

## ON THE GEOMETRY AND TOPOLOGY OF INITIAL DATA SETS WITH HORIZONS\*

LARS ANDERSSON<sup>†</sup>, MATTIAS DAHL<sup>‡</sup>, GREGORY J. GALLOWAY<sup>§</sup>, AND  
DANIEL POLLACK<sup>¶</sup>

**Abstract.** We study the relationship between initial data sets with horizons and the existence of metrics of positive scalar curvature. We define a Cauchy Domain of Outer Communications (CDOC) to be an asymptotically flat initial set  $(M, g, K)$  such that the boundary  $\partial M$  of  $M$  is a collection of Marginally Outer (or Inner) Trapped Surfaces (MOTSs and/or MITTs) and such that  $M \setminus \partial M$  contains no MOTSs or MITTs. This definition is meant to capture, on the level of the initial data sets, the well known notion of the domain of outer communications (DOC) as the region of spacetime outside of all the black holes (and white holes). Our main theorem establishes that in dimensions  $3 \leq n \leq 7$ , a CDOC which satisfies the dominant energy condition and has a strictly stable boundary has a positive scalar curvature metric which smoothly compactifies the asymptotically flat end and is a Riemannian product metric near the boundary where the cross sectional metric is conformal to a small perturbation of the initial metric on the boundary  $\partial M$  induced by  $g$ . This result may be viewed as a generalization of Galloway and Schoen’s higher dimensional black hole topology theorem [17] to the exterior of the horizon. We also show how this result leads to a number of topological restrictions on the CDOC, which allows one to also view this as an extension of the initial data topological censorship theorem, established in [10] in dimension  $n = 3$ , to higher dimensions.

**Key words.** Initial data set, marginally outer trapped surfaces, Jang’s equation.

**Mathematics Subject Classification.** 53C21, 83C57.

**1. Introduction.** One of the interesting features of general relativity is that it does not a priori impose any restrictions on the topology of space. In fact, as was shown in [22], given a compact manifold  $M$  of arbitrary topology and a point  $p \in M$ , there always exists an asymptotically flat solution to the vacuum Einstein constraint equations on  $M \setminus \{p\}$ . However, according to the principle of topological censorship, the topology of the domain of outer communications (DOC), that is the region outside of all black holes (and white holes), should, in a certain sense, be simple. The rationale for this is roughly as follows. Known results [18, 23] suggest that nontrivial topology tends to induce gravitational collapse. In the standard collapse scenario, based on the weak cosmic censorship conjecture, the process of gravitational collapse leads to the formation of an event horizon which shields the singularities from view. As a result, according to the viewpoint of topological censorship, the nontrivial topology gets hidden behind the event horizon, and hence the DOC should have simple topology. There have been a number of results supporting this point of view, the most basic of which establishes the simple connectivity of the DOC in asymptotically flat spacetimes obeying suitable energy and causality conditions [14, 15]. However, all the results alluded to here are spacetime results, that is, they involve conditions that are essentially global in time.

---

\*Received September 16, 2016; accepted for publication November 22, 2016.

<sup>†</sup>Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Golm, Germany (laan@aei.mpg.de)

<sup>‡</sup>Institutionen för Matematik, Kungliga Tekniska Höskolan, 100 44 Stockholm, Sweden (dahl@math.kth.se).

<sup>§</sup>Department of Mathematics, University of Miami, Coral Gables, FL 33124, U.S.A. (galloway@math.miami.edu).

<sup>¶</sup>Department of Mathematics, University of Washington, Seattle, WA 98195, U.S.A. (pollack@math.washington.edu).

In [10] a result on topological censorship was obtained at the pure initial data level for asymptotically flat initial data sets, thereby circumventing difficult questions of global evolution; see [10, Theorem 5.1]. This result, which establishes, under appropriate conditions, the topological simplicity of 3-dimensional asymptotically flat initial data sets with horizons, relies heavily on deep results in low dimensional topology, in particular the resolution of the Poincaré and the geometrization conjectures. The aim of the present paper is to obtain some results of a similar spirit, but in higher dimensions. While similar in spirit, the methods we employ here are entirely different. Given an asymptotically flat initial data set which satisfies the dominant energy condition and contains an inner horizon (marginally outer trapped surface), we use Jang's equation ([34, 2]), and other techniques, to deform the metric to one of positive scalar curvature on the manifold obtained by compactifying the end, such that the metric has a special structure near the horizon. As we shall discuss, one can then use known obstructions to the existence of such positive scalar curvature metrics to obtain restrictions on the topology of the original initial data manifold. Related approaches to the topology of asymptotically flat initial data sets *without* horizons have been considered in [33, 29]. Here we must overcome a number of difficulties due to the presence of a horizon.

An initial data set  $(M, g, K)$  for Einstein's equations consists of an  $n$ -dimensional manifold  $M$ , a Riemannian metric  $g$  on  $M$ , and a symmetric 2-tensor  $K$  on  $M$ . The energy density  $\mu$  and the momentum density  $J$  of  $(M, g, K)$  are computed through

$$2\mu = R^g - |K|_g^2 + (\operatorname{tr}^g K)^2 \quad \text{and} \quad J = \operatorname{div}^g K - d(\operatorname{tr}^g K).$$

The initial data set  $(M, g, K)$  satisfies the dominant energy condition if

$$\mu \geq |J|.$$

Let  $\Sigma$  be a compact 2-sided hypersurface in an initial data set  $(M, g, K)$ . Then  $\Sigma$  admits a smooth unit normal field  $\nu$  in  $M$ . By convention, refer to such a choice as outward pointing. Then the outgoing and ingoing null expansion scalars  $\theta_{\pm}$  are defined in terms of the initial data as  $\theta_{\pm} = P \pm H$ , where  $P = \operatorname{tr}_{\Sigma} K$  is the partial trace of  $K$  along  $\Sigma$  and  $H$  is the mean curvature of  $\Sigma$ , which, by our conventions, is the divergence of  $\nu$  along  $\Sigma$ . We call  $\Sigma$  outer trapped if  $\theta_+ < 0$  on  $\Sigma$ , while if  $\theta_- < 0$ ,  $\Sigma$  is inner trapped. We call  $\Sigma$  a marginally outer trapped surface (MOTS) if  $\theta_+ = 0$ , while if  $\theta_- = 0$ , we call  $\Sigma$  a marginally inner trapped surface (MITS). The distinction between MOTS and MITS is only meaningful when a choice, natural or otherwise, has been made between the notions of “outside” and “inside”.

Galloway and Schoen have proved the following extension of Hawking's black hole topology theorem to higher dimensions.

**THEOREM 1.1** ([17]). *Let  $(M, g, K)$  be an  $n$ -dimensional,  $n \geq 3$  initial data set satisfying the dominant energy condition. If  $\Sigma$  is a stable MOTS, in particular if  $\Sigma$  is outermost, then, apart from certain exceptional circumstances,  $\Sigma$  is of positive Yamabe type.*

In the situation of this theorem, let  $h$  denote the induced metric on  $\Sigma$ . The conclusion is then that  $h$  is conformal to a metric of positive scalar curvature. The “exceptional circumstances” can be ruled out in various ways [17, 16], in particular if  $\Sigma$  is assumed to be *strictly* stable. See [2] and references therein for the notion of MOTS stability.

The main result of the present paper is an extension of the above theorem for asymptotically flat initial data sets, stating that the positive scalar curvature metric on  $\Sigma$  can be extended to the one-point compactification of the asymptotically flat manifold consisting of the exterior of  $\Sigma$ .

**DEFINITION 1.1.** *A Cauchy Domain of Outer Communications (CDOC) is an asymptotically flat initial set  $(M, g, K)$  such that the boundary of  $M$ ,  $\partial M$  is a collection of MOTSs and/or MITs and such that  $M \setminus \partial M$  contains no MOTSs or MITs.*

The precise form of asymptotic flatness we require is given in Definition 2.1.

