and K' coincide and are located in $S_\lambda \times [0,1]$ at lower elevations than the pieces represented.

These elevation conventions greatly simplify the computation of the quantum trace $\mathrm{Tr}^\omega_\lambda\big([K]\big)$, because we do not have to worry about correction factors coming from biangles. The construction of the quantum trace in [7]

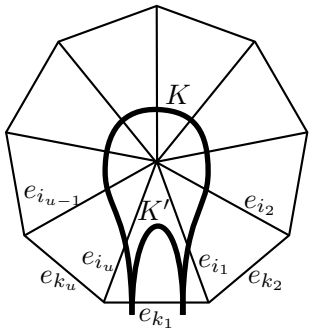


Figure 3. Sweeping through a puncture

then gives that

$$\begin{aligned}
\mathrm{Tr}^{\omega}_{\lambda}\big([K]\big) =& B_{++} Z_{k_1} \Big(\omega^{3-u} Z_{i_1} Z_{i_2} \dots Z_{i_u} \\
&\qquad + \omega^{3-u} \sum_{j=1}^{u-1} Z_{i_1} Z_{i_2} \dots Z_{i_j} Z^{-1}_{i_{j+1}} Z^{-1}_{i_{j+2}} \dots Z^{-1}_{i_u} \Big) Z_{k_1} \\
&+ B_{+-} Z_{k_1} \Big(\omega^{5-u} \sum_{j=1}^{u-1} Z_{i_1} Z_{i_2} \dots Z_{i_j} Z^{-1}_{i_{j+1}} Z^{-1}_{i_{j+2}} \dots Z^{-1}_{i_u} \Big) Z^{-1}_{k_1} \\
&+ B_{-+} Z^{-1}_{k_1} \Big(\omega^{1-u} Z_{i_1} Z_{i_2} \dots Z_{i_u} + \omega^{1-u} Z^{-1}_{i_1} Z^{-1}_{i_2} \dots Z^{-1}_{i_u} \\
&\qquad + \omega^{1-u} \sum_{j=1}^{u-1} Z_{i_1} Z_{i_2} \dots Z_{i_j} Z^{-1}_{i_{j+1}} Z^{-1}_{i_{j+2}} \dots Z^{-1}_{i_u} \Big) Z_{k_1} \\
&+ B_{--} Z^{-1}_{k_1} \Big(\omega^{3-u} Z^{-1}_{i_1} Z^{-1}_{i_2} \dots Z^{-1}_{i_u} \\
&\qquad + \omega^{3-u} \sum_{j=1}^{u-1} Z_{i_1} Z_{i_2} \dots Z_{i_j} Z^{-1}_{i_{j+1}} Z^{-1}_{i_{j+2}} \dots Z^{-1}_{i_u} \Big) Z^{-1}_{k_1}
\end{aligned}$$

where the terms $B_{\pm\pm}$ are contributions of the parts of the link K that are not represented on Figure 3. The domino diagrams of Exercises 8.5–8.7 and 10.14 in [4] may be here useful to list all possible terms. The order of terms is dictated by our condition on relative elevations.

We now move all terms Z_{k_1} together. Because of our hypothesis that λ is combinatorial, there are no extraneous identifications between the edges represented in Figure 3. It follows that Z_{k_1} commutes with all Z_{i_j} with $1 < j < u$, and that $Z_{k_1} Z_{i_1} = \omega^{-2} Z_{i_1} Z_{k_1}$ and $Z_{k_1} Z_{i_u} = \omega^2 Z_{i_u} Z_{k_1}$. This

gives:

$$\begin{aligned}\mathrm{Tr}^{\omega}_{\lambda}([K]) = B_{++}Z_{k_1}^2\Big(&\omega^{3-u}Z_{i_1}Z_{i_2}\dots Z_{i_u}\\ &+\omega^{7-u}\sum_{j=1}^{u-1}Z_{i_1}Z_{i_2}\dots Z_{i_j}Z_{i_{j+1}}^{-1}Z_{i_{j+2}}^{-1}\dots Z_{i_u}^{-1}\Big)\\ + B_{+-}\Big(&\omega^{1-u}\sum_{j=1}^{u-1}Z_{i_1}Z_{i_2}\dots Z_{i_j}Z_{i_{j+1}}^{-1}Z_{i_{j+2}}^{-1}\dots Z_{i_u}^{-1}\Big)\\ + B_{-+}\Big(&\omega^{1-u}Z_{i_1}Z_{i_2}\dots Z_{i_u}+\omega^{1-u}Z_{i_1}^{-1}Z_{i_2}^{-1}\dots Z_{i_u}^{-1}\\ &+\omega^{5-u}\sum_{j=1}^{u-1}Z_{i_1}Z_{i_2}\dots Z_{i_j}Z_{i_{j+1}}^{-1}Z_{i_{j+2}}^{-1}\dots Z_{i_u}^{-1}\Big)\\ + B_{--}Z_{k_1}^{-2}\Big(&\omega^{3-u}Z_{i_1}^{-1}Z_{i_2}^{-1}\dots Z_{i_u}^{-1}\\ &+\omega^{-1-u}\sum_{j=1}^{u-1}Z_{i_1}Z_{i_2}\dots Z_{i_j}Z_{i_{j+1}}^{-1}Z_{i_{j+2}}^{-1}\dots Z_{i_u}^{-1}\Big).\end{aligned}$$

We now recognize several of the terms in this sum. For instance, the central element $H_v \in \mathcal{Z}^{\omega}(\lambda)$ associated to the vertex v is

$$H_v = [Z_{i_1}Z_{i_2}\dots Z_{i_u}] = \omega^{2-u}Z_{i_1}Z_{i_2}\dots Z_{i_u}.$$

Similarly,

$$H_v^{-1} = [Z_{i_1}^{-1}Z_{i_2}^{-1}\dots Z_{i_u}^{-1}] = \omega^{2-u}Z_{i_1}^{-1}Z_{i_2}^{-1}\dots Z_{i_u}^{-1}.$$

Also.

$$\begin{aligned}&\sum_{j=1}^{u-1}Z_{i_1}Z_{i_2}\dots Z_{i_j}Z_{i_{j+1}}^{-1}Z_{i_{j+2}}^{-1}\dots Z_{i_u}^{-1}\\ &\qquad=\Big(\sum_{j=1}^{u-1}\omega^{-4(j-1)}Z_{i_1}^2Z_{i_2}^2\dots Z_{i_j}^2\Big)Z_{i_1}^{-1}Z_{i_2}^{-1}\dots Z_{i_u}^{-1}\\ &\qquad=\Big(\sum_{j=2}^{u}\omega^{-4(j-2)}Z_{i_1}^2Z_{i_2}^2\dots Z_{i_{j-1}}^2\Big)\omega^{u-2}H_v^{-1}\\ &\qquad=\omega^{u+2}(Q_v-1)H_v^{-1}\end{aligned}$$

where Q_v is the off-diagonal term of the vertex v defined in §4.2.

This gives

$$\begin{aligned}\mathrm{Tr}^\omega_\lambda\big([K]\big) =& B_{++}Z_{k_1}^2\big(\omega H_v + \omega^9(Q_v-1)H_v^{-1}\big) + B_{+-}\omega^3(Q_v-1)H_v^{-1}\\ &+ B_{-+}\big(\omega^{-1}H_v + \omega^{-1}H_v^{-1} + \omega^7(Q_v-1)H_v^{-1}\big)\\ &+ B_{--}Z_{k_1}^{-2}\big(\omega H_v^{-1} + \omega(Q_v-1)H_v^{-1}\big).\end{aligned}$$

Note that all the terms arising in this expression belong to the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$. We can therefore apply the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$. Remembering that $\mu_\lambda(H_v) = -\omega^4\,\mathrm{Id}_{E_\lambda}$,

$$\begin{aligned}\rho_\lambda\big([K]\big) =& \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\big([K]\big)\\ =& \mu_\lambda(B_{++}Z_{k_1}^2)\circ\big(-\omega^5\mu_\lambda(Q_v)\big) + \mu_\lambda(B_{+-})\circ\big(-\omega^{-1}\mu_\lambda(Q_v) + \omega^{-1}\mathrm{Id}_{E_\lambda}\big)\\ &+ \mu_\lambda(B_{-+})\circ\big(-\omega^{-5}\,\mathrm{Id}_{E_\lambda} - \omega^3\mu_\lambda(Q_v)\big)\\ &+ \mu_\lambda(B_{--}Z_{k_1}^{-2})\circ\big(-\omega^{-3}\mu_\lambda(Q_v)\big).\end{aligned}$$

This expression greatly simplifies when we restrict it to the total off-diagonal kernel $F_\lambda \subset \ker\mu_\lambda(Q_v)$, and

$$\rho_\lambda\big([K]\big)_{|F_\lambda} = \omega^{-1}\mu_\lambda(B_{+-})_{|F_\lambda} - \omega^{-5}\mu_\lambda(B_{-+})_{|F_\lambda}.$$

We now perform the same computations for the skein $[K'] \in \mathcal{S}^A(S)_0)$. The principles are the same, but everything is much simpler because the framed knot K' meets many fewer edges of λ. In particular, the expression of $\mathrm{Tr}^\omega_\lambda\big([K']\big)$ is much less cumbersome, and gives

$$\mathrm{Tr}^\omega_\lambda\big([K']\big) = B_{+-}Z_{k_1}(\omega^{-1})Z_{k_1}^{-1} + B_{-+}Z_{k_1}^{-1}(-\omega^{-5})Z_{k_1} = \omega^{-1}B_{+-} - \omega^{-5}B_{-+}$$

where the terms $B_{\pm\pm}$ are contributions of the parts of the link K' that are not represented on Figure 3, and are the same as those that appeared in the computation of $\mathrm{Tr}^\omega_\lambda\big([K]\big)$ since these "hidden parts" of the links K and K' coincide.

As a consequence,

$$\rho_\lambda\big([K']\big)_{|F_\lambda} = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\big([K']\big)_{|F_\lambda} = \omega^{-1}\mu_\lambda(B_{+-})_{|F_\lambda} - \omega^{-5}\mu_\lambda(B_{-+})_{|F_\lambda}.$$

Comparing the two formulas, we see that $\rho_\lambda\big([K]\big)_{|F_\lambda} = \rho_\lambda\big([K']\big)_{|F_\lambda}$, which completes the proof of Proposition 17. □

Remark 18. As in Remark 16, the proof of Proposition 17 is valid under a weaker hypothesis than the requirement that the triangulation λ be combinatorial. Indeed, the following condition is sufficent for the statement to hold: no edge of λ connects the vertex v to itself, and no two distinct edges connect v to the same vertex of λ. We will use this observation in §5.1.

A consequence of Proposition 17 is that the representation $\rho_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ induces a representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ by the property that

$$\check{\rho}_\lambda\big([K]\big) = \rho_\lambda\big([K]\big)_{|F_\lambda} \in \mathrm{End}(F_\lambda).$$

Proposition 19. *The above representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ has classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, in the sense that*

$$T_N\big(\check{\rho}_\lambda([K])\big) = -\mathrm{Tr}\, r(K)\, \mathrm{Id}_{F_\lambda}$$

for every framed knot $K \subset S_\lambda \times [0,1]$ whose projection to S_λ has no crossing and whose framing is vertical.

Proof. Let $K \subset S_\lambda \times [0,1]$ be a framed knot whose projection to S_λ has no crossing and whose framing is vertical. By definition of the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and of $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda$, and in particular by Condition (4) of Proposition 8,

$$T_N\big(\rho_\lambda([K])\big) = -\mathrm{Tr}\, r(K)\, \mathrm{Id}_{E_\lambda}.$$

In particular, by restriction to the off-diagonal kernel F_λ,

$$T_N\big(\check{\rho}_\lambda([K])\big) = T_N\big(\rho_\lambda([K])\big)_{|F_\lambda} = -\mathrm{Tr}\, r(K)\, \mathrm{Id}_{F_\lambda}. \qquad \square$$

At this point, it looks like we are almost done with the proof of Theorem 2. The only problem is that we do not know that the total off-diagonal kernel $F_\lambda = \bigcap_{v \in V_\lambda} F_v$ is non-trivial. In fact, we don't even know that any of the off-diagonal kernels F_v is non-trivial. The rest of the article is devoted to estimating the dimension of F_λ. At the same time, we will prove that the representation $\check{\rho}_\lambda$ is, up to isomorphism, independent of all the choices that we have made.

5. Changing triangulations

In this section, we introduce two moves that modify the triangulation λ without changing the isomorphism class of the representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ constructed above. We will then use these moves to prove that, up to isomorphism and sign-reversal symmetry, $\check{\rho}_\lambda$ is independent of the choice of λ and of the λ–enhancement ξ.

Unlike in the previous section, the triangulations that we are considering here are not assumed any more to be combinatorial.

5.1. Subdividing faces. Let λ be a triangulation of the surface S. Let λ' be the triangulation obtained from λ by subdividing the face T into three triangles as in Figure 4. In particular, the vertex set of λ' consists of the vertices of λ plus one vertex v_0 in the interior of T.

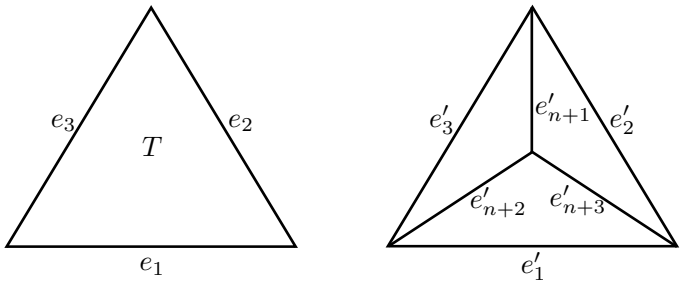


Figure 4. Subdividing a face

For convenience in the notation, index the edges $e_1, e_2, \dots, e_n$ of λ and the edges $e'_1, e'_2, \dots, e'_{n+3}$ of λ' in such a way that:

(1) the sides of the face T of λ are e_1, e_2, e_3, in this order as one goes counterclockwise around T;

(2) for $i \leqslant n$, the edge e'_i of λ' coincides with the edge e_i of λ;

(3) e'_{n+1}, e'_{n+2}, e'_{n+3} are the "new" edges of λ' that are not edges of λ, and each e'_{n+j} is opposite e'_j in T as in Figure 4(b), in the sense that no face of λ' contains both e'_{n+j} and e'_j.

We assume that we are given a λ'–enhancement $\xi' \colon \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$ for the homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. By restriction, ξ' defines a λ–enhancement $\xi \colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ for r.

We want to compare the two irreducible representations $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$ respectively associated to the enhanced homomorphisms (r, ξ) and (r, ξ') by Proposition 8. For this, we first construct a natural algebra homomorphism $\mathcal{Z}^\omega(\lambda) \to \mathcal{Z}^\omega(\lambda')$.

Let $\Phi \colon \mathcal{Z}^\omega(\lambda) \to \mathcal{Z}^\omega(\lambda')$ be the linear map defined by the property that

$$\Phi\big([Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}]\big) = \big[{Z'_1}^{k_1} {Z'_2}^{k_2} \dots {Z'_n}^{k_n} {Z'_{n+1}}^{\frac{k_2+k_3-k_1}{2}} {Z'_{n+2}}^{\frac{k_1+k_3-k_2}{2}} {Z'_{n+3}}^{\frac{k_1+k_2-k_3}{2}}\big]$$

for every monomial $[Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}] \in \mathcal{Z}^\omega(\lambda)$ (where $[\ \ ]$ denotes the Weyl quantum ordering). By definition of the Weyl quantum ordering, this is

equivalent to the property that

$$\Phi\big(Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}\big) = \omega^{k_1k_3 - k_1k_2 - k_2k_3} Z_1'^{k_1} Z_2'^{k_2} \dots Z_n'^{k_n} \big[Z_{n+1}'^{\frac{k_2+k_3-k_1}{2}} Z_{n+2}''^{\frac{k_1+k_3-k_2}{2}} Z_{n+3}'^{\frac{k_1+k_2-k_3}{2}}\big],$$

which is a little easier to handle since this second formula does not require us to consider the skew-commutativity properties of the Z_i and Z_i' with $1 \leqslant i \leqslant n$. The definition may look rather mysterious at this point, but will become clearer with the proof of Lemma 20 below.

Note that the definition makes sense because, if the exponents $(k_1, k_2, \dots, k_n)$ satisfy the parity condition required for $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}$ to be in the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$, the parity of the exponents

$$(k_1, k_2, \dots, k_n, \tfrac{k_2+k_3-k_1}{2}, \tfrac{k_1+k_3-k_2}{2}, \tfrac{k_1+k_2-k_3}{2})$$

also guarantees that $Z_1'^{k_1} Z_2'^{k_2} \dots Z_n'^{k_n} Z_{n+1}'^{\frac{k_2+k_3-k_1}{2}} Z_{n+2}'^{\frac{k_1+k_3-k_2}{2}} Z_{n+3}'^{\frac{k_1+k_2-k_3}{2}}$ belongs to $\mathcal{Z}^\omega(\lambda')$.

Lemma 20. *The map $\Phi\colon \mathcal{Z}^\omega(\lambda) \to \mathcal{Z}^\omega(\lambda')$ is an algebra homomorphism.*

Proof. This is a simple consequence of the description, given in [10, §2.2], of the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ in terms of the Thurston intersection form on a train track τ_λ associated to λ. The train track $\tau_\lambda \subset S$ is defined by the property that, on each face of λ, it consists of three edges as in Figure 5(a). In particular, there is a one-to-one correspondence between the switches of τ_λ and the edges e_i of λ.

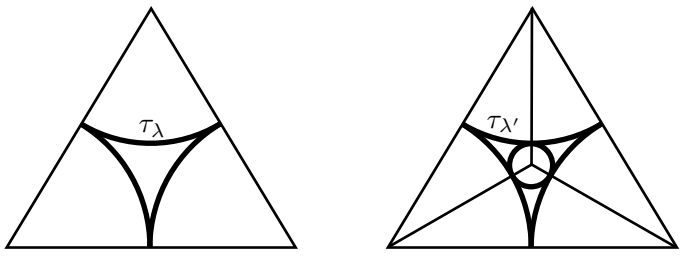


Figure 5. The train tracks τ_λ and $\tau_{\lambda'}$

In [10, §2], we interpret the monomials $[Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}] \in \mathcal{Z}^\omega(\lambda)$ as integer edge weight systems for τ_λ satisfying the usual switch conditions. Namely, at the switch of τ_λ located on the edge e_i of λ, the weights of the

two edges of τ_λ incoming on one side of that switch are required to add up to the same number k_i as the weights of the two edges outgoing on the other side. The exponents $(k_1, k_2, \dots, k_n)$ of $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}$ satisfy the parity condition required for $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}$ to be in the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ if and only if they are associated in this way to an integer edge weight system for τ_λ; in addition, the edge weight system is then uniquely determined.

This enables us to identify the set $\mathcal{W}(\tau_\lambda; \mathbb{Z})$ of integer edge weight systems for τ_λ to the set of exponent n–tuples $\mathbf{k} = (k_1, k_2, \dots, k_n)$ satisfying the required parity condition, and therefore to the set of Weyl quantum ordered monomials $[Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}] \in \mathcal{Z}^\omega(\lambda)$.

The set $\mathcal{W}(\tau_\lambda; \mathbb{Z})$ of edge weight systems for a train track carries a natural bilinear form, the *Thurston intersection form*, which provides an antisymmetric bilinear form

$$\Omega \colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \times \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{Z}.$$

Lemma 10 of [10] then describes the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ by the property that

$$[Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}]\, [Z_1^{k'_1} Z_2^{k'_2} \dots Z_n^{k'_n}] = \omega^{2\Omega(\mathbf{k}, \mathbf{k}')} [Z_1^{k_1 + k'_1} Z_2^{k_2 + k'_2} \dots Z_n^{k_n + k'_n}]$$

for every $\mathbf{k} = (k_1, k_2, \dots, k_n)$ and $\mathbf{k}' = (k'_1, k'_2, \dots, k'_n) \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$.

