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Dynamics and pattern formation of a diffusive predator–prey model with predator-taxis

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We propose a new reaction–diffusion predator–prey model system with predator-taxis in which the preys could move in the opposite direction of predator gradient. A similar situation also occurs when susceptible population avoids the infected ones in epidemic spreading. The global existence and boundedness of solutions of the system in bounded domains of arbitrary spatial dimension and any predator-taxis sensitivity coefficient are proved. It is also shown that such predator-taxis does not qualitatively affect the existence and stability of coexistence steady state solutions in many cases. For diffusive predator–prey system with diffusion-induced instability, it is shown that the presence of predator-taxis may annihilate the spatial patterns.

Keywords: Reaction–diffusion system; predator–prey model; predator-taxis; global existence; boundedness; Turing instability; non-constant steady states.

AMS Subject Classification: 35K57, 35K59, 35B45, 35B32, 92D25, 92D30

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1. Introduction

Predators pursuing preys are one of fundamental binary interactions in a complex ecosystem, and various mathematical models have been established to describe such predator–prey relation to predict long term outcome and impact on the entire ecosystem.[39,50] Structurally similar models also appear in consumer–resource interaction or activator–inhibitor relation.[17,38]

It is known that spatial heterogeneity of the environment could affect the predator–prey dynamics.[18,22,44] Spatial predator–prey dynamics could be modeled by combining the kinetic dynamics and the diffusive movement of the predator and prey individuals,[14,51] and the dynamic behavior of such reaction–diffusion predator–prey models has been extensively studied in Refs. 8, 12, 15, 59, 62 and 81.

The diffusive predator–prey model is based on the assumption that predators and preys move randomly in the habitat, and the random movement is modeled by the passive diffusion. In reality, the spatial movement of predators and preys can be pursuit and evasion between them, that is, predators pursuing preys and preys escaping from predators. Such movement is not random but directed: predators move toward the gradient direction of prey distribution (called "prey-taxis"), and/or preys move opposite to the gradient of predator distribution (called "predator-taxis"). The following reaction–diffusion model with prey-taxis has been considered in Refs. 1, 27, 25, 32, 56, 60, 61, 75 and 77:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (q(u)\nabla v) + c\phi(u,v) - g(u), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = d\Delta v + f(v) - \phi(u,v), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where the global existence, dynamical behavior and steady states have been considered.

In this paper, we consider a diffusive predator–prey model with predator-taxis, that is, the preys try to evade the predators and move away from the direction of the higher predator density:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + c\phi(u,v,x) - g(u,x), & x \in \Omega,\ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (q(v)\nabla u) + f(v,x) - \phi(u,v,x), & x \in \Omega,\ \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where the habitat of both species Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$; and homogeneous Neumann boundary condition is imposed; $u(x,t)$ and $v(x,t)$ represent the densities of predator and prey at the location x and time t, respectively; $d > 0$ is the rescaled diffusion coefficient for the predator and the diffusion coefficient of the prey is now rescaled as 1; the function $f(v,x)$ is the growth rate of prey, and the function $g(u,x)$ represents the mortality rate of the predator; the function $\phi(u,v,x)$ measures the predation rate, and the positive parameter c is the conversion rate; and the term $\xi\nabla\cdot(q(v)\nabla u)$ shows the tendency of prey moving toward the opposite direction of the increasing predator gradient direction.

System (1.2) can be derived similarly as (1.1) following the approach for prey-taxis in Ref. 27. A reaction–diffusion model of predator–prey pursuit–evasion with both prey-taxi in (1.1) and predator-taxi in (1.2) has been proposed in Refs. 57 and 58, and the traveling wave solutions for such models were considered. Very recently, an equation in the form of (1.2) modeling the avoidance behaviors of prey has also been proposed in Ref. 69, and the pattern formation for several classes of functional responses for the model was studied. The last study is motivated by the anti-predator behavior which causes the reproduction rate reduction of song-sparrows.[65,68,82]

In (1.2), the prey growth rate $f(v,x)$, the predator mortality rate $g(u,x)$, the predation rate $\phi(u,v,x)$ and the sensitivity function $q(v)$ are similar to the ones in Ref. 77, and examples are shown in Ref. [77]. In this paper, we assume these functions satisfy the following more general hypotheses:

- (H_0) The functions $g:[0,\infty)\times\overline{\Omega}\to[0,\infty)$, $q:[0,\infty)\to[0,\infty)$, $f:[0,\infty)\times\overline{\Omega}\to\mathbb{R}$ and $\phi:[0,\infty)\times[0,\infty)\times\overline{\Omega}\to[0,\infty)$ are continuously differentiable; $f(0,x)=0$, $g(0,x)=0$ for $x\in\overline{\Omega}$; $q(0)=0$; and $\phi(u,0,x)=0$ and $\phi(0,v,x)=0$ for any $u,v\geq 0$ and $x\in\overline{\Omega}$;
- (H_1) There exists $B>0$ such that $\phi(u,v,x)\leq Bu$ for any $u,v\geq 0$ and $x\in\overline{\Omega}$;
- (H_2) There exists $C>0$ such that $q(v)\leq Cv$ for any $v\geq 0$ and $x\in\overline{\Omega}$;
- (H_3) There exists $D>0$ such that $g(u,x)\geq Du$ for any $u\geq 0$ and $x\in\overline{\Omega}$;
- (H_4) There exists $E,F>0$ such that $f(v,x)\leq Ev-Fv^2$ for any $v\geq 0$ and $x\in\overline{\Omega}$.

Our main results on the global existence and boundedness of solutions of system (1.2) are as follows.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n\geq 1)$ with smooth boundary $\partial\Omega$. Suppose that $d,c>0$, $\xi\geq 0$, $f(v,x)$, $g(u,x)$, $q(v)$ and $\phi(u,v,x)$ satisfy (H_0)–(H_4). For any $(u_0,v_0)\in[W^{1,p}(\Omega)]^2$ where $p>n$, satisfying $u_0(x)\geq 0, v_0(x)\geq 0$ for $x\in\overline{\Omega}$, the system (1.2) possesses a unique global classical solution $(u(x,t),v(x,t))$ satisfying $(u,v)\in(C([0,\infty);W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$, and $(u(x,t),v(x,t))$ is uniformly bounded in $\Omega\times(0,\infty)$, i.e. there is a constant $M_1(u_0,v_0)>0$ such that $\|u(\cdot,t)\|_\infty+\|v(\cdot,t)\|_\infty\leq M_1(u_0,v_0)$ for all $t\in[0,\infty)$.*

The repulsive predator-taxis (chemotaxis) in (1.2) also occurs in modeling spatial epidemics, as healthy people may want to stay away from infective people. Motivated by the diffusive SIS model proposed in Ref. 3, we propose the following reaction–diffusion SIS epidemic model with "infected-taxis":

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + \phi(u,v,x) - g(u,x), & x \in \Omega, \;\; t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi\nabla\cdot(q(v)\nabla u) + g(u,x) - \phi(u,v,x), & x \in \Omega, \;\; t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \;\; t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.3}$$

Here $u(x,t)$ and $v(x,t)$ represent the density of infected and susceptible individuals at location x and time t; d is the diffusion coefficient of the infected population, and the diffusion coefficient of the susceptible population is normalized to be 1; The function $\phi(u,v,x)$ represents the infection rate which may depend on the population densities and the spatial location x; and the function $g(u,x)$ denotes the rate of recovery from infected class back to the susceptible class. We propose here a repulsive taxis term $\xi\nabla\cdot(q(v)\nabla u)$ which shows the tendency of healthy people staying away from infective people. For (1.3), we assume that these functions satisfy (H_0)–(H_2), and $f(v,x)$ is replaced by $g(u,x)$ in (1.2). When $\xi = 0$, the model in (1.3) reduces to a more general form of the SIS epidemic reaction–diffusion model in Ref. 3, and in that case there has been many studies on the dynamics and steady states.[3,10,11,45–47,80]

We have a global existence and boundedness of solutions of system (1.3) as follows.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$. Suppose that $d > 0$, $\xi \geq 0$, $g(u,x)$, $q(v)$ and $\phi(u,v,x)$ satisfy (H_0)–(H_2). For any $(u_0,v_0) \in [W^{1,p}(\Omega)]^2$ where $p > n$, satisfying $u_0(x) \geq 0, v_0(x) \geq 0$ for $x \in \Omega$, the system (1.3) possesses a unique global classical solution $(u(x,t),v(x,t))$ satisfying $(u,v) \in (C([0,\infty);W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2$, and $(u(x,t),v(x,t))$ is uniformly bounded in $\Omega\times(0,\infty)$, i.e. there is a constant $M_1(u_0,v_0) > 0$ such that $\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \leq M_1(u_0,v_0)$ for all $t \in [0,\infty)$.*

Remark 1.1. (1) The global existence and boundedness results in Theorems 1.1 and 1.2 for predator-taxis model (1.2) and infected-taxis model (1.3) do not require any conditions on the sensitivity coefficient ξ or the spatial dimension n. For the corresponding prey-taxis model (1.1), the global existence and boundedness result in Ref. 77 is for small prey-taxis sensitivity coefficient $\chi > 0$ and arbitrary dimension n, while the result in Ref. 25 is for arbitrary prey-taxis sensitivity coefficient $\chi > 0$ and $n = 2$. More recently the existence of global weak solution for a prey-taxis system like (1.2) with $n \leq 5$ was proved in Ref. 75. The global existence and boundedness of solutions to (1.1) for arbitrary $\chi > 0$ and arbitrary n are still not known, but the same question for (1.2) is completely resolved in Theorem 1.1.

(2) The global bound shown in Theorems 1.1 and 1.2 depends on the initial condition (u_0, v_0). It is an interesting question whether a uniform bound independent of initial conditions could be obtained for large time t. For (1.1), such ultimate boundedness is known for small χ and arbitrary n.[77]

(3) The results in Theorems 1.1 and 1.2 also hold if $q(v)$ is replaced by $q(u, v)$ satisfying (as in (H_2)) $q(u, v) \leq Cv$ for some $C > 0$ and any $u, v \geq 0$. For example $q(u, v) = v/(1 + \alpha u)^2$ as in Ref. 70.

The second part of our paper is devoted to study the dynamical behavior of (1.2) and (1.3). For that purpose, some more specific conditions need to be imposed for the functions f, g, ϕ, q. Indeed we assume that all functions are independent of x, and have the following form:

$$f(v, x) \equiv f(v), \quad g(u, x) \equiv uk(u), \quad \phi(u, v, x) \equiv u\Phi(v). \tag{1.4}$$

Hence we consider the more special form of (1.2):

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + cu\Phi(v) - uk(u), & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi\nabla \cdot (v\nabla u) + f(v) - u\Phi(v), & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.5}$$

and we have the following additional conditions:

(H_5) The function $\Phi : [0, \infty) \to [0, \infty)$ is continuously differentiable, $\Phi(0) = 0$, $\Phi'(v) > 0$ for any $v \geq 0$, and there exists $N_* > 0$ such that $v^2\Phi'(v) \geq N_*[\Phi(v)]^2$ for $v \geq 0$;

(H_6) Define $\psi(v) = \frac{f(v)}{\Phi(v)}$. Then $\psi : [0, \infty) \to [0, \infty)$ is continuously differentiable, $\psi(0) > 0$, there exists a constant N such that when $v \not\equiv N$ for $v > 0$, $\psi(v)(v - N) < 0$ and $\psi(N) = 0$; there exists $M \in (0, N)$ such that $\psi'(v) > 0$ when $[0, M)$, $\psi'(v) < 0$ when $[M, N)$.

Note that biologically $\Phi(v)$ is the predator functional response with respect to the prey population, and $k(u)$ is the predator mortality rate.

Then we have the following results regarding the dynamical behavior of (1.5) with constant predator mortality rate.

Theorem 1.3. *Suppose that* $d, c > 0$, $\xi \geq 0$, $f(v)$, $g(u) = uk(u)$, $q(v)$ *and* $\Phi(v)$ *satisfy* (H_0), (H_5), (H_6) *and*

(H_7) $k(u) \equiv k > 0$, *where* k *is a constant*; *and there exists* $\lambda > 0$ *such that* $\Phi(\lambda) = k/c$.

Then:

(1) $(0,0)$ *is an unstable constant steady state of* (1.5) *for all* $\lambda > 0$; $(0,N)$ *is a locally asymptotically stable constant steady state of* (1.5) *when* $\lambda > N$ *and it is unstable when* $0 < \lambda < N$; *and* $(\psi(\lambda),\lambda)$ *is a positive constant steady state of* (1.5) *only when* $0 < \lambda < N$; *it is locally asymptotically stable when* $M < \lambda < N$ *and it is unstable when* $0 < \lambda < M$.