Definition 1.1 is meant to capture, strictly on the level of initial data sets, the well known notion of the DOC as the region of spacetime outside of all the black holes (and white holes). More precisely, it is meant to model an asymptotically flat (partial) Cauchy surface within the DOC, with boundary on the event horizon (in the equilibrium case) or perhaps somewhat inside the event horizon (in the dynamic case). Our main theorem, as noted above, establishes the existence of a particular type of positive scalar curvature metric on a CDOC.

**THEOREM 1.2.** *Let  $(M, g, K)$  be an  $n$ -dimensional,  $3 \leq n \leq 7$ , CDOC whose boundary  $\Sigma = \partial M$  is connected and is a strictly stable MOTS. Suppose further that the initial data  $(g, K)$  extends to a slightly larger manifold  $N$  (which contains  $M$  and a collar neighborhood of  $\Sigma$ ) such that the dominant energy condition (DEC) holds on  $N$ ,  $\mu \geq |J|$ .*

*Let  $\tilde{M}$  denote  $M$  with the asymptotically flat end compactified by a point. Let  $h$  denote the metric on  $\partial M$  induced from  $g$ . Then  $\tilde{M}$  admits a positive scalar curvature metric  $\tilde{g}$*

- (1) *whose induced metric on the boundary  $\partial M$  is conformal to a small perturbation of  $h$ , and*
- (2) *is a Riemannian product metric in a collar neighborhood of  $\partial M$ .*

**REMARK 1.2.** *Theorem 1.2 is stated for simplicity for a CDOC with a single outermost MOTS  $\Sigma$ . The analogous statement for the case where  $\Sigma$  is a collection of outermost MOTSs and MITs can easily be proved along the same lines.*

**REMARK 1.3.** *The proof in the  $n = 3$  case of Theorem 1.2 requires a modification from the general case when  $n > 3$ . This is addressed in Remarks 2.2 and 3.1. The restriction that  $n \leq 7$  in Theorem 1.2 is the result of our use of existence results for smooth solutions of Jang's equation (see in particular Theorems 2.1 and 2.2 below). This restriction is closely related to the partial regularity imposed in higher dimensions by the existence of the Simons cone, a singular area minimizing hypersurface in  $\mathbb{R}^8$ .*

The existence of a positive scalar curvature metric on the compactification of  $M$  (and its double, which follows immediately from the product structure near the boundary) gives restrictions on the topology of  $M$ . In Section 4 we discuss such restrictions in more detail.

The black ring spacetime of Emparan and Reall [13], which is an asymptotically flat, stationary solution to the vacuum Einstein equations, illustrates certain features of our results. Let  $M$  be the closure of a Cauchy surface for the domain of outer communications of the black ring. The boundary of  $M$  coincides with the bifurcate horizon, which has topology  $S^2 \times S^1$ . Further, as shown in [5] (see also [1]), the compactification  $\tilde{M}$  of  $M$  has topology  $S^2 \times D^2$ . This is consistent with Theorem 1.2, as well as standard results on topological censorship, which require the domain of

outer communications to be simply connected. Note that, while  $\pi_1(\check{M})$  is trivial,  $H_2(\check{M}, \mathbb{Z}) = \mathbb{Z} \neq 0$ .

An essential part of the argument is to show that we can specialize to the case in which dominant energy condition holds strictly,  $\mu > |J|$ . This involves a perturbation of the initial data, as discussed in Section 2. It is here that we need the assumption that  $\Sigma$  is *strictly* stable.

This paper is a contribution to the long history of results tying the existence of metrics of positive scalar curvature to the analysis of initial data sets in general relativity. One of the earliest and most important examples of this is the transition from Schoen and Yau's work on topological obstructions to positive scalar curvature metrics [30, 32] to their proof, using minimal hypersurfaces, of the positive mass theorem [31, 34]. The results here, like Theorem 1.1, make strong ties between the dominant energy condition, the presence of marginally trapped surfaces and metrics of positive scalar curvature.

**Acknowledgements.** The authors wish to thank Michael Eichmair, Lan-Hsuan Huang and Anna Sakovich for many helpful comments on the work in this paper.

**2. Deforming to strict dominant energy condition.** We start by introducing an appropriate notion of asymptotically flat initial data. We shall use the conventions of [11] for function spaces, which agree with the conventions of [4]. All sections of bundles are assumed to be smooth unless otherwise stated. The following definition is an adaption of [11, Definition 3] to our situation.

**DEFINITION 2.1.** *Let  $(M, g, K)$  be an initial data set of dimension  $n \geq 3$ . Let  $k$  be an integer,  $k \geq 3$ . Further, let  $p > n$ ,  $q \in ((n-2)/2, n-2)$ ,  $q_0 > 0$ ,  $\alpha \in (0, 1-n/p]$ . We say that  $(M, g, K)$  is asymptotically flat (of type  $k, p, q, q_0, \alpha$ ) if there is a compact set  $Y \subset M$  and a  $C^{k+1, \alpha}$  diffeomorphism identifying  $M \setminus Y$  with  $\mathbb{R}^n \setminus B$  for some closed ball  $B \subset \mathbb{R}^n$ , for which*

$$(g - \delta, K) \in W_{-q}^{k, p}(\mathbb{R}^n \setminus B) \times W_{-q-1}^{k-1, p}(\mathbb{R}^n \setminus B) \quad (2.1)$$

where  $\delta$  is the standard flat metric on  $\mathbb{R}^n$ , and

$$(\mu, J) \in C_{-n-q_0}^{k-2, \alpha}. \quad (2.2)$$

In order to avoid certain technical problems our definition of asymptotic flatness differs from that of [11] by assuming higher regularity. Note that our assumptions imply pointwise estimates for two derivatives of  $g - \delta$  which is not valid under the assumptions of [11]. Note also that the condition (2.2) which is adapted from [11, Definition 3] implies additional fall-off for  $(\mu, J)$  over that implied by (2.1).

We shall make use of the weighted Sobolev and Hölder spaces in the setting of manifolds with boundary. Definition 2.1 extends immediately to this situation.

The aim of this section is to establish the following perturbation result.

**THEOREM 2.1.** *Let  $(M, g, K)$  be an initial data set of dimension  $n$ ,  $4 \leq n \leq 7$ , which is asymptotically flat in the sense of Definition 2.1 and such that  $M$  is a manifold with boundary, whose boundary  $\Sigma = \partial M$  is a connected strictly stable MOTS. Suppose further:*

- (i) *The initial data  $(g, K)$  extends to a slightly larger manifold  $N$  (which contains  $M$  and a collar neighborhood of  $\Sigma$ ) such that the dominant energy condition (DEC) holds on  $N$ ,  $\mu \geq |J|$ .*

(ii) *There are no MOTSs or MITSs in  $M \setminus \Sigma$ .*

*Then for  $\epsilon > 0$  there is an asymptotically flat initial data  $(\hat{g}, \hat{K})$  such that*

$$\|\hat{g} - g\|_{W_{-q}^{k,p}} + \|\hat{K} - K\|_{W_{-1-q}^{k-1,p}} < \epsilon$$

*and a manifold  $\hat{M} \subset N$  diffeomorphic to  $M$ , with MOTS boundary  $\hat{\Sigma} = \partial\hat{M}$ , which is a small (in  $C^{2,\alpha}$ ) perturbation of  $\Sigma$ , such that the following statements hold.*

(1) *The dominant energy condition holds strictly on  $(\hat{M}, \hat{g}, \hat{K})$ , that is*

$$\hat{\mu} > |\hat{J}|.$$

(2) *There exists a smooth solution to Jang's equation  $u : \hat{M} \setminus \hat{\Sigma} \rightarrow \mathbb{R}$ , such that*

$$u \in W_{1-q}^{k+1,p} \quad (2.3)$$

*and  $u \rightarrow \infty$  on approach to  $\hat{\Sigma}$ .*

REMARK 2.2. *In the statement of Theorem 2.1 we have excluded the case  $n = 3$ . The reason for this is that in the proof we are making use of the density theorem [11, Theorem 22], which yields deformed data  $(\hat{g}, \hat{K})$  satisfying the strict dominant energy condition, and with the same asymptotic behavior as  $(g, K)$ , in particular  $\text{tr}^{\hat{g}} \hat{K} \in W_{-1-q}^{k-1,p}$ , which in case  $n = 3$  is in general incompatible with having a bounded solution (near infinity) to Jang's equation. This problem does not arise for  $n \geq 4$  in which case  $\text{tr}^{\hat{g}} \hat{K} = O(|x|^{-\gamma})$  for some  $\gamma > 2$ . In Theorem 2.2 below, which does not rely on the just mentioned density theorem, we have avoided this technical point by including the additional assumption (2.4). In Remark 3.1 below we describe the modifications necessary to prove Theorem 1.2 in the case  $n = 3$ .*

The proof involves several elements. We begin with some comments about Jang's equation. Schoen and Yau [34] studied in detail the existence and regularity of solutions to Jang's equation in their proof of the positive mass theorem in the general (not time-symmetric) case. They interpreted Jang's equation geometrically as a prescribed mean curvature equation, and discovered that the only possible obstruction to global existence are MOTSs in the initial data, where, in fact, the solution may have cylindrical blow-ups.