The key observation is now that there is a natural embedding $\tau_\lambda \to \tau_{\lambda'}$, identifying τ_λ to the complement in $\tau_{\lambda'}$ of the three edges that are adjacent to the central vertex $v_0 \in T$. This embedding provides a map $\varphi \colon W(\tau_\lambda; \mathbb{Z}) \to \mathcal{W}(\tau_{\lambda'}; \mathbb{Z})$, which is expressed in terms of switch weights as

$$\varphi(k_1, k_2, \dots, k_n) = \big(k_1, k_2, \dots, k_n, \tfrac{k_2 + k_3 - k_1}{2}, \tfrac{k_1 + k_3 - k_2}{2}, \tfrac{k_1 + k_2 - k_3}{2}\big).$$

for every $(k_1, k_2, \dots, k_n) \in \mathcal{W}(\lambda)$. As a consequence, identifying each element $(k_1, k_2, \dots, k_n) \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$ to the corresponding monomial $[Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}] \in \mathcal{Z}^\omega(\lambda)$, the map $\Phi \colon \mathcal{Z}^\omega(\lambda) \to \mathcal{Z}^\omega(\lambda')$ is the unique linear extension of φ.

Because $\varphi \colon W(\tau_\lambda; \mathbb{Z}) \to \mathcal{W}(\tau_{\lambda'}; \mathbb{Z})$ is induced by the embedding $\tau_\lambda \to \tau_{\lambda'}$, the classical homological interpretation of the Thurston intersection form as a homological intersection number in an orientation covering (see for instance Lemma 28 of [10]) shows that φ sends the Thurston form of τ_λ to the Thurston form of $\tau_{\lambda'}$. From the description of the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ and $\mathcal{Z}^\omega(\lambda')$ in terms of Thurston intersection forms, it follows that Φ is an algebra homomorphism. □

Now that we know that Φ is an algebra homomorphism, we can consider the composition

$$\mu_{\lambda'} \circ \Phi \colon \mathcal{Z}^{\omega}(\lambda) \to \mathrm{End}(E_{\lambda'})$$

of Φ with the representation $\mu_{\lambda'} \colon \mathcal{Z}^{\omega}(\lambda') \to \mathrm{End}(E_{\lambda'})$ associated by Proposition 8 to the λ'–enhancement ξ'.

Recall that v_0 is the vertex of λ' that is not a vertex of λ, namely the one that was added in the interior of the face T of λ.

Lemma 21. *The image $\mu_{\lambda'} \circ \Phi\big(\mathcal{Z}^{\omega}(\lambda)\big) \subset \mathrm{End}(E_{\lambda'})$ respects the off-diagonal kernel $F'_{v_0} \subset E_{\lambda'}$ of the vertex $v_0 \in V_{\lambda'} - V_{\lambda}$.*

Proof. By the exponent parity condition defining $\mathcal{Z}^{\omega}(\lambda)$, each monomial of $\mathcal{Z}^{\omega}(\lambda)$ is a product of constants, monomials of the form $Z_1 Z_2 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n}$, $Z_1 Z_3 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n}$, $Z_2 Z_3 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n} \in \mathcal{Z}^{\omega}(\lambda')$ and their inverses. It therefore suffices to show that the image of each of these elementary monomials respects F'_{v_0}.

Consider for instance $Z = Z_1 Z_2 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n}$. Its image under Φ is

$$\Phi(Z) = \Phi\big(Z_1 Z_2 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n}\big) = \omega^{-1} Z'_1 Z'_2 {Z'_4}^{k_4} {Z'_5}^{k_5} \dots {Z'_n}^{k_n} Z'_{n+3}.$$

As a consequence, $\Phi(Z)$ commutes with each of the elements ${Z'_{n+1}}^2$, ${Z'_{n+2}}^2$ and ${Z'_{n+3}}^2 \in \mathcal{Z}^{\omega}(\lambda')$ associated to the edges of λ' emanating from the vertex v_0. In particular, $\Phi(Z)$ commutes with the off-diagonal term $Q_{v_0} = 1 + \omega^{-4} {Z'_{n+1}}^2 + \omega^{-8} {Z'_{n+1}}^2 {Z'_{n+2}}^2$ of the vertex v_0. It follows that $\mu_{\lambda'} \circ \Phi(Z)$ respects the kernel F'_{v_0} of $\mu_{\lambda'}(Q_{v_0})$.

The same argument holds for the other two monomial types $Z_1 Z_3 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n}$ and
$Z_2 Z_3 Z_4^{k_4} Z_5^{k_5} \dots Z_n^{k_n}$, and proves the required result. □

The following result plays a critical role in our arguments. Its proof uses the non-quantum context, and in particular the off-diagonal equality of Lemma 11, in a crucial way.

Lemma 22. *The dimension of the off-diagonal kernel $F'_{v_0} \subset E_{\lambda'}$ is equal to the dimension $N^{3g+p_\lambda-3}$ of E_λ, where g is the genus of the surface S and p_λ is the number of vertices of the triangulation λ.*

Proof. By construction of the representation $\mu_{\lambda'}$ by Proposition 8, $\dim E_{\lambda'} = N^{3g+p_{\lambda'}-3}$. Since $p_\lambda = p_{\lambda'} - 1$, it therefore suffices to show that F'_{v_0} has dimension $\frac{1}{N} \dim E_{\lambda'}$.

Consider the off-diagonal term

$$Q_{v_0} = 1 + \omega^{-4} Z'^2_{n+1} + \omega^{-8} Z'^2_{n+1} Z'^2_{n+2}.$$

Because Z'^2_{n+1} ω^8–commutes with $Z'^2_{n+1}Z'^2_{n+2}$ and because $\omega^8 = A^{-4}$ is a primitive N–root of unity (we here use the fact that N is odd), the Quantum Binomial Formula (see for instance [24, §IV.2]) gives that

$$(Q_{v_0} - 1)^N = \left(\omega^{-4} Z'^2_{n+1} + \omega^{-8} Z'^2_{n+1} Z'^2_{n+2}\right)^N = Z'^{2N}_{n+1} + Z'^{2N}_{n+1} Z'^{2N}_{n+2}.$$

Applying $\mu_{\lambda'}$ then gives that

$$\begin{aligned}\mu_{\lambda'}(Q_{v_0} - 1)^N &= \mu_{\lambda'}(Z'^{2N}_{n+1}) + \mu_{\lambda'}(Z'^{2N}_{n+1} Z'^{2N}_{n+2}) \\ &= x'_{n+1}\,\mathrm{Id}_{E_{\lambda'}} + x'_{n+1}x'_{n+2}\mathrm{Id}_{E_{\lambda'}} = -\mathrm{Id}_{E_{\lambda'}},\end{aligned}$$

where the $x'_i \in \mathbb{C}^*$ are the crossratio weights associated by the enhancement ξ' to the edges e'_i of λ', and where the last equality comes from Lemma 11.

It follows that $\mu_{\lambda'}(Q_{v_0} - 1) \in \mathrm{End}(E_{\lambda'})$ is diagonalizable, and that its eigenvalues are N–roots of -1, namely are all of the form $-\omega^{8k}$ with $k \in \mathbb{Z}$.

Now consider the element $Z'^2_2 \in \mathcal{Z}^\omega(\lambda')$ associated to the edge e'_2 of λ'. Since $Z'^2_2(Q_{v_0} - 1) = \omega^{-8}(Q_{v_0} - 1)Z'^2_2$, the linear isomorphism $\mu_{\lambda'}(Z'^2_2) \in \mathrm{End}(E_{\lambda'})$ sends the $(-\omega^{8k})$–eigenspace of $\mu_{\lambda'}(Q_{v_0} - 1)$ to the $(-\omega^{8k+8})$–eigenspace. It follows that all numbers $-\omega^{8k}$ occur as eigenvalues of $\mu_{\lambda'}(Q_{v_0} - 1)$, and that the corresponding eigenspaces all have the same dimension. Since there are N such eigenspaces, their dimension is $\frac{1}{N}\dim E_{\lambda'}$.

In particular, $F'_{v_0} = \ker \mu_{\lambda'}(Q_{v_0})$ has dimension $\frac{1}{N}\dim E_{\lambda'}$, since it is the (-1)–eigenspace of $\mu_{\lambda'}(Q_{v_0} - 1)$. This concludes the proof. □

At this point, we have two representations $\mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(E_{\lambda'})$. The first one is our usual

$$\rho_{\lambda'} = \mu_{\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'} \colon \mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(E_{\lambda'}).$$

The second representation comes from the composition

$$\mathcal{S}^A(S_{\lambda'}) \xrightarrow{I} \mathcal{S}^A(S_\lambda) \xrightarrow{\mathrm{Tr}^\omega_\lambda} \mathcal{Z}^\omega(\lambda) \xrightarrow{\Phi} \mathcal{Z}^\omega(\lambda') \xrightarrow{\mu_{\lambda'}} \mathrm{End}(E_{\lambda'}),$$

where the first homomorphism $I \colon \mathcal{S}^A(S_{\lambda'}) \to \mathcal{S}^A(S_\lambda)$ is induced by the inclusion map $S_{\lambda'} \to S_\lambda$. This gives a new representation

$$\widehat{\rho}_{\lambda'} = \mu_{\lambda'} \circ \Phi \circ \mathrm{Tr}^\omega_\lambda \circ I \colon \mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(E_{\lambda'}).$$

Note that Lemma 20 is here required to guarantee that $\widehat{\rho}_{\lambda'}$ is an algebra homomorphism.

The images of these two representations respect the off-diagonal kernel $F'_{v_0} \subset E_{\lambda'}$, by Lemma 21 for $\widehat{\rho}_{\lambda'}$ and by Proposition 15 for $\rho_{\lambda'}$. Actually, because λ' is not necessarily combinatorial, we need to refer here to the strengthened version of Proposition 15 provided by Remark 16.

As a consequence, $\rho_{\lambda'}$ and $\widehat{\rho}_{\lambda'}$ induce two representations $\mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(F'_{v_0})$. We now show that these induced representations coincide.

Lemma 23. *The two representations $\rho_{\lambda'}$, $\widehat{\rho}_{\lambda'} \colon \mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(E_{\lambda'})$ above are such that*

$$\rho_{\lambda'}\big([K]\big)_{|F'_{v_0}} = \widehat{\rho}_{\lambda'}\big([K]\big)_{|F'_{v_0}}$$

for every framed link $K \subset S_{\lambda'} \times [0,1]$.

Proof. As in the proof of Proposition 15, we can arrange that the projection of K to S meets the face T of λ along a family of arcs a_1, a_2, …, a_t, where a_l meets only the edges $e_{\sigma_l(1)}$, $e_{\sigma_l(2)}$, $e'_{n+\sigma_l(3)}$ of λ', for a cyclic permutation σ_l of the indices $\{1,2,3\}$. Namely, the situation is as illustrated in Figure 2 with $u = 3$. Then, still as in the proof of Proposition 15, the quantum trace $\mathrm{Tr}^{\omega}_{\lambda'}\big([K]\big) \in \mathcal{Z}^{\omega}(\lambda')$ is a sum of terms of the form

$$A'_1 A'_2 \dots A'_t B' C' \in \mathcal{Z}^{\omega}(\lambda')$$

where each term correspond to a state for the boundary of $K \cap T \times [0,1]$, where C' is equal to $C' = [Z'^{-1}_{i'_1} Z'^{-1}_{i'_2} \dots Z'^{-1}_{i'_{w'}}]$ if K crosses the edges $e'_{i'_1}$, $e'_{i'_2}$, …, $e'_{i'_{w'}}$ of λ', where B' involves only generators Z'_i with $4 \leqslant i \leqslant n$ (corresponding to edges e'_i of λ' contained in the complement of T), and where A'_l is the contribution of the arc a_l and is of one of the following three types:

$$\begin{aligned} A'_l &= Z'^{2}_{\sigma_l(1)} Z'^{2}_{\sigma_l(2)} Z'^{2}_{n+\sigma_l(3)} \\ \text{or } A'_l &= \omega^{-4} Z'^{2}_{\sigma_l(2)} Z'^{2}_{n+\sigma_l(3)} + Z'^{2}_{\sigma_l(2)} \\ \text{or } A'_l &= 1. \end{aligned}$$

Similarly, $\mathrm{Tr}^{\omega}_{\lambda}\big([K]\big) \in \mathcal{Z}^{\omega}(\lambda)$ is a sum of terms

$$A_1 A_2 \dots A_t B C \in \mathcal{Z}^{\omega}(\lambda)$$

corresponding to states for the boundary of $K \cap T \times [0,1]$, where C is equal to $C = [Z^{-1}_{i_1} Z^{-1}_{i_2} \dots Z^{-1}_{i_w}]$ if K crosses the edges e_{i_1}, e_{i_2}, …, e_{i_w} of λ, where

B involves only generators Z_i with $4 \leqslant i \leqslant n$ (corresponding to edges e_i of λ contained in the complement of T), and where A_l is the contribution of the arc a_l and is of one of the following three types:

$$\begin{aligned} A_l &= \omega^4 Z^2_{\sigma_l(1)} Z^2_{\sigma_l(2)} \\ \text{or } A_l &= Z^2_{\sigma_l(2)} \\ \text{or } A_l &= 1. \end{aligned}$$

In order to show that $\rho_{\lambda'}\big([K]\big)_{|F'_{v_0}} = \mu_{\lambda'} \circ \mathrm{Tr}^{\omega}_{\lambda'}\big([K]\big)_{|F'_{v_0}}$ is equal to $\widehat{\rho}_{\lambda'}\big([K]\big)_{|F'_{v_0}} = \mu_{\lambda'} \circ \Phi \circ \mathrm{Tr}^{\omega}_{\lambda}\big([K]\big)_{|F'_{v_0}}$, we will compare the respective contributions to these quantities of the terms $A'_1 A'_2 \dots A'_t B' C'$ and $A_1 A_2 \dots A_t B C$ associated to the same state for the boundary of $K \cap T \times [0,1]$.

Let $A'_1 A'_2 \dots A'_t B' C' \in \mathcal{Z}^{\omega}(\lambda')$ and $A_1 A_2 \dots A_t B C \in \mathcal{Z}^{\omega}(\lambda)$ be the terms of $\mathrm{Tr}^{\omega}_{\lambda'}\big([K]\big)$ and $\mathrm{Tr}^{\omega}_{\lambda}\big([K]\big)$ respectively associated to the same state for the boundary of $K \cap T \times [0,1]$.

From the definition of the homomorphism $\Phi\colon \mathcal{Z}^{\omega}(\lambda) \to \mathcal{Z}^{\omega}(\lambda')$, is is immediate that $\Phi(B) = B'$. From the observation that each arc a_l contributes a monomial $Z'^{-1}_{\sigma_l(1)} Z'^{-1}_{\sigma_l(2)} Z'^{-1}_{n+\sigma_l(3)}$ to C' and a monomial $Z^{-1}_{\sigma_l(1)} Z^{-1}_{\sigma_l(2)}$ to C, it also easily follows from the definition of Φ that $\Phi(C) = C'$. As a consequence, $\mu_{\lambda'}(B')_{|F'_{v_0}} = \mu_{\lambda'} \circ \Phi(B)_{|F'_{v_0}}$ and $\mu_{\lambda'}(C')_{|F'_{v_0}} = \mu_{\lambda'} \circ \Phi(C)_{|F'_{v_0}}$

We need to compare each $\mu_{\lambda'}(A'_l)_{|F'_{v_0}}$ to the corresponding $\mu_{\lambda'} \circ \Phi(A_l)_{|F'_{v_0}}$.

In the case when $A'_l = 1$, then $A_l = 1$ and of course $\Phi(A_l) = \Phi(1) = 1 = A'_l$. In particular, $\mu_{\lambda'}(A'_l)_{|F'_{v_0}}$ is equal to the corresponding $\mu_{\lambda'} \circ \Phi(A_l)_{|F'_{v_0}}$ in this simple case.

The case where $A'_l = Z'^2_{\sigma_l(1)} Z'^2_{\sigma_l(2)} Z'^2_{n+\sigma_l(3)}$ is barely more complicated, as the corresponding term is $A_l = \omega^4 Z^2_{\sigma_l(1)} Z^2_{\sigma_l(2)}$. Indeed, $\Phi(A_l) = A'_l$, so that $\mu_{\lambda'}(A'_l)_{|F'_{v_0}} = \mu_{\lambda'} \circ \Phi(A_l)_{|F'_{v_0}}$ in this case as well.

The case where $A'_l = \omega^{-4} Z'^2_{\sigma_l(2)} Z'^2_{n+\sigma_l(3)} + Z'^2_{\sigma_l(2)}$ and $A_l = Z^2_{\sigma_l(2)}$ is much more interesting, because $\Phi(A_l) = Z'^2_{\sigma_l(2)} [Z'_{n+\sigma_l(1)} Z'^{-1}_{n+\sigma_l(2)} Z'_{n+\sigma_l(3)}]$ looks very different from A'_l. We can rewrite these terms as

$$A'_l = Z'^2_{\sigma_l(2)} (1 + \omega^{-4} Z'^2_{n+\sigma_l(3)}) = Z'^2_{\sigma_l(2)} Q'_{v_0} - \omega^{-8} Z'^2_{\sigma_l(2)} Z'^2_{n+\sigma_l(3)} Z'^2_{n+\sigma_l(1)}$$

and $\Phi(A_l) = \omega^{-4} H'^{-1}_{v_0} Z'^2_{\sigma_l(2)} Z'^2_{n+\sigma_l(3)} Z'^2_{n+\sigma_l(1)}$

for the off-diagonal term $Q'_{v_0} = 1 + \omega^{-4} Z'^2_{n+\sigma_l(3)} + \omega^{-8} Z'^2_{n+\sigma_l(3)} Z'^2_{n+\sigma_l(1)}$ and the central element $H'_{v_0} = [Z'_{n+\sigma_l(1)} Z'_{n+\sigma_l(2)} Z'_{n+\sigma_l(3)}]$ associated to the vertex v_0. Using the properties that $\mu_{\lambda'}(Q_{v_0})_{|F'_{v_0}} = 0$ and $\mu_{\lambda'}(H'_{v_0}) =$

$-\omega^4 \mathrm{Id}_{E_{\lambda'}}$, it follows that

$$\mu_{\lambda'}(A'_l)_{|F'_{v_0}} = \mu_{\lambda'}(-\omega^{-8} Z'^2_{\sigma_l(2)} Z'^2_{n+\sigma_l(3)} Z'^2_{n+\sigma_l(1)})_{|F'_{v_0}} = \mu_{\lambda'}\big(\Phi(A_l)\big)_{|F'_{v_0}}.$$

This proves that

$$\mu_{\lambda'}(A'_1 A'_2 \dots A'_t B' C')_{|F'_{v_0}} = \mu_{\lambda'} \circ \Phi(A_1 A_2 \dots A_t B C)_{|F'_{v_0}}$$

whenever the terms $A'_1 A'_2 \dots A'_t B' C' \in \mathcal{Z}^\omega(\lambda')$ of $\mathrm{Tr}^\omega_{\lambda'}\big([K]\big)$ and $A_1 A_2 \dots A_t B C \in \mathcal{Z}^\omega(\lambda)$ of $\mathrm{Tr}^\omega_{\lambda}\big([K]\big)$ correspond to the same state for the boundary of $K \cap T \times [0,1]$. As a consequence,

$$\rho_{\lambda'}\big([K]\big)_{|F'_{v_0}} = \mu_{\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'}\big([K]\big)_{|F'_{v_0}} = \mu_{\lambda'} \circ \Phi \circ \mathrm{Tr}^\omega_{\lambda}\big([K]\big)_{|F'_{v_0}} = \widehat{\rho}_{\lambda'}\big([K]\big)_{|F'_{v_0}}. \ \square$$

We now return to the irreducible representations $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$, respectively associated by Proposition 8 to the λ–enhancement $\xi \colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ and the λ'–enhancement $\xi' \colon \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$ for the homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Recall that ξ is just the restriction of ξ' to $\widetilde{V}_\lambda \subset \widetilde{V}_{\lambda'}$.