(2) *If in addition* (H_1)–(H_4) *are also satisfied,* $\lambda \in (M_*, N]$ *where* $M_* \in (M,N)$ *such that* $\psi(M_*) = \psi(0)$, *and*

$$0 \leq \xi < \sqrt{\frac{4d\psi(\lambda)}{cN_*\Phi(\lambda)N^2}}, \tag{1.6}$$

then $(\psi(\lambda),\lambda)$ *is globally asymptotically stable for system* (1.5) *with any* $u_0, v_0 \geq 0$ *and* $v_0 \not\equiv 0$.

(3) *When* $0 < \lambda < M$, *system* (1.5) *has a spatially homogenous positive periodic orbit.*

Remark 1.2. (1) When $\xi = 0$, system (1.5) with (H_7) is the classical diffusive Rosenzweig–MacArthur predator–prey system, for which the dynamics has been thoroughly studied in Refs. 62 and 81. Theorem 1.3 shows that the addition of repulsive predator-taxis does not affect the local stability of constant steady states $(0,0)$, $(0,N)$ and $(\psi(\lambda),\lambda)$.

(2) The local stability results in Theorem 1.3 hold for any diffusion coefficient $d > 0$ and predator-taxis coefficient $\xi \geq 0$, hence neither diffusion nor predator-taxis induces (Turing-type) instability for (1.5) when $k(u) \equiv k$.

(3) The global stability of $(\psi(\lambda),\lambda)$ when $\lambda \in (M_*, N]$ and $\xi = 0$ has been proved in Refs. 62 and 81, and a similar global stability result is also known for the prey-taxis system (1.1).[25] It is interesting to ask whether such global stability still holds for (1.5) with large $\xi > 0$, as well as for (1.1) with large $\chi > 0$.

(4) We conjecture that the boundary equilibrium $(0,N)$ is indeed globally asymptotically stable for (1.5) when $\lambda > N$. This was proved for the case of $\xi = 0$ in Refs. 62 and 81, and it was also proved for the case of (1.1) when $\chi > 0$ is small.

Contrast to the constant predator mortality case in Theorem 1.3, a Turing type instability in (1.5) does occur for the purely diffusive case (without predator-taxis) when $k(u)$ is a linear increasing function (see Refs. 30 and 37): the positive constant equilibrium becomes unstable when the diffusion coefficient d is sufficiently large. Our next result shows how the predator-taxis affects this Turing instability.

Theorem 1.4. *Suppose that* $d, c > 0$, $\xi \geq 0$, $f(v)$, $g(u) = uk(u)$, $q(v)$ *and* $\Phi(v)$ *satisfy* (H_0), (H_5), (H_6) *and*

(H_8) $k(u) = k + lu$, *where* $k > 0$ *and* $l > 0$.

Suppose that $(\psi(\lambda), \lambda)$ *is a positive constant steady state solution of* (1.5) *satisfying*

$$\psi'(\lambda) < \min\left\{\frac{l\psi(\lambda)}{\Phi(\lambda)}, \frac{c\Phi'(\lambda)}{l}\right\}. \tag{1.7}$$

Then:

(1) *There exists* $\xi^*(d, \lambda) \in \mathbb{R}$ *which is independent of* Ω *such that when* $\xi > \xi^*(d, \lambda)$, *the positive constant steady state* $(\psi(\lambda), \lambda)$ *is locally asymptotically stable with respect to the dynamics of* (1.5);
(2) *For a given bounded domain* Ω, *there exists* $\xi_*(d, \lambda, \Omega)$ *such that when* $\xi < \xi_*(d, \lambda, \Omega)$, *the positive constant steady state* $(\psi(\lambda), \lambda)$ *is unstable with respect to the dynamics of* (1.5);
(3) *If in addition all eigenvalues of* $-\Delta$ *in* Ω *with Neumann boundary condition are simple, then there exists a sequence* $\{\xi_j : j \in \mathbb{N}\}$ *such that when* $\xi_j \neq \xi_i$ *for any* $i \neq j$, $\xi = \xi_j$ *is a bifurcation point of* (1.5) *such that a smooth curve* Γ_j *of non-constant steady state solutions bifurcates from the line of constant steady states* $\{(\xi, \psi(\lambda), \lambda) : \xi \in \mathbb{R}\}$; *moreover* Γ_j *is contained in a connected component* C_j *of* Σ, *which is the closure of the set of positive non-constant steady state solutions of* (1.5), *and either* C_j *is unbounded or* C_j *contains another* $(\xi_i, \psi(\lambda), \lambda)$ *with* $\xi_i \neq \xi_j$.

It is evident that $\xi_*(d, \lambda, \Omega) < \xi^*(d, \lambda)$ and the values of these critical sensitivity coefficients can be either positive or negative. Indeed the precise values of $\xi^*(d, \lambda)$ and $\xi_*(d, \lambda, \Omega)$ can be determined which will be shown in Sec. 4.2. Theorem 1.4 shows that a larger predator-taxis effect (larger $\xi > 0$) will stabilize the positive constant steady state, and spatial pattern formation cannot be achieved in that case; but a smaller (or even negative) predator-taxis effect will destabilize the positive constant steady state and generate non-constant spatial pattern. Note that in Sec. 4.2, we show that it is possible that $\xi_*(d, \lambda, \Omega) > 0$, hence the spatial pattern formation can still be achieved for $\xi \in (0, \xi_*(d, \lambda, \Omega))$ (see numerical simulations in Sec. 4.2). Spatial pattern formation for similar prey-taxis diffusive predator–prey system (1.1) has been studied in Refs. 60 and 61, and the pattern formations in these work always occur for negative $\chi < 0$, which means the predators are repulsive to the preys due to group defense or other unusual mechanisms. In these work, spatial patterns do not exists when the prey-taxis is absent, and in our case, the spatial patterns also exist without the predator-taxis and they persist when the predator-taxis is not strong.

The predator-taxis considered here is partially motivated by repulsive chemotaxis models:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \nabla \cdot (\xi(u)\nabla v), & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - \alpha v + \beta u, & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0. \end{cases} \tag{1.8}$$

The global boundedness and convergence of the solution of (1.8) have been established in Refs. 53 and 67. Repulsive chemotaxis is one of possible interactions between cells and chemical signals in biological tissues. More well-studied chemotaxis models are the attractive ones such as the Keller–Segel type models,[28] and work on these models has been recently surveyed in Ref. 7. The global existence, boundedness and blow-up of solutions to the attractive chemotaxis systems are more delicate.[20,40,42,71–74] In many biological processes, it is also possible to have attractive and repulsive chemotaxis simultaneously, see for example, Refs. 23, 33, 34, 35, 24, 54, 78 and 79. Pattern formation and dynamics of attractive chemotactic system with logistic growth has also been investigated in many recent works, see for example, Refs. 16, 26, 29, 31, 43, 55, 64, 72 and 74.

The organization of the remaining part of the paper is as follows. In Sec. 2, we recall some analytic tools and obtain some preliminary results. The global existence and uniform boundedness of the solutions to the predator-taxis system are proved in Sec. 3. The dynamical behavior and pattern formation of the predator-taxis system are studied in Sec. 4, and the dynamical behavior of the infected-taxis epidemic model is considered in Sec. 5. In this paper, we use $\|\cdot\|_p$ as the norm of $L^p(\Omega)$, $1 \le p \le \infty$; and $\|\cdot\|_{m,p}$ as the norm of $W^{m,p}(\Omega)$, $m = 1, 2$, $1 \le p \le \infty$. And $\|(u,v)\|_*$ is understood as $\|u\|_* + \|v\|_*$.

2. Local Existence and Preliminaries

First we state the local-in-time existence result of a classical solution of (1.2), which can be proved by using the abstract theory of quasilinear parabolic systems in Ref. 4.

Lemma 2.1. *Assume that the initial data* $(u_0, v_0) \in (W^{1,p}(\Omega))^2$ *for* $p > n$, $u_0 \ge 0, v_0 \ge 0$, *and the conditions* (H_0), (H_1), (H_3) *and* (H_4) *hold. Then*:

(1) *There exists a positive constant* $T_{\max}$ (*the maximal existence time*) *such that the system* (1.2) *has a unique non-negative classical solution* $(u(x,t), v(x,t))$ *satisfying* $(u,v) \in (C([0,T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})))^2$.

(2) *The total mass of* $u(x,t)$ *and* $v(x,t)$ *satisfies*

$$\int_\Omega u(x,t)dx \le C_0, \quad \int_\Omega v(x,t)dx \le C_1 \quad \text{for all } t \in (0, T_{\max}), \tag{2.1}$$

where $C_0 = \max\{\frac{(E+D)c}{D} C_1, \int_\Omega (u_0 + cv_0)\}$, $C_1 = \max\{\|v_0\|_1, \frac{E|\Omega|}{F}\}$.

(3) *There exists a constant* $Q > 0$ *such that* u, v *satisfy*

$$0 \le u(x,t) \le Q, \quad v(x,t) \ge 0, \quad x \in \overline{\Omega}, \quad 0 \le t < T_{\max}. \tag{2.2}$$

(4) *If for each* $T > 0$ *there exists a constant* $M_0(T)$ *such that*

$$\|(u(t), v(t))\|_\infty \le M_0(T), \quad 0 < T < \min\{T, T_{\max}\}, \tag{2.3}$$

where $M_0(T)$ *is a constant depending on* T *and* $\|(u_0, v_0)\|_{1,p}$, *then* $T_{\max} = +\infty$.

Proof. Let $\omega = (u, v)$. Then the system (1.2) can be rewritten as

$$\begin{cases} \omega_t = \nabla \cdot (a(\omega)\nabla\omega) + \Phi(\omega), & x \in \Omega, \ t > 0, \\ \dfrac{\partial \omega}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ \omega(\cdot, 0) = (u_0, v_0), & x \in \Omega, \end{cases} \tag{2.4}$$

where

$$a(\omega) = \begin{pmatrix} 1 & 0 \\ \xi q(v) & d \end{pmatrix}, \quad \Phi(\omega) = \begin{pmatrix} c\phi(u,v,x) - g(v,x) \\ f(v,x) - \phi(u,v,x) \end{pmatrix}.$$

Then from Theorems 14.4 and 14.6 in Ref. 4, we obtain the local existence of $(u(x,t), v(x,t))$ in part (1). Next we show that the solution $(u(x,t), v(x,t))$ is bounded in $L^1(\Omega)$. Let $\int_\Omega u(x,t)dx = Q_1(t)$ and $\int_\Omega v(x,t)dx = Q_2(t)$. Integrating the second equation in (1.2) and from (H_4), we have

$$\frac{dQ_2}{dt} = \int_\Omega f(v,x) \le \int_\Omega (Ev - Fv^2). \tag{2.5}$$

By the Hölder inequality, we obtain

$$\int_\Omega v^2 \ge \frac{1}{|\Omega|} \left(\int_\Omega v \right)^2. \tag{2.6}$$

Integrating (2.5) over Ω, and combining (2.5) with (2.6), we have

$$\int_\Omega v(x,t)dx = Q_2 \le C_1, \tag{2.7}$$

where $C_1 = \max\{\|v_0\|_1, \frac{E|\Omega|}{F}\} > 0$. Then we have

$$\begin{aligned} (Q_1 + cQ_2)_t = \frac{dQ_1}{dt} + c\frac{dQ_2}{dt} &= \int_\Omega u_t + c\int_\Omega v_t = c\int_\Omega f(v,x) - \int_\Omega g(u,x) \\ &\le Ec\int_\Omega v - D\int_\Omega u = -DQ_1 + EcQ_2 \\ &= -D(Q_1 + cQ_2) + c(E+D)Q_2. \end{aligned}$$

Since $Q_2(t) \le C_1$, it gets

$$\int_\Omega u(x,t)dx = Q_1(t) < Q_1(t) + cQ_2(t) \le \max\left\{\frac{(E+D)c}{D}C_1, \int_\Omega (u_0 + cv_0)\right\} := C_0.$$

This completes the proof of part (2).

To prove (2.2), we rewrite the second equation of (1.2) as follows:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + \xi q'(v)\nabla u \cdot \nabla v + \xi q(v)\Delta u + f(v,x) - \phi(u,v,x), & x \in \Omega, \ t > 0, \\ \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ v(x,0) = v_0(x) \ge 0, & x \in \Omega. \end{cases} \tag{2.8}$$

Treating (2.8) as a scalar linear equation in u, and using (H_0), we find that $\underline{v} = 0$ is a lower solution to (2.8), therefore we can apply the maximum principle for parabolic equations to obtain that $v(x,t) \ge 0$. Similarly we can obtain that $u(x,t) \ge 0$. Also from (1.2) and $u \ge 0$, we obtain that

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u = c\phi(u,v,x) - g(u,x) \le c\phi(u,v) \le cBu, & x \in \Omega, \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.9}$$

Using the comparison principle and Theorem 3.1 in Ref. 2, we have

$$u(x,t) \le \max\{1, \|u_0\|_\infty, \|u\|_1\} \le \max\{1, \|u_0\|_\infty, C_0\} := Q, \tag{2.10}$$

which proves part (3). Since the system (2.4) is a lower triangular system, then part (4) follows from Theorem 15.5 in Ref. 5, so we have $T_{\max} = \infty$. □

Similarly we prove the local-in-time existence of a classical solution of the SIS system (1.3).