Given an initial data set  $(M, g, K)$ , consider graphs of functions  $u : M \rightarrow \mathbb{R}$  in the initial data set  $(\bar{M}, \bar{g}, \bar{K})$  of one dimension higher, where  $\bar{M} = M \times \mathbb{R}$ ,  $\bar{g} = g + dt^2$ , and  $\bar{K}$  is the pullback of  $K$  to  $\bar{M}$  by the projection to  $M$ . Jang's equation may then be written as

$$H(u) - \text{tr } \bar{K}(u) = 0,$$

where  $H(u)$  is the mean curvature of  $\text{graph}(u)$ , with respect to the downward pointing normal, in  $(\bar{M}, \bar{g})$  and  $\text{tr } \bar{K}(u)$  is the partial trace of  $\bar{K}$  over the tangent spaces of  $\text{graph}(u)$ .

The fundamental existence result of Schoen and Yau [34, Proposition 4] for Jang's equation may now be applied. We also rely on the work of Metzger [25] to allow for an interior barrier, and the regularity theory (up to dimension 7) of Eichmair [7, 8] (see also [12, 9]). This together yields the following existence result for Jang's equation in our setting.

**THEOREM 2.2.** *Let  $(M, g, K)$  be an initial data set of dimension  $n$ ,  $3 \leq n \leq 7$  which is asymptotically flat in the sense of definition 2.1. In case  $n = 3$ , we require that  $\text{tr}^g K$  satisfies the additional decay condition*

$$\text{tr}^g K = O(|x|^{-\gamma}) \quad (2.4)$$

for some  $\gamma > 2$ . Further, we assume that  $M$  is a manifold with boundary, whose boundary  $\Sigma = \partial M$  is a compact connected outer trapped surface.

Then there exist open pairwise disjoint sets  $\Omega$ ,  $\Omega_+$  and  $\Omega_-$ , with  $\Omega$  containing a neighborhood of infinity, and an extended-real valued function  $u$  whose domain includes the union  $\Omega \cup \Omega_+ \cup \Omega_-$  (and is realized as a limit of solutions to the capillarity regularized Jang equation (2.5)) such that

- (1)  $M = \bar{\Omega} \cup \bar{\Omega}_+ \cup \bar{\Omega}_-$ .
- (2)  $u = +\infty$  on  $\Omega_+$ , where  $\Omega_+$  contains a neighborhood of  $\Sigma$ , and  $u = -\infty$  on  $\Omega_-$ .
- (3) Each boundary component  $\Sigma_a^+$  of  $\Omega_+$  is a MOTS (except for  $\Sigma$ ), and each boundary component  $\Sigma_b^-$  of  $\Omega_-$  is a MITS. (Here “outside” is determined by the outward normal to these open sets.)
- (4)  $u : \Omega \rightarrow (-\infty, \infty)$  is a smooth solution to Jang’s equation such that  $u(x) \rightarrow 0$  as  $x \rightarrow \infty$ ,  $u(x) \rightarrow +\infty$  as  $x \rightarrow \partial\Omega_+$ , and  $u(x) \rightarrow -\infty$  as  $x \rightarrow \partial\Omega_-$ . The boundary components of  $\Omega$  are smooth and form a subcollection of the MOTSs  $\Sigma_a^+$  and MITSs  $\Sigma_b^-$  in point (3).

**REMARK 2.3.** To prove Theorem 2.2 one considers the capillarity regularized Jang equation

$$H(u_\tau) - \text{tr}\bar{K}(u_\tau) = \tau u_\tau, \quad (2.5)$$

and studies the limit as  $\tau \rightarrow 0$ . This regularized equation satisfies an a priori height estimate that allows one to construct a smooth global solution  $u_\tau$  on  $M \setminus \Sigma$  such that  $u_\tau \rightarrow 0$  on the asymptotically flat end (uniformly in  $\tau$ ), and  $u_\tau \rightarrow \infty$  as  $\tau \rightarrow 0$  on a fixed neighborhood of  $\Sigma$ ; see [34, 3, 7, 25, 12]. To get smooth convergence up to dimension 7, one applies the method of regularity introduced in the study of MOTSs by Eichmair [7], based on the  $C$ -minimizing property. By a calibration argument, the graphs  $G_\tau = \text{graph}(u_\tau)$  obey the  $C$ -minimizing property (this is true in general for graphs of bounded mean curvature). By the compactness and regularity theory of  $C$ -minimizers as described in [7, Appendix A], a subsequence of these graphs converges to a smooth hypersurface  $G \subset \bar{M}$ , consisting of cylindrical components (which occur at the intersection of  $\partial\Omega_+$  and  $\partial\Omega_-$ , where  $u$  is not defined) and graphical components which are also  $C$ -minimizing. It is this hypersurface  $G$  that determines the open sets  $\Omega$ ,  $\Omega_+$  and  $\Omega_-$ . Considering translations of the graphical components of  $G$  gives rise to further cylinders obeying the  $C$ -minimizing property. The collections of MOTSs  $\Sigma_a^+$  and MITSs  $\Sigma_b^-$  in part (3) of Theorem 2.2 arise from the intersection of all these cylinders with  $M = M \times \{0\}$  in  $\bar{M}$ . Of further importance to us, as observed in [7], the  $C$ -minimizing property of these cylinders descends to the collection of MOTSs  $\Sigma_a^+$  and MITSs  $\Sigma_b^-$ .

**2.1. Proof of Theorem 2.1.** Since  $\Sigma$  is strictly stable in  $(M, g, K)$ , assumption (i) of the theorem enables us to construct an enlarged manifold  $M' = M \cup V \subset N$ , where  $V$  is an exterior collar  $V \approx [0, \epsilon] \times \Sigma$  attached to  $\Sigma = \partial M$ , such that  $\Sigma_t = \{t\} \times \Sigma$  is outer trapped for all  $t \in (0, \epsilon]$  and  $\Sigma_0 = \Sigma$ , see Figure 2.1. For more details, see the

discussion following Definition 3.1 in [2]. Then  $(M', g, K)$  is an asymptotically flat initial data set with boundary  $\partial M' = \Sigma' := \Sigma_\epsilon$ . On the asymptotically flat end we let  $\Sigma(r)$  denote the radial sphere  $|x| = r$ .

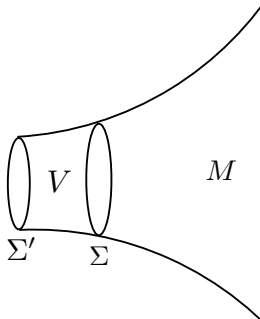


FIG. 2.1.  $M' = M \cup V$ .