By Lemma 21, the composition $\mu_{\lambda'} \circ \Phi \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_{\lambda'})$ respects the off-diagonal kernel $F'_{v_0} \subset E_{\lambda'}$, and therefore induces a representation $\bar{\mu}_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(F'_{v_0})$ by the property that $\bar{\mu}_\lambda(Z) = \mu_{\lambda'} \circ \Phi(Z)_{F'_{v_0}}$ for every $Z \in \mathcal{Z}^\omega(\lambda)$.

Lemma 24. *After pre-composing μ_λ with the action of a sign-reversal symmetry of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if necessary, the representations $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\bar{\mu}_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(F'_{v_0})$ are isomorphic.*

Proof. By the uniqueness statement in Proposition 8, it suffices to check that $\bar{\mu}_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(F'_{v_0})$ satisfies the following four properties, which characterize μ_λ:

(1) $\dim F'_{v_0} = N^{3g+p_\lambda-3}$;

(2) $\bar{\mu}_\lambda(Z_i^{2N}) = x_i \,\mathrm{Id}_{F'_{v_0}}$ for each $i = 1, 2, \dots, n$, where x_i is the crossratio weight associated to the edge e_i of λ by the λ–enhancement ξ;

(3) $\bar{\mu}_\lambda(H_v) = -\omega^4 \,\mathrm{Id}_{F'_{v_0}}$ for every vertex v of λ;

(4) $T_N\Big(\bar{\mu}_\lambda \circ \mathrm{Tr}^\omega_\lambda\big([K]\big)\Big) = -\mathrm{Tr}\, r(K) \,\mathrm{Id}_{F'_{v_0}}$ for every framed knot $K \subset S_\lambda \times [0,1]$ whose projection to K has no crossing and whose framing is vertical.

The first property (1) is proved in Lemma 22.

For the second property (2) the case where $i > 3$, namely the case where the edge e_i is not a side of the face T that is being subdivided, is somewhat trivial. Indeed, $\bar\mu_\lambda(Z_i^{2N}) = \mu_{\lambda'} \circ \Phi(Z_i^{2N}) = \mu_{\lambda'}(Z_i'^{\,2N}) = x_i' \operatorname{Id}_{F'_{v_0}} = x_i \operatorname{Id}_{F'_{v_0}}$ as the enhancements ξ and ξ' associate the same crossratio weight $x_i = x_i'$ to the edge e_i.

The cases where $i \leqslant 3$ require a geometric argument. For instance,

$$\begin{aligned}\bar\mu_\lambda(Z_1^{2N}) &= \mu_{\lambda'} \circ \Phi(Z_1^{2N})_{|F'_{v_0}} = \mu_{\lambda'}\big([Z_1'^{\,2N} Z_{n+1}'^{\,-N} Z_{n+2}'^{\,N} Z_{n+3}'^{\,N}]\big)_{|F'_{v_0}} \\ &= \mu_{\lambda'}\big(Z_1'^{\,2N} Z_{n+1}'^{\,-2N} H_{v_0}'^{\,N}\big)_{|F'_{v_0}} = -x_1' x_{n+1}'^{\,-1} \operatorname{Id}_{F'_{v_0}}\end{aligned}$$

since $\mu_{\lambda'}(Z_1'^{\,2N}) = x_1' \operatorname{Id}_{E'_\lambda}$, $\mu_{\lambda'}(Z_{n+1}'^{\,2N}) = x_{n+1}' \operatorname{Id}_{E'_\lambda}$, $\mu_{\lambda'}(H'_{v_0}) = -\omega^4 \operatorname{Id}_{E'_\lambda}$ and $\omega^{4N} = 1$. Going back to the definition of the crossratio weights, a computation shows that $x_1' x_{n+1}'^{\,-1} = -x_1$. It follows that $\bar\mu_\lambda(Z_1^{2N}) = x_1 \operatorname{Id}_{F'_{v_0}}$, as required.

Identical computations show that $\bar\mu_\lambda(Z_2^{2N}) = x_2 \operatorname{Id}_{F'_{v_0}}$ and $\bar\mu_\lambda(Z_3^{2N}) = x_3 \operatorname{Id}_{F'_{v_0}}$, and complete the proof of (2) in all cases.

By definition of the homomorphism Φ, it sends the central element $H_v \in \mathcal{Z}^\omega(\lambda)$ associated to a vertex v of λ to the central element $H'_v \in \mathcal{Z}^\omega(\lambda')$ associated to v considered as a vertex of λ'. It follows that $\bar\mu_\lambda(H_v) = \mu_{\lambda'}(H'_v) = -\omega^4 \operatorname{Id}_{F'_{v_0}}$. This proves the third property (3).

Finally, (4) is a consequence of Lemma 23. Indeed, for every framed knot $K \subset S_{\lambda'} \times [0,1]$ whose projection to K has no crossing and whose framing is vertical

$$\begin{aligned}T_N\Big(\bar\mu_\lambda \circ \operatorname{Tr}_\lambda^\omega\big([K]\big)\Big) &= \bar\mu_\lambda \circ \operatorname{Tr}_\lambda^\omega\Big(T_N\big([K]\big)\Big) = \widehat\rho_{\lambda'}\Big(T_N\big([K]\big)\Big)_{|F'_{v_0}} \\ &= \rho_{\lambda'}\Big(T_N\big([K]\big)\Big)_{|F'_{v_0}} = \mu_{\lambda'} \circ \operatorname{Tr}_{\lambda'}^\omega\Big(T_N\big([K]\big)\Big)_{|F'_{v_0}} \\ &= T_N\Big(\mu_{\lambda'} \circ \operatorname{Tr}_{\lambda'}^\omega\big([K]\big)\Big)_{|F'_{v_0}} = -\operatorname{Tr} r(K) \operatorname{Id}_{F'_{v_0}}\end{aligned}$$

where the first and fifth equalities come from the fact that all maps involved are algebra homomorphisms, where the second equality comes from the definitions of the representations $\widehat\rho_{\lambda'}$ and $\bar\mu_\lambda$, where the third equality is provided by Lemma 23, and where the last equality is part of the definition of $\mu_{\lambda'}$ by Proposition 8.

This proves that the representation $\bar\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \operatorname{End}(F'_{v_0})$ satisfies the properties (1–4) listed above. By Proposition 8, it follows that $\bar\mu_\lambda$ is isomorphic to μ_λ. □

Our last step is to show that the isomorphism provided by Lemma 24 is compatible with off-diagonal kernels.

Lemma 25. *For a vertex v of the triangulation λ, let $F_v \subset E_\lambda$ and $F'_v \subset E_{\lambda'}$ be the respective off-diagonal kernels of v for the representations μ_λ and $\mu_{\lambda'}$, defined by considering v as a vertex of both λ and λ'. Then, the isomorphism $E_\lambda \to F'_{v_0}$ between the representations $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\bar{\mu}_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(F'_{v_0})$ provided by Lemma 24 sends F_v to $F'_v \cap F'_{v_0}$.*

Proof. We first simplify the situation a little. The representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ is only defined up to isomorphism and up to sign-reversal symmetry by Proposition 8. Modifying it by a sign-reversal symmetry if necessary, and by the isomorphism of Lemma 24, we can consequently assume that it is equal, not just isomorphic, to $\bar{\mu}_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(F'_{v_0})$. In particular, $E_\lambda = F'_{v_0}$, $\mu_\lambda = \bar{\mu}_\lambda$ and the isomorphism is the identity.

Note that the modification of μ_λ by a sign-reversal symmetry does not change the off-diagonal kernel F_v, as the off-diagonal term $Q_v \in \mathcal{Z}^\omega(\lambda)$ involves only even powers of the generators Z_i. We consequently have to show that $F_v = F'_v \cap F'_{v_0}$ once we have arranged that the representations μ_λ and $\bar{\mu}_\lambda$ coincide.

If v is not one of the vertices of the face T of λ that is being subdivided, the expression of the off-diagonal term $Q_v \in \mathcal{Z}^\omega(\lambda)$ involves only generators Z_i with $i > 3$, and $\Phi(Q_v) \in \mathcal{Z}^\omega(\lambda')$ is obtained from $Q_v \in \mathcal{Z}^\omega(\lambda)$ by replacing each generator Z_i by Z'_i. As a consequence, we can choose the off-diagonal term $Q'_v \in \mathcal{Z}^\omega(\lambda')$ to be equal to $\Phi(Q_v)$. Then,

$$\begin{aligned} F_v &= \ker \mu_\lambda(Q_v) = \ker \bar{\mu}_\lambda(Q_v) = \ker \mu_{\lambda'} \circ \Phi(Q_v)_{|F'_{v_0}} \\ &= \ker \mu_{\lambda'}(Q'_v)_{|F'_{v_0}} = \big(\ker \mu_{\lambda'}(Q'_v)\big) \cap F'_{v_0} = F'_v \cap F'_{v_0}. \end{aligned}$$

When v belongs to the face T of λ, this case splits into three subcases according to whether v corresponds to 1, 2 or 3 vertices of the triangle T. We restrict our discussion to the subcase where v corresponds to two vertices of T. The other two subcases are very similar.

Without loss of generality, we can choose the edge indexing of Figure 4 so that both endpoints of the edge e_1 are equal to the vertex v. Then, the off-diagonal term of v starting at the edge e_2 can be written as

$$Q_v = 1 + \omega^{-4} Z_2^2 + Z_2^2 Z_1^2 B + Z_2^2 Z_1^2 C Z_1^2 + Z_2^2 Z_1^2 C Z_1^2 Z_3^2 D$$

where B, C and D are polynomials in the variables Z_i^2 with $4 \leqslant i \leqslant n$; namely, these Z_i^2 correspond to edges of λ that are not contained in the face T.

Similarly, if we start from the edge e'_2, the off-diagonal term of the vertex v in $\mathcal{Z}^\omega(\lambda')$ is

$$\begin{aligned} Q'_v = 1 &+ \omega^{-4} {Z'_2}^2 + \omega^{-8} {Z'_2}^2 Z'^2_{n+3} + \omega^{-4} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 B' \\ &+ \omega^{-4} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 + \omega^{-8} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 Z'^2_{n+2} \\ &+ \omega^{-8} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 Z'^2_{n+2} {Z'_3}^2 D' \end{aligned}$$

where B', C', $D' \in \mathcal{Z}^\omega(\lambda')$ are respectively obtained from B, C, $D \in \mathcal{Z}^\omega(\lambda)$ by replacing each Z_i^2 with ${Z'_i}^2$.

By definition of the homomorphism $\Phi\colon \mathcal{Z}^\omega(\lambda) \to \mathcal{Z}^\omega(\lambda')$,

$$\begin{aligned} \Phi(Z_2^2) &= \omega^3 {Z'_2}^2 Z'_{n+1} Z'^{-1}_{n+2} Z'_{n+3} & \Phi(\omega^{-4} Z_2^2 Z_1^2) &= \omega^{-8} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 \\ \Phi(Z_1^2) &= \omega^3 {Z'_1}^2 Z'_{n+3} Z'^{-1}_{n+1} Z'_{n+2} & \Phi(\omega^{-4} Z_1^2 Z_3^2) &= \omega^{-8} {Z'_1}^2 Z'^2_{n+2} {Z'_3}^2 \\ \Phi(B) &= B' \qquad \Phi(C) = C' & \Phi(D) &= D'. \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(Q_v) = 1 &+ \omega^{-1} {Z'_2}^2 Z'_{n+1} Z'^{-1}_{n+2} Z'_{n+3} + \omega^{-4} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 B' \\ &+ \omega^{-1} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 Z'_{n+3} Z'^{-1}_{n+1} Z'_{n+2} \\ &+ \omega^{-8} {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 Z'^2_{n+2} {Z'_3}^2 D'. \end{aligned}$$

The above expressions of Q'_v and $\Phi(Q_v)$ share several terms, and their difference can therefore be expressed as

$$\begin{aligned} Q'_v - \Phi(Q_v) &= {Z'_2}^2 (\omega^{-4} + \omega^{-8} Z'^2_{n+3} - \omega^{-1} Z'_{n+1} Z'^{-1}_{n+2} Z'_{n+3}) \\ &\quad + {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 (\omega^{-4} + \omega^{-8} Z'^2_{n+2} - \omega^{-1} Z'_{n+3} Z'^{-1}_{n+1} Z'_{n+2}) \\ &= {Z'_2}^2 Z'^{-2}_{n+2} (\omega^{-4} Z'^2_{n+2} + \omega^{-8} Z'^2_{n+2} Z'^2_{n+3} - \omega^{-3} Z'_{n+2} Z'_{n+1} Z'_{n+3}) \\ &\quad + {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 Z'^{-2}_{n+1} (\omega^{-4} Z'^2_{n+1} + \omega^{-8} Z'^2_{n+1} Z'^2_{n+2} \\ &\qquad\qquad - \omega^{-3} Z'_{n+1} Z'_{n+3} Z'_{n+2}) \\ &= {Z'_2}^2 Z'^{-2}_{n+2} (Q'_{v_0} - 1 - \omega^{-4} H'_{v_0}) \\ &\qquad + {Z'_2}^2 Z'^2_{n+3} {Z'_1}^2 C' {Z'_1}^2 Z'^{-2}_{n+1} (Q''_{v_0} - 1 - \omega^{-4} H'_{v_0}) \end{aligned}$$

for

$$\begin{aligned} Q'_{v_0} &= 1 + \omega^{-4} Z'^2_{n+2} + \omega^{-8} Z'^2_{n+2} Z'^2_{n+3} \\ Q''_{v_0} &= 1 + \omega^{-4} Z'^2_{n+1} + \omega^{-8} Z'^2_{n+1} Z'^2_{n+2} \\ H'_{v_0} &= \omega^{-1} Z'_{n+2} Z'_{n+1} Z'_{n+3} = \omega^{-1} Z'_{n+1} Z'_{n+3} Z'_{n+2}. \end{aligned}$$

Note that Q'_{v_0} and $Q''_{v_0} \in \mathcal{Z}^\omega(\lambda')$ are two off-diagonal terms for the vertex v_0, corresponding to different indexings of the edges around this vertex. As a consequence, $\mu_{\lambda'}(Q'_{v_0})_{|F'_{v_0}} = \mu_{\lambda'}(Q''_{v_0})_{|F'_{v_0}} = 0$ while $\mu_{\lambda'}(H'_{v_0})_{|F'_{v_0}} = -\omega^4 \operatorname{Id}_{F'_{v_0}}$. It consequently follows from the above computation that

$$\mu_{\lambda'}(Q'_v)_{|F'_{v_0}} - \mu_{\lambda'} \circ \Phi(Q_v)_{|F'_{v_0}} = 0.$$

Then, as in the first case considered,

$$\begin{aligned} F_v &= \ker \mu_\lambda(Q_v) = \ker \bar{\mu}_\lambda(Q_v) = \ker \mu_{\lambda'} \circ \Phi(Q_v)_{F'_{v_0}} \\ &= \ker \mu_{\lambda'}(Q'_v)_{|F'_{v_0}} = \big(\ker \mu_{\lambda'}(Q'_v)\big) \cap F'_{v_0} = F'_v \cap F'_{v_0}. \end{aligned}$$

This concludes the proof of Lemma 25 when the vertex v corresponds to two vertices of the triangle T. The cases where it corresponds to one or three vertices of T are very similar, and we omit the corresponding proofs. □

We now gather the results of this section in the following statement, which we state in inductive form for later use in §5.3.

It is convenient to introduce some terminology. If the representation $\rho_\lambda \colon \mathcal{S}^A(S_\lambda) \to \operatorname{End}(E_\lambda)$ respects the total off-diagonal kernel $F_\lambda \subset E_\lambda$, we say that ρ_λ *induces a representation* $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \operatorname{End}(F_\lambda)$ if $\rho_\lambda\big([K]\big)_{|F_\lambda} = \rho_\lambda\big([K']\big)_{|F_\lambda}$ whenever the two framed links K, $K' \subset S_\lambda \times [0,1]$ are isotopic in $S \times [0,1]$.

For instance, when the triangulation λ is combinatorial, Proposition 15 shows that the representation $\rho_\lambda \colon \mathcal{S}^A(S_\lambda) \to \operatorname{End}(E_\lambda)$ respects F_λ, and Proposition 17 implies that ρ_λ induces a representation $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \operatorname{End}(F_\lambda)$.

Proposition 26. *Let λ' be obtained from the triangulation λ of the surface S by subdividing a face into three triangles as in Figure 4, let $\xi' \colon \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$ be a λ'–enhancement for the homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, and let $\xi \colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ be the λ–enhancement defined by restriction of ξ' to $\widetilde{V}_\lambda \subset \widetilde{V}_{\lambda'}$. Let $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \operatorname{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \operatorname{End}(E_{\lambda'})$ be the representations respectively associated to ξ and ξ' by Proposition 8.*

Suppose in addition that $\rho_{\lambda'} = \mu_{\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'} \colon \mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(E_{\lambda'})$ *respects the total off-diagonal kernel* $F_{\lambda'} \subset E_{\lambda'}$ *of* $\mu_{\lambda'}$*, and induces a representation* $\check\rho_{\lambda'} \colon \mathcal{S}^A(S) \to \mathrm{End}(F_{\lambda'})$ *as above. Then,* $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ *respects the total off-diagonal kernel* $F_\lambda \subset E_\lambda$ *of* μ_λ*, and induces a representation* $\check\rho_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$*. Moreover,* $\check\rho_\lambda$ *is isomorphic to* $\check\rho_{\lambda'}$ *after a possible pre-composition with the action of a sign-reversal symmetry of* $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ *on* $\mathcal{S}^A(S)$*.*

Proof. After pre-composition with the action of a sign-reversal symmetry of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ on $\mathcal{Z}^\omega(\lambda)$, Lemma 24 provides an isomorphism between the two representations $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\bar\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(F'_{v_0})$. Note that this modification of μ_λ does not change its total off-diagonal kernel F_λ, as a sign-reversal symmetry respects each off-diagonal term $Q_v \in \mathcal{Z}^\omega(\lambda)$.

As in the beginning of the proof of Lemma 25, we can arrange without loss of generality that this isomorphism is the identity, so that $\mu_\lambda = \bar\mu_\lambda$. Under these conditions, we want to prove that $F_\lambda = F_{\lambda'}$, and that $\check\rho_\lambda = \check\rho_{\lambda'}$.

We first compare the two total off-diagonal kernels F_λ and $F_{\lambda'}$. Lemma 25 shows that $F_v = F'_v \cap F'_{v_0}$ for every vertex v of λ. Then

$$F_\lambda = \bigcap_{v \in V_\lambda} F_v = \bigcap_{v \in V_\lambda} (F'_v \cap F'_{v_0}) = F'_{v_0} \cap \bigcap_{v \in V_\lambda} F'_v = \bigcap_{v' \in V_{\lambda'}} F'_{v'} = F_{\lambda'}.$$

Also, by our assumption that the isomorphism between μ_λ and $\bar\mu_\lambda$ is the identity, $E_\lambda = F'_{v_0} \subset E_{\lambda'}$. For every framed link $K \subset S_{\lambda'} \times [0,1]$, the fact that $\mu_\lambda = \bar\mu_\lambda$ and the definition of $\bar\mu_\lambda$ imply that

$$\rho_\lambda\big([K]\big) = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\big([K]\big) = \mu_{\lambda'} \circ \Phi \circ \mathrm{Tr}^\omega_\lambda\big([K]\big)_{|F'_{v_0}} = \rho_{\lambda'}\big([K]\big)_{|F'_{v_0}},$$

where the last equality is provided by Lemma 23, and where we use the same notation for the skeins $[K] \in \mathcal{S}^A(S_{\lambda'})$ and $[K] = I\big([K]\big) \in \mathcal{S}^A(S_\lambda)$.