Lemma 2.2. *Assume that the initial data* $(u_0, v_0) \in (W^{1,p}(\Omega))^2$ *for* $p > n$, $u_0 \ge 0$, $v_0 \ge 0$, *and the conditions* (H_0)–(H_2) *hold. Then*:

(1) *There exists a positive constant* $T_{\max}$ (*the maximal existence time*) *such that the system* (1.3) *has a unique non-negative classical solution* $(u(x,t), v(x,t))$ *satisfying* $(u,v) \in (C([0,T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^2$.
(2) *There exist constants* $C_0 > 0, C_1 > 0$ *such that*

$$\int_\Omega u(x,t)dx \le C_0, \quad \int_\Omega v(x,t)dx \le C_1 \quad \text{for all } t \in (0, T_{\max}). \tag{2.11}$$

(3) *There exists a constant* $Q > 0$ *such that* u, v *satisfy*

$$0 \le u(x,t) \le Q, \quad v(x,t) \ge 0, \quad x \in \overline{\Omega}, \quad 0 \le t < T_{\max}. \tag{2.12}$$

(4) *If for each* $T > 0$ *there exists a constant* $M_0(T)$ *such that*

$$\|(u(t), v(t))\|_\infty \le M_0(T), \quad 0 < T < \min\{T, T_{\max}\}, \tag{2.13}$$

where $M_0(T)$ *is a constant depending on* T *and* $\|(u_0, v_0)\|_{1,p}$, *then* $T_{\max} = +\infty$.

Proof. For the system (1.3), we apply the maximum principle and comparison principle for parabolic equation to obtain that $v(x,t) \ge 0$. Similarly we can obtain that $u(x,t) \ge 0$. Let $\int_\Omega (u_0 + v_0) := N_1$ be the total number of individuals in Ω at $t = 0$. Adding the u-equation with the v-equation and integrating over Ω, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega (u(x,t) + v(x,t)) &= \int_\Omega \Delta(u(x,t) + v(x,t)) \\ &= \int_{\partial\Omega} \frac{\partial}{\partial \nu}(u(x,t) + v(x,t)) = 0, \quad t > 0,\end{aligned}$$

then we have

$$\int_\Omega (u(x,t) + v(x,t)) = N_1, \quad t \ge 0. \tag{2.14}$$

Therefore, from (2.14), $\|u\|_1$ is bounded. Then using the comparison principle and Theorem 3.1 in Ref. 2 again, from (2.9), we have $u(x,t) \le Q$. □

Next we recall some preliminary estimates which will be used in our proof. First we review some well-known estimates for the diffusion semigroup with homogeneous Neumann boundary conditions (see Ref. 20). For $p \in (1, \infty)$, let A denote the sectorial operator defined by

$$Au := -\Delta u \quad \text{for } u \in D(A) := \left\{\omega \in W^{2,p}(\Omega) : \frac{\partial \omega}{\partial n} = 0 \text{ on } \partial\Omega\right\}. \tag{2.15}$$

Similarly we let $A_d u = -d\Delta u$ which satisfies the same properties as A with a scaling. Then we only collect properties of A here while the same properties for A_d will be applied in the following analysis.

Lemma 2.3. *Assume that* $m \in \{0, 1\}$, $p \in [1, \infty]$ *and* $q \in (1, \infty)$. *Then there exists some positive constant* c_1, *such that*

$$\|u\|_{m,p} \le c_1 \|(A+1)^\theta u\|_q, \tag{2.16}$$

for any $u \in D((A+1)^\theta)$ *where* $\theta \in (0, 1)$ *satisfies*

$$m - \frac{n}{p} < 2\theta - \frac{n}{q}.$$

If in addition $q \geq p$, *then there exist* $c_2 > 0$ *and* $\gamma > 0$ *such that for any* $u \in L^p(\Omega)$,

$$\|(A+1)^{\theta} e^{-t(A+1)} u\|_q \leq c_2 t^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\gamma t} \|u\|_p, \tag{2.17}$$

where the associated diffusion semigroup $\{e^{-t(A+1)}\}_{t \geq 0}$ *maps* $L^p(\Omega)$ *into* $D((A+1)^{\theta})$. *Moreover, for any* $p \in (1,\infty)$ *and* $\varepsilon > 0$, *there exist* $c_3 > 0$ *and* $\mu > 0$ *such that*

$$\|(A+1)^{\theta} e^{-tA} \nabla \cdot u\|_p \leq c_3 t^{-\theta - \frac{1}{2} - \varepsilon} e^{-\mu t} \|u\|_p \tag{2.18}$$

is valid for all $\mathbb{R}^n$*-valued* $u \in L^p(\Omega)$.

The following Gagliardo–Nirenberg inequality also plays a key role in our proof (see Ref. 41 for detail).

Lemma 2.4. *Let* $u \in L^p(\Omega)$ *and* $D^k u \in L^q(\Omega)$ *where* $p, q \in [1,\infty]$. *Then for the derivatives* $D^i u$, $i \in [0,k)$, *there exists a constant* $c_4 > 0$ *such that*

$$\|D^i u\|_h \leq c_4(\|D^k u\|_q^{\lambda} \|u\|_p^{1-\lambda} + \|u\|_m), \tag{2.19}$$

where

$$\frac{1}{h} - \frac{i}{n} = \lambda\left(\frac{1}{q} - \frac{k}{n}\right) + (1-\lambda)\frac{1}{p}, \quad m > 0,$$

and λ *satisfies*

$$\frac{i}{k} \leq \lambda \leq 1.$$

Moreover, if $q \in (1,\infty)$ *and* $k - i - \frac{n}{q}$ *is a non-negative integer, then the Gagliardo–Nirenberg inequality* (2.19) *holds for*

$$\frac{i}{k} \leq \lambda < 1.$$

The following Sobolev inequality will be used in forthcoming proofs.

Lemma 2.5. *Let*

$$2^* = \begin{cases} \infty, & n \leq 2, \\ \dfrac{n}{n-2}, & n > 2. \end{cases}$$

Then for any $1 < \alpha \leq 2^*$ *and* $k > 0$ *there exists a positive constant* M_0 *such that*

$$\left(\int_{\Omega} u^{(k+1)\alpha} dx\right)^{\frac{1}{\alpha}} \leq M_0 \int_{\Omega} (|\nabla(u^{\frac{k+1}{2}})|^2 + u^{k+1}) dx. \tag{2.20}$$

Finally we recall the following elementary inequality.[76]

Lemma 2.6. *Assume that* $y, z \in \mathbb{R}$, $y, z \geq 0$ *and* $r > 0$, *then we have*

$$(y+z)^r \leq 2^r(y^r + z^r). \tag{2.21}$$

3. Global Existence and Boundedness

In this section, we prove the global existence and boundedness of solutions in Theorem 1.1. The main step toward the result is to establish a uniform bound of $v(x,t)$ in $L^k(\Omega)$ for any $k \in [2, \infty)$.

Lemma 3.1. *Let (u, v) be a solution of (1.2). Assume that (H_0)–(H_4) are satisfied, then for any $k \geq 2$, there exists a positive constant $G > 0$ such that*

$$\|v(\cdot, t)\|_k \leq G \quad \textit{for } t \in (0, T_{\max}). \tag{3.1}$$

Proof. First we show that for any $\tau \in (0, T_{\max})$, there exists a constant $H(\tau) > 0$ such that

$$\|u(\cdot, t)\|_{1,\infty} \leq H(\tau) \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.2}$$

Let $\tau \in (0, T_{\max})$ be given such that $\tau < 1$, and choose $q > n$ and $\theta \in (\frac{1}{2}(1 + \frac{n}{q}), 1)$. The first equation of (1.2) can be rewritten as

$$u_t = d\Delta u - u + \varphi(u, v, x), \tag{3.3}$$

where $\varphi(u, v, x) = c\phi(u, v, x) - g(u, x) + u$. Then from the variation of constants formula for (3.3), we have

$$u(\cdot, t) = e^{-t(A_d+1)}u_0 + \int_0^t e^{-(t-s)(A_d+1)}\varphi(u(\cdot, t), v(\cdot, t))ds.$$

From (2.16) and (2.17) we have

$$\begin{aligned}
\|u(\cdot, t)\|_{1,\infty} &\leq c_1\|(A_d + 1)^\theta u(\cdot, t)\|_q \\
&\leq C_2 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \|c\phi(u(\cdot, t), v(\cdot, t)) + u(\cdot, t) - g(u(\cdot, t))\|_q ds \\
&\quad + C_2 t^{-\theta} e^{-\gamma t}\|u_0\|_q \\
&\leq C_2 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \|u(\cdot, t)\|_\infty ds + C_2 t^{-\theta} e^{-\gamma t}\|u_0\|_q \\
&\leq C_2 t^{-\theta} + C_2 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} ds \leq C_2 t^{-\theta} + C_2 \int_0^\infty \sigma^{-\theta} e^{-\gamma\sigma} d\sigma \\
&\leq C_2(\tau^{-\theta} + 1) := H(\tau) \quad \text{for all } t \in (\tau, T_{\max}),
\end{aligned} \tag{3.4}$$

where C_2 denotes a generic constant that may vary from line to line. For any $k \geq 2$, from (1.2), (3.4) and Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v^k &= k\int_\Omega v^{k-1} v_t \\
&\leq k\int_\Omega v^{k-1}\Delta v + k\int_\Omega v^{k-1}\xi\nabla \cdot (q(v)\nabla u) + k\int_\Omega v^{k-1} f(v)
\end{aligned}$$

$$
\begin{aligned}
&= -k(k-1)\int_\Omega v^{k-2}|\nabla v|^2 - k(k-1)\xi\int_\Omega v^{k-2}q(v)\nabla u\cdot\nabla v \\
&\quad + k\int_\Omega v^{k-1}f(v) \\
&\le \frac{-4(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + Ik(k-1)\int_\Omega v^{k-2}q(v)|\nabla v| + Ek\int_\Omega v^k \\
&\le \frac{-4(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + CIk(k-1)\int_\Omega v^{k-1}|\nabla v| + Ek\int_\Omega v^k \\
&= \frac{-4(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + CIk(k-1)\frac{2}{k}\int_\Omega v^{\frac{k}{2}}|\nabla v^{\frac{k}{2}}| + Ek\int_\Omega v^k \\
&\le \frac{-4(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + CI(k-1)\left(\frac{2}{CIk}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + \frac{CIk}{2}\int_\Omega v^k\right) \\
&\quad + Ek\int_\Omega v^k \\
&\le \frac{-2(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + \left(\frac{C^2I^2k(k-1)}{2} + Ek\right)\int_\Omega v^k, \qquad (3.5)
\end{aligned}
$$

where $I = \xi H(\tau)$ is a positive constant. Then we have

$$
\frac{d}{dt}\int_\Omega v^k \le \frac{-2(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + \left(\frac{C^2I^2k(k-1)}{2} + Ekc\right)\int_\Omega v^k. \qquad (3.6)
$$

From Lemmas 2.4 and 2.6, we find that

$$
\begin{aligned}
\int_\Omega v^k = \|v^{\frac{k}{2}}\|_2^2 &\le c_4(\|\nabla v^{\frac{k}{2}}\|_2^\lambda \|v^{\frac{k}{2}}\|_{\frac{2}{k}}^{1-\lambda} + \|v^{\frac{k}{2}}\|_{\frac{2}{k}})^2 \\
&\le c_4(\|\nabla v^{\frac{k}{2}}\|_2^\lambda \|1+v\|_1^{\frac{k}{2}(1-\lambda)} + \|1+v\|_1^{\frac{k}{2}})^2 \\
&\le c_4(\|\nabla v^{\frac{k}{2}}\|_2^\lambda (|\Omega| + C_1)^{\frac{k}{2}(1-\lambda)} + (|\Omega| + C_1)^{\frac{k}{2}})^2 \\
&\le C_3(\|\nabla v^{\frac{k}{2}}\|_2^{2\lambda} + 1), \qquad (3.7)
\end{aligned}
$$

where

$$
\lambda = \frac{kn-n}{2+kn-n} \in (0,1), \qquad (3.8)
$$

for any $k \ge 2$. Since (3.8) implies that $2\lambda < 2$, then from (3.7) we obtain

$$
\int_\Omega v^k \le C_4(\|\nabla v^{\frac{k}{2}}\|_2^2 + 1). \qquad (3.9)
$$

By using Young's inequality and (3.9), we obtain

$$
\left(\frac{C^2I^2k(k-1)}{2} + Ekc + 1\right)\int_\Omega v^k \le \frac{2(k-1)}{k}\int_\Omega |\nabla v^{\frac{k}{2}}|^2 + C_5, \qquad (3.10)
$$

for some $C_5 > 0$. Combining (3.6) and (3.10), we have

$$\frac{d}{dt}\int_\Omega v^k + \int_\Omega v^k \le C_5. \tag{3.11}$$

Integrating (3.11), we arrive at

$$\int_\Omega v^k \le \max\left\{\int_\Omega v_0^k, C_5\right\} := R, \tag{3.12}$$

which is the desired result. □

Next we establish the L^∞ bound for $v(x,t)$ using the result in Lemma 3.1.