LEMMA 2.3. *With the assumptions of Theorem 2.1, there exists a sequence of initial data sets  $(M', g_i, K_i)$  such that  $(g_i, K_i)$  converges to  $(g, K)$  in  $W_{-q}^{k,p} \times W_{-1-q}^{k-1,p}$  and such that the following holds.*

- (1) *The dominant energy condition holds strictly,  $\mu_i > |J_i|_{g_i}$ .*
- (2) *For each data set  $(M', g_i, K_i)$ ,  $\Sigma_t$  is outer trapped for all  $\frac{1}{i} \leq t \leq \epsilon$ .*
- (3) *There exists  $r_0 > 0$  such that for all  $r \geq r_0$ ,  $\Sigma(r)$  is inner trapped ( $\theta_- < 0$ ) and outer untrapped ( $\theta_+ > 0$ ) with respect to each  $(g_i, K_i)$ .*

*Proof.* The proof of Lemma 2.3 is based on [11, Theorem 22]. By considering the double of  $M'$ , one sees that  $M'$  can be compactly “filled-in” beyond its boundary to obtain a complete manifold  $N'$  without boundary. Extend the data  $(g, K)$  arbitrarily, but smoothly to  $N'$ . Then  $(N', g, K)$  is an asymptotically flat manifold such that dominant energy condition holds on  $M' \subset N'$ . Thus, by [11, Theorem 22]<sup>1</sup> and the remark following the statement of [11, Theorem 18] concerning higher regularity, there exists a sequence of asymptotically flat initial data sets  $(M', g_i, K_i)$  satisfying

$$\|g - g_i\|_{W_{-q}^{k,p}} \leq \frac{1}{i} \quad \text{and} \quad \|K - K_i\|_{W_{-1-q}^{k-1,p}} \leq \frac{1}{i}$$

such that part 2.3 of Lemma 2.3 holds. Moreover, as follows from [11, Equation (40)], for each  $i$ ,  $(g_i, K_i)$  can be made sufficiently  $C^2$ -close to  $(g, K)$  on the collar  $V$  so that part 2.3 holds.

Now consider the null mean curvatures  $\theta_\pm(r) = \text{tr}_{\Sigma(r)} K \pm H$  (resp.,  $\theta_{i\pm}(r)$ ) of the coordinate spheres  $\Sigma(r)$  in the initial data set  $(g, K)$  (resp.,  $(g_i, K_i)$ ). Since the null mean curvatures  $\theta_\pm(r)$  are polynomials in  $g$  and its first derivatives, and  $K$  (and similarly for  $\theta_{i\pm}(r)$  with respect to  $g_i$  and  $K_i$ ) the weighted Sobolev embedding  $W_{-q}^{k,p} \subset C_{-q}^{m,\alpha}$  provided  $m + \alpha < k - \frac{n}{p}$ , implies that

$$\|\theta_{i\pm}(r) - \theta_\pm(r)\|_{C_{-q-1}^{0,\alpha}} \leq \frac{C}{i}$$

<sup>1</sup>For our situation we are actually applying a small refinement of [11, Theorem 22], whereby one does not require the dominant energy condition to hold everywhere. Essentially, the construction in [11] gives strictness at points where the dominant energy condition holds in the original data. We thank Lan-Hsuan Huang [21] for clarification on this point.



for a constant  $C$  independent of  $i$ . This implies that

$$|\theta_{i\pm}(r) - \theta_{\pm}(r)| = O(r^{-q-1}).$$

Since the mean curvature of large spheres falls off linearly with the radius this implies that part 2.3 of Lemma 2.3 holds. This concludes the proof of the lemma.  $\square$

We now apply Theorem 2.2 to each initial data set  $(M', g_i, K_i)$  guaranteed by Lemma 2.3. Thus, for each  $i$  there exist open sets  $\Omega_i$ ,  $\Omega_{i+}$  and  $\Omega_{i-}$  and an extended-real valued function  $u_i$  as in the theorem. In particular,  $u_i : \Omega_i \rightarrow \mathbb{R}$  is a smooth solution to Jang's equation. Let  $\Omega_i^{ext}$  be the component of  $\Omega_i$  containing the asymptotically flat end. We are primarily interested in the smooth solutions  $u_i : \Omega_i^{ext} \rightarrow \mathbb{R}$ . The boundary  $S_i := \partial\Omega_i^{ext}$  consists of MOTSs  $S_{i,a}^+$  and MITs  $S_{i,b}^-$ . (Consistent with Theorem 2.2, the “outside” is determined by the normal pointing into  $\Omega_i^{ext}$ .) Here we use indices  $a, b$  to enumerate the MOTS and MITS components of  $S_i$ . We have  $u_i \rightarrow +\infty$  on approach to the MOTSs  $S_{i,a}^+$  and  $u_i \rightarrow -\infty$  on approach to the MITs  $S_{i,b}^-$ .

Let  $\Omega'_{i+}$  be the component of  $\Omega_{i+}$  containing  $\Sigma'$ . From Theorem 2.2,  $\Omega'_{i+} \neq \emptyset$  for all  $i$ . In fact, as in [25], the maximum principle implies that  $\Omega'_{i+} \supset V_i := \cup_{t \in [\frac{1}{4}, \epsilon]} \Sigma_t$ . This implies, in particular, that  $S_i \neq \emptyset$  for all  $i$ . Moreover, by part 2.3 of Lemma 2.3 and the maximum principle,  $S_i \subset M'(r_0)$ , where  $M'(r_0) \subset M'$  is the compact region bounded by  $\Sigma(r_0)$ . By the convergence of the data  $(g_i, K_i)$  to  $(g, K)$  on  $M'(r_0)$ , the sequence  $S_i$  obeys a uniform  $C$ -minimizing property, see Remark 2.3. Hence, by the compactness theory presented in [7, 8] (which provides area bounds, curvature bounds and injectivity bounds), by passing to a subsequence if necessary, the sequence  $S_i$  converges in  $C^{2,\alpha}$  to  $S$  which is a combination of MOTSs and MITs in  $(M, g, K)$ . Note that no component of  $S$  enters the collar region exterior to  $M$  in  $M'$  since  $S_i \cap V_i = \emptyset$  for all  $i$ .

We shall now prove that  $S = \Sigma$ . By the above there is a unique smallest collection  $\mathring{S}_i$  of components of  $S_i$  surrounding  $\Sigma'$  in the sense that  $\mathring{S}_i$  separates  $\Sigma'$  from infinity, and hence a unique smallest collection  $\mathring{S}$  of components of  $S$  surrounding  $\Sigma$ . Since  $\mathring{S}$  consists of MOTSs and MITs, and since our assumptions exclude any MOTSs or MITs in the exterior of  $\Sigma$ , it follows that  $S = \mathring{S}$  and that each component of  $\mathring{S}$  meets  $\Sigma$  at some point.

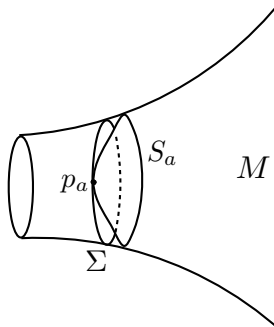


FIG. 2.2.  $S_a$  meets  $\Sigma$  at  $p_a$ .

Let now  $S_a$  be one of the components of  $S$ , which by the above must meet  $\Sigma$  at some point  $p_a$ , see Figure 2.2. Suppose that  $S_a$  is a MOTS. Since the outward pointing normal of  $S_a$  must agree with that of  $\Sigma$ , the maximum principle implies



that in this case  $S_a = \Sigma$  and we are done. It remains to consider the case when  $S_a$  is a MITS. By construction,  $S_a$  is the limit of a sequence  $S_{i,a}^-$  of components of the boundary of  $\Omega_i^{ext}$ , each of which is separated from  $\Sigma_{1/i}$  by a part of  $\Omega_{i-}$ . Pulling back slightly from the limit, i.e. for very large  $i$  and for values of  $\tau$  sufficiently small, in a small neighborhood  $U_a$  of  $p_a$ , the capillarity regularized Jang graph, graph  $u_{i,\tau}$ , and its vertical translates come uniformly close to the vertical cylinders over the MITS  $S_{i,a}^-$  and a MITS component  $T_i$  of  $\Omega_{i-}$  inside  $S_{i,a}^-$ , which converges to  $\Sigma$  (see Figure 2.3). For  $i$  very large and values of  $\tau$  sufficiently small, this can be seen to lead to a violation of the uniform (in both  $i$  and  $\tau$ )  $C$ -minimizing property of these capillarity regularized Jang graphs [7, 8] (for example by gluing in a tube.) Thus, the case that  $S_a$  is a MITS meeting  $\Sigma$  is precluded by this  $C$ -minimizing property. Hence we find that  $S = \Sigma$ . (We remark that instead of considering the limits  $\tau \searrow 0$  and  $i \nearrow \infty$  separately, a simultaneous limit in  $\tau, i$  can be taken. Arguing along the same lines as above, and making use of the uniformity of the  $C$ -minimizing property with respect to  $\tau$ , one concludes that the graphs  $u_{\tau,i}$  have a smooth subsequential limit which satisfies the  $C$ -minimizing property. This again precludes the MITS  $S_a$  meeting  $\Sigma$ .)