In particular, since $\rho_{\lambda'}\big([K]\big) \in \mathrm{End}(E_{\lambda'})$ respects the total off-diagonal kernel $F_{\lambda'}$ by hypothesis, then $\rho_\lambda\big([K]\big) = \rho_{\lambda'}\big([K]\big)_{|F'_{v_0}}$ respects F_λ since $F_\lambda = F_{\lambda'} \subset F'_{v_0} \subset E_{\lambda'}$.

The same equality $\rho_\lambda\big([K]\big) = \rho_{\lambda'}\big([K]\big)_{|F'_{v_0}}$ shows that $\rho_\lambda\big([K]\big)_{|F_\lambda} = \rho_{\lambda'}\big([K]\big)_{|F_\lambda}$ since $F_\lambda \subset F'_{v_0}$. Therefore, if K, $K' \subset S_{\lambda'} \times [0,1]$ are isotopic in $S \times [0,1]$,

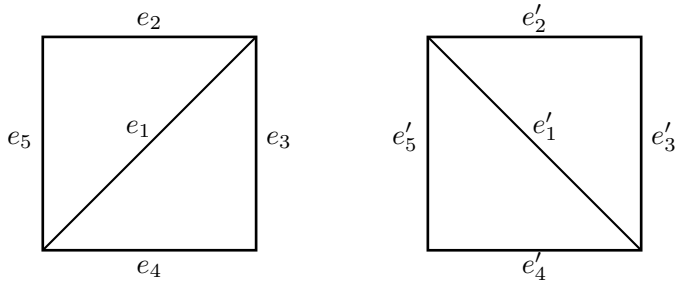
$$\rho_\lambda\big([K]\big)_{|F_\lambda} = \rho_{\lambda'}\big([K]\big)_{|F_\lambda} = \rho_{\lambda'}\big([K']\big)_{|F_\lambda} = \rho_\lambda\big([K']\big)_{|F_\lambda}$$

where the second equality comes from the hypothesis that $\rho_{\lambda'}$ induces a representation $\check{\rho}_{\lambda'}\colon \mathcal{S}^A(S) \to \mathrm{End}(F_{\lambda'})$ and the fact that $F_\lambda = F_{\lambda'}$. As a consequence, ρ_λ induces a representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$.

Finally, the properties that $F_\lambda = F_{\lambda'}$ and $\rho_\lambda\big([K]\big)_{|F_v} = \rho_{\lambda'}\big([K]\big)_{|F_v}$ show that $\check{\rho}_\lambda = \check{\rho}_{\lambda'}$. □

5.2. Diagonal exchanges. Diagonal exchanges (also called flips) are triangulation moves that occur in many different contexts. The arguments in this section are very similar to those used for earlier results in quantum Teichmüller theory [14, 13, 5, 28]. In particular, this section is conceptually and technically much simpler than the previous one.

Let λ and λ' be two triangulations of S which have the same vertices, and which differ only in one edge. We can index the edges of λ as e_1, e_2, …, e_n, and the edges of λ' as e'_1, e'_2, …, e'_n in such a way that $e_i = e'_i$ when $i \geqslant 2$. Then, the two faces of λ containing the edge e_1 form a "square" Q as in Figure 6, and e'_1 is the other diagonal of the square Q. In this case, we say that λ and λ' differ by a *diagonal exchange*.



The triangulation λ The triangulation λ'

Figure 6.

Let $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ be a λ–enhancement for the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$.

We assume that the following conditions are satisfied:

(1) The λ–enhancement ξ is also a λ'–enhancement for r. Since the triangulations λ and λ' have the same vertex sets $V_\lambda = V_{\lambda'} \subset S$, this just means that $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ assigns distinct values to the endpoints of an arbitrary lift of e'_1.

(2) The four sides of the square Q, formed by the two faces of λ containing the edge e_1, correspond to distinct edges of λ.

The second condition is not essential, but will simplify our exposition by dispensing us from the need to consider many cases, as was required in [5, 28]. Note that we are allowing identifications between the corners of Q which, for instance, could very well correspond to the same vertex of λ. The first condition is really critical.

This second condition enables us to index the edges of λ and λ' so that the sides of the square Q are $e_2 = e'_2$, $e_3 = e'_3$, $e_4 = e'_4$, $e_5 = e'_5$, as in Figure 6.

In [21, §6], Chris Hiatt constructs a natural isomorphism $\Theta_{\lambda\lambda'} \colon \widehat{\mathcal{Z}}^\omega(\lambda') \to \widehat{\mathcal{Z}}^\omega(\lambda)$ between the fraction algebras $\widehat{\mathcal{Z}}^\omega(\lambda')$ and $\widehat{\mathcal{Z}}^\omega(\lambda)$ of the balanced Chekhov-Fock algebras $\mathcal{Z}^\omega(\lambda')$ and $\mathcal{Z}^\omega(\lambda)$. The elements of $\widehat{\mathcal{Z}}^\omega(\lambda)$ are formal fractions UV^{-1} with U, $V \in \mathcal{Z}^\omega(\lambda)$ and $V \neq 0$, and are manipulated with the usual rules for fractions (except that the noncommutative context can greatly complicate computations, in particular when one needs to reduce two fractions to a common denominator in order to add them).

The homomorphism $\Theta_{\lambda\lambda'} \colon \widehat{\mathcal{Z}}^\omega(\lambda') \to \widehat{\mathcal{Z}}^\omega(\lambda)$ is defined as follows. Because of the exponent parity condition defining the balanced Chekhov-Fock algebra, every monomial Z' of $\mathcal{Z}^\omega(\lambda')$ can be uniquely written as

$$Z' = (Z'_1 Z'_2 Z'_4)^{k_1} (Z'_2 Z'_3)^{k_2} (Z'_4 Z'_5)^{k_3} {Z'_2}^{2k_4} {Z'_4}^{2k_5} B'$$

for exponents $k_i \in \mathbb{Z}$ and for a monomial B' involving only generators Z'_i with $i > 5$. Then $\Theta_{\lambda\lambda'}$ is uniquely determined by the property that

$$\Theta_{\lambda\lambda'}(Z') = \big(\omega^4 Z_1 Z_2 Z_4 + Z_1^{-1} Z_2 Z_4\big)^{k_1} (\omega Z_1 Z_2 Z_3)^{k_2} (\omega Z_1 Z_4 Z_5)^{k_3} \\ \big(Z_2^2 + \omega^4 Z_1^2 Z_2^2\big)^{k_4} \big(Z_4^2 + \omega^4 Z_1^2 Z_4^2\big)^{k_5} B$$

where B is obtained from B' by replacing each generator Z'_i with $i > 5$ by Z_i. The fact that this really defines an algebra homomorphism $\Theta_{\lambda\lambda'} \colon \widehat{\mathcal{Z}}^\omega(\lambda') \to \widehat{\mathcal{Z}}^\omega(\lambda)$ is proved in [21].

Lemma 27.

$$\Theta_{\lambda\lambda'}({Z'_1}^2) = Z_1^{-2} \qquad \Theta_{\lambda\lambda'}({Z'_2}^2) = \big(1 + \omega^4 Z_1^2\big) Z_2^2$$
$$\Theta_{\lambda\lambda'}({Z'_3}^2) = \big(1 + \omega^4 Z_1^{-2}\big)^{-1} Z_3^2 \qquad \Theta_{\lambda\lambda'}({Z'_4}^2) = \big(1 + \omega^4 Z_1^2\big) Z_4^2$$
$$\Theta_{\lambda\lambda'}({Z'_5}^2) = \big(1 + \omega^4 Z_1^{-2}\big)^{-1} Z_5^2 \quad \textit{and } \Theta_{\lambda\lambda'}({Z'_i}^2) = Z_i^2 \textit{ for every } i > 5.$$

Proof. This is a simple computation based on the formula defining $\Theta_{\lambda\lambda'}$: Use the property that ${Z'_1}^2 = \omega^{a+4} (Z'_1 Z'_2 Z'_4)^2 {Z'_2}^{-2} {Z'_4}^{-2}$, ${Z'_3}^2 = \omega^{b-6} (Z'_2 Z'_3)^2 {Z'_2}^{-2}$

and $Z_5'^2 = \omega^{c-6}(Z_4'Z_5')^2 Z_4'^{-2}$, where the integers a, b, $c \in \mathbb{Z}$ are determined by the faces of λ that are adjacent to several sides of the square Q and are not contained in Q (and contribute additional terms to the skew-commutativity relations between the Z_i', and between the Z_i). See also [21, §6], which explains that $\Theta_{\lambda\lambda'}$ was designed as a 'square root' of the Chekhov-Fock coordinate change of [14, 13, 5, 28]. □

Let $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$ be the representations associated to the enhanced character (r, ξ) by Proposition 8. We would like to consider the representation $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_\lambda)$. But this composition is not immediately defined, as $\Theta_{\lambda\lambda'}$ is valued in the fraction algebra $\widehat{\mathcal{Z}}^\omega(\lambda)$, whereas μ_λ is only defined on the Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda) \subset \widehat{\mathcal{Z}}^\omega(\lambda)$.

Lemma 28. *There is a well-defined representation $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_\lambda)$, defined as follows.*

(1) *For every $Z' \in \mathcal{Z}^\omega(\lambda')$, there exists U_1, V_1, U_2, $V_2 \in \mathcal{Z}^\omega(\lambda)$ such that*

$$\Theta_{\lambda\lambda'}(Z') = U_1 V_1^{-1} = V_2^{-1} U_2 \in \widehat{\mathcal{Z}}^\omega(\lambda)$$

and $\mu_\lambda(V_1)$ and $\mu_\lambda(V_2)$ are invertible in $\mathrm{End}(E_\lambda)$.

(2) *For every decomposition $\Theta_{\lambda\lambda'}(Z') = U_1 V_1^{-1} = V_2^{-1} U_2$ as above,*

$$\mu_\lambda \circ \Theta_{\lambda\lambda'}(Z') = \mu_\lambda(U_1)\mu_\lambda(V_1)^{-1} = \mu_\lambda(V_2)^{-1}\mu_\lambda(U_2) \in \mathrm{End}(E_\lambda).$$

Proof. When Z' is a monomial $(Z_1'Z_2'Z_4')^{k_1}(Z_2'Z_3')^{k_2}(Z_4'Z_5')^{k_3} Z_3'^{2k_4} Z_5'^{2k_5} B'$, the non-monomial terms occurring in the definition of $\Theta_{\lambda\lambda'}(Z')$ can be written as

$$\begin{aligned} \omega^4 Z_1 Z_2 Z_4 + Z_1^{-1} Z_2 Z_4 &= (1 + \omega^4 Z_1^2) Z_1^{-1} Z_2 Z_4 \\ Z_2^2 + \omega^4 Z_1^2 Z_2^2 &= (1 + \omega^4 Z_1^2) Z_2^2 \\ Z_4^2 + \omega^4 Z_1^2 Z_4^2 &= (1 + \omega^4 Z_1^2) Z_4^2 \end{aligned}$$

The skew commutativity properties then enable us to write $\Theta_{\lambda\lambda'}(Z') = U_1 V_1^{-1} = V_2^{-1} U_2$ for some U_1, V_1, U_2, $V_2 \in \mathcal{Z}^\omega(\lambda)$ where the denominators V_1 and V_2 are products of terms $1 + \omega^{4k} Z_1^2$ with $k \in \mathbb{Z}$.

The same holds for any $Z' \in \mathcal{Z}^\omega(\lambda')$ by decomposing Z' as a sum of monomials, applying the above argument to each monomial, and reducing to a common denominator. (The reduction to a common denominator is here trivial, because all denominators commute with each other.)

By definition of the representation μ_λ in Proposition 8, $\mu_\lambda(Z_1^2)^N = x_1 \operatorname{Id}_{E_\lambda}$ where x_1 is the crossratio weight associated to the edge e_1 of λ by the enhancement ξ. In particular, $\mu_\lambda(Z_1^2)$ is diagonalizable and its eigenvalues are N–roots of x_1. Also, x_1 is different from -1 because ξ sends the end points of each lift of the edge e'_1 to different points of $\mathbb{CP}^1$. Because N is odd, it follows that the eigenvalues of $\mu_\lambda(Z_1^2)$ are never of the form $-\omega^{-4k}$ with $k \in \mathbb{Z}$, and therefore that $\mu_\lambda(1 + \omega^{4k} Z_1^2)$ is invertible for each such k.

This proves that the image of each $Z' \in \mathcal{Z}^\omega(\lambda)$ under $\Theta_{\lambda\lambda'}$ can be decomposed as

$$\Theta_{\lambda\lambda'}(Z') = U_1 V_1^{-1} = V_2^{-1} U_2 \in \widehat{\mathcal{Z}}^\omega(\lambda)$$

for some U_1, V_1, U_2, $V_2 \in \mathcal{Z}^\omega(\lambda)$ with $\mu_\lambda(V_1)$ and $\mu_\lambda(V_2)$ invertible in $\operatorname{End}(E_\lambda)$.

An elementary algebraic manipulation shows that $\mu_\lambda(U_1)\mu_\lambda(V_1)^{-1}$ is equal to $\mu_\lambda(V_2)^{-1}\mu_\lambda(U_2)$ in $\operatorname{End}(E_\lambda)$, and that this endomorphism is independent of the above decomposition. We can therefore define a map $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \operatorname{End}(E_\lambda)$ by the property that

$$\mu_\lambda \circ \Theta_{\lambda\lambda'}(Z') = \mu_\lambda(U_1)\mu_\lambda(V_1)^{-1} = \mu_\lambda(V_2)^{-1}\mu_\lambda(U_2).$$

for every such $Z' \in \mathcal{Z}^\omega(\lambda')$.

The property that the map $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \operatorname{End}(E_\lambda)$ is an algebra homomorphism easily follows from a couple more easy algebraic manipulations. □

Lemma 29. *For a vertex v of the triangulations λ and λ', consider its associated central elements $H_v \in \mathcal{Z}^\omega(\lambda)$ and $H'_v \in \mathcal{Z}^\omega(\lambda')$. Then, the triangulation change homomorphism $\Theta_{\lambda\lambda'} \colon \widehat{\mathcal{Z}}^\omega(\lambda') \to \widehat{\mathcal{Z}}^\omega(\lambda)$ sends $H'_v \in \mathcal{Z}^\omega(\lambda')$ to $H_v \in \mathcal{Z}^\omega(\lambda)$.*

Proof. If we index the edges of λ' meeting v as $e_{i'_1}$, $e_{i'_2}$, …, $e_{i'_u}$ in counterclockwise order around v and if we suitably choose the starting point of this indexing, each corner of the square Q that is equal to v contributes a block $Z'_2 Z'_3$, $Z'_3 Z'_1 Z'_4$, $Z'_4 Z'_5$ or $Z'_5 Z'_1 Z'_2$ to the expression

$$H'_v = \omega^{-u+2} Z'_{i_1} Z'_{i_2} \dots Z'_{i_u}$$

provided by Lemma 12.

A computation using the formula defining $\Theta_{\lambda\lambda'}$ then shows that $\Theta_{\lambda\lambda'}(H'_v)$ is obtained from this expression by replacing each block $Z'_2 Z'_3$, $Z'_3 Z'_1 Z'_4$,

$Z'_4Z'_5$ or $Z'_5Z'_1Z'_2$ by $\omega^{-1}Z_2Z_1Z_3$, ωZ_3Z_4, $\omega^{-1}Z_4Z_1Z_5$ or ωZ_5Z_2, respectively, and by replacing each Z'_i with $i > 5$ by Z_i. For instance, for a block $Z'_3Z'_1Z'_4$,

$$Z'_3Z'_1Z'_4 = \omega^{a-2}(Z'_1Z'_2Z'_4)(Z'_2Z'_3){Z'_2}^{-2}$$

where $a \in \mathbb{Z}$ depends on the q–commutativity of Z'_2, Z'_3, Z'_4, namely on the faces of λ that are adjacent to several faces of the square Q and are not contained in Q (and is equal to 0 if there are no such faces). In the expression for $\Theta_{\lambda\lambda'}(H'_v)$, this block $\omega^{a-2}(Z'_1Z'_2Z'_4)(Z'_2Z'_3){Z'_2}^{-2}$ is then replaced by

$$\begin{aligned}\omega^{a-2}&(\omega^4 Z_1Z_2Z_4 + Z_1^{-1}Z_2Z_4)(\omega Z_1Z_2Z_3)(Z_2^2+\omega^4Z_1^2Z_2^2)^{-1}\\ &= \omega^{a-1}(\omega^4Z_1^2+1)Z_1^{-1}Z_2Z_4Z_1Z_2Z_3Z_2^{-2}(1+\omega^4Z_1^2)^{-1}\\ &= \omega^{a-1}Z_1^{-1}Z_2Z_4Z_1Z_2Z_3Z_2^{-2}\\ &= \omega Z_3Z_4.\end{aligned}$$

The computation for the other blocks $Z'_2Z'_3$, $Z'_3Z'_1Z'_4$, $Z'_4Z'_5$ or $Z'_5Z'_1Z'_2$ is very similar.

The result immediately follows from this computation, and from the application of Lemma 12 to H_v. □

Lemma 30. *After pre-composing $\mu_{\lambda'}$ with the action on $\mathcal{Z}^\omega(\lambda')$ of a sign-reversal symmetry of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if necessary, the representations $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$ are isomorphic.*

Proof. By Proposition 8, the representation $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$ is characterized up to isomorphism and sign-reversal symmetry by the following properties.

(1) The dimension of $E_{\lambda'}$ is equal to $N^{3g+p_{\lambda'}-3}$, where g is the genus of the surface S and where $p_{\lambda'}$ is the number of vertices of the triangulation λ';

(2) For every edge e'_i of λ', let $x'_i \in \mathbb{C}^*$ be the crossratio weight associated to e'_i by the enhancement ξ, and let Z'_i be the corresponding generator of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$. Then,

$$\mu_{\lambda'}({Z'_i}^{2N}) = x'_i \,\mathrm{Id}_{E_{\lambda'}}.$$

(3) For every vertex v of λ', with associated central element $H'_v \in \mathcal{Z}^\omega(\lambda')$,

$$\mu_{\lambda'}(H'_v) = -\omega^4 \,\mathrm{Id}_{E_{\lambda'}}.$$

(4) The representation $\rho_{\lambda'} = \mu_{\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'} \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ has classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, in the sense that

$$T_N\big(\rho_{\lambda'}([K])\big) = -\mathrm{Tr}\, r(K)\, \mathrm{Id}_{E_{\lambda'}}$$

for every knot $K \subset S_{\lambda'} \times [0,1]$ whose projection to $S_{\lambda'}$ has no crossing and whose framing is vertical.

It therefore suffices to show that the representation $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_\lambda)$ satisfies the same conditions.

The triangulations λ and λ' have the same vertex set, so that $p_\lambda = p_{\lambda'}$. The dimension of the space E_λ is equal to $N^{3g+p_\lambda-3} = N^{3g+p_{\lambda'}-3}$ by Proposition 8 applied to μ_λ, which proves the first condition.