Lemma 3.2. *Let* (u, v) *be a solution of* (1.2). *Assume that* (H_0)–(H_4) *hold, then there exists a positive constant* M *such that*

$$\|v(\cdot,t)\|_\infty \le M \quad \textit{for all } t \in (0, T_{\max}). \tag{3.13}$$

Proof. We use semigroup arguments (e.g. see Refs. 20, 71 and 70) to get the L^∞-bound of v. First, by using the variation of constants formula, we have

$$\begin{aligned} v(\cdot,t) &= e^{-t(A+1)}v_0 - \xi\int_0^t e^{-(t-s)(A+1)}\nabla\cdot(q(v(\cdot,t))\nabla u(\cdot,t))ds \\ &\quad + \int_0^t e^{-(t-s)(A+1)}\psi(u(\cdot,t), v(\cdot,t))ds \\ &:= V_1 + V_2 + V_3, \end{aligned} \tag{3.14}$$

where $\psi(u(\cdot,t), v(\cdot,t)) = f(v(\cdot,t)) - \phi(u(\cdot,t), v(\cdot,t)) + v(\cdot,t)$. Then we estimate the L^∞-bound for each of V_1, V_2 and V_3 separately. We also choose $\tau < 1$ as done in Lemma 3.1.

For V_1, we find that

$$\|V_1(\cdot,t)\|_\infty \le c_2\tau^\vartheta e^{-\epsilon t}\|v_0\|_q \le \|v_0\|_\infty \quad \text{for all } t \in (\tau, T_{\max}), \tag{3.15}$$

where $\vartheta \in (\frac{n}{2q}, 1)$, $q > n$ and $\epsilon > 0$.

For V_2, set $m = 0$, $q > n$ and $p = \infty$ in Lemma 2.3, so we can choose $\theta \in (\frac{n}{2q}, \frac{1}{2})$. In this case, we have $\varepsilon \in (0, \frac{1}{2} - \theta)$. Then there exist positive constants C_6 and μ such that

$$\begin{aligned} \|V_2(\cdot,t)\|_\infty &\le c_1\|(A+1)^\theta V_2(\cdot,t)\|_q \\ &\le \xi c_1 c_3 \int_0^t \|(A+1)^\theta e^{-(t-s)(A+1)}\nabla\cdot(q(v(\cdot,t))\nabla u(\cdot,t))\|_q ds \\ &\le \xi c_1 c_3 \int_0^t e^{-(t-s)}\|(A+1)^\theta e^{-(t-s)A}\nabla\cdot(q(v(\cdot,t))\nabla u(\cdot,t))\|_q ds \\ &\le C_6 \int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon} e^{-(\mu+1)(t-s)}\|q(v(\cdot,t))\nabla u(\cdot,t)\|_q ds \end{aligned} \tag{3.16}$$

for all $t \in (0, T_{\max})$. From (3.4), we have

$$\|\nabla u(\cdot,t)\|_\infty \le H(\tau) \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.17}$$

Hence, there exists $C_7 > 0$ such that

$$\|q(v(\cdot,t))\nabla u(\cdot,t)\|_q \le C_7 \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.18}$$

Therefore, we obtain $C_8 > 0$ such that

$$\begin{aligned} \|V_2(\cdot,t)\|_\infty &\le C_6 C_7 \int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon} e^{-(\mu+1)(t-s)} ds \\ &\le C_6 C_7 \int_0^\infty \sigma^{-\theta-\frac{1}{2}-\varepsilon} e^{-(\mu+1)\sigma} d\sigma \\ &\le C_8 \Gamma\left(\frac{1}{2}-\theta-\varepsilon\right) \quad \text{for all } t \in (\tau, T_{\max}), \end{aligned} \tag{3.19}$$

where $\Gamma(x)$ is the Gamma function. Since $\frac{1}{2}-\theta-\varepsilon > 0$, then $\Gamma(\frac{1}{2}-\theta-\varepsilon)$ is positive and real-valued.

Finally, for V_3, by using (2.16) and (2.17), we have

$$\begin{aligned} \|V_3(\cdot,t)\|_{1,p} &\le c_1 \|(A+1)^\theta V_3(\cdot,t)\|_q \\ &\le c_1 c_2 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \|\psi(u(\cdot,t), v(\cdot,t))\|_q ds \\ &\le c_1 c_2 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \|f(v(\cdot,t)) - \phi(u(\cdot,t), v(\cdot,t)) + v(\cdot,t)\|_q ds \\ &\le C_9 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \left(\|u(\cdot,t)\|_q + \|v(\cdot,t)\|_q\right) ds \\ &\le C_9 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} ds \\ &\le C_9 \int_0^\infty \sigma^{-\theta} e^{-\gamma\sigma} d\sigma \le C_9 \Gamma(1-\theta) \quad \text{for all } t \in (\tau, T_{\max}), \end{aligned} \tag{3.20}$$

where C_9 denotes a generic constant that may vary from line to line, and $\Gamma(1-\theta) > 0$ for $1-\theta > 0$. For $p > n$, from the Sobolev embedding theorem, we have

$$\|V_3(\cdot,t)\|_\infty \le C_{10} \Gamma(1-\theta) \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.21}$$

Therefore, by (3.15), (3.19) and (3.21), we obtain that $\|v(\cdot,t)\|_\infty$ is bounded for $t \in (0, T_{\max})$. Along with Lemma 2.1(2), this proves that $T_{\max} = \infty$ and therefore $(u(x,t), v(x,t))$ is bounded for $(x,t) \in \Omega \times (0,\infty)$. □

Now we complete the proof of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. With the results established in Lemmas 2.1 and 3.2, we obtain the desired result of Theorem 1.1. The method of proving the global existence and boundedness of system (1.2) can also be applied to system (1.3) directly, which we omit here. □

4. Dynamical Behavior of Predator-Taxis System

In this section, we study the dynamic behavior of the system (1.5), which is a special form of (1.2). We will consider the local and global stability of the constant equilibrium solutions. Note that from the principle of linearized stability for quasilinear parabolic problems (see, for example, Refs. 13 and 48), if all the eigenvalues of the linearized equation of (1.5) at an equilibrium are of negative real parts, then the equilibrium is locally asymptotically stable in $W^{1,p}(\Omega)$. Hence in the following the local stability will be considered in the sense of linear stability. The linearized problem of (1.5) at any equilibrium $e_* = (u, v)$ can be expressed by

$$\begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix} = L(\xi) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = D \begin{pmatrix} \Delta\phi \\ \Delta\psi \end{pmatrix} + J_{(u,v)} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{4.1}$$

where

$$D = \begin{pmatrix} d & 0 \\ \xi v & 1 \end{pmatrix}, \quad J_{(u,v)} = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix} \tag{4.2}$$

and

$$\begin{aligned} &A(u,v) = c\Phi(v) - k(u) - k'(u)u, \quad B(u,v) = cu\Phi'(v), \\ &C(u,v) = -\Phi(v), \quad D(u,v) = f'(v) - u\Phi'(v). \end{aligned} \tag{4.3}$$

The stability of e_* is determined by the eigenvalue problem:

$$L(\xi) \begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \beta \begin{pmatrix} \phi \\ \varphi \end{pmatrix},$$

that is

$$\begin{cases} d\Delta\phi + A\phi + B\varphi = \beta\phi, & x \in \Omega, \\ \Delta\varphi + \xi v \Delta\phi + C\phi + D\varphi = \beta\varphi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\psi}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Recall that $-\Delta$ under Neumann boundary condition has eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ and $\lim_{i\to\infty} \mu_i = \infty$. Let $\phi_i(x)$ be the normalized eigenfunction corresponding to μ_i. Assume that β is an eigenvalue of (4.4) with corresponding eigenfunction (ϕ, φ). Then by using Fourier expansion, there exists $\{a_n\}, \{b_n\}$ such that

$$\phi(x) = \sum_{n=0}^{\infty} a_n \phi_n(x), \quad \varphi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x).$$

With a straightforward analysis, we obtain that

$$L_n(\xi)\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \beta \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad n = 0, 1, 2, \ldots,$$

where

$$L_n(\xi) = \begin{pmatrix} A - d\mu_n & B \\ C - \xi v \mu_n & D - \mu_n \end{pmatrix}. \tag{4.5}$$

It follows that the eigenvalues of (4.4) are given by the eigenvalues of L_n for $n = 0, 1, 2, \ldots$. The characteristic equation of L_n is

$$\gamma^2 - T_n \gamma + D_n = 0, \tag{4.6}$$

where

$$\begin{aligned} T_n(\xi) &= A + D - (d+1)\mu_n, \\ D_n(\xi) &= d\mu_n^2 - (A + dD - B\xi v)\mu_n + AD - BC. \end{aligned} \tag{4.7}$$

Notice that the constant steady states of (1.5) are equivalent to the constant steady states of (1.5) without taxis. It is clear that $(0, 0)$ and $(0, N)$ are two trivial constant steady states of (1.5). If (u_*, v_*) is a positive constant steady state solution of (1.5), then it is in the form of $(\psi(\lambda), \lambda)$, where λ satisfies

$$\psi(\lambda) = \frac{f(\lambda)}{\Phi(\lambda)}, \quad c\Phi(\lambda) = k(\psi(\lambda)). \tag{4.8}$$

In the following, we consider two cases of different predator mortality rates: (i) constant $k(u) = k$; and (ii) linear $k(u) = k + lu$. Note that linear increasing mortality rate is a result of intraspecific competition.

4.1. *Constant mortality case*

In this subsection, we assume that $k(u) = k$, then we study the dynamical behavior of the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + cu\Phi(v) - ku, & x \in \Omega, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u) + f(v) - u\Phi(v), & x \in \Omega, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{4.9}$$

First we show the local stability of non-negative constant steady states of the system (4.9).

Lemma 4.1. *Suppose that $d > 0$, $\xi \geq 0$, (H_0) and (H_5) hold. Then for system* (4.9),

(1) $(0, 0)$ *is unstable for all* $\lambda > 0$;
(2) $(0, N)$ *is locally asymptotically stable for* $\lambda > N$ *and is unstable for* $0 < \lambda < N$;
(3) $(\psi(\lambda), \lambda)$ *is locally asymptotically stable for* $M < \lambda < N$ *and is unstable for* $0 < \lambda < M$.

Proof. (1) At $e_* = (0, 0)$, then $L_n(\xi) = \begin{pmatrix} -k - d\mu_n & 0 \\ 0 & \Phi'(0)\psi(0) - \mu_n \end{pmatrix}$. If $n = 0$, then one of the eigenvalues of L_n is positive, so $(0, 0)$ is unstable.

(2) At $e_* = (0, N)$, $L_n(\xi) = \begin{pmatrix} \Phi(N) - k - d\mu_n & 0 \\ -\Phi(N) - \xi N\mu_n & \Phi(N)\psi'(N) - \mu_n \end{pmatrix}$. If $\lambda > N$, then $\Phi(N) - k < 0$, so for all $n \geq 0$,

$$T_n(\xi) = -(d + 1)\mu_n + \Phi(N) - k + \Phi(N)\psi'(N) < 0,$$

$$D_n(\xi) = (\mu_n d - \Phi(N) + k)(\mu_n - \Phi(N)\psi'(N)) > 0.$$

So $(0, N)$ is locally asymptotically stable. On the other hand, if $\lambda < N$, then $\Phi(N) - k > 0$. For $n = 0$, $D_0(\xi) = (\Phi(N) - k)\Phi(N)\psi'(N) < 0$, which implies that L_n has at least one root with positive real part. So $(0, N)$ is unstable.