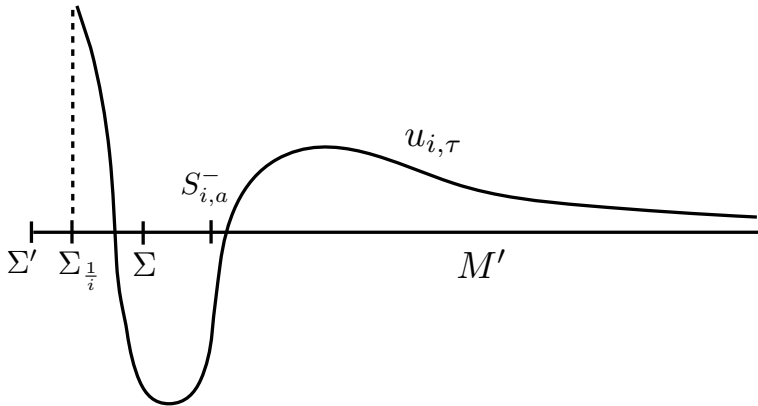


FIG. 2.3. The capillarity regularized Jang graph for  $(M', g_i, K_i)$ .

It follows from the above that for all sufficiently large  $i$ ,  $S_i$  must have only one component which is a MOTS surrounding  $\Sigma'$ . In fact, by the conclusions of the compactness results for MOTS in [8] we know that, for all  $i$  large,  $S_i$  must be a graph over  $\Sigma$ . To prove Theorem 2.1 we set  $\hat{M} = \overline{\Omega_i^{ext}}$  and  $\hat{\Sigma} = S_i$  for  $i$  sufficiently large. It remains to prove the regularity claimed in point (2.1). This follows by writing Jang's equation in the form

$$\Delta^g u = (1 + \nabla^k u \nabla_k u)^{1/2} g^{ij} K_{ij} + (1 + \nabla^k u \nabla_k u)^{-1} \nabla^i u \nabla^j u \nabla_i \nabla_j u - \nabla^i u \nabla^j u K_{ij},$$

making use of the fact that the barrier argument used in constructing the solution to Jang's equation yields  $u = O(|x|^{-\beta})$  for some  $\beta > 0$  and using elliptic estimates. This completes the proof of Theorem 2.1.

**3. Proof of Theorem 1.2.** In this section we prove our main theorem. Assume that  $(M, g, K)$  is an initial data set of dimension  $n$  as in Theorem 1.2. For technical reasons we first restrict to the case  $4 \leq n \leq 7$ . The extension of the proof to the case  $n = 3$  is discussed in Remark 3.1 below.

The proof is broken up into a number of steps.

**Step 1: Apply Theorem 2.1 to deform to strict DEC.** By Theorem 2.1,  $(M, g, K)$  may be deformed to an initial data set satisfying the dominant energy condition with strict inequality,  $E := \mu - |J| > 0$ , while preserving the MOTS boundary, and so that there is a solution  $u$  to Jang's equation which blows up at the MOTS boundary  $\Sigma := \partial M$  and has no further blow-up. We denote the deformed data set again by  $(M, g, K)$ .

Let  $\hat{M} \subset M \times \mathbb{R}$  be the graph of  $u$ , and let  $\hat{g}$  be the induced Riemannian metric on  $\hat{M}$ . Then  $(\hat{M}, \hat{g})$  is asymptotically flat and near  $\Sigma$  it is asymptotic to the cylinder  $(\Sigma \times \mathbb{R}, h + dt^2)$ . From the Schoen-Yau identity [2, Section 3.6] it follows that

$$\int_{\hat{M}} (2|\nabla \phi|_{\hat{g}}^2 + R^{\hat{g}} \phi^2) d\mu^{\hat{g}} \geq \int_{\hat{M}} 2E\phi^2 d\mu^{\hat{g}} \quad (3.1)$$

for every compactly supported smooth function  $\phi$  on  $\hat{M}$ .

**Step 2: Deform the metric to exactly cylindrical ends.** Near  $\Sigma$  the Jang graph  $(\hat{M}, \hat{g})$  is asymptotic to the cylinder  $(\Sigma \times \mathbb{R}, h + dt^2)$ , where  $h = g|_{\Sigma}$ .

We can write  $\hat{M}$  as  $\hat{M}_0 \cup \hat{M}_{\text{cyl}}$  where  $\hat{M}_{\text{cyl}} = \Sigma \times [t_0, \infty)$ . Using the normal exponential map of  $\Sigma \times [t_0, \infty)$  in  $M \times \mathbb{R}$  the asymptotically cylindrical end  $(\hat{M}_{\text{cyl}}, \hat{g})$  can be written as a graph of a function  $U : \Sigma \times [t_0, \infty) \rightarrow \mathbb{R}$ . In [34, Corollary 2] it is proven that for every  $\epsilon > 0$  there is a  $t_{\epsilon} \geq t_0$  so that

$$|U(p, t)| + |\nabla^h U(p, t)| + |(\nabla^h)^2 U(p, t)| \leq \epsilon$$

for  $p \in \Sigma$  and  $t \geq t_{\epsilon}$ . The result we refer to is stated in dimension 3, but its proof holds in all dimensions.

By deforming the function  $U$  to be identically zero for large  $t$  we can replace  $\hat{g}$  by a metric, which we still denote by  $\hat{g}$ , such that  $\hat{g} = h + dt^2$  on  $\Sigma \times [t_1, \infty)$ , for  $t_1 > t_0$ . Under the deformation of  $U$  the inequality (3.1) is almost preserved, so we get, for  $t_0$  sufficiently large,

$$\int_{\hat{M}} ((2 + \epsilon)|\nabla \phi|_{\hat{g}}^2 + R^{\hat{g}} \phi^2) d\mu^{\hat{g}} \geq \int_{\hat{M}} E\phi^2 d\mu^{\hat{g}} \quad (3.2)$$

for some small  $\epsilon > 0$  and all smooth compactly supported functions  $\phi$ .

*In each of the remaining steps the metric  $\hat{g}$  is replaced by a modified metric  $\check{g}$ , which is then renamed as  $\hat{g}$ .*

**Step 3: Deform the metric to be flat on the asymptotically flat end.** Working in the asymptotically flat end of  $(\hat{M}, \hat{g})$ , let  $h$  denote the difference between  $\hat{g}$  and the flat background metric  $\delta$ . By construction, the metric on the Jang graph is  $\hat{g} = g + du^2$ , where  $u$  is the solution of Jang's equation which in our situation satisfies (2.3). It follows that if we write  $\hat{g} = \delta + h$ , then  $h \in W_{-q}^{k,p}$ .

Let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth cut-off function such that  $\chi(t) = 0$  for  $t \leq 1$ ,  $\chi(t) = 1$  for  $t \geq 2$ . For  $(M, g)$  asymptotically flat we define  $\chi_{\rho}(x) = \chi(r(x)/\rho)$  where  $r$  is the Euclidean radial coordinate. This is defined for  $\rho$  large and  $x$  sufficiently far out in the asymptotically flat end, and then extended to all of  $M$  by  $\chi_{\rho} = 0$  inside the end. We have that  $\nabla \chi_{\rho}$  is supported in the annulus  $A_{\rho} := \{x \in M \mid \rho \leq r(x) \leq 2\rho\}$ . Let  $\mathbb{1}_{A_{\rho}}$  denote the characteristic function of  $A_{\rho}$ .