The second condition is checked by several computations. The first elementary computation is that the crossratio weights x_i and x'_i respectively associated to the edges of λ and λ' by the enhancement ξ are related by the property that x'_i is equal to x_1^{-1} if $i = 1$, to $(1 + x_1)x_2$ if $i = 2$, to $x_1(1 + x_1)^{-1}x_3$ if $i = 3$, to $(1 + x_1)x_4$ if $i = 4$, to $x_1(1 + x_1)^{-1}x_5$ if $i = 5$, and to x_i if $i > 5$. See for instance [28, §2] or [5, §8].

Then, if $a \in \{0, -1\}$ is such that $Z'_2 Z'_3 = \omega^{2(a+1)} Z'_3 Z'_2$ (so that $a = -1$ only when there is a face of λ that contains the edges e'_2 and e'_3 and is not contained in the square Q),

$$\begin{aligned}
\mu_\lambda \circ \Theta_{\lambda\lambda'}({Z'_3}^{2N}) &= \mu_\lambda \circ \Theta_{\lambda\lambda'}\big(\omega^{-2N(2N+1)(a+1)}(Z'_2 Z'_3)^{2N} {Z'_2}^{-2N}\big) \\
&= \mu_\lambda\big(\omega^{-2N(2N+1)(a+1)}(\omega Z_1 Z_2 Z_3)^{2N}(Z_2^2 + \omega^4 Z_1^2 Z_2^2)^{-N}\big) \\
&= \mu_\lambda\big(Z_1^{2N} Z_2^{2N} Z_3^{2N}(Z_2^{2N} + Z_1^{2N} Z_2^{2N})^{-1}\big) \\
&= x_1 x_2 x_3 (x_2 + x_1 x_2)^{-1} \mathrm{Id}_{E_\lambda} = x'_3 \mathrm{Id}_{E_\lambda},
\end{aligned}$$

where the third equality uses the relation $Z_2^2(Z_1^2 Z_2^2) = \omega^4 (Z_1^2 Z_2^2) Z_2^2$, the Quantum Binomial Formula [24, §IV.2] and the fact that ω^4 is a primitive N–root of unity.

Similar computations show that $\mu_\lambda \circ \Theta_{\lambda\lambda'}({Z'_i}^{2N}) = x'_i \mathrm{Id}_{E_\lambda}$ for every i. See also [5, §§7–8]. This proves the second condition.

The third condition is an immediate consequence of Lemma 29.

Finally, the fourth condition is a consequence of the property, proved in Theorem 28 of [7], that $\Theta_{\lambda\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'} = \mathrm{Tr}^\omega_\lambda$. □

Because the triangulations λ and λ' have the same vertex sets $V_\lambda = V_{\lambda'}$, the associated punctured surfaces $S_\lambda = S - V_\lambda$ and $S_{\lambda'} = S - V_{\lambda'}$ are equal. As a consequence, the homomorphisms $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda$ and $\rho_{\lambda'} = \mu_{\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'}$

associated to the enhanced character (r, ξ) provide representations of the same skein algebra $\mathcal{S}^A(S_\lambda) = \mathcal{S}^A(S_{\lambda'})$.

Corollary 31. *After pre-composing $\mu_{\lambda'}$ with the action on $\mathcal{Z}^\omega(\lambda')$ of a sign-reversal symmetry of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if necessary, the representations $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ and $\rho_{\lambda'} = \mu_{\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'} \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ are isomorphic.*

Proof. This is an immediate consequence of Lemma 30 and of the fact, proved in Theorem 28 of [7], that $\Theta_{\lambda\lambda'} \circ \mathrm{Tr}^\omega_{\lambda'} = \mathrm{Tr}^\omega_\lambda$. □

Lemma 32. *Every isomorphism $\varphi \colon E_\lambda \to E_{\lambda'}$ between the representations $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$, as in Lemma 30, sends the off-diagonal kernel $F_v \subset E_\lambda$ of each vertex $v \in V_\lambda = V_{\lambda'}$ to the off-diagonal kernel $F'_v \subset E_{\lambda'}$.*

Proof. There are again several cases to consider according to which corners of the square Q correspond to the vertex v. We will give the proof in the case when v corresponds to two corners of Q, the one where e_2 and e_3 meet and the corner where e_5 and e_2 meet. The reader may return to Figure 6 to remember the indexing convention for the edges e_i.

As usual, index the edges of λ' around the vertex v as $e'_{i_1}, e'_{i_2}, \dots, e'_{i_u}$, in counterclockwise order around v. We can choose the starting point of the indexing at e'_2, so that $e'_{i_1} = e'_2$, $e'_{i_2} = e'_3$, $e'_{i_{s-1}} = e'_5$, $e'_{i_s} = e'_1$, and $e'_{i_{s+1}} = e'_2$ for some index s. To avoid having to worry about whether $s+1 = u$ or not, it is convenient to shift the indexing by 1 and to consider the off-diagonal element

$$Q'_v = 1 + \omega^{-4} Z'^2_{i_2} + \omega^{-8} Z'^2_{i_2} Z'^2_{i_3} + \dots + \omega^{-4u} Z'^2_{i_2} Z'^2_{i_3} \dots Z'^2_{i_u} \in \mathcal{Z}^\omega(\lambda').$$

Then,

$$\begin{aligned} Z'^2_2 Q'_v = Z'^2_{i_1} Q'_v &= Z'^2_{i_1} + \omega^{-4} Z'^2_{i_1} Z'^2_{i_2} + \omega^{-8} Z'^2_{i_1} Z'^2_{i_2} Z'^2_{i_3} + \dots + \omega^{-4u} Z'^2_{i_1} Z'^2_{i_2} Z'^2_{i_3} \dots Z'^2_{i_u} \\ &= Z'^2_2 + \omega^{-4} Z'^2_2 Z'^2_3 B' \\ &\quad + \omega^{-4(s-2)} Z'^2_2 Z'^2_3 Z'^2_{i_3} \dots Z'^2_{i_{s-2}} (Z'^2_5 + \omega^{-4} Z'^2_5 Z'^2_1 + \omega^{-8} Z'^2_5 Z'^2_1 Z'^2_2) \\ &\quad + \omega^{-4s} Z'^2_2 Z'^2_3 Z'^2_{i_3} \dots Z'^2_{i_{s-2}} Z'^2_5 Z'^2_1 Z'^2_2 C' \end{aligned}$$

where B', $C' \in \mathcal{Z}^\omega(\lambda')$ are polynomials in the variables Z'^2_i with $i > 5$, corresponding to edges of λ' that are not contained in the square Q.

Similarly, we can index the edges of λ counterclockwise around v as e_{j_1}, e_{j_2}, …, e_{j_u}, in such a way that $e_{j_1} = e_2$, $e_{j_2} = e_1$, $e_{j_3} = e_3$, $e_{i_s} = e_5$, and $e_{i_{s+1}} = e_2$. Then,

$$\begin{aligned} Z_2^2 Q_v = Z_{j_1}^2 Q_v = Z_2^2 &+ \omega^{-4} Z_2^2 Z_1^2 + \omega^{-8} Z_2^2 Z_1^2 Z_3^2 B \\ &+ \omega^{-4(s-1)} Z_2^2 Z_1^2 Z_3^2 Z_{j_4}^2 \dots Z_{j_{s-1}}^2 (Z_5^2 + \omega^{-4} Z_5^2 Z_2^2) \\ &+ \omega^{-4s} Z_2^2 Z_1^2 Z_3^2 Z_{j_4}^2 \dots Z_{j_{s-1}}^2 Z_5^2 Z_2^2 C \end{aligned}$$

where B, $C \in \mathcal{Z}^\omega(\lambda)$ are respectively obtained from B', $C' \in \mathcal{Z}^\omega(\lambda')$ by replacing each term $Z_i'^2$ (with $i > 5$) with the corresponding Z_i^2.

The computations of Lemma 27 show that

$$\begin{aligned} &\Theta_{\lambda\lambda'}(Z_2'^2) = Z_2^2 + \omega^{-4} Z_2^2 Z_1^2 && \Theta_{\lambda\lambda'}(Z_2'^2 Z_3'^2) = \omega^{-4} Z_2^2 Z_1^2 Z_3^2 \\ &\Theta_{\lambda\lambda'}(Z_5'^2 + \omega^{-4} Z_5'^2 Z_1'^2) = Z_5^2 && \Theta_{\lambda\lambda'}(Z_5'^2 Z_1'^2 Z_2'^2) = \omega^4 Z_5^2 Z_2^2, \end{aligned}$$

while $\Theta_{\lambda\lambda'}(B') = B$ and $\Theta_{\lambda\lambda'}(C') = C$ since $\Theta_{\lambda\lambda'}(Z_i'^2) = Z_i^2$ whenever $i > 5$. It follows that $\Theta_{\lambda\lambda'}(Z_2'^2 Q_v') = Z_2^2 Q_v$.

As a consequence, the isomorphism φ sends the kernel of $\mu_\lambda(Z_2^2 Q_v) = \mu_\lambda \circ \Theta_{\lambda\lambda'}(Z_2'^2 Q_v')$ to the kernel of $\mu_{\lambda'}(Z_2'^2 Q_v')$. Since Z_2^2 is invertible in $\mathcal{Z}^\omega(\lambda)$, the kernel of $\mu_\lambda(Z_2^2 Q_v) = \mu_\lambda(Z_2^2) \circ \mu_\lambda(Q_v)$ is equal to the kernel of $\mu_\lambda(Q_v)$, namely to the off-diagonal kernel $F_v \subset E_\lambda$. Similarly, the kernel of $\mu_{\lambda'}(Z_2'^2 Q_v')$ is equal to the off-diagonal kernel $F_v' \subset E_{\lambda'}$.

This concludes the proof in the case when v corresponds to the corners of Q where e_2 and e_3 meet as well as to the corner where e_5 and e_2 meet. The other cases are essentially identical to this one. □

We summarize the discussion and results of this section in the following statement. Let the triangulations λ and λ' differ from each other by a diagonal exchange as in Figure 6. Recall that we are assuming that the sides of the square Q where the diagonal exchange takes place are distinct; however, the triangulations λ and λ' are not necessarily assumed to be combinatorial. Since λ and λ' have the same vertex set $V_\lambda = V_{\lambda'}$, the punctured surfaces $S_\lambda = S - V_\lambda$ and $S_{\lambda'} = S - V_{\lambda'}$ are equal.

Proposition 33. *Let the triangulations λ and λ' differ from each other by a diagonal exchange as in Figure 6, let $\xi\colon \widetilde{V}_\lambda = \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$ be simultaneously a λ– and a λ'–enhancement for the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, and let $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'}\colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$ be the representations associated to this data by Proposition 8.*

Suppose in addition that $\rho_{\lambda'} = \mu_{\lambda'} \circ \mathrm{Tr}^{\omega}_{\lambda'} \colon \mathcal{S}^A(S_{\lambda'}) \to \mathrm{End}(E_\lambda)$ *respects the total off-diagonal kernel* $F_{\lambda'} \subset E_{\lambda'}$ *of* $\mu_{\lambda'}$ *and induces a representation* $\check{\rho}_{\lambda'} \colon \mathcal{S}^A(S) \to \mathrm{End}(F_{\lambda'})$*, as defined above Proposition* 26*. Then,* $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^{\omega}_{\lambda} \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_{\lambda'})$ *respects the total off-diagonal kernel* $F_\lambda \subset E_\lambda$ *of* μ_λ *and induces a representation* $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$*. Moreover,* $\check{\rho}_\lambda$ *is isomorphic to* $\check{\rho}_{\lambda'}$ *after a possible pre-composition with the action of a sign-reversal symmetry of* $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ *on* $\mathcal{S}^A(S)$*.*

Proof. Lemma 30 provides an isomorphism $\varphi \colon E_\lambda \to E_{\lambda'}$ between the Chekhov-Fock algebra representations $\mu_\lambda \circ \Theta_{\lambda\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_\lambda)$ and $\mu_{\lambda'} \colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$. Also, Theorem 28 of [7] states that the quantum trace homomorphisms $\mathrm{Tr}^{\omega}_{\lambda} \colon \mathcal{S}^A(S_\lambda) \to \mathcal{Z}^\omega(\lambda)$ and $\mathrm{Tr}^{\omega}_{\lambda'} \colon \mathcal{S}^A(S_\lambda) \to \mathcal{Z}^\omega(\lambda')$ are compatible with the isomorphism $\Theta_{\lambda\lambda'} \colon \widehat{\mathcal{Z}}^\omega(\lambda') \to \widehat{\mathcal{Z}}^\omega(\lambda)$ in the sense that $\Theta_{\lambda\lambda'} \circ \mathrm{Tr}^{\omega}_{\lambda'} = \mathrm{Tr}^{\omega}_{\lambda}$. Therefore, $\varphi \colon E_\lambda \to E_{\lambda'}$ provides an isomorphism between the representations $\mu_\lambda \circ \Theta_{\lambda\lambda'} \circ \mathrm{Tr}^{\omega}_{\lambda'} = \mu_\lambda \circ \mathrm{Tr}^{\omega}_{\lambda} = \rho_\lambda$ and $\mu_{\lambda'} \circ \mathrm{Tr}^{\omega}_{\lambda'} = \rho_{\lambda'}$.

By Lemma 32, the isomorphism $\varphi \colon E_\lambda \to E_{\lambda'}$ sends the total off-diagonal kernel $F_\lambda = \bigcap_{v \in V_\lambda} F_v$ to the total off-diagonal kernel $F_{\lambda'} = \bigcap_{v \in V_\lambda} F'_v$. Since the representation $\rho_{\lambda'}$ respects $F_{\lambda'}$ by hypothesis, it follows that ρ_λ respects F_λ.

Finally, the property that $\rho_{\lambda'}$ induces a representation $\bar{\rho}_{\lambda'} \colon \mathcal{S}^A(S) \to \mathrm{End}(F_{\lambda'})$ means that $\rho_{\lambda'}\big([K]\big)_{|F_\lambda} = \rho_{\lambda'}\big([K']\big)_{|F_\lambda}$ whenever the two framed links K, $K' \subset S_\lambda \times [0,1]$ are isotopic in $S \times [0,1]$. The isomorphism φ again shows that the same property holds for ρ_λ. □

5.3. Constructing representations of the skein algebra of a closed surface using arbitrary triangulations.

Lemma 34. *Let* λ *be a triangulation of* S*. Then one can apply to* λ *a sequence of face subdivisions and diagonal exchanges, as in* §§5.1 *and* 5.2*, to obtain a new triangulation* λ' *that is combinatorial, in the sense that each edge of* λ' *has distinct endpoints and no two edges have the same endpoints.*

In addition, any λ*–enhancement* $\xi \colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ *for the group homomorphism* $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ *can be extended to a* λ'*–enhancement* $\xi' \colon \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$*. (Note that the vertex set* V_λ *of* λ *is contained in the vertex set* $V_{\lambda'}$*).*

Proof. By subdividing a few faces if necessary, we can arrange that any two faces of λ have at most one edge in common.

After this preliminary step, let λ'' be obtained from λ by subdividing each face, and let λ' be obtained by performing a diagonal exchange along

each edge of λ'' that is also an edge of λ; see Figure 7. All edges of a face of λ are distinct by our general convention for triangulations, and we had arranged at the beginning of the proof that any two faces have at most one edge in common. It easily follows that each edge of the resulting triangulation λ' joins, either a vertex of V_λ to a vertex of $V_{\lambda'} - V_\lambda$, or two distinct vertices of $V_{\lambda'} - V_\lambda$. As a consequence, λ' is combinatorial.

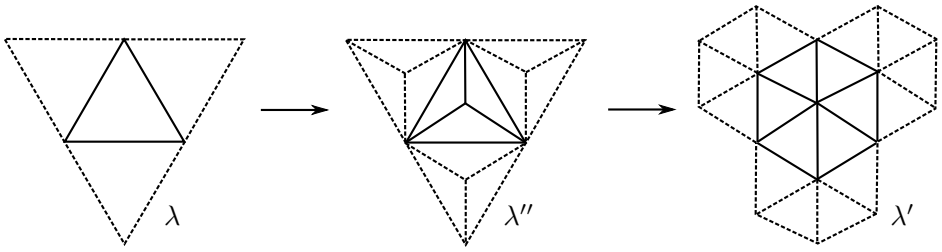


Figure 7.

Using the properties that λ' is combinatorial and that each edge of λ' touches at most one edge of V_λ, the inductive process of the proof of Lemma 7 then proves the second statement. □

Lemma 34 and the results of §§5.1 and 5.2 enable us to extend Proposition 15 and 17 to triangulations that are not necessarily combinatorial.

Theorem 35. *Given a triangulation λ of the surface S and a λ–enhancement $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ for the group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, let $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ be the irreducible representation associated to this data by Proposition 8. Then, the total off-diagonal kernel $F_\lambda \subset E_\lambda$ of μ_λ is invariant under the representation $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ constructed in §3.4, and ρ_λ induces a representation $\check\rho_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$.*

Proof. By Lemma 34, there exists a sequence of triangulations $\lambda = \lambda_0$, λ_1, …, λ_{n-1}, $\lambda_n = \lambda'$ such that λ' is combinatorial, and such that each λ_{i+1} is obtained from λ_i by a face subdivision or by a diagonal exchange. In addition, for every i, the λ–enhancement $\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ can be extended to a λ_i–enhancement $\xi_i\colon \widetilde{V}_{\lambda_i} \to \mathbb{CP}^1$ for r, in such a way that each ξ_{i+1} restricts to ξ_i on $\widetilde{V}_{\lambda_i}$.

Since λ' is combinatorial, the property sought holds for λ' by Propositions 15 and 17. Propositions 26 and 33 assert that the property will also hold for λ_i if it holds for λ_{i+1}. The result then follows by induction. □

Theorem 36. *The representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ provided by Theorem 35 has classical shadow equal to the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ represented by the group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, in the sense that*

$$T_N\big(\check{\rho}_\lambda([K])\big) = -\mathrm{Tr}\, r(K)\, \mathrm{Id}_{F_\lambda}$$

for every knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical.

Proof. We again use a sequence of triangulations $\lambda = \lambda_0$, λ_1, …, λ_{n-1}, $\lambda_n = \lambda'$ and λ_i–enhancements $\xi_i\colon \widetilde{V}_{\lambda_i} \to \mathbb{CP}^1$ such that λ' is combinatorial, each λ_{i+1} is obtained from λ_i by a face subdivision or by a diagonal exchange, and each ξ_{i+1} restricts to ξ_i on $\widetilde{V}_{\lambda_i}$.

Since λ' is combinatorial, Proposition 19 shows that $\check{\rho}_{\lambda'} = \check{\rho}_{\lambda_n}$ has classical shadow equal to $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. Propositions 26 and 33 then inductively show that the $\check{\rho}_{\lambda_i}$ are all isomorphic, and consequently also have classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. In particular, $\check{\rho}_\lambda = \check{\rho}_{\lambda_0}$ has classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. □

5.4. Independence of choices. We now prove that the construction of the representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ of Theorem 35 is very natural.

Lemma 37. *Let λ and λ' be two triangulations of S whose vertex sets are disjoint, and let $\xi : \widetilde{V}_\lambda \to \mathbb{CP}^1$ and $\xi' : \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$ be λ– and λ'–enhancements, respectively, for the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Then λ and λ' can be connected by a sequence of triangulations $\lambda = \lambda_0$, λ_1, …, λ_{n-1}, $\lambda_n = \lambda'$, each equipped with a λ_i–enhancement $\xi_i : V_{\lambda_i} \to \mathbb{CP}^1$ for r, such that:*

1. *each λ_{i+1} is obtained from λ_i by a face subdivision as introduced in §5.1, the inverse of a face subdivision, or a diagonal exchange as in §5.2;*
2. *$\xi_0 = \xi$ and $\xi_n = \xi'$;*
3. *for every i, ξ_i and ξ_{i+1} coincide on the intersection $\widetilde{V}_{\lambda_i} \cap \widetilde{V}_{\lambda_{i+1}}$.*

For the third condition, note that the vertex sets V_{λ_i} and $V_{\lambda_{i+1}}$ differ by at most one vertex, so that $\widetilde{V}_{\lambda_i}$ and $\widetilde{V}_{\lambda_{i+1}}$ differ by at most one $\pi_1(S)$–orbit.