(3) At $e_* = (\psi(\lambda), \lambda)$, $L_n(\xi) = \begin{pmatrix} -d\mu_n & c\psi(\lambda)\Phi'(\lambda) \\ -\Phi(\lambda) - \xi\lambda\mu_n & \Phi(\lambda)\psi'(\lambda) - \mu_n \end{pmatrix}$. If $M \leq \lambda \leq N$, then from assumption (H_7), $\Phi(\lambda)\psi'(\lambda) < 0$. For all $n \geq 0$,

$$T_n(\xi) = -(d + 1)\mu_n + \Phi(\lambda)\psi'(\lambda) < 0,$$

$$D_n(\xi) = d\mu_n(\mu_n - \Phi(\lambda)\psi'(\lambda)) + c\psi(\lambda)\Phi'(\lambda)(\xi\lambda\mu_n + \Phi(\lambda)) > 0.$$

So $(\psi(\lambda), \lambda)$ is locally asymptotically stable. On the other hand, if $0 \leq \lambda \leq M$, then $\Phi(\lambda)\psi'(\lambda) > 0$ and for $n = 0$, $T_0(\xi) = \Phi(\lambda)\psi'(\lambda) > 0$, which implies that L_0 has at least one eigenvalue with positive real part. So $(\psi(\lambda), \lambda)$ is unstable. □

For the results in Lemma 4.1, the case of $\xi = 0$ has been obtained in Ref. 62, and here we prove that the same stability result holds for $\xi > 0$. Hence the predator-taxis does not affect the local stability of constant steady states.

Next we prove the global stability of the positive steady state $(\psi(\lambda), \lambda)$.

Theorem 4.1. *Assume that (H_0)–(H_6) hold. If $\lambda \in (M_*, N]$ where $M_* \in (M, N)$ such that $\psi(M_*) = \psi(0)$, and ξ satisfies* (1.6), *then $(\psi(\lambda), \lambda)$ is globally asymptotically stable in $L^\infty(\Omega)$ for system* (4.9) *with $u_0, v_0 \geq 0$ and $v_0 \not\equiv 0$.*

Proof. Define $E : X \times X \to \mathbb{R}$, $X = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0\}$ and

$$E(u, v) = \frac{1}{c}\int_\Omega \left(u - \psi(\lambda) - \psi(\lambda)\ln\frac{u}{\psi(\lambda)}\right) + \int_\Omega \left(\int_\lambda^v \frac{\Phi(s) - \Phi(\lambda)}{\Phi(s)} ds\right). \tag{4.10}$$

Let $(u(\cdot,t),v(\cdot,t))$ be a solution of the system (4.9). Then we have

$$\begin{aligned}\frac{d}{dt}E(u,v) &= \frac{1}{c}\int_\Omega\left(1-\frac{\psi(\lambda)}{u}\right)u_t+\int_\Omega\frac{\Phi(v)-\Phi(\lambda)}{\Phi(v)}v_t\\ &= -d\frac{\psi(\lambda)}{c}\int_\Omega\left|\frac{\nabla u}{u}\right|^2-\Phi(\lambda)\int_\Omega\Phi'(v)\left|\frac{\nabla v}{\Phi(v)}\right|^2\\ &\quad -\xi\int_\Omega v\Phi(\lambda)\frac{\Phi'(v)}{\Phi^2(v)}\nabla u\cdot\nabla v+\frac{1}{c}\int_\Omega\left(1-\frac{\psi(\lambda)}{u}\right)(cu\Phi(v)-ku)\\ &\quad +\int_\Omega(\Phi(v)-\Phi(\lambda))\left(\frac{f(v)}{\Phi(v)}-u\right).\end{aligned}\tag{4.11}$$

The first three terms of the right-hand side of (4.11) can be written as

$$\begin{aligned}A_1 :&= -d\frac{\psi(\lambda)}{c}\int_\Omega\left|\frac{\nabla u}{u}\right|^2-\Phi(\lambda)\int_\Omega\Phi'(v)\left|\frac{\nabla v}{\Phi(v)}\right|^2-\xi\int_\Omega v\Phi(\lambda)\frac{\Phi'(v)}{\Phi^2(v)}\nabla u\cdot\nabla v\\ &= -\begin{bmatrix}\nabla u\\ \nabla v\end{bmatrix}^T\Lambda\begin{bmatrix}\nabla u\\ \nabla v\end{bmatrix},\end{aligned}\tag{4.12}$$

where

$$\Lambda=\begin{pmatrix}\dfrac{d\psi(\lambda)}{cu^2} & \dfrac{\xi\Phi(\lambda)v\Phi'(v)}{2\Phi^2(v)}\\ \dfrac{\xi\Phi(\lambda)v\Phi'(v)}{2\Phi^2(v)} & \dfrac{\Phi(\lambda)\Phi'(v)}{\Phi^2(v)}\end{pmatrix}.\tag{4.13}$$

From (H_5) and (1.6), we obtain that the matrix Λ is positive semidefinite, which implies that $A_1\le 0$. On the other hand, the last two terms of the right-hand side of (4.11) can be evaluated as

$$\begin{aligned}A_2 :&= \frac{1}{c}\int_\Omega\left(1-\frac{\psi(\lambda)}{u}\right)(cu\Phi(v)-ku)+\int_\Omega(\Phi(v)-\Phi(\lambda))\left(\frac{f(v)}{\Phi(v)}-u\right)\\ &= \int_\Omega(u-\psi(\lambda))(\Phi(v)-\Phi(\lambda))+\frac{1}{c}\int_\Omega(u-\psi(\lambda))(c\Phi(\lambda)-k)\\ &\quad +\int_\Omega(\Phi(v)-\Phi(\lambda))\left(\frac{f(v)}{\Phi(v)}-u\right)\\ &= \frac{1}{c}\int_\Omega(u-\psi(\lambda))(c\Phi(\lambda)-k)+\int_\Omega(\Phi(v)-\Phi(\lambda))\left(\frac{f(v)}{\Phi(v)}-\psi(\lambda)\right)\\ &= \int_\Omega(\Phi(v)-\Phi(\lambda))(\psi(v)-\psi(\lambda))\le 0.\end{aligned}\tag{4.14}$$

Here the last inequality holds since $\lambda\in(M_*,N]$ and $\psi(v)$ is decreasing in (M,N). Combining (4.12) with (4.14), we have

$$\frac{d}{dt}E(u,v)\le 0,\tag{4.15}$$

and $\frac{d}{dt}E(u,v) = 0$ if and only if when $u = \psi(\lambda)$ and $v = \lambda$. Now from the LaSalle's invariance principle,[19] the global existence of solution proved in Theorem 1.1, and the argument in the proof of Lemma 3.3 in Ref. 6, we conclude that $(\psi(\lambda), \lambda))$ is globally asymptotically stable in $L^\infty(\Omega)$ for all non-negative initial conditions (u_0, v_0) with $v_0 \neq 0$. □

The application of the Lyapunov function $E(u,v)$ in the above proof to chemotaxis or prey-taxis reaction–diffusion system was first introduced in Ref. 6, see also Ref. 25. It has been previously used to prove the global stability for reaction–diffusion systems without chemotaxis or prey-taxis.[21,81] Note that Lemma 4.1 and Theorem 4.1 give parts (1) and (2) of Theorem 1.3, and part (3) of Theorem 1.3 follows from the corresponding results for ODE models (see Refs. 62 and 81).

We apply the results above to the diffusive Rosenzweig–MacArthur predator–prey model with predator-taxis:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + \dfrac{Buv}{h+v} - ku, & x \in \Omega,\ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (u \nabla v) + Ev\left(1 - \dfrac{v}{N}\right) - \dfrac{Buv}{h+v}, & x \in \Omega,\ \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{4.16}$$

Note that when $\xi = 0$, (4.16) has been studied in Ref. 81 and the global existence of solutions follows from Ref. 2. For the case of $\xi > 0$, now we have the following corollary of Theorems 1.1 and 4.1.

Corollary 4.1. *Consider the system* (4.16), *and assume that* $\xi, d, h, k, B, E, N > 0$. *Then the results in Theorem* 1.1 *hold for* (4.16). *Moreover, if*

$$0 \leq \xi < \sqrt{\frac{4dB(BN - kN - k)}{hkN^3(B-k)^2}}, \quad \textit{and} \quad \frac{B(N-h)}{N} < k < \frac{BN}{h+N}, \tag{4.17}$$

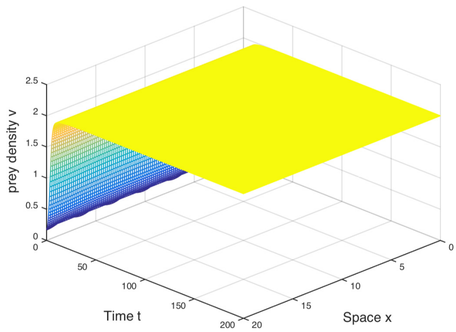
then the positive constant steady state solution $(\psi(\lambda), \lambda)$ *of* (4.16) *is global asymptotically stable, where*

$$\lambda = \frac{kh}{B-k}, \quad \psi(\lambda) = \frac{E}{BN}(N-\lambda)(h+\lambda).$$

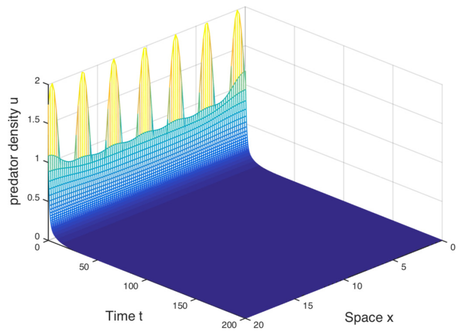
Some numerical simulations of (4.16) are shown in Fig. 1, where we use $D = B = h = \xi = 1$ and $N = 2$. Note that in this case the two threshold values of k in (4.17) are given by

$$k_1 = \frac{B(N-h)}{N} = \frac{1}{2}, \quad k_2 = \frac{BN}{h+N} = \frac{2}{3}.$$

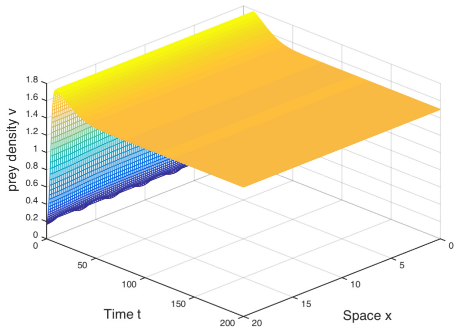
If $k = 0.8 > k_2$, from Figs. 1(a) and 1(b), we find that the solution tends to the boundary steady state $(0, N) = (0, 2)$ as $t \to \infty$; if $k = 0.6 \in (k_1, k_2)$,



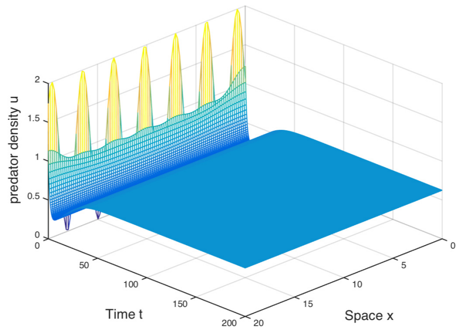
(a) Prey density when $k = 0.8$



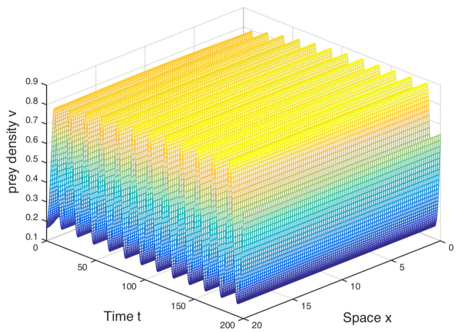
(b) Predator density when $k = 0.8$



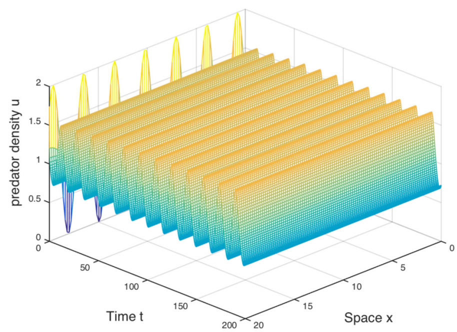
(c) Prey density when $k = 0.6$



(d) Predator density when $k = 0.6$



(e) Prey density when $k = 0.3$



(f) Predator density when $k = 0.3$

Fig. 1. Numerical simulation of solutions (u, v) of the system (4.16). Here $n = 1$, $E = B = h = \xi = 1$, $N = 2$, $\Omega = (0, 20)$ with the initial data $u_0 = v_0 = 1 + \sin(2x)$, and the solutions are integrated with Matlab PDE solver PDEPE.

from Figs. 1(c) and 1(d), the solution goes to the positive constant steady state $(\psi(\lambda), \lambda) = (1.5, 0.625)$ as $t \to \infty$; and if $k = 0.3 < k_1$, from Figs. 1(e) and 1(f), the solution approaches to a periodic solution as $t \to \infty$. This confirms the results in Theorems 1.1 and 4.1, and Corollary 4.1. These results show that the predator-taxis does not have effect on the asymptotical behavior of solutions of (4.16) as the solutions have same behavior when $\xi = 0$.