For  $\rho$  large let

$$\check{g} := \delta + (1 - \chi_{\rho})h = \hat{g} - \chi_{\rho}h.$$

Then we have  $\check{g} = \hat{g}$  for  $r < \rho$  and  $\check{g} = \delta$  for  $r > 2\rho$ , while in  $A_\rho$  we have the estimates

$$\|\partial^m \check{g}\|_{C^0(A_\rho)} \leq C\rho^{-(n-2)/2-m}, \quad m = 1, \dots, k-1.$$

In particular, we have that

$$\|R^{\check{g}}\|_{C^0(A_\rho)} \leq C\rho^{-(n-2)/2-2}$$

and hence

$$\left| R^{\check{g}} - R^{\hat{g}} \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} \right| \leq C \mathbb{1}_{\{r \geq \rho\}} r^{-(n-2)/2-2}$$

where  $\frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}}$  is the function such that  $d\mu^{\hat{g}} = \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} d\mu^{\check{g}}$ . From (3.2) we have

$$\begin{aligned} \int_{\hat{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\check{g}} \phi^2) d\mu^{\check{g}} &\geq \int_{\hat{M}} (a_n - (2 + \epsilon)) |\nabla \phi|_{\check{g}}^2 d\mu^{\check{g}} \\ &\quad + (2 + \epsilon) \int_{\hat{M}} \left( |\nabla \phi|_{\check{g}}^2 - |\nabla \phi|_{\check{g}}^2 \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} \right) d\mu^{\check{g}} \\ &\quad + \int_{\hat{M}} \left( R^{\check{g}} - R^{\hat{g}} \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} \right) \phi^2 d\mu^{\check{g}} \\ &\quad + \int_{\hat{M}} E \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} \phi^2 d\mu^{\check{g}}, \end{aligned} \quad (3.3)$$

where

$$a_n := \frac{4(n-1)}{n-2}$$

is chosen so that the left hand side of (3.3) is conformally invariant. The integrand in the second term on the right hand side of (3.3) can be estimated in terms of  $C\rho^{-(n-2)/2} |\nabla \phi|_{\check{g}}^2$ . Using  $a_n > 4$  we get

$$\int_{\hat{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\check{g}} \phi^2) d\mu^{\check{g}} \geq \int_{\hat{M}} \left( |\nabla \phi|_{\check{g}}^2 + \hat{E} \phi^2 \right) d\mu^{\check{g}} \quad (3.4)$$

for  $\rho$  large enough, where

$$\hat{E} := E \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} + \left( R^{\check{g}} - R^{\hat{g}} \frac{\sqrt{\hat{g}}}{\sqrt{\check{g}}} \right).$$

We note that  $\hat{E}$  may be negative for  $r \geq \rho$  due to the contribution from the difference of the scalar curvatures. By construction, we have

$$E \leq Cr^{-n}$$

for  $r \geq R_0$ , and

$$|\hat{E}| \leq Cr^{-(n-2)/2-2} \quad (3.5)$$

for some constant  $C$ .

Next we will estimate the right hand side of (3.4) using a weighted Hardy inequality. Let  $\chi_\rho$  be the smooth cut-off function on  $\mathbb{R}^n$  defined by  $\chi_\rho(x) := \chi(|x|/\rho)$  where

$\chi$  is the function introduced above. The following lemma follows by the standard Hardy inequality (see for example Section 2.1.6. of [24]) applied to  $\chi_\rho u$ , followed by an application of the Cauchy-Schwartz inequality.

LEMMA 3.1. *There is a constant  $C_n > 0$  depending only on  $n$ , so that*

$$C_n \int_{\mathbb{R}^n} u^2 \chi_\rho^2 r^{-2} d\mu \leq \int_{\mathbb{R}^n} \left( \chi_\rho^2 |\nabla u|_\delta^2 + \frac{1}{\rho^2} \mathbb{1}_{A_\rho} u^2 \right) d\mu$$

for all  $u \in C_0^\infty(\mathbb{R}^n)$ .

If  $(M, g)$  is asymptotically flat we have

$$\int_M \chi_\rho^2 |\nabla u|_g^2 d\mu^g \geq C \int_{\mathbb{R}^n} \chi_\rho^2 |\nabla u|_\delta^2 d\mu$$

for  $\rho$  sufficiently large from the identification of the end with  $\mathbb{R}^n \setminus B$ . Applying Lemma 3.1 gives the following corollary.

COROLLARY 3.2. *Let  $(M, g)$  be an asymptotically flat Riemannian manifold of dimension  $n$ , as in Definition 2.1 (with  $K = 0$ ). For  $R_0$  sufficiently large and  $\rho > R_0$ , there is a constant  $C_n > 0$  depending only on  $n$  such that*

$$C_n \int_M \chi_\rho^2 r^{-2} u^2 d\mu^g \leq \int_M \left( |\nabla u|_g^2 + \frac{1}{\rho^2} \mathbb{1}_{A_\rho} u^2 \right) d\mu^g$$

for all  $u \in C_0^\infty(M)$ .

Fix some large  $\rho_0$ . Corollary 3.2 with  $\rho = \rho_0$  gives us the estimate

$$\int_{\hat{M}} \left( \lambda |\nabla \phi|_{\check{g}}^2 + \hat{E} \phi^2 \right) d\mu^{\check{g}} \geq \int_{\hat{M}} \left( \lambda C_n \chi_{\rho_0}^2 r^{-2} + \hat{E} - \lambda \rho_0^{-2} \mathbb{1}_{A_{\rho_0}} \right) \phi^2 d\mu^{\check{g}}$$

for  $\lambda > 0$ . With  $\lambda = \rho^{-1/4}$ , where  $\rho$  is the parameter in the definition of  $\check{g}$ , the inequality

$$\lambda C_n \chi_{\rho_0}^2 r^{-2} + \hat{E} = \rho^{-1/4} C_n \chi_{\rho_0}^2 r^{-2} + \hat{E} > 0$$

holds trivially on the region inside  $r = \rho$  (since  $\check{g} = \hat{g}$  there) and by (3.5) it holds on the region  $r \geq \rho$  if we choose  $\rho$  sufficiently large. Similarly, keeping  $\rho_0$  fixed and choosing  $\rho$  sufficiently large, we have

$$\hat{E} - \lambda \rho_0^{-2} \mathbb{1}_{A_{\rho_0}} = \hat{E} - \rho^{-1/4} \rho_0^{-2} \mathbb{1}_{A_{\rho_0}} > 0$$

in  $A_{\rho_0}$ , since then  $\hat{E} = E$  on  $A_{\rho_0}$  so  $\hat{E}$  is positive and independent of  $\rho$  there.

Redefining  $\hat{E}$  as

$$\lambda C_n \chi_{\rho_0}^2 r^{-2} + \hat{E} - \lambda \rho_0^{-2} \mathbb{1}_{A_{\rho_0}} = \rho^{-1/4} C_n \chi_{\rho_0}^2 r^{-2} + \hat{E} - \rho^{-1/4} \rho_0^{-2} \mathbb{1}_{A_{\rho_0}}$$

we have that  $\hat{E} > 0$  by the above choices. Redefining  $\hat{g}$  as  $\check{g}$  we get from (3.4) that

$$\int_{\hat{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\check{g}} \phi^2) d\mu^{\check{g}} \geq \int \hat{E} \phi^2 d\mu^{\hat{g}} \quad (3.6)$$

for all compactly supported smooth functions  $\phi$ . By construction we now have that  $\hat{g}$  is flat on the asymptotically flat end and

$$\hat{E} \geq Cr^{-2}$$

for  $r$  large.