Proof. By Lemma 34, we can assume without loss of generality that λ and λ' are combinatorial. The existence of the sequence $\lambda = \lambda_0$, λ_1, …, λ_{n-1}, $\lambda_n = \lambda'$ in this combinatorial setup is then the 2–dimensional case of Pachner's theorem [30, 31] (which of course predates the full generality of

Pachner's theorem by many decades). In addition, the λ_i provided by this statement are all combinatorial.

To construct the enhancements ξ_i, note that $\pi_1(S)$ acts on each $\widetilde{V}_{\lambda_i}$, and therefore on the union $\bigcup_{i=1}^n \widetilde{V}_{\lambda_i}$. Extend ξ and ξ' to an r–equivariant map $\xi''\colon \bigcup_{i=1}^n \widetilde{V}_{\lambda_i} \to \mathbb{CP}^1$ (this is where we use the fact that the vertex sets V_λ and $V_{\lambda'}$ are disjoint), orbit by orbit as in the proof of Lemma 7. In this construction, we can require that distinct $\pi_1(S)$–orbits in $\bigcup_{i=1}^n \widetilde{V}_{\lambda_i}$ have disjoint images in $\mathbb{CP}^1$, since we only need to avoid countably many values at each step. Then, the restriction ξ_i of ξ'' to $\widetilde{V}_{\lambda_i}$ is a λ_i–enhancement for r; indeed, because λ_i is combinatorial, the endpoints of each edge of its lift $\widetilde{\lambda}_i$ to the universal cover $\widetilde{S}$ belong to distinct $\pi_1(S)$–orbits, and in particular have distinct images under ξ''. □

Theorem 38. *Up to isomorphism and up to the action of a sign-reversal symmetry of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ on $\mathcal{S}^A(S)$, the representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ provided by Theorem 35 depends only on the group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, not on the triangulation λ or the λ–enhancement ξ used in the construction.*

Proof. Consider two triangulations λ and λ', with respective enhancements $\xi : \widetilde{V}_\lambda \to \mathbb{CP}^1$ and $\xi' : \widetilde{V}_{\lambda'} \to \mathbb{CP}^1$ for the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Modifying λ' by a small isotopy does not change the associated representations $\mu_{\lambda'}\colon \mathcal{Z}^\omega(\lambda') \to \mathrm{End}(E_{\lambda'})$ and $\check{\rho}_{\lambda'}\colon \mathcal{S}^A(S) \to \mathrm{End}(F_{\lambda'})$, so we can assume that the vertex sets V_λ and $V_{\lambda'}$ are disjoint. We can then consider the sequences of triangulations $\lambda = \lambda_0, \lambda_1, \dots, \lambda_{n-1}, \lambda_n = \lambda'$ and λ_i–enhancement $\xi_i : V_{\lambda_i} \to \mathbb{CP}^1$ provided by Lemma 37. Theorem 35 associates to each triangulation λ_i and λ_i–enhancement ξ_i a representation $\check{\rho}_{\lambda_i}\colon \mathcal{S}^A(S) \to \mathrm{End}(F_i)$. Propositions 26 and 33 show that each $\check{\rho}_{\lambda_i}\colon \mathcal{S}^A(S) \to \mathrm{End}(F_i)$ is isomorphic to $\check{\rho}_{\lambda_{i+1}}\colon \mathcal{S}^A(S) \to \mathrm{End}(F_{i+1})$ after possible composition with a sign-reversal symmetry. It follows that $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ is isomorphic to $\check{\rho}_{\lambda'}\colon \mathcal{S}^A(S) \to \mathrm{End}(F_{\lambda'})$ after possible composition with a sign-reversal symmetry. □

Remark 39. Conjugating the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ by an element $\theta \in \mathrm{SL}_2(\mathbb{C})$ also leaves the isomorphism class of the representation $\check{\rho}_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ invariant; indeed, the λ–enhancement $\theta\xi\colon \widetilde{V}_\lambda \to \mathbb{CP}^1$ for $\theta r \theta^{-1}$ induces the same edge weights x_i as ξ in the construction of $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ in Proposition 8. For irreducible homomorphisms $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$, being conjugate by an element of $\mathrm{SL}_2(\mathbb{C})$ is equivalent

to defining the same character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. However, for reducible homomorphisms r, we do not know if the representation $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ depends only on the induced character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or on subtler properties of the conjugacy class of $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$.

6. The dimension of the total off-diagonal kernel

We now have associated to each group homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ as representation $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ of the skein algebra $\mathcal{S}^A(S)$, with classical shadow equal to the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ represented by r. This construction is very natural as, up to isomorphism, $\check{\rho}_\lambda$ is independent of the triangulation λ and of the λ–enhancement ξ.

However, we still do not know that this representation is non-trivial, namely that the total off-diagonal kernel F_λ is non-trivial. This section is devoted to proving the non-triviality of F_λ, and to estimate its dimension.

Theorem 40. *Let $\check{\rho}_\lambda \colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ be the representation of the Kauffman bracket skein algebra of the closed oriented surface S associated to the group homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ by Theorem 35. Then, the dimension of the off-diagonal kernel F_λ is such that*

$$\dim F_\lambda \geqslant \begin{cases} N^{3(g-1)} & \text{if } g \geqslant 2 \\ N & \text{if } g = 1 \\ 1 & \text{if } g = 0 \end{cases}$$

where g is the genus of S. In addition, the above inequality is an equality when the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ represented by r is sufficiently generic, in the sense that it belongs to an explicit Zariski open dense subset of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

When the surface S is not the sphere, the proof of Theorem 40 is based on explicit computations for triangulations λ that have only one vertex. In particular, these triangulations cannot be combinatorial. This proof is the only reason why we struggled to include non-combinatorial triangulations in the previous sections.

6.1. Proof of Theorem 40 when the surface S has genus $g \geqslant 2$. Let λ be a triangulation of the surface S with only one vertex v. In particular, every edge of λ is a loop. Because S has genus $g \geqslant 2$, we can choose λ so that, in addition, there is an edge e_{i_0} of λ that separates the surface S into

two subsurfaces S_1 and S_2. Because of our conventions for triangulations, the three sides of each face of λ are distinct, and an Euler characteristic argument shows that each of S_1 and S_2 has positive genus.

We first consider the case where the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ admits a λ–enhancement ξ. By Lemma 7, this is equivalent to the property that $r(e_i) \neq \pm\mathrm{Id}$ for every edge e_i of the triangulation λ. Let $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ be the representation associated to the enhanced homomorphism (r, ξ) by Proposition 8, and consider the representation $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$ as in §3.4. Note that S_λ is here the punctured surface $S - \{v\}$, obtained by removing from S the vertex v of λ. In particular, the total off-diagonal kernel $F_\lambda \subset E_\lambda$ of μ_λ is equal to the off-diagonal kernel F_v of v.

Let $K_1 \subset S_1$ be the closed curve obtained by pushing the edge loop e_{i_0} inside of the subsurface S_1, and let $K_2 \subset S_2$ be similarly defined. In particular, K_1 and K_2 are both contained in the punctured surface $S_\lambda = S - \{v\}$. When endowed with the vertical framing, K_1 and K_2 define skeins $[K_1]$, $[K_2] \in \mathcal{S}^A(S_\lambda)$.

Lemma 41. *The off-diagonal kernel $F_\lambda = \ker \mu_\lambda(Q_v)$ is equal to the kernel of $\rho_\lambda\big([K_1]\big) - \rho_\lambda\big([K_2]\big)$.*

Proof. Since K_1 and K_2 are isotopic in $S \times [0,1]$, Theorem 35 shows that the restrictions $\rho_\lambda\big([K_1]\big)_{|F_\lambda} = \rho_\lambda\big([K_2]\big)_{|F_\lambda}$ coincide. The off-diagonal kernel F_λ is therefore contained in the kernel of $\rho_\lambda\big([K_1]\big) - \rho_\lambda\big([K_2]\big)$.

Because λ is not combinatorial, our proof of Theorem 35 relied on the "drill, baby, drill" strategy to reduce the problem to a combinatorial triangulation, where we could apply Proposition 17. To prove that F_λ is exactly equal to the kernel of $\rho_\lambda\big([K_1]\big) - \rho_\lambda\big([K_2]\big)$, we will here use a careful examination of an analogue of Proposition 17 for the non-combinatorial triangulation λ.

We first need to compute $\mathrm{Tr}^\omega_\lambda\big([K_1]\big)$ and $\mathrm{Tr}^\omega_\lambda\big([K_2]\big)$. For this, index the edges around the vertex v as e_{i_0}, e_{i_1}, e_{i_2}, $\dots e_{i_t}$, e_{i_0}, e_{j_1}, e_{j_2}, $\dots e_{j_u}$, counterclockwise in this order, so that all edges e_{i_k} are contained in the subsurface S_1 and all edges e_{j_k} are contained in S_2. See Figure 8(a).

The computation of $\mathrm{Tr}^\omega_\lambda\big([K_1]\big)$ given by [7] can be somewhat complicated, because the projection of K_1 to S cuts some edges of λ more than once; this usually introduces correction factors in bigon neighborhoods of these edges. A convenient way to avoid these correction factors is to isotop K_1 to a framed knot $K_1' \subset S_\lambda \times [0,1]$ whose projection to S coincides with

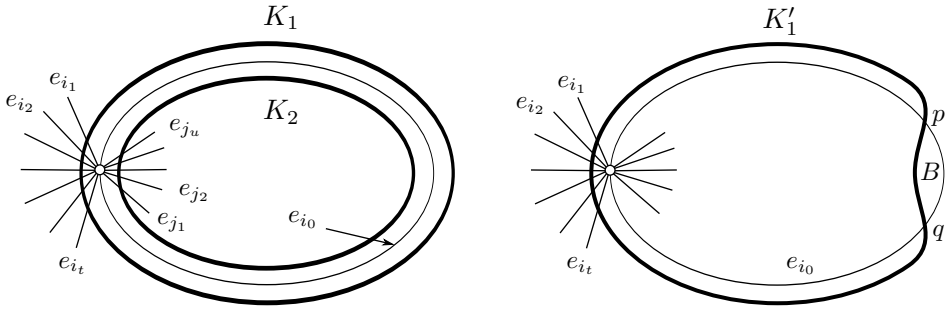


Figure 8.

the projection of K_1 for most of its length, except for a small interval that is pushed across the edge e_{i_0} to create a small bigon $B \subset S_2$ bounded by an arc in e_{i_0} and an arc in the projection of K_1'. In particular, the projection of K_1' to S cuts the edge e_{i_0} in two points p and q occuring in this order for the orientation of e_{i_0} coming from the boundary orientation of K_1. See Figure 8(b).

In addition, we can arrange that the elevation on $K_1 \subset S_\lambda \times [0,1]$ steadily increases as one goes around K_1 from p to q, crossing the preimage of the edges e_{i_1}, e_{i_2}, $\dots e_{i_t}$, and then steeply goes down from q to p along the bigon B to return to the starting point p.

Then, there is no need for correction terms, except for the contribution of the bigon B. More precisely, the construction of the quantum trace in [7] gives in this case:

$$\begin{aligned}\mathrm{Tr}^\omega_\lambda\big([K_1]\big) = \mathrm{Tr}^\omega_\lambda\big([K_1']\big) &= \omega^{-1}\sum_{k=0}^{t}\omega^{-t+1}Z_{i_0}Z_{i_1}\dots Z_{i_k}Z^{-1}_{i_{k+1}}Z^{-1}_{i_{k+2}}\dots Z^{-1}_{i_t}Z^{-1}_{i_0}\\ &= \omega^{-t}\sum_{k=0}^{t}Z_{i_0}Z_{i_1}\dots Z_{i_k}Z^{-1}_{i_{k+1}}Z^{-1}_{i_{k+2}}\dots Z^{-1}_{i_t}Z^{-1}_{i_0}\end{aligned}$$

where the factor ω^{-1} is the contribution of the bigon B.

We will use the computation of Weyl quantum orderings in Lemma 13 to rearrange this expression. By the first case of Lemma 13,

$$\begin{aligned}&[Z_{i_0}Z_{i_1}\dots Z_{i_k}] = \omega^{-k}Z_{i_0}Z_{i_1}\dots Z_{i_k},\\ &[Z_{i_0}Z_{i_1}\dots Z_{i_k}]^{-1} = [Z^{-1}_{i_0}Z^{-1}_{i_1}\dots Z^{-1}_{i_k}] = \omega^{-k}Z^{-1}_{i_0}Z^{-1}_{i_1}\dots Z^{-1}_{i_k},\\ \text{and } &[Z_{i_0}Z_{i_1}\dots Z_{i_k}]^2 = [Z^2_{i_0}Z^2_{i_1}\dots Z^2_{i_k}] = \omega^{-4k}Z^2_{i_0}Z^2_{i_1}\dots Z^2_{i_k}\end{aligned}$$

It follows that

$$\begin{aligned} Z_{i_0}Z_{i_1}\dots Z_{i_k} &= \omega^k[Z_{i_0}Z_{i_1}\dots Z_{i_k}] \\ &= \omega^k[Z_{i_0}Z_{i_1}\dots Z_{i_k}]^2[Z_{i_0}Z_{i_1}\dots Z_{i_k}]^{-1} \\ &= \omega^{-4k}Z_{i_0}^2Z_{i_1}^2\dots Z_{i_k}^2Z_{i_0}^{-1}Z_{i_1}^{-1}\dots Z_{i_k}^{-1}. \end{aligned}$$

This enables us to write

$$\begin{aligned} \mathrm{Tr}_\lambda^\omega\big([K_1]\big) &= \omega^{-t}\left(\sum_{k=0}^{t}\omega^{-4k}Z_{i_0}^2Z_{i_1}^2\dots Z_{i_k}^2\right)\left(Z_{i_0}^{-1}Z_{i_1}^{-1}\dots Z_{i_t}^{-1}Z_{i_0}^{-1}\right) \\ &= \left(\sum_{k=0}^{t}\omega^{-4k}Z_{i_0}^2Z_{i_1}^2\dots Z_{i_k}^2\right)Z_{i_0}^{-1}[Z_{i_1}^{-1}\dots Z_{i_t}^{-1}]Z_{i_0}^{-1}, \end{aligned}$$

where the second equality follows from an application of the third case of Lemma 13 to $[Z_{i_1}^{-1}\dots Z_{i_t}^{-1}] = \omega^{-t}Z_{i_1}^{-1}\dots Z_{i_t}^{-1}$.

Consider the term $[Z_{i_1}Z_{i_2}\dots Z_{i_t}]$. First of all, note that its exponents satisfy the parity condition required to belong to the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$. Also, $[Z_{i_1}Z_{i_2}\dots Z_{i_t}]$ commutes with Z_{i_0}; indeed, the only Z_{i_k} that do not commute with Z_{i_0} are Z_{i_1}, Z_{i_t}, and a pair of consecutive elements $Z_{i_{k_1}} = Z_{i_1}$ and $Z_{i_{k_1+1}} = Z_{i_t}$ corresponding to the third vertex of the face of λ that is contained in the subsurface S_1 and is adjacent to the edge e_{i_0}. It also commutes with all generators Z_{j_l} since the corresponding edges are located in the interior of the surface S_2.

Similarly $[Z_{j_1}Z_{j_2}\dots Z_{j_u}]$ is an element of $\mathcal{Z}^\omega(\lambda)$ that commutes with Z_{i_0} and with all Z_{i_k}.

In particular, $[Z_{i_1}Z_{i_2}\dots Z_{i_t}]$, $[Z_{j_1}Z_{j_2}\dots Z_{j_u}]$ and Z_{i_0} commute with each other, and the central element H_v associated to the vertex v is equal to

$$H_v = [Z_{i_0}Z_{i_1}\dots Z_{i_t}Z_{i_0}Z_{j_1}\dots Z_{j_u}] = Z_{i_0}^2[Z_{i_1}Z_{i_2}\dots Z_{i_t}][Z_{j_1}Z_{j_2}\dots Z_{j_u}].$$

Returning to $\mathrm{Tr}_\lambda^\omega\big([K_1]\big)$ and remembering that H_v and $[Z_{j_1}Z_{j_2}\dots Z_{j_u}]$

commute with all Z_{i_k}, we conclude that

$$\begin{aligned}\mathrm{Tr}_\lambda^\omega\big([K_1]\big) &= \Big(\sum_{k=0}^{t}\omega^{-4k}Z_{i_0}^2Z_{i_1}^2Z_{i_2}^2\dots Z_{i_k}^2\Big)Z_{i_0}^{-2}[Z_{i_1}Z_{i_2}\dots Z_{i_t}]^{-1}\\ &= \Big(\sum_{k=0}^{t}\omega^{-4k}Z_{i_0}^2Z_{i_1}^2Z_{i_2}^2\dots Z_{i_k}^2\Big)H_v^{-1}[Z_{j_1}Z_{j_2}\dots Z_{j_u}]\\ &= H_v^{-1}[Z_{j_1}Z_{j_2}\dots Z_{j_u}]\Big(\sum_{k=0}^{t}\omega^{-4k}Z_{i_0}^2Z_{i_1}^2Z_{i_2}^2\dots Z_{i_k}^2\Big)\\ &= [Z_{i_1}Z_{i_2}\dots Z_{i_t}]^{-1}Z_{i_0}^{-2}\Big(\sum_{k=0}^{t}\omega^{-4k}Z_{i_0}^2Z_{i_1}^2Z_{i_2}^2\dots Z_{i_k}^2\Big).\end{aligned}$$

The same arguments applied to the framed knot K_2 give

$$\mathrm{Tr}_\lambda^\omega\big([K_2]\big) = [Z_{j_1}Z_{j_2}\dots Z_{j_u}]^{-1}Z_{i_0}^{-2}\Big(\sum_{l=0}^{u}\omega^{-4l}Z_{i_0}^2Z_{j_1}^2Z_{j_2}^2\dots Z_{j_l}^2\Big).$$

In particular,

$$[Z_{i_1}Z_{i_2}\dots Z_{i_t}]\mathrm{Tr}_\lambda^\omega\big([K_1]\big) = 1+\sum_{k=1}^{t}\omega^{-4k}Z_{i_1}^2Z_{i_2}^2\dots Z_{i_k}^2$$

and

$$\begin{aligned}&[Z_{i_1}Z_{i_2}\dots Z_{i_t}]\mathrm{Tr}_\lambda^\omega\big([K_2]\big)\\ &\quad= Z_{i_0}^{-2}[Z_{j_1}Z_{j_2}\dots Z_{j_u}]^{-1}[Z_{i_1}Z_{i_2}\dots Z_{i_t}]\Big(\sum_{l=0}^{u}\omega^{-4l}Z_{i_0}^2Z_{j_1}^2Z_{j_2}^2\dots Z_{j_l}^2\Big)\\ &\quad= H_v^{-1}[Z_{i_1}Z_{i_2}\dots Z_{i_t}]^2\Big(\sum_{l=0}^{u}\omega^{-4l}Z_{i_0}^2Z_{j_1}^2Z_{j_2}^2\dots Z_{j_l}^2\Big)\\ &\quad= H_v^{-1}\Big(\sum_{l=0}^{u}\omega^{-4t-4l}Z_{i_1}^2Z_{i_2}^2\dots Z_{i_t}^2Z_{i_0}^2Z_{j_1}^2Z_{j_2}^2\dots Z_{j_l}^2\Big)\end{aligned}$$

by using again the third case of Lemma 13, in addition to the fact that $[Z_{i_1}Z_{i_2}\dots Z_{i_t}]$, $[Z_{j_1}Z_{j_2}\dots Z_{j_u}]$ and Z_{i_0} commute with each other.