4.2. *Linearly increasing mortality rate*

In this subsection, we assume that $k(u) = k + lu$ with $k, l > 0$, then we study the dynamical behavior of the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + cu\Phi(v) - ku - lu^2, & x \in \Omega, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u) + f(v) - u\Phi(v), & x \in \Omega, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{4.18}$$

We still assume that $\Phi(v)$ and $f(v)$ satisfy (H_5) and (H_6). Under these conditions, (4.18) may have multiple positive constant steady states. But we will focus on possible diffusion and predator-taxis induced instability phenomenon for (4.18), so we assume (see Ref. 37) that (4.18) has a positive constant steady state $(\psi(\lambda), \lambda)$ (λ satisfying (4.8)) which is locally asymptotically stable for the corresponding ODE dynamics:

$$\begin{cases} \dfrac{du}{dt} = cu\Phi(v) - ku - lu^2, \\ \dfrac{dv}{dt} = f(v) - u\Phi(v) = \Phi(v)(\psi(v) - u). \end{cases} \tag{4.19}$$

From (4.3), we know that the Jacobian matrix at $(\psi(\lambda), \lambda)$ is

$$J = \begin{pmatrix} -l\psi(\lambda) & c\psi(\lambda)\Phi'(\lambda) \\ -\Phi(\lambda) & \psi'(\lambda)\Phi(\lambda) \end{pmatrix} := \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{4.20}$$

and the trace and the determinant of J are given by

$$T_0 = -l\psi(\lambda) + \psi'(\lambda)\Phi(\lambda), \quad D_0 = \psi(\lambda)\Phi(\lambda)[c\Phi'(\lambda) - l\psi'(\lambda)]. \tag{4.21}$$

We have the following result regarding the local stability of a positive constant steady state $(\psi(\lambda), \lambda)$ of (4.18).

Theorem 4.2. *Assume that* (H_0), (H_5) *and* (H_6) *hold,* $\{\mu_i\}$ *are the positive eigenvalues of* $-\Delta$ *on* $H^1(\Omega)$, *and* $A(\lambda), B(\lambda)$, $C(\lambda)$, $D(\lambda)$, T_0 *and* D_0 *are defined as in* (4.20) *and* (4.21). *Suppose that* $(\psi(\lambda), \lambda)$ *is a positive constant steady state solution of* (4.18) *satisfying*

$$\psi'(\lambda) < \min\left\{ \frac{l\psi(\lambda)}{\Phi(\lambda)}, \frac{c\Phi'(\lambda)}{l} \right\}. \tag{4.22}$$

(1) $(\psi(\lambda), \lambda)$ *is locally asymptotically stable with respect to the dynamics of* (4.19).

(2) *If the parameters* ξ, d *satisfy*

$$\xi > \frac{A(\lambda) + dD(\lambda) - 2\sqrt{dD_0}}{\lambda B(\lambda)}, \tag{4.23}$$

then $(\psi(\lambda), \lambda)$ *is locally asymptotically stable with respect to the dynamics of* (4.18).

(3) *If*

$$\xi > -\frac{l}{c\lambda\Phi'(\lambda)} = \frac{A(\lambda)}{\lambda B(\lambda)}, \tag{4.24}$$

then $(\psi(\lambda), \lambda)$ *is unstable with respect to the dynamics of* (4.18) *if and only if* $\mathcal{A} := \{i \in \mathbb{N} : D(\lambda) - \mu_i > 0\} \neq \emptyset$, *and*

$$d > \min_{i \in \mathcal{A}} \frac{\mu_i(B(\lambda)\lambda\xi - A(\lambda)) + D_0}{\mu_i(D(\lambda) - \mu_i)} := d_*(\xi, \Omega), \tag{4.25}$$

(4) *If* $d > 0$, *then* $(\psi(\lambda), \lambda)$ *is unstable with respect to the dynamics of* (4.18) *if and only if*

$$\xi < \max_{i \in \mathbb{N}} \frac{\mu_i(A(\lambda) + dD(\lambda)) - d\mu_i^2 - D_0}{\lambda B(\lambda)\mu_i} := \xi_*(d, \Omega), \tag{4.26}$$

Proof. (1) It is easy to see that $(\psi(\lambda), \lambda)$ is locally asymptotically stable with respect to the dynamics of (4.19) if and only if (4.22) holds. That is, $T_0 < 0$ and $D_0 > 0$.

(2) With respect to the dynamics of (4.18), we have

$$T_n(\xi) = -(d+1)\mu_n + A(\lambda) + D(\lambda),$$

$$D_n(\xi) = d\mu_n^2 - (A(\lambda) + dD(\lambda) - \lambda B(\lambda)\xi)\mu_n + D_0.$$

Then $(\psi(\lambda), \lambda)$ is locally asymptotically stable with respect to the dynamics of (4.18) if $T_n(\xi) < 0$ and $D_n(\xi) > 0$ for all $n \in \mathbb{N}$. Since $T_0 < 0$, then $T_n(\xi) = -(d+1)\mu_n + T_0 < 0$. To prove $D_n(\xi) > 0$, we define

$$\xi(p) = \frac{dp^2 - (A(\lambda) + dD(\lambda))p + D_0}{-B(\lambda)\lambda p}, \quad p > 0. \tag{4.27}$$

Then we have, for any $p > 0$,

$$\begin{aligned} \xi(p) &= \frac{A(\lambda) + dD(\lambda)}{B(\lambda)\lambda} - \frac{1}{B(\lambda)\lambda}\left(dp + \frac{D_0}{p}\right) \\ &\leq \frac{A(\lambda) + dD(\lambda) - 2\sqrt{dD_0}}{B(\lambda)\lambda}. \end{aligned} \tag{4.28}$$

If $D_n(\xi) = 0$ for some $n \in \mathbb{N}$, then $\xi(\mu_n) = 0$. Hence when (4.23) is satisfied, (4.28) cannot hold thus $D_n(\xi) > 0$ for all $n \in \mathbb{N}$ and $(\psi(\lambda), \lambda)$ is locally asymptotically stable with respect to the dynamics of (4.18).

(3) If $\mathcal{A} = \emptyset$ and $\mu_n - D(\lambda) > 0$ for all $n \in \mathbb{N}$, then $D_n(\xi) \geq d\mu_n^2 - (A(\lambda) - \lambda B(\lambda)\xi)\mu_n + D_0 > 0$ from (4.24) and $D_0 > 0$. Thus when $\mathcal{A} = \emptyset$, the constant steady state $(\psi(\lambda), \lambda)$ is locally asymptotically stable with respect to the dynamics of (4.18). But if $\mathcal{A} \neq \emptyset$, then there exists $n \in \mathcal{A}$, such that

$$\frac{\mu_n(B(\lambda)\lambda\xi - A(\lambda)) + D_0}{\mu_n(D(\lambda) - \mu_n)} = d_*(\xi, \Omega).$$

Then when $d > d_*(\xi, \Omega) > 0$, $D_n(\xi) < 0$ hence the matrix $L_n(\xi)$ has a positive eigenvalue and $(\psi(\lambda), \lambda)$ is unstable with respect to the dynamics of (4.18). The proof of part (4) is similar to that of part (3) and we omit it. □

Remark 4.1. (1) The relation (4.23) provides a condition for the stability of $(\psi(\lambda), \lambda)$ for any bounded domain Ω, but the conditions for instability such as (4.25) and (4.26) depend on μ_i which is determined by Ω.

(2) The result in part (3) of Theorem 4.2 shows a generalized Turing-type instability. When $\xi = 0$ and $d > d_*(0, \Omega)$, the constant steady state $(\psi(\lambda), \lambda)$ becomes unstable so a potential stable non-constant steady state exists. The critical diffusion coefficient $d_*(0, \Omega)$ is a threshold for the spatial pattern formation. Now for positive predator-taxis sensitivity coefficient $\xi > 0$, a similar critical diffusion coefficient $d_*(\xi, \Omega)$ still exists, but $d_*(\xi, \Omega) > d_*(0, \Omega)$ as $B(\lambda) > 0$. Hence the pattern formation for $\xi > 0$ is achieved with a larger predator diffusion rate. In that sense, the predator-taxis in (4.18) makes the pattern formation harder to happen.

(3) The result in parts (3) and (4) of Theorem 4.2 holds for not only positive $\xi > 0$ but also for negative ξ satisfying (4.24). Hence for $\xi \in (-l/(c\lambda\Phi'(\lambda)), 0)$, the critical diffusion coefficient $d_*(\xi, \Omega) < d_*(0, \Omega)$, hence an attractive predator-taxis for the prey will enhance the formation of spatial patterns, and spatial pattern formation could occur if one decreases the predator-taxis sensitivity coefficient ξ. But an attractive predator-taxis is not biologically reasonable.

(4) The set $\mathcal{A}$ is independent of ξ so the predator-taxis does not change the unstable modes.

Next we show that when $(\psi(\lambda), \lambda)$ is unstable, there exist positive non-constant steady state solutions of (4.18). To show that we use bifurcation theory to prove the existence of positive non-constant steady state solutions. The bifurcations can be shown with parameter d or ξ as shown in Theorem 4.2. To observe the effect of the predator-taxis, we use the predator-taxis sensitivity coefficient ξ as the bifurcation parameter. From the relation given in (4.26), we define the potential bifurcation points

$$\xi_i = \frac{\mu_i(A(\lambda) + dD(\lambda)) - d\mu_i^2 - D_0}{\lambda B(\lambda)\mu_i}, \quad i \in \mathbb{N}, \tag{4.29}$$

where $A(\lambda), B(\lambda)$, $D(\lambda)$ and D_0 are defined as in (4.20) and (4.21). Apparently $\lim_{i\to\infty}\xi_i = -\infty$.

Theorem 4.3. *Assume that* $d > 0$, (H_0), (H_5) *and* (H_6) *hold, and for some* $j \in N$, μ_j *is a simple eigenvalue of* $-\Delta$ *in* Ω *with Neumann boundary condition and the corresponding eigenfunction is* $\phi_j(x)$. *Let* $(\psi(\lambda), \lambda)$ *be the unique constant steady state solution of* (4.18) *satisfying* (4.22), *and let* ξ_j *be defined as in* (4.29) *such that* $\xi_j \neq \xi_i$ *for any* $i \neq j$. *Then*:

(1) *Near* $(\xi_j, \psi(\lambda), \lambda)$, *the set of positive non-constant steady state solutions of* (4.18) *is a smooth curve* $\Gamma_j = \{\xi, u(s), v(s) : s \in (-\varepsilon, \varepsilon)\}$, *where*

$$\begin{cases} u(s) = \psi(\lambda) + sa_j\phi_j(x) + sh_{1,j}(s), \\ v(s) = \lambda + sb_j\phi_j(x) + sh_{2,j}(s) \end{cases}$$

for some continuous function $h_{1,j}(s)$, $h_{2,j}(s)$ *such that* $h_{1,j}(0) = h_{2,j}(0) = 0$, *and* (a_j, b_j) *satisfies*

$$L_j(\xi_j)\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(2) *The smooth curve* Γ_j *in part* (1) *is contained in a connected component* C_j *of* Σ, *which is the closure of the set of positive non-constant steady state solutions of* (4.18), *and either* C_j *is unbounded or* C_j *contains another* $(\xi_i, \psi(\lambda), \lambda)$ *with* $\xi_i \neq \xi_j$.

Proof. We use a global bifurcation theorem formulated in Shi and Wang (Ref. 52, Theorem 4.3) which is based on almost the same conditions of the local bifurcation theorem from a simple eigenvalue due to Crandall and Rabinowitz (Ref. 9, Theorem 1.7), and it is also a generalization of the classical Rabinowitz global bifurcation theorem.[49]

Let $p > n$, $Z = H^p_N(\Omega) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$, $Y = L^p(\Omega)$. Define a nonlinear mapping $F : \mathbb{R} \times Z \times Z \to Y \times Y$ by

$$F(\xi, u, v) = \begin{pmatrix} -d\Delta u - cu\Phi(v) + uk(u) \\ -\Delta v - \xi\nabla \cdot (v\nabla u) - f(v) + u\Phi(v) \end{pmatrix}.$$

Then $F(\xi, u, v) = 0$ is equivalent to (4.18). It is observed that $F \in C^1(\mathbb{R} \times Z \times Z)$; $F(\xi, \psi(\lambda), \lambda) = 0$ for all $\xi \in \mathbb{R}$. For any fixed $(\xi, u_1, v_1) \in \mathbb{R} \times Z \times Z$, the Fréchet derivative is given by

$$D_{(u,v)}F(\xi, u_1, v_1)(u, v) = \begin{pmatrix} -d\Delta u - [\Phi(v_1) - k(u_1) - u_1k'(u_1)]u + cu_1v \\ -d_2\Delta v - H(u, v) - \Phi(v_1)u - (f'(v_1) - u_1\Phi'(v_1))v \end{pmatrix},$$

where

$$H(u, v) = \xi v_1\Delta u + \xi\nabla v_1\nabla u + \xi\Delta u_1 v + \xi\nabla u_1\nabla v.$$

We show that the conditions for Theorem 4.3 in Ref. 52 are satisfied in several steps.