**Step 4: Conformal compactification of the asymptotically flat end.** The flat background metric  $\delta$  compactifies to the standard round metric on  $S^n$  by the conformal change

$$\left(\frac{2}{1+r^2}\right)^2 \delta = \left(\frac{2}{1+r^2}\right)^2 (dr^2 + r^2 \sigma) = d\theta^2 + \sin^2 \theta \sigma$$

where  $2r/(1+r^2) = \sin \theta$  and  $\sigma$  is the round metric on  $S^{n-1}$ . Define

$$\alpha := \left(\frac{2}{1+r^2}\right)^{(n-2)/2}$$

in coordinates on the asymptotically flat end and extend  $\alpha$  to a positive function on all of  $\hat{M}$  with  $\alpha = 1$  on the cylindrical end. Set  $\check{g} = \alpha^{4/(n-2)} \hat{g}$  and let  $\check{M}$  be  $\hat{M}$  with the asymptotically flat end compactified by adding a point  $P_\infty$  at infinity. Then  $(\check{M}, \check{g})$  is isometric to the standard round metric on  $S^n$  in a neighbourhood of the new point at infinity.

The conformal Laplacians  $L^{\hat{g}} = a_n \Delta^{\hat{g}} + R^{\hat{g}}$  and  $L^{\check{g}} = a_n \Delta^{\check{g}} + R^{\check{g}}$  are related by

$$L^{\hat{g}} \phi = \alpha^{\frac{n+2}{n-2}} L^{\check{g}} (\alpha^{-1} \phi).$$

Further, we have

$$d\mu^{\hat{g}} = \alpha^{-\frac{2n}{n-2}} d\mu^{\check{g}}.$$

These equations, together with inequality (3.6) gives us

$$\begin{aligned} \int_{\hat{M}} (\alpha^{-1} \phi) L^{\check{g}} (\alpha^{-1} \phi) d\mu^{\check{g}} &= \int_{\hat{M}} \phi L^{\hat{g}} \phi d\mu^{\hat{g}} \\ &= \int_{\hat{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\hat{g}} \phi^2) d\mu^{\hat{g}} \\ &\geq \int_{\hat{M}} \hat{E} \phi^2 d\mu^{\hat{g}} \\ &= \int_{\hat{M}} \hat{E} \alpha^{-4/(n-2)} (\alpha^{-1} \phi)^2 d\mu^{\check{g}}. \end{aligned}$$

So with  $\check{E} := \hat{E} \alpha^{-4/(n-2)}$  it holds that

$$\begin{aligned} \int_{\check{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\check{g}} \phi^2) d\mu^{\check{g}} &\geq \int_{\check{M}} \hat{E} \alpha^{-4/(n-2)} \phi^2 d\mu^{\check{g}} \\ &= \int_{\check{M}} \check{E} \phi^2 d\mu^{\check{g}} \end{aligned}$$

for all smooth compactly supported functions  $\phi$  on  $\check{M}$  whose support does not contain  $P_\infty$ .

In terms of the spherical radial coordinate  $\theta$  at  $P_\infty$ , we have that  $\hat{E} = O(\theta^2)$  and  $\alpha^{-4/(n-2)} = O(\theta^{-4})$ . This means that  $\check{E} = O(\theta^{-2})$  which is compatible with the fact that we made use of the Hardy inequality in the construction of  $\hat{E}$ . We can now modify  $\check{E}$  by decreasing its values in a neighborhood of the point at infinity, and thereby replace it by a bounded smooth function which is uniformly positive on  $\check{M}$ . We finally get the inequality

$$\int_{\check{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\check{g}} \phi^2) d\mu^{\check{g}} \geq \int_{\check{M}} \check{E} \phi^2 d\mu^{\check{g}} \quad (3.7)$$

for  $\phi \in C_0^\infty(\check{M} \setminus P_\infty)$ . A cut-off function argument shows that (3.7) is valid for  $\phi \in C_0^\infty(\check{M})$ .

We redefine  $(\hat{M}, \hat{g})$  and  $\hat{E}$  as  $(\check{M}, \check{g})$  and  $\check{E}$ . This is then a metric with the asymptotically flat end compactified by a point, and an exact cylindrical end, such that

$$\int_{\hat{M}} (a_n |\nabla \phi|_{\hat{g}}^2 + R^{\hat{g}} \phi^2) d\mu^{\hat{g}} \geq \int_{\hat{M}} \hat{E} \phi^2 d\mu^{\hat{g}} \quad (3.8)$$

for  $\phi \in C_0^\infty(\hat{M})$ .

**Step 5: Conformal change to positive scalar curvature on the cylindrical end.** The next two steps are motivated by results in [6].

The estimate (3.8) holds in particular for functions of compact support on the cylindrical end  $(\Sigma \times [0, \infty), h + dt^2)$  of  $(\hat{M}, \hat{g})$ . On the cylinder we have that  $\hat{E}$  is larger than a constant  $C$ , so

$$\int_{\Sigma \times \mathbb{R}} (a_n |\nabla \phi|_{h+dt^2}^2 + R^h \phi^2) d\mu^{h+dt^2} \geq C \int_{\Sigma \times \mathbb{R}} \phi^2 d\mu^{h+dt^2} \quad (3.9)$$

for functions  $\phi$  with compact support on  $\Sigma \times \mathbb{R}$ . Let  $\chi(t)$  be smooth compactly supported function on  $\mathbb{R}$  with

$$\int_{\mathbb{R}} (\chi(t))^2 dt = 1, \quad a_n \int_{\mathbb{R}} (\chi'(t))^2 dt \leq \frac{C}{2},$$

and let  $v$  be any smooth function on  $\Sigma$ . If we set  $\phi = \chi v$  in (3.9) we get

$$\begin{aligned} & \int_{\Sigma} \int_{\mathbb{R}} (a_n (\chi'(t))^2 v^2 + a_n (\chi(t))^2 |\nabla v|_h^2 + R^h (\chi(t))^2 v^2) dt d\mu^h \\ & \geq C \int_{\Sigma} \int_{\mathbb{R}} (\chi(t))^2 v^2 dt d\mu^h, \end{aligned}$$

which by the properties of  $\chi(t)$  gives us

$$\int_{\Sigma} (a_n |\nabla v|_h^2 + R^h v^2) d\mu^h \geq \frac{C}{2} \int_{\Sigma} v^2 d\mu^h.$$

This means that the operator  $\mathcal{L}^h := -a_n \Delta^h + R^h$  on  $\Sigma$  has a spectrum consisting only of positive eigenvalues.

Let  $v_0$  be a positive eigenfunction corresponding to the smallest eigenvalue  $\mu_0$  of  $\mathcal{L}^h$ , that is  $\mathcal{L}^h v_0 = \mu_0 v_0$ . Let  $v$  be a positive function on  $\hat{M}$  which is equal to  $v_0$  on  $\Sigma \times [0, \infty)$  and set

$$\check{g} := v^{4/(n-2)} \hat{g}.$$

Then  $R^{\check{g}} = v^{-\frac{n+2}{n-2}} L^{\hat{g}} v$ , so on  $\Sigma \times [0, \infty)$  we have

$$R^{\check{g}} = v_0^{-\frac{n+2}{n-2}} L^{\hat{g}} v_0 = v_0^{-\frac{n+2}{n-2}} \mathcal{L}^h v_0 = \mu_0 v_0^{-4/(n-2)} \geq c > 0.$$

The estimate (3.8) and conformal invariance gives us

$$\int_{\hat{M}} (a_n |\nabla \phi|_{\check{g}}^2 + R^{\check{g}} \phi^2) d\mu^{\check{g}} \geq \int_{\hat{M}} \hat{E} v^{-4/(n-2)} \phi^2 d\mu^{\check{g}}.$$

We redefine  $\hat{g}$  as  $\check{g}$  and  $\hat{E}$  as  $\hat{E} v^{-4/(n-2)}$ . Then  $(\hat{M}, \hat{g})$  has positive scalar curvature on the cylindrical end, it satisfies all the properties from step 4, and

$$\int_{\hat{M}} (a_n |\nabla \phi|_{\hat{g}}^2 + R^{\hat{g}} \phi^2) d\mu^{\hat{g}} \geq \int_{\hat{M}} \hat{E} \phi^2 d\mu^{\hat{g}} \quad (3.10)$$

for  $\phi \in C_0^\infty(\hat{M})$ .

**Step 6: Conformal change to positive scalar curvature everywhere.** We follow the argument in the proof of Proposition 4.6 in [6].