This is beginning to look a lot like the off-diagonal term $Q_v \in \mathcal{Z}^\omega(\lambda)$ associated to the counterclockwise indexing of the edges of λ around v as $e_{i_1}, e_{i_2}, \dots, e_{i_t}, e_{i_0}, e_{j_1}, e_{j_2}, \dots, e_{j_u}, e_{i_0}$. Indeed, this off-diagonal term can

be written as

$$Q_v = 1 + \sum_{k=1}^{t} \omega^{-4k} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_k}^2 + \sum_{l=0}^{u} \omega^{-4t-4l+4} Z_{i_1}^2 Z_{i_2}^2 \dots Z_{i_t}^2 Z_{i_0}^2 Z_{j_1}^2 Z_{j_2}^2 \dots Z_{j_l}^2$$

If we apply the representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ and remember that $\mu_\lambda(H_v^{-1}) = -\omega^{-4}\mathrm{Id}_{E_\lambda}$, this proves that

$$\mu_\lambda\Big([Z_{i_1} Z_{i_2} \dots Z_{i_t}]\Big) \circ \left(\mu_\lambda\Big(\mathrm{Tr}_\lambda^\omega\big([K_1]\big)\Big) - \mu_\lambda\Big(\mathrm{Tr}_\lambda^\omega\big([K_1]\big)\Big)\right) = \mu_\lambda(Q_v).$$

Since $[Z_{i_1} Z_{i_2} \dots Z_{i_t}]$ is invertible in $\mathcal{Z}^\omega(\lambda)$, the linear map $\mu_\lambda\Big([Z_{i_1} Z_{i_2} \dots Z_{i_t}]\Big) \in \mathrm{End}(E_\lambda)$ is invertible. It follows that the kernel of $\mu_\lambda\Big(\mathrm{Tr}_\lambda^\omega\big([K_1]\big)\Big) - \mu_\lambda\Big(\mathrm{Tr}_\lambda^\omega\big([K_1]\big)\Big)$ is equal to the kernel of $\mu_\lambda(Q_v)$, namely to the off-diagonal kernel $F_\lambda = F_v \subset E_\lambda$.

Since $\rho_\lambda = \mu_\lambda \circ \mathrm{Tr}_\lambda^\omega$, this completes the proof of Lemma 41. □

We now consider the algebraic structure of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ and of the irreducible representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$.

Let λ_1 and λ_2 be the triangulations of the surfaces S_1 and S_2 respectively induced by the triangulation λ. Define the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda_1)$ as the subalgebra of $\mathcal{Z}^\omega(\lambda)$ generated by all monomials in the generators Z_{i_1}, Z_{i_2}, …, Z_{i_t} satisfying the appropriate exponent parity condition. Similarly, $\mathcal{Z}^\omega(\lambda_2) \subset \mathcal{Z}^\omega(\lambda)$ is generated by all monomials in the generators Z_{j_1}, Z_{j_2}, …, Z_{j_u} with the appropriate exponent parity condition.

Because each Z_{i_k} (with $k > 0$) commutes with each Z_{j_l} and because the element H_v is central, the inclusion maps $\mathcal{Z}^\omega(\lambda_1) \to \mathcal{Z}^\omega(\lambda)$, $\mathcal{Z}^\omega(\lambda_2) \to \mathcal{Z}^\omega(\lambda)$ and $\mathbb{C}[H_v^{\pm 1}] \to \mathcal{Z}^\omega(\lambda)$ define an algebra homomorphism

$$\mathcal{Z}^\omega(\lambda_1) \otimes \mathcal{Z}^\omega(\lambda_2) \otimes \mathbb{C}[H_v^{\pm 1}] \to \mathcal{Z}^\omega(\lambda).$$

Lemma 42. *The above homomorphism defines an isomorphism*

$$\mathcal{Z}^\omega(\lambda) \cong \mathcal{Z}^\omega(\lambda_1) \otimes \mathcal{Z}^\omega(\lambda_2) \otimes \mathbb{C}[H_v^{\pm 1}].$$

Proof. We need to show that the algebra homomorphism above is a linear isomorphism.

The key observation for this is the following. For every monomial

$$Z_{i_0}^n Z_{i_1}^{m_1} Z_{i_2}^{m_2} \dots Z_{i_t}^{m_t} Z_{j_1}^{n_1} Z_{j_2}^{n_2} \dots Z_{j_u}^{n_u}$$

of $\mathcal{Z}^\omega(\lambda)$, the exponent parity condition defining the balanced Chekhov-Fock algebra implies that the exponent n of Z_{i_0} is even, because the edge e_0 separates the surface S. As a consequence, such a monomial can be split as the product of a monomial of $\mathcal{Z}^\omega(\lambda_1)$, a monomial of $\mathcal{Z}^\omega(\lambda_2)$, and a power of $H_v = [Z_{i_0}^2 Z_{i_1} Z_{i_2} \dots Z_{i_t} Z_{j_1} Z_{j_2} \dots Z_{j_u}]$; in addition, this splitting is unique up to multiplicative constants.

Since these monomials $Z_{i_0}^n Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_t}^{n_t} Z_{j_1}^{m_1} Z_{j_2}^{m_2} \dots Z_{j_u}^{m_u}$ form a basis for $\mathcal{Z}^\omega(\lambda)$, while the monomials $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_t}^{n_t}$ form a basis for $\mathcal{Z}^\omega(\lambda_1)$ and the monomials $Z_{j_1}^{m_1} Z_{j_2}^{m_2} \dots Z_{j_u}^{m_u}$ form a basis for $\mathcal{Z}^\omega(\lambda_1)$, the result immediately follows. □

The structure theorem provided by Lemma 42 enables us to split the irreducible representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ as a tensor product. Indeed, by elementary linear algebra (see for instance [5, §4]) or a careful analysis of the proof of Proposition 8, there exists irreducible representations $\mu_1 \colon \mathcal{Z}^\omega(\lambda_1) \to \mathrm{End}(E_1)$, $\mu_2 \colon \mathcal{Z}^\omega(\lambda_2) \to \mathrm{End}(E_2)$, $\mu_0 \colon \mathbb{C}[H_v^{\pm 1}] \to \mathrm{End}(\mathbb{C})$, and an isomorphism $E_\lambda \cong E_1 \otimes E_2$ for which μ_λ corresponds to

$$\mu_1 \otimes \mu_2 \otimes \mu_0 \colon \mathcal{Z}^\omega(\lambda_1) \otimes \mathcal{Z}^\omega(\lambda_2) \otimes \mathbb{C}[H_v^{\pm 1}] \to \mathrm{End}(E_1 \otimes E_2 \otimes \mathbb{C}) = \mathrm{End}(E_\lambda).$$

In fact, since $\mu_\lambda(H_v) = -\omega^4 \mathrm{Id}_{E_\lambda}$, μ_0 is the unique algebra homomorphism such that $\mu_0(H_v) = -\omega^4 \mathrm{Id}_{\mathbb{C}}$.

We now return to the knots $K_1, K_2 \subset S \times [0,1]$. The knot K_1 is contained in $S_1 \times [0,1]$, so that the quantum trace $\mathrm{Tr}_\lambda^\omega\big([K_1]\big)$ belongs to the subalgebra $\mathcal{Z}^\omega(\lambda_1) \subset \mathcal{Z}^\omega(\lambda)$ corresponding to the subsurface S_1. In particular,

$$\rho_\lambda\big([K_1]\big) = \mu_\lambda\big(\mathrm{Tr}_\lambda^\omega\big([K_1]\big)\big) = \mu_1\big(\mathrm{Tr}_\lambda^\omega\big([K_1]\big)\big) \otimes \mathrm{Id}_{E_2}$$

in $\mathrm{End}(E_\lambda) = \mathrm{End}(E_1 \otimes E_2) = \mathrm{End}(E_1) \otimes \mathrm{End}(E_2)$. Similarly,

$$\rho_\lambda\big([K_2]\big) = \mu_\lambda\big(\mathrm{Tr}_\lambda^\omega\big([K_2]\big)\big) = \mathrm{Id}_{E_1} \otimes \mu_2\big(\mathrm{Tr}_\lambda^\omega\big([K_2]\big)\big).$$

By Lemma 41, the off-diagonal kernel $F_\lambda \subset E_\lambda$ is equal to the kernel of $\rho_\lambda\big([K_1]\big) - \rho_\lambda\big([K_2]\big)$. The following statement is then an immediate consequence of the above observations.

Lemma 43. *The off-diagonal kernel $F_\lambda \subset E_\lambda = E_1 \otimes E_2$ is equal to*

$$F_\lambda = \bigoplus_{a \in \mathbb{C}} E_1^{(a)} \otimes E_2^{(a)}$$

where, for each $a \in \mathbb{C}$, $E_i^{(a)} = \big\{w \in E_i, \mu_i\big(\mathrm{Tr}_\lambda^\omega\big([K_i]\big)\big)(w) = aw\big\}$ is the eigenspace of $\mu_i\big(\mathrm{Tr}_\lambda^\omega\big([K_i]\big)\big) \in \mathrm{End}(E_\lambda)$ corresponding to a if a is an eigenvalue of this endomorphism, and is 0 otherwise. □

This reduces the problem to the determination of the eigenvalues and eigenspaces of the homomorphisms $\mu_1\big(\mathrm{Tr}^\omega_\lambda([K_1])\big)$ and $\mu_2\big(\mathrm{Tr}^\omega_\lambda([K_2])\big)$.

Let us focus attention on the first homomorphism. The eigenvalues and eigenspaces of $\mu_1\big(\mathrm{Tr}^\omega_\lambda([K_1])\big)$ are easily deduced from those of $\rho_\lambda\big([K_1]\big) = \mu_1\big(\mathrm{Tr}^\omega_\lambda([K_1])\big) \otimes \mathrm{Id}_{E_2}$.

Lemma 44. *Suppose that the homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ is generic enough that $\mathrm{Tr}\, r(e_{i_0}) \neq \pm 2$. Then the homomorphism $\rho_\lambda\big([K_1]\big) \in \mathrm{End}(E_\lambda)$ is diagonalizable, its eigenvalues are the N distinct solutions of the equation $T_N(x) = -\mathrm{Tr}\, r(K_1)$, and all of its eigenspaces have the same dimension $\frac{1}{N} \dim E_\lambda$.*

Proof. We begin with a simple observation about the Chebyshev polynomial $T_N(x)$. If $y \neq \pm 2$, the equation $T_N(x) = y$ has N distinct solutions. Indeed, if we write y as $y = b + b^{-1}$ for some b, the solutions to the equation $T_N(x) = y$ are of the form $x = a + a^{-1}$ as a ranges over all N–roots of b. A little algebraic manipulation shows that these solutions are all distinct unless $b = \pm 1$, which is excluded by our hypothesis that $y \neq \pm 2$.

The fact that $\rho_\lambda\big([K_1]\big) \in \mathrm{End}(E_\lambda)$ is diagonalizable is then an immediate consequence of this observation and of the property, provided by Conclusion (4) of Proposition 8, that

$$T_N\big(\rho_\lambda([K_1])\big) = -\mathrm{Tr}\, r(K_1)\, \mathrm{Id}_{E_\lambda} = -\mathrm{Tr}\, r(e_{i_0})\, \mathrm{Id}_{E_\lambda}.$$

Indeed, this relation and our hypothesis that $\mathrm{Tr}\, r(e_{i_0}) \neq \pm 2$ show that the minimal polynomial of $\rho_\lambda\big([K_1]\big)$ has simple roots, so that $\rho_\lambda\big([K_1]\big)$ is diagonalizable. In addition, all eigenvalues of $\rho_\lambda\big([K_1]\big)$ are solutions of the equation $T_N(x) = -\mathrm{Tr}\, r(e_{i_0})$.

Showing that all solutions of the above equation occur as eigenvalues, and computing the dimension of the corresponding eigenspaces, will require a more elaborate argument.

By Complement 10, if we vary the enhanced homomorphism (r, ξ) over a small open subset in the space of such pairs, the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ can be chosen so that, for every monomial $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n} \in \mathcal{Z}^\omega(\lambda)$,

$$\mu_\lambda\big([Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}]\big) = u_1^{k_1} u_2^{k_2} \dots u_n^{k_n}\, A_{k_1 k_2 \dots k_n}$$

where each $u_i = \sqrt[2N]{x_i}$ is a local determination of the $2N$–root of the crossratio weight x_i defined by (r, ξ), and where the endomorphisms $A_{k_1 k_2 \dots k_n} \in \mathrm{End}(E_\lambda)$ are independent of (r, ξ).

We can now reverse the process and add more generality to it in order to give ourselves some flexibility. Consider the space $\mathcal{W} = (\mathbb{C} - \{0\})^n$ of weight systems $\sqcap$ assigning a weight u_i to each edge e_i of λ, with no specific relation between these edge weights. (The edge weights $u_i = \sqrt[2N]{x_i}$ associated to an enhanced homomorphism (r, ξ) that we considered so far were constrained by the relations of §4.1.) An edge weight system $\mathbf{u} \in \mathcal{W}$ determines a representation $\mu_\lambda^{\mathbf{u}} \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ by the property that

$$\mu_\lambda^{\mathbf{u}}(Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}) = u_1^{k_1} u_2^{k_2} \dots u_n^{k_n} A_{k_1 k_2 \dots k_n}$$

for every monomial $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n} \in \mathcal{Z}^\omega(\lambda)$, where the endomorphisms $A_{k_1 k_2 \dots k_n} \in \mathrm{End}(E_\lambda)$ are the ones occurring above.

This associates to $\mathbf{u} \in \mathcal{W}$ a representation $\rho_\lambda^{\mathbf{u}} = \mu_\lambda^{\mathbf{u}} \circ \mathrm{Tr}_\lambda^\omega \colon \mathcal{S}^A(S_\lambda) \to \mathrm{End}(E_\lambda)$, and the miraculous cancellations of [8] (as used in [10, §4]) provide a homomorphism $r^{\mathbf{u}} \colon \pi_1(S_\lambda) \to \mathrm{SL}_2(\mathbb{C})$ such that

$$T_N\big(\rho_\lambda^{\mathbf{u}}([K])\big) = T_N\big(\mu_\lambda^{\mathbf{u}} \circ \mathrm{Tr}_\lambda^\omega([K])\big) = -\mathrm{Tr}\, r^{\mathbf{u}}(K)\, \mathrm{Id}_{E_\lambda}$$

for every framed knot $K \subset S_\lambda \times [0,1]$ whose projection to S_λ has no crossing and whose framing is vertical.

In particular, if we return to the formula

$$\mathrm{Tr}_\lambda^\omega([K_1]) = \omega^{-t} \sum_{k=0}^{t} Z_{i_0} Z_{i_1} \dots Z_{i_k} Z_{i_{k+1}}^{-1} Z_{i_{k+2}}^{-1} \dots Z_{i_t}^{-1} Z_{i_0}^{-1}$$

used in the proof of Lemma 41, this gives

$$\rho_\lambda^{\mathbf{u}}([K_1]) = \mu_\lambda^{\mathbf{u}} \circ \mathrm{Tr}_\lambda^\omega([K_1]) = \sum_{k=0}^{t} u_{i_1} u_{i_2} \dots u_{i_k} u_{i_{k+1}}^{-1} u_{i_{k+2}}^{-1} \dots u_{i_t}^{-1} A_k$$

where $A_k \in \mathrm{End}(E_\lambda)$ is the product of a suitable term $A_{k_1 k_2 \dots k_n}$ with a power of ω. In particular, we will use the observation that for $k = t$

$$u_{i_1} u_{i_2} \dots u_{i_t} A_t = \mu_\lambda^{\mathbf{u}}\big(\omega^{-t} Z_{i_0} Z_{i_1} Z_{i_2} \dots Z_{i_t} Z_{i_0}^{-1}\big) = \mu_\lambda^{\mathbf{u}}\big([Z_{i_1} Z_{i_2} \dots Z_{i_t}]\big)$$

where the quantum ordering computation comes from Lemma 13 and the fact that Z_{i_0} commutes with $Z_{i_1} Z_{i_2} \dots Z_{i_t}$ in $\mathcal{T}^\omega(\lambda)$. Similarly, for $k = 0$,

$$u_{i_1}^{-1} u_{i_2}^{-1} \dots u_{i_t}^{-1} A_0 = \mu_\lambda^{\mathbf{u}}\big(\omega^{-t} Z_{i_0} Z_{i_1}^{-1} Z_{i_2}^{-1} \dots Z_{i_t}^{-1} Z_{i_0}^{-1}\big) = \mu_\lambda^{\mathbf{u}}\big([Z_{i_1}^{-1} Z_{i_2}^{-1} \dots Z_{i_t}^{-1}]\big),$$

from which it follows that $A_0 = A_t^{-1}$.

Also, by our determination of the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ in [10, §2.2] (and in particular Lemma 10 of that article), $[Z_{i_1}^N Z_{i_2}^N \dots Z_{i_t}^N]$ is central

in $\mathcal{Z}^\omega(\lambda)$ since $\omega^{4N} = 1$. By irreducibility of the representation $\mu_\lambda^{\mathbf{u}}$, there consequently exists a number $x \in \mathbb{C}^*$ such that $\mu_\lambda^{\mathbf{u}}\big([Z_{i_1}^N Z_{i_2}^N \dots Z_{i_t}^N] = x\,\mathrm{Id}_{E_\lambda}$. Taking the square of this equation and using the property that $\mu_\lambda^{\mathbf{u}}(Z_i^{2N}) = u_i^{2N}\,\mathrm{Id}_{E_\lambda}$, we conclude that $x = \pm u_{i_1}^N u_{i_2}^N \dots u_{i_t}^N$ and that

$$A_t^N = u_{i_1}^{-N} u_{i_2}^{-N} \dots u_{i_t}^{-N} \mu_\lambda^{\mathbf{u}}\big([Z_{i_1}^N Z_{i_2}^N \dots Z_{i_t}^N]\big) = \pm \mathrm{Id}_{E_\lambda}.$$

After these preliminary observations, we now return to the main line of our proof. If $\mathrm{Tr}\, r^{\mathbf{u}}(K_1) \neq \pm 2$, the same argument as before shows that $\rho_\lambda^{\mathbf{u}}\big([K_1]\big)$ is diagonalizable, and that all its eigenvalues are solutions of the equation $T_N(x) = -\mathrm{Tr}\, r^{\mathbf{u}}(K_1)$. Our strategy will be to determine the dimension of the eigenspaces of $\rho_\lambda^{\mathbf{u}}\big([K_1]\big)$ for one specific value of $\mathbf{u}$, and then to conclude by a connectedness property that these dimensions are the same for all $\mathbf{u} \in \mathcal{W}$ with $\mathrm{Tr}\, r^{\mathbf{u}}(K_1) \neq \pm 2$.