Step 1. For any fixed $(u_1, v_1) \in Z \times Z$, $D_{(u,v)}F(\xi, u_1, v_1)(u, v) : Z \times Z \to Y \times Y$ is a Fredholm operator with index zero.

Note that the leading order part of $D_{(u,v)}F(\xi, u_1, v_1)$ is given by

$$\begin{pmatrix} -d\Delta u \\ -\Delta v - \xi v_1 \Delta u \end{pmatrix} = -\begin{pmatrix} d & 0 \\ \xi v_1 & 1 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}.$$

Thus in the notation of Remark 2.5 of Case 3 in Ref. 52, $i = j = 1$, $\alpha_{11} = 1$, $N = 2$ and $a(x) = \left(\begin{smallmatrix} d & 0 \\ \xi v_1 & 1 \end{smallmatrix}\right)$ whose eigenvalues are $d > 0$ and $1 > 0$. The boundary condition is $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, which is equivalent to

$$\overrightarrow{0} = a(x) \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} \gamma_1(x) \quad (\text{so } b_1(x) = a(x)\alpha_{11}(x)\gamma_1(x)),$$

where $\gamma_1(x)$ is the unit outer normal of $\partial\Omega$. Note that $\delta(x) = I_{2\times 2}$ at $\partial\Omega$. So the condition in Remark 2.5 Ref. 52 of case "$(I - \delta(x))a(x)\delta(x) = 0, \ \forall x \in \partial\Omega$" is satisfied. It now follows that the operator $D_{(u,v)}F(\xi, u, v)$ and the Neumann boundary operator satisfy Agmon's condition for all angles in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Now by Corollary 2.11 in Ref. 52, $D_{(u,v)}F(\xi, u, v)$ is a Fredholm operator of index zero.

Step 2. $\dim N(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda)) = 1$.

From the definition of ξ_j, it is easy to verify that $D_j(\xi_j) = 0$ hence zero is an eigenvalue of L_j (defined in (4.5)) with an eigenvector $(a_j, b_j) = (-B(\lambda), A(\lambda) - d\mu_j)$. Then $V_j = (a_j, b_j)\phi_j$ is an eigenfunction of $L(\xi_j)$ (defined in (4.1) and evaluated at $(\psi(\lambda), \lambda)$) with eigenvalue zero. Indeed $-L(\xi) = D_{(u,v)}F(\xi, \psi(\lambda), \lambda)$. Since μ_j is a simple eigenvalue of $-\Delta$ and $\mu_j \neq \mu_i$, then the eigenvector is unique up to a constant multiple. Thus one has $N(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda)) = \text{span}\{V_j\}$ which is one-dimensional. Note that from Step 1, we also have that $\text{codim} R(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda)) = 1$ as $D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda)$ is Fredholm with index zero.

Step 3. $D_{(u,v)\xi}F(\xi_j, \psi(\lambda), \lambda)(V_j) \notin R(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda))$.

It is easy to see that the adjoint operator of $-L(\xi)$ is given by

$$-L^*(\xi) \begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \begin{pmatrix} -d\Delta - A(\lambda) & \xi\lambda\Delta - C(\lambda) \\ B(\lambda) & -\Delta - D(\lambda) \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix},$$

and an eigenvector of $L^*(\xi_j)$ corresponding to zero eigenvalue is $V_j^* = (a_j^*, b_j^*) = (-\mu_j + D(\lambda), B(\lambda))\phi_j$.

Now if $(h_1, h_2) \in R(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda))$, then there exists $(\phi_1, \varphi_1) \in Z \times Z$ such that

$$D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda) \begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix} = -L(\xi_j) \begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Then we have

$$\begin{aligned}\langle (h_1, h_2), (a_j^*, b_j^*)\phi_j \rangle &= \langle L(\xi_j)[(\phi_1, \varphi_1)], (a_j^*, b_j^*)\phi_j \rangle \\ &= \langle (\phi_1, \varphi_1), L^*(\xi_j)[(a_j^*, b_j^*)\phi_j] \rangle = \langle (\phi_1, \varphi_1), 0 \rangle = 0,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner-product in $[L^2(\Omega)]^2$. This shows that

$$R(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda)) = \left\{ (h_1, h_2) \in Z \times Z : \int_\Omega (a_j^* h_1 + b_j^* h_2)\phi_j dx = 0 \right\}.$$

Notice that

$$D_{(u,v)\xi}F(\xi_j, \psi(\lambda), \lambda)(V_j) = \begin{pmatrix} 0 \\ \lambda b_j \Delta \phi_j \end{pmatrix} = \begin{pmatrix} 0 \\ -\mu_j \lambda b_j \phi_j \end{pmatrix},$$

and

$$\begin{aligned}\int_\Omega \left(a_j^* \cdot 0 + b_j^* \cdot (-\mu_j \lambda b_j \phi_j) \right) \phi_j dx &= -\int_\Omega \mu_j \lambda b_j b_j^* \phi_j^2 dx \\ &= -\int_\Omega \mu_j \lambda (A(\lambda) - d\mu_j) B(\lambda) \phi_j^2 dx > 0,\end{aligned}$$

as $A(\lambda) < 0$ and $B(\lambda) > 0$. Hence

$$D_{(u,v)\xi}F(\xi_j, \psi(\lambda), \lambda)(V_j) \notin R(D_{(u,v)}F(\xi_j, \psi(\lambda), \lambda)).$$

Now we can apply Theorem 4.3 in Ref. 52 to obtain the existence of Γ_j and C_j which bifurcate from $(\xi_j, \psi(\lambda), \lambda)$. The solutions of (4.18) on C_j near the bifurcation point are apparently positive. We claim that any solution on C_j is positive. In fact, if this is not true, then from the maximum principle, C_j contains either $(\xi^*, 0, 0)$ or $(\xi^*, 0, N)$ for some $\xi^* \in \mathbb{R}$. However, from the linearization around $(0, 0)$ or $(0, N)$, any solutions bifurcating from $(u, v) = (0, 0)$ or $(0, N)$ are not positive near bifurcation points, and hence the positive solution branches cannot be connected to $(0, 0)$ or $(0, N)$. Therefore, either C_j is unbounded or C_j contains another $(\xi_i, \psi(\lambda), \lambda)$ with $\xi_i \neq \xi_j$. □

Note that Theorem 1.4 follows from Theorems 4.2 and 4.3. Some further discussions of the dynamics of (4.18) and (4.9) are given in the following remark.

Remark 4.2. (1) It is possible to show that all positive steady state solutions of (4.18) or (4.9) are *a priori* bounded for $\xi \in \mathbb{R}$. Hence the alternative of C_j being bounded implies that the projection of C_j onto ξ-axis contains either (ξ_j, ∞) or $(-\infty, \xi_j)$. We conjecture that it contains $(-\infty, \xi_j)$ as the local stability of $(\xi, \psi(\lambda), \lambda)$ holds for all large ξ (see Theorem 4.2(2)). Note that for one-dimensional Keller–Segel-type model, the non-constant steady states exist for $\chi \in (\chi_j, \infty)$ where χ_j is a steady state bifurcation point.[36,63]

(2) For either (4.18) or (4.9), a Hopf bifurcation analysis can be given to show that the system has a spatially homogenous time-periodic solution for small $\lambda > 0$ (recall that λ is the v-coordinate of constant steady state $(\psi(\lambda), \lambda)$),

and one can also show the existence of spatially non-homogeneous periodic solutions using the methods in Ref. 81. Note that the trace of L_n is given by $T_n = -(d+1)\mu_n + A(\lambda) + D(\lambda)$ which is independent of ξ, hence the predator-taxis does not affect these Hopf bifurcations, and we omit the details of Hopf bifurcations and existence of periodic orbits as they are similar to the ones in Ref. 81.

We apply the above results to the following example: a diffusive Rosenzweig–MacArthur predator–prey model with predator-taxis and mortality due to competition:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + \dfrac{Buv}{h+v} - ku - lu^2, & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u) + Ev\left(1 - \dfrac{v}{N}\right) - \dfrac{Buv}{h+v}, & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{4.30}$$

Note that when $\xi = 0$, Turing patterns of (4.30) bifurcated from a positive constant steady state have been studied in Ref. 37 and the global existence of solutions follows from Ref. 2. Now for the case of $\xi > 0$, we have the following corollary from the results above.

Corollary 4.2. *Consider the system* (4.30), *and assume that* $\xi, l, d, h, k, B, E, N > 0$, *then the results in Theorem* 1.1 *hold for* (4.30). *Moreover, let* $(\psi(\lambda), \lambda)$ *be a positive constant steady state solution of* (4.30) *satisfying*

$$\frac{E(N-h-2\lambda)}{BN} < \min\left\{\frac{lE(N-\lambda)(h+\lambda)^2}{B^2N\lambda}, \frac{Bh}{l(h+\lambda)^2}\right\}, \tag{4.31}$$

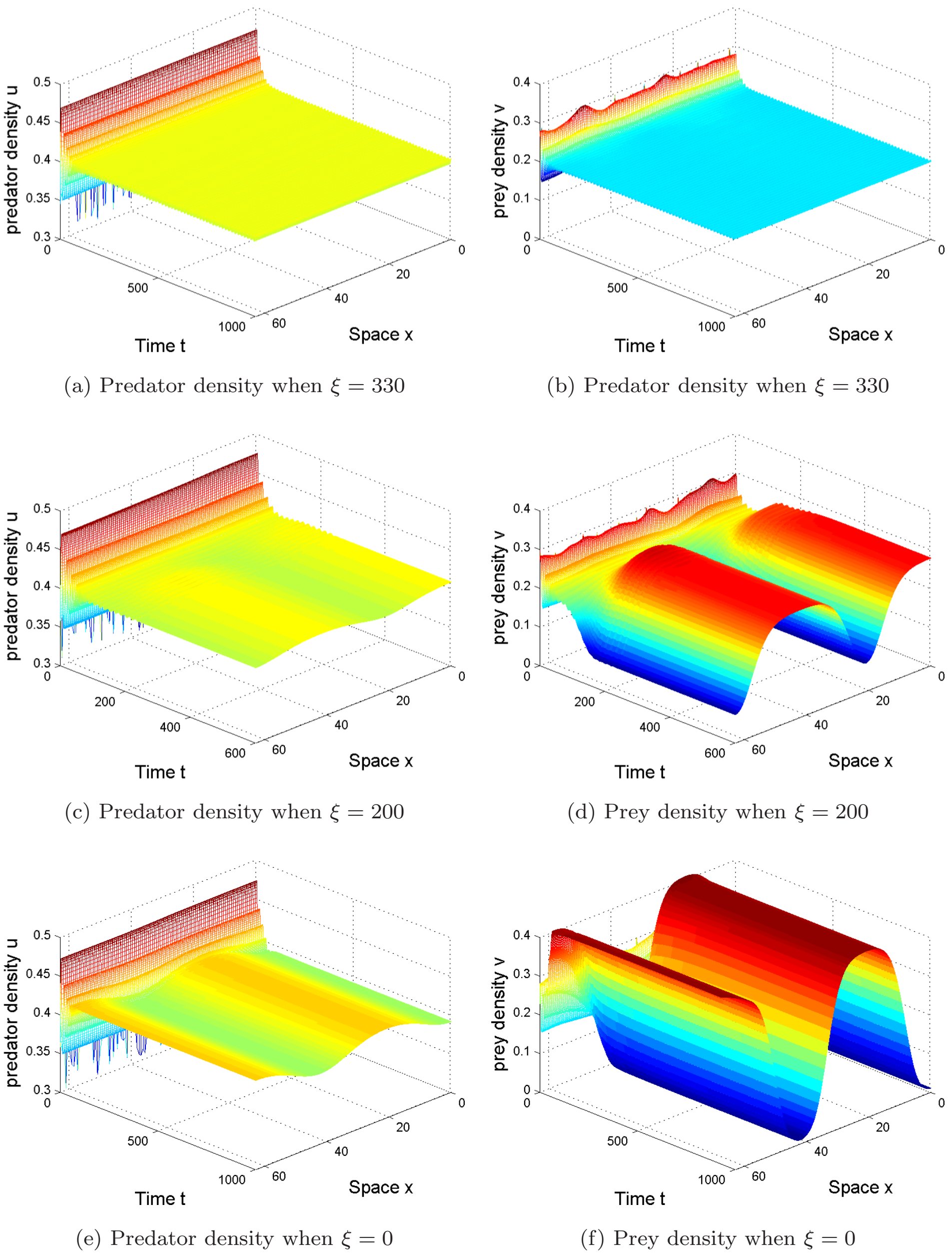
where $(\psi(\lambda), \lambda)$ *satisfies*

$$\frac{B\lambda}{h+\lambda} - k = \frac{lE(N-\lambda)(h+\lambda)}{BN}, \quad \psi(\lambda) = \frac{E(N-\lambda)(h+\lambda)}{BN}. \tag{4.32}$$

Then the results of stability/instability in Theorem 4.2 *and bifurcation of positive solutions in Theorem* 4.3 *hold for* $(\psi(\lambda), \lambda)$ *given the conditions in theorems are satisfied.*

Note that the stability/instability conditions (4.24)–(4.26) on ξ, d in Theorem 4.2 can be specified for (4.30), but we will not list the conditions here for simplicity. The condition (4.31) can be easily achieved when $(N-h)/2 < \lambda < N$.