For this, we borrow two distinct ideas from Julien Roger [36, 35]. The first one is a result of [36, Lemma 19], where Roger considers monomials of $\mathcal{Z}^\omega(\lambda)$ associated to simple closed curves in the punctured surface S_λ. In the case of K_1, the corresponding monomial is $[Z_{i_1} Z_{i_2} \dots Z_{i_t}]$ and Roger produces a monomial $B \in \mathcal{T}^\omega(\lambda_1)$ such that $B[Z_{i_1} Z_{i_2} \dots Z_{i_t}] = \omega^4 [Z_{i_1} Z_{i_2} \dots Z_{i_t}] B$. Taking the square B^2 to make sure that we have an element of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$, applying the representation $\mu_\lambda^{\mathbf{u}} \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ associated to $\mathbf{u} \in \mathcal{W}$, and remembering that $\mu_\lambda^{\mathbf{u}}\big([Z_{i_1} Z_{i_2} \dots Z_{i_t}]\big) = u_{i_1} u_{i_2} \dots u_{i_t} A_t$, it follows that $\mu_\lambda^{\mathbf{u}}(B^2) A_t = \omega^8 A_t\, \mu_\lambda^{\mathbf{u}}(B^2)$. As a consequence, $\mu_\lambda^{\mathbf{u}}(B^2)$ sends the eigenspace of A_t corresponding to the eigenvalue a to the eigenspace corresponding to the eigenvalue $\omega^8 a$. Since we observed that $A_t^N = \pm \mathrm{Id}_{E_\lambda}$ and since ω^8 is a primitive N–root of unity (as $A = \omega^2$ is a primitive N–root of -1 and N is odd), it follows that the eigenvalues of A_t are all N–roots of ± 1, and that its eigenspaces have the same dimension $\frac{1}{N} \dim E_\lambda$.

We now follow another idea first exploited in [36, §2.2] and [35, Appendix B], except that the broader context of $\mathcal{W}$ enables us to use an explicit argument without having to rely on results of [36, 35]. To construct a suitable edge weight system $\widehat{\mathbf{u}} \in \mathcal{W}$, pick an arbitrary number $u_0 \in \mathbb{C} - \{0\}$ such that $u_0^{4N} \neq 1$, and another number $\varepsilon \in \mathbb{C} - \{0\}$ close to 0. Then define $\widehat{\mathbf{u}}$ to assign weight $\widehat{u}_{i_1} = u_0 \varepsilon$ to the edge e_{i_1}, weight $\widehat{u}_{i_t} = \varepsilon^{-1}$ to e_{i_t}, and weight $\widehat{u}_i = 1$ to all other e_i. Remember from the proof of Lemma 41 that there exists an index k_1 such that the edge $e_{i_{k_1}}$ is equal to e_{i_1} and $e_{i_{k_1+1}}$ is equal to e_{i_t} (as seen by consideration of the face of λ whose sides are e_0,

e_{i_1} and e_{i_t}). In particular, $\widehat{u}_{i_{k_1}} = u_0\varepsilon$ and $\widehat{u}_{i_{k_1+1}} = \varepsilon^{-1}$. It follows that

$$\widehat{u}_{i_1}\dots\widehat{u}_{i_k}\widehat{u}^{-1}_{i_{k+1}}\widehat{u}^{-1}_{i_{k+2}}\dots\widehat{u}^{-1}_{i_t} = \begin{cases} u_0^{-2} & \text{if } k=0 \\ \varepsilon^2 & \text{if } 0<k<k_1 \\ u_0^2\varepsilon^4 & \text{if } k=k_1 \\ u_0^2\varepsilon^2 & \text{if } k_1<k<t \\ u_0^2 & \text{if } k=t \end{cases}$$

Then, if ε is sufficiently small,

$$\rho^{\widehat{\mathbf{u}}}_\lambda\big([K_1]\big) = \sum_{k=0}^{t} \widehat{u}_{i_1}\dots\widehat{u}_{i_k}\widehat{u}^{-1}_{i_{k+1}}\widehat{u}^{-1}_{i_{k+2}}\dots\widehat{u}^{-1}_{i_t}A_k$$

is very close to

$$C = u_0^{-2}A_0 + u_0^2A_t = u_0^{-2}A_t^{-1} + u_0^2A_t \in \mathrm{End}(E_\lambda).$$

We proved that the eigenvalues of A_t are all N–roots of ±1, where $\pm$ is the sign such that $A_t^N = \pm\mathrm{Id}_{E_\lambda}$. Therefore, the eigenvalues of $C = u_0^{-2}A_t^{-1} + u_0^2A_t$ are the numbers $\pm(u_0^{-2}\omega^{-4k} + u_0^2\omega^{4k})$ with $k = 0, 1, \dots, N-1$. These N numbers are distinct by our assumption that $u_0^{4N} \neq 1$. The eigenspaces of C are the eigenspaces of A_t, which we proved all have the same dimension $\frac{1}{N}\dim E_\lambda$.

Therefore, for $\widehat{\mathbf{u}} \in \mathcal{W}$ associated to u_0 and ε as above, with ε small enough, the diagonalizable endomorphism $\rho^{\widehat{\mathbf{u}}}_\lambda\big([K_1]\big) \in \mathrm{End}(E_\lambda)$ has N distinct eigenvalues and the corresponding eigenspaces all have dimension $\frac{1}{N}\dim E_\lambda$.

In the space $\mathcal{W} \cong (\mathbb{C}^*)^n$ of edge weight systems for λ, the subspace $\mathcal{W}'$ consisting of those $\mathbf{u} \in \mathcal{W}$ with $\mathrm{Tr}\, r^{\mathbf{u}}(K_1) \neq \pm2$ is connected, since its complement has complex codimension 1. Note that the above point $\widehat{\mathbf{u}}$ belongs to $\mathcal{W}'$ since

$$\mathrm{Tr}\, r^{\widehat{\mathbf{u}}}(K_1)\,\mathrm{Id}_{E_\lambda} = T_N\big(\rho^{\widehat{\mathbf{u}}}_\lambda\big([K_1]\big)\big)$$

is very close to

$$T_N(u_0^{-2}A_t^{-1} + u_0^2A_t) = u_0^{-2N}A_t^{-N} + u_0^{2N}A_t^N = \pm(u_0^{-2N} + u_0^{2N})\mathrm{Id}_{E_\lambda}.$$

Therefore, the trace $\mathrm{Tr}\, r^{\widehat{\mathbf{u}}}(K_1)$ is very close to $\pm(u_0^{-2N} + u_0^{2N})$, and is consequently different from ±2 by our assumption that $u_0^{4N} \neq 1$.

We saw that, for all $\mathbf{u} \in \mathcal{W}'$, the endomorphism $\rho_\lambda^{\mathbf{u}}\big([K_1]\big)$ is diagonalizable and its eigenvalues are solutions of the equation $T_N(x) = -\operatorname{Tr} r^{\mathbf{u}}(K_1)$. Since the solutions of that equation are always simple for $\mathbf{u} \in \mathcal{W}'$, the dimension of the eigenspaces is a locally constant function of $\mathbf{u}$, and is therefore constant by connectedness of $\mathcal{W}'$. We found one point $\widehat{\mathbf{u}} \in \mathcal{W}$ such that all eigenspaces of $\rho_\lambda^{\widehat{\mathbf{u}}}\big([K_1]\big)$ have dimension $\frac{1}{N} \dim E_\lambda$. Therefore, the eigenspaces of $\rho_\lambda^{\mathbf{u}}\big([K_1]\big)$ have dimension $\frac{1}{N} \dim E_\lambda$ for every $\mathbf{u} \in \mathcal{W}'$.

In particular, this property holds for $\mathbf{u} \in \mathcal{W}'$ defined by the edge weights $u_i = \sqrt[2N]{x_i}$ associated to the enhanced homomorphism (r, ξ) satisfying the hypotheses of Lemma 44, which proves this statement. □

Proposition 45. *Let S be a closed oriented surface of genus $g \geqslant 2$, and consider a homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Suppose that there exists a triangulation λ_0 of S with exactly one vertex v and with at least one separating edge e_{i_0}, such that $r(e_i) \neq \pm\mathrm{Id}$ for every edge e_i of λ_0 and $\operatorname{Tr} r(e_{i_0}) \neq \pm 2$. Then, for every triangulation λ of S and every λ–enhancement ξ for r, the off-diagonal kernel F_λ associated to this data has dimension*

$$\dim F_\lambda = N^{3(g-1)}.$$

Proof. The hypotheses on r and λ_0 guarantee that, by Lemma 7, r admits at east one λ_0–enhancement ξ_0. By Theorem 38, the total off-diagonal kernel F_{λ_0} is isomorphic to F_λ and we can consequently restrict attention to the case where $\lambda = \lambda_0$ and $\xi = \xi_0$.

Namely, we assume that λ has exactly one vertex v, and that r admits a λ–enhancement ξ; in addition, an edge e_{i_0} of λ separates S into two subsurfaces S_1 and S_2, and $\operatorname{Tr} r(e_{i_0}) \neq \pm 2$. In this case and with the notations of this section, recall that we have split the representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ associated to the enhanced homomorphism (r, ξ) as a tensor product

$$\mu_1 \otimes \mu_2 \otimes \mu_0\colon \mathcal{Z}^\omega(\lambda_1) \otimes \mathcal{Z}^\omega(\lambda_2) \otimes \mathbb{C}[H_v^{\pm 1}] \to \mathrm{End}(E_1 \otimes E_2 \otimes \mathbb{C}) = \mathrm{End}(E_\lambda).$$

of three irreducible representations $\mu_1\colon \mathcal{Z}^\omega(\lambda_1) \to \mathrm{End}(E_1)$, $\mu_2\colon \mathcal{Z}^\omega(\lambda_2) \to \mathrm{End}(E_2)$, $\mu_0\colon \mathbb{C}[H_v^{\pm 1}] \to \mathrm{End}(\mathbb{C})$, for isomorphisms $\mathcal{Z}^\omega(\lambda) \cong \mathcal{Z}^\omega(\lambda_1) \otimes \mathcal{Z}^\omega(\lambda_2) \otimes \mathbb{C}[H_v^{\pm 1}]$ and $E_\lambda \cong E_1 \otimes E_2$. In addition, μ_0 is the unique algebra homomorphism such that $\mu_0(H_v) = -\omega^4 \mathrm{Id}_{\mathbb{C}}$.

By Lemma 43

$$\dim F_\lambda = \sum_{a \in \mathbb{C}} \dim E_1^{(a)} \dim E_2^{(a)}.$$

where $E_i^{(a)}$ is the eigenspace of $\mu_i\big(\mathrm{Tr}^\omega_{\lambda_i}\big([K_i]\big)\big)$ corresponding to the eigenvalue a (and is 0 if a is not an eigenvalue).

Since $\rho_\lambda\big([K_1]\big) = \mu_\lambda\big(\mathrm{Tr}^\omega_\lambda\big([K_1]\big)\big) = \mu_1\big(\mathrm{Tr}^\omega_\lambda\big([K_1]\big)\big) \otimes \mathrm{Id}_{E_2}$, the a–eigenspace of $\rho_\lambda\big([K_1]\big)$ is equal to the tensor product $E_1^{(a)} \otimes E_2$. By Lemma 44, we conclude that

$$\dim E_1^{(a)} \dim E_2 = \tfrac{1}{N} \dim E_\lambda = \tfrac{1}{N} \dim E_1 \dim E_2$$

when $T_N(a) = -\mathrm{Tr}\, r(K_1)$, and $E_1^{(a)} = 0$ otherwise. As a consequence, $\dim E_1^{(a)}$ is equal to $\frac{1}{N} \dim E_1$ if $T_N(a) = -\mathrm{Tr}\, r(K_1)$ and to 0 otherwise.

Similarly, the dimension $\dim E_2^{(a)}$ is equal to $\frac{1}{N} \dim E_2$ if $T_N(a) = -\mathrm{Tr}\, r(K_2)$ and to 0 otherwise.

By hypothesis, $\mathrm{Tr}\, r(K_1) = \mathrm{Tr}\, r(K_2) \neq \pm 2$, so there are exactly N values of a that have non-zero contributions to the sum

$$\dim F_\lambda = \sum_{a \in \mathbb{C}} \dim E_1^{(a)} \dim E_2^{(a)} = N(\tfrac{1}{N} \dim E_1)(\tfrac{1}{N} \dim E_2) = \tfrac{1}{N} \dim E_\lambda = N^{3g-3}$$

since $\dim E_\lambda = N^{3g-2}$ by Proposition 8. □

Remark 46. If we fix a triangulation λ_0 with exactly one vertex and with at least one separating edge e_{i_0}, the homomorphisms r satisfying the hypotheses of Proposition 45 form a Zariski open dense subset of the space of all group homomorphisms $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. Indeed, for a simple closed curve γ, many possible arguments show that the set of characters $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that $\mathrm{Tr}\, r(\gamma) \neq \pm 2$ is Zariski open and dense in $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$.

Proposition 47. *Let S be a closed oriented surface of genus $g \geqslant 2$. Then, for every homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and for every triangulation λ such that r admits a λ–enhancement ξ, the total off-diagonal kernel F_λ defined by this enhanced homomorphism (r, ξ) has dimension*

$$\dim F_\lambda \geqslant N^{3(g-1)}.$$

Proof. By Theorem 38, the dimension of F_λ depends only on the group homomorphism r, not on the triangulation λ or the enhancement ξ. In particular, we can assume without loss of generality that λ is combinatorial, so that every homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ admits an enhancement ξ by Lemma 7. If we locally vary r, the proof of Lemma 7 shows that we can choose the enhancement ξ so that it varies continuously with r. Then, the

representation $\mu_\lambda\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ associated to (r, ξ) by Proposition 8 depends continuously on r by Complement 10.

The total off-diagonal kernel F_λ is defined as an intersection of kernels $\ker \mu_\lambda(Q_v)$. Its dimension is therefore a lower semi-continuous function of the representation μ_λ, thus of the homomorphism r. Proposition 45 (see also Remark 46) asserts that the dimension of F_λ is equal to N^{3g-3} for generic homomorphisms $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. By lower semi-continuity, it follows that $\dim F_\lambda \geqslant N^{3g-3}$ for all r. □

6.2. Proof of Theorem 40 when the surface S is the torus.

Proposition 48. *Suppose that the surface S is a torus, and that the image of the homomorphism $\bar r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ induced by $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ has more than two elements. Then, for every triangulation λ of S and every λ–enhancement ξ for r, the associated off-diagonal kernel has dimension*

$$\dim F_\lambda = N.$$

Proof. By Theorem 38, the dimension of F_λ is independent of the triangulation λ and of the enhancement ξ. This provides us with flexibility in the choice of λ to perform computations.

By hypothesis, the image of $\bar r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is neither trivial nor isomorphic to $\mathbb{Z}_2$. A simple algebraic manipulation then provides a set of generators a_1, a_2 of $\pi_1(S) \cong \mathbb{Z}^2$ such that $\bar r(a_1)$, $\bar r(a_2)$ and $\bar r(a_1a_2) \in \mathrm{PSL}_2(\mathbb{C})$ are non-trivial. Then there exists a triangulation λ with one vertex v, and whose edges e_1, e_2 and e_3 respectively represent the classes a_1, a_2 and a_1a_2 in $\pi_1(S)$. By Lemma 7, this guarantees that there exists a λ–enhancement ξ for r.

In the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$, let Z_1, Z_2, Z_3 be the generators respectively associated to the edges e_1, e_2 and e_3. Exchanging the rôles of e_1 and e_2 if necessary, we can assume that e_1, e_2, e_3 arise in this order clockwise around each of the two faces of λ. Then the skew-commutativity relations satisfied by the Z_i are that $Z_iZ_{i+1} = \omega^4 Z_{i+1}Z_i$ for every i (considering indices modulo 3).

The central element $H_v \in \mathcal{Z}^\omega(\lambda)$ associated to the vertex v is equal to

$$H_v = [Z_1^2Z_2^2Z_3^2] = \omega^{-8} Z_1^2Z_2^2Z_3^2.$$

while its off-diagonal term is

$$\begin{aligned} Q_v &= 1 + \omega^{-4} Z_1^2 + \omega^{-8} Z_1^2 Z_2^2 + \omega^{-12} Z_1^2 Z_2^2 Z_3^2 \\ &\qquad + \omega^{-12} Z_1^2 Z_2^2 Z_3^2 Z_1^2 + \omega^{-12} Z_1^2 Z_2^2 Z_3^2 Z_1^2 Z_2^2 \\ &= (1 + \omega^{-12} Z_1^2 Z_2^2 Z_3^2)(1 + \omega^{-4} Z_1^2 + \omega^{-8} Z_1^2 Z_2^2) \\ &= (1 + \omega^{-4} H_v)(1 + \omega^{-4} Z_1^2 + \omega^{-8} Z_1^2 Z_2^2). \end{aligned}$$

The representation $\mu_\lambda \colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(E_\lambda)$ associated to the enhanced homomorphism (r, ξ) by Proposition 8 has dimension $\dim E_\lambda = N$, and sends H_v to $-\omega^4 \mathrm{Id}_{E_\lambda}$. The above computation shows that $\mu_\lambda(Q_v) = 0 \in \mathrm{End}(E_\lambda)$. Therefore, the off-diagonal kernel is equal to

$$F_\lambda = \ker \mu_\lambda(Q_v) = \ker 0 = E_\lambda$$

and has dimension N. □

The hypotheses of Proposition 48 are realized on a Zariski open dense subset of the space of homomorphisms $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. The same lower semi-continuity argument used in the proof of Proposition 47 gives the following general statement.

Proposition 49. *Suppose that the surface S is a torus. Then, for every triangulation λ of the torus and every λ–enhancement ξ for r, the associated off-diagonal kernel F_λ has dimension at least N.* □

6.3. Proof of Theorem 40 when the surface S is the sphere. In this case, every homomorphism $r \colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ is of course trivial.

Proposition 50. *Suppose that the surface S is a sphere. Then, for every triangulation λ of S and every λ–enhancement ξ for the trivial homomorphism, the associated total off-diagonal kernel F_λ has dimension equal to* 1.

Proof. By Theorem 38, it suffices to check this for any triangulation λ for which the trivial homomorphism admits a λ–enhancement; in this case, this just means that every edge of λ has distinct endpoints. We use the smallest such triangulation λ, with exactly three vertices and two faces glued along their boundary.

For this triangulation, the generators Z_1, Z_2, Z_3 of the (unbalanced) Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ commute, and the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ is isomorphic to the Laurent polynomial algebra $\mathbb{C}[H_1^{\pm 1}, H_2^{\pm 1}, H_3^{\pm 1}]$, where the $H_i = Z_{i+1} Z_{i+2}$ are the central elements associated to the three

vertices of λ (counting indices modulo 3). In particular, the representation μ_λ provided by Proposition 8 is 1–dimensional, and is the unique representation sending each H_i to $-\omega^4 \mathrm{Id}_{E_\lambda}$.

Each off-diagonal term is of the form $Q_i = 1 + \omega^{-4} Z_i^2 = 1 + \omega^{-4} H_{i+1} H_{i+2} H_i^{-1}$. The off-diagonal kernel of each vertex is therefore $\ker \mu_\lambda(Q_i) = \ker 0 = E_\lambda$, and the total off-diagonal kernel F_λ has dimension

$$\dim F_\lambda = \dim \bigcap_{i=1}^{3} \ker \mu_\lambda(Q_i) = \dim E_\lambda = 1 \qquad \square$$

The combination of Propositions 45, 47, 48, 49 and 50 completes the proof of Theorem 40.

6.4. Proof of the Realization Theorem 2. We are now ready to complete the proof of the Realization Theorem 2.

Given a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and a combinatorial triangulation λ of the surface S, Proposition 19 provided a representation $\check\rho_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(F_\lambda)$ whose classical shadow is equal to the character $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. Theorem 40 shows that F_λ is different from 0, so that this representation is non-trivial. The representation $\check\rho_\lambda$ may or may not be irreducible, but it admits at least one irreducible component $\rho_r\colon \mathcal{S}^A(S) \to \mathrm{End}(E)$ with $E \subset F_\lambda$. This irreducible representation satisfies the conclusions of Theorem 2.

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Department of Mathematics, University of Southern California, Los Angeles CA 90089-2532, U.S.A.

e-mail: fbonahon@usc.edu

Department of Mathematics, Claremont McKenna College, Claremont CA 91711, U.S.A.

e-mail: Helen.Wong@ClaremontMcKenna.edu