Some numerical simulations of (4.30) are shown in Fig. 2, where we use $E = B = N = 1$, $h = 0.3$, $k = 0.2$, $l = 0.5$, $d = 400$ and $\Omega = (0, 20\pi)$ (one-dimensional space). We can calculate that $(0.4, 0.2)$ is the unique positive constant steady state solution, and $(0.4, 0.2)$ is locally asymptotically stable with respect to the ODE



(a) Predator density when $\xi = 330$

(b) Predator density when $\xi = 330$

(c) Predator density when $\xi = 200$

(d) Prey density when $\xi = 200$

(e) Predator density when $\xi = 0$

(f) Prey density when $\xi = 0$

Fig. 2. Numerical simulation of solutions (u, v) of the system (4.30). Here $n = 1$, $E = B = N = 1$, $h = 0.3$, $k = 0.2$, $l = 0.5$, $d = 400$, $\Omega = (0, 20\pi)$ with the initial data (u_0, v_0) being a small random perturbation of $(0.3, 0.2)$, and the solutions are integrated with Matlab PDE solver PDEPE.

dynamics as (4.22) is satisfied. Indeed for the steady state $(0.4, 0.2)$, the Jacobian matrix is given by

$$J(0.4, 0.2) = \begin{pmatrix} -0.2 & 0.48 \\ -0.4 & 0.12 \end{pmatrix}.$$

Then the condition (4.23) implies that when $\xi > 327.134$, the steady state $(0.4, 0.2)$ is locally asymptotically stable. From Figs. 2(a) and 2(b), we find that when $\xi = 330$, the solution of (4.30) converges to the positive constant steady state $(0.4, 0.2)$ as $t \to \infty$. In Figs. 2(c)–2(f), the solutions of (4.30) with $\xi = 200$ and $\xi = 0$ are plotted. In either case, the solution approaches to a spatial pattern (non-constant steady state) as $t \to \infty$. From Theorems 4.2 and 4.3, one can estimate the wave length or unstable mode of the spatial patterns from the bifurcation points. Indeed, the bifurcation points in (4.29) are given by

$$\xi_j = \frac{5975\mu_j - 50000\mu_j^2 - 21}{12\mu_j}, \quad \mu_j = \frac{j^2}{400}, \quad j \in \mathbb{N}.$$

It is easy to verify that $\xi_3 \approx 326.389 = \max_{j \in \mathbb{N}} \xi_j$. So indeed when $\xi > \xi_3 \approx 326.389$, the constant steady state $(0.4, 0.2)$ is locally asymptotically stable for the domain $\Omega = (0, 20\pi)$, but a steady state bifurcation occurs at $\xi = \xi_3$ with a spatial pattern with mode $j = 3$ and eigenmode $\cos(3x/20)$ emerging. In Figs. 2(c)–2(f), a spatial pattern with mode $j = 3$ and eigenmode $\cos(3x/20)$ is observed for $\xi = 200$ and $\xi = 0$. One can see that the amplitude of the pattern for $\xi = 0$ is significantly larger than the one for $\xi = 200$. Hence one can interpret it as the pattern vanishes as ξ increases to the bifurcation point ξ_3, and the increasing predator-taxis has the effect of eliminating spatial pattern.

5. Diffusive Epidemic Model with Repulsive Infected-Taxis

In this section, we consider some further dynamic behavior of the SIS epidemic reaction–diffusion model (1.3). For simplicity, we consider the following special case which was first proposed in Ref. 3 without the infected-taxis:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + \dfrac{\beta(x)uv}{u+v} - \gamma(x)u, & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u) + \gamma(x)u - \dfrac{\beta(x)uv}{u+v}, & x \in \Omega, \ \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, \ \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{5.1}$$

Here $u(x,t)$ and $v(x,t)$ represent the densities of the infected and susceptible individuals at location x and time t, respectively; $\beta(x)$ and $\gamma(x)$ are positive Hölder continuous functions and they account for the rates of disease transmission and disease recovery at x respectively. Note that the global existence and boundedness of solutions of (5.1) have been shown in Theorem 1.2, and in the following we discuss the dynamic behavior of (5.1). By adding the first equation to the second equation of the system (5.1) and integrating over Ω, we obtain that the total population size is a constant:

$$\int_\Omega [u(x,t)+v(x,t)]dx = \int_\Omega [u(x,0)+v(x,0)]dx := N, \quad t \geq 0. \tag{5.2}$$

As in Ref. 3, we say that (u,v) is a disease-free equilibrium (DFE) if (u,v) is a nonnegative equilibrium solution to (5.1) in which $v = 0$ for every $x \in \Omega$; and (u,v) is an endemic equilibrium (EE) if $v > 0$ for some $x \in \Omega$. Similar to Lemma 2.1 in Ref. 3, we know that (5.1) has a unique DFE $(\bar{u}, 0) = (N/|\Omega|, 0)$. Linearizing the system (5.1) around the DFE, we have the following linear eigenvalue problem:

$$\begin{cases} \Delta\phi - \beta(x)\psi + \gamma(x)\psi + \xi\bar{u}\Delta\psi + \mu\phi = 0, & x \in \Omega, \\ d\Delta\psi + \beta(x)\psi - \gamma(x)\psi + \mu\psi = 0, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

Then by using the methods in Refs. 3 and 66, we find that the basic reproduction number $\mathbf{R}_0$ of (5.1) is the same as the one for $\xi = 0$ in Ref. 3:

$$\mathbf{R}_0 = \sup_{\varphi \in H^1(\Omega),\ \varphi \neq 0} \left\{ \frac{\int_\Omega \beta\varphi^2}{\int_\Omega d|\nabla\varphi|^2 + \gamma\varphi^2} \right\}. \tag{5.4}$$

From the expression of $\mathbf{R}_0$ and by adapting the arguments in Lemmas 2.1–2.5 in Ref. 3, we have the following results regarding the dynamics of (5.1). Since the proofs are basically same as the ones in Ref. 3, we omit the proofs here.

Proposition 5.1. *Suppose that $d > 0$, $\xi \geq 0$ and $N > 0$ (as defined in (5.2)) is fixed, and $\beta, \gamma : \overline{\Omega} \to \mathbb{R}^+$ are Hölder continuous. Let $\mathbf{R}_0$ be defined as in (5.4). Then:*

(1) *Equation (5.1) has a unique DFE $(\bar{u}, 0) = (N/|\Omega|, 0)$, and if $\mathbf{R}_0 < 1$ then $(\bar{u}, 0)$ is globally asymptotically stable.*

(2) *$\mathbf{R}_0 = \mathbf{R}_0(d)$ is a monotone decreasing function of d such that*

$$\lim_{d\to 0} \mathbf{R}_0(d) = \max\left\{ \frac{\beta(x)}{\gamma(x)} : x \in \overline{\Omega} \right\}, \quad \lim_{d\to\infty} \mathbf{R}_0(d) = \frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx}.$$

(3) *If $\int_\Omega \beta < \int_\Omega \gamma$, then there exists a threshold value $d^* \in (0,\infty)$ such that $\mathbf{R}_0 > 1$ for $d < d^*$ and $\mathbf{R}_0 < 1$ for $d > d^*$ where d^* is defined as*

$$d^* = \sup\left\{ \frac{\int_\Omega (\beta-\gamma)\varphi^2}{\int_\Omega |\nabla\varphi|^2} : \varphi \in H^1(\Omega) \text{ and } \int_\Omega (\beta-\gamma)\varphi^2 > 0 \right\}. \tag{5.5}$$

(4) *If $\int_\Omega \beta > \int_\Omega \gamma$, then $\mathbf{R}_0 > 1$ for all $d > 0$.*

When $\mathbf{R}_0 > 1$, one can show that an EE exists for system (5.1), and the uniqueness and asymptotic stability of the EE for $\xi = 0$ is only known for some special cases.[3,46] Here we show that a global stability result of EE in Ref. 46 for $\xi = 0$ still holds for small $\xi > 0$.

Proposition 5.2. *Assume that the conditions in Proposition* 5.1 *hold. If* $\beta(x) = r\gamma(x)$ *on* $\overline{\Omega}$ *for some positive constant* $r > 1$, *then* (5.1) *has a constant EE* $(u_*, v_*) = ((r-1)N/(r|\Omega|), N/(r|\Omega|))$. *Moreover, if*

$$0 \le \xi < 2\sqrt{\frac{d}{r-1}\frac{r|\Omega|}{Q}}, \tag{5.6}$$

where Q is defined in (2.10), *then* (u_*, v_*) *is globally asymptotically stable.*

Proof. It is easy to see that $(u_*, v_*) = ((r-1)N/(r|\Omega|), N/(r|\Omega|))$ is a constant EE of (5.1) and $\mathbf{R}_0 > 1$ in this case. Define a Lyapunov function $E : X \times X \to \mathbb{R}$ by

$$E(u,v) = \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*}\right). \tag{5.7}$$

If $(u(\cdot,t), v(\cdot,t))$ is a solution of (5.1), then we have

$$\begin{aligned}\frac{d}{dt}E(u(\cdot,t),v(\cdot,t)) &= \int_\Omega \left(1 - \frac{u_*}{u}\right) u_t + \int_\Omega \left(1 - \frac{v_*}{v}\right) v_t \\ &= -d\int_\Omega \frac{u_*}{u^2}|\nabla u|^2 - \int_\Omega \frac{v_*}{v^2}|\nabla v|^2 - \xi\int_\Omega \frac{v_*}{v}\nabla u\cdot\nabla v \\ &\quad + \int_\Omega \beta u\left(\frac{\gamma}{\beta} - \frac{v}{u+v}\right)\left(\frac{u_*}{u} - \frac{v_*}{v}\right) \\ &:= A_1 + A_2.\end{aligned} \tag{5.8}$$

Now

$$\begin{aligned}A_2 &= \int_\Omega \beta u\left(\frac{\gamma}{\beta} - \frac{v}{u+v}\right)\left(\frac{u_*}{u} - \frac{v_*}{v}\right) = \int_\Omega \beta u\left(\frac{v_*}{u_*+v_*} - \frac{v}{u+v}\right)\left(\frac{u_*}{u} - \frac{v_*}{v}\right) \\ &= -\int_\Omega \frac{\beta(x)u^2 v}{(u_*+v_*)(u+v)}\left(\frac{u_*}{u} - \frac{v_*}{v}\right)^2 \le 0.\end{aligned} \tag{5.9}$$

Here we use

$$\frac{\gamma(x)}{\beta(x)} = \frac{1}{r} = \frac{v_*}{u_*+v_*}.$$

On the other hand,

$$\begin{aligned}A_1 &= -d\int_\Omega \frac{u_*}{u^2}|\nabla u|^2 - \int_\Omega \frac{v_*}{v^2}|\nabla v|^2 - \xi\int_\Omega \frac{v_*}{v}\nabla u\cdot\nabla v \\ &= -\begin{bmatrix}\nabla u \\ \nabla v\end{bmatrix}^T \Pi \begin{bmatrix}\nabla u \\ \nabla v\end{bmatrix},\end{aligned} \tag{5.10}$$

where

$$\Pi = \begin{pmatrix} \dfrac{du_*}{u^2} & \dfrac{\xi v_*}{2v} \\ \dfrac{\xi v_*}{2v} & \dfrac{v_*}{v^2} \end{pmatrix}.$$

When (5.6) is satisfied, the matrix Π is non-negative definite, then we have $A_1 \le 0$. Combined with (5.9), we have $\frac{d}{dt}E(u(\cdot,t),v(\cdot,t)) \le 0$, and only when $u = u_*, v = v_*$, $\frac{d}{dt}E(u(\cdot,t),v(\cdot,t)) = 0$. From Theorem 1.2, we know that any solution $(u(x,t),v(x,t))$ is bounded in $\overline{\Omega} \times (0,\infty)$. Therefore, (u_*,v_*) is globally asymptotically stable for all positive initial conditions from the LaSalle's invariance principle. □

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