We first prove that the conclusions of Lemma 4.5 in [6] follow from (3.10). The first of these conclusions is that the  $L^2$ -spectrum of  $L^{\hat{g}} = -a_n \Delta^{\hat{g}} + R^{\hat{g}}$  is contained in  $[0, \infty)$ , which clearly follows from (3.10).

The second conclusion is that  $L^{\hat{g}}$  does not have zero as an eigenvalue. Assume that  $u$  is an  $L^2$  function with  $L^{\hat{g}} u = 0$ . Since  $R^{\hat{g}} \geq c > 0$  on the cylindrical end of  $(\hat{M}, \hat{g})$  the “tangential part” of the operator  $L^{\hat{g}}$  on the cylindrical end has a spectrum consisting only of positive eigenvalues. Separation of variables tells us that the  $L^2$  function  $u$  must have exponential decay on the cylindrical end. Since  $u$  has exponential decay we can multiply with cut-off functions, insert in (3.10) and integrate by parts to conclude that  $u = 0$ , since  $\hat{E}$  is strictly positive.

Let  $\theta$  be a positive function which is equal to  $R^{\hat{g}}$  outside a large compact set. We want to solve the equation

$$L^{\hat{g}} f = \theta$$

for a positive function  $f$  which tends to 1 on the cylindrical end. Set  $f = 1 + \alpha$ , so that

$$L^{\hat{g}} \alpha = L^{\hat{g}}(f - 1) = \theta - R^{\hat{g}} =: \tilde{\theta}$$

where  $\tilde{\theta}$  has compact support. In the proof of Proposition 4.6 in [6] there is an argument using barrier functions to show that there is a solution  $\alpha$ , such that  $f = 1 + \alpha > 0$ .

Making a conformal change with  $f$  we get the metric  $f^{4/(n-2)} \hat{g}$  which has scalar curvature

$$R^{f^{4/(n-2)} \hat{g}} = f^{-\frac{n+2}{n-2}} L^{\hat{g}} f = f^{-\frac{n+2}{n-2}} \theta > 0.$$

This metric does not have an exact cylindrical end, but since  $\alpha$  decays exponentially we can deform it to be zero outside a large compact set, so that  $f = 1$  outside this compact set, while preserving positivity of scalar curvature. Finally, we set  $\check{g} = f^{4/(n-2)} \hat{g}$  with the modified function  $f$ , and we cut off the cylindrical end of  $\hat{M}$  to get a manifold  $(\check{M}, \check{g})$  with boundary satisfying all the stated properties.



This completes the proof of Theorem 1.2 for the case  $4 \leq n \leq 7$ . The following remark deals with the case  $n = 3$ .

**REMARK 3.1.** *In the case  $n = 3$ , the decay of the initial data  $(g_i, K_i)$  provided by the density theorem [11, Theorem 22] is not compatible with solving Jang's equation, due to the slow decay of the mean curvature  $\text{tr}^{g_i} K_i$ . A cut-off argument similar to that used in Step 3 above can be used to modify this data near infinity to get  $\text{tr}^{g_i} K_i = O(|x|^{-\gamma})$  for  $\gamma > 2$ . The modified data, however, fails to have strict DEC in a neighborhood of infinity. Performing the cut-off at a sufficiently large radius, one finds upon constructing a solution to Jang's equation for this modified data, that the Hardy inequality argument used in Step 3 can be applied again to recover the inequality (3.6). This approach allows us to extend the result of Theorem 1.2 to the case  $n = 3$ . We leave the details to the reader.*

**4. Obstructions to positive scalar curvature.** In this section we will discuss conclusions about the topology of manifold  $M$  which can be drawn from the Theorem 1.2.

**4.1. Three dimensions.** First we consider the case when the dimension  $n = 3$ . Assume that  $M$  is a connected oriented 3-manifold, with connected boundary, satisfying the conclusion of Theorem 1.2. Then its boundary  $\partial M$  is diffeomorphic to a 2-sphere. From [27, Theorem 3.4] we know that the positive scalar curvature metric  $h$  on  $\partial M$  is isotopic to the standard metric on  $S^2$ . Using the isotopy of metrics we get a positive scalar curvature metric on the cylinder  $S^2 \times I$  if the interval  $I$  is long enough (so that the isotopy is run through very slowly). Capping off the cylinder with the standard metric on the hemisphere we get a positive scalar curvature metric on the 3-ball  $B$  which has  $h$  as its induced metric on the boundary and is product near the boundary. By a classical result of Gromov and Lawson [19], and the positive resolution of the Poincaré conjecture,  $\tilde{M} = \tilde{M} \cup_{\partial M} B$  must be a connected sum of spherical space forms (manifolds of the form  $S^3/\Gamma$ , where  $\Gamma$  is a finite group of isometries of  $S^3$ ) and copies of  $S^2 \times S^1$ . We thus have the following.

**PROPOSITION 4.1.** *Let  $(M, g, K)$  be a 3-dimensional initial data set satisfying the hypotheses of Theorem 1.2, and assume  $M$  is orientable. Then  $M$  is diffeomorphic to  $\mathbb{R}^3 \# N \setminus B$ , where  $N$  is a connected sum (possibly empty) of spherical space forms and copies of  $S^2 \times S^1$ , and  $B$  is an open Euclidean ball.*

A key assumption in Proposition 4.1 is that there are no MOTS/MITS in  $M \setminus \partial M$ . In [10] it was shown that  $M \approx \mathbb{R}^3 \setminus B$ , under the stronger assumption that there are no *immersed* MOTS (as defined in [10]) in  $M \setminus \partial M$ . It remains an interesting open question whether the same conclusion can be reached under the assumption of no MOTS/MITS.

**4.2. Index obstructions.** In general dimensions there are obstructions to the existence of positive scalar curvature metrics coming from the index of Dirac operators on the manifold. The setting which is relevant here is with a compact spin manifold  $M$  with boundary  $\partial M$  and a given metric of positive scalar curvature  $h$  defined on the boundary. Actually, the metric on the boundary is only required to have invertible Dirac operator, which holds also when  $h$  is conformal to a positive scalar curvature metric. The metric  $h$  is then extended to a metric  $g$  on  $M$  with the only requirement that  $g$  is a product  $h + dt^2$  in a neighbourhood of the boundary. Using  $g$  a Dirac operator on  $M$  is defined. The index  $\text{ind}(M, h)$  of this Dirac operator defines an

element of  $K_n(C_r^*\pi_1(M))$ , that is of the  $K$ -theory of the reduced  $C^*$ -algebra  $C_r^*\pi_1(M)$  of the fundamental group  $\pi_1(M)$ , see for example [28] or [26]. The index depends only on the pair  $(M, h)$  up to cobordism, meaning that if  $(M', h')$  is cobordant to  $(M, h)$  through a manifold with corners, and the induced cobordism from  $\partial M$  to  $\partial M'$  is equipped with a positive scalar curvature metric which restricts to  $h$ , resp.  $h'$ , then  $\text{ind}(M', h') = \text{ind}(M, h)$ . Again, it is actually only required that the induced bordism between the boundaries has invertible Dirac operator. If the metric  $h$  can be extended to a metric on  $M$  which has positive scalar curvature and is a product near the boundary, then  $\text{ind}(M, h) = 0$  by the Schrödinger-Lichnerowicz formula.

Applied to our setting we thus have the index  $\text{ind}(\check{M}, h) \in K_n(C_r^*\pi_1(M))$  defined for the compactified exterior  $\check{M}$  of any strictly stable MOTS in  $M$ , since by Theorem 1.1 the induced metric on the MOTS boundary is conformal to a metric of positive scalar curvature. In case the MOTS is outermost and the initial data set is spin and satisfies the dominant energy condition we conclude from Theorem 1.2 that  $\text{ind}(\check{M}, h)$  vanishes. The conclusion is actually that the index with respect to the boundary metric of  $\check{g}$  vanishes, but since this is conformal to a small perturbation of  $h$  the index is the same for the two boundary metrics.

**PROPOSITION 4.2.** *Let  $(M, g, K)$  be an  $n$ -dimensional,  $3 \leq n \leq 7$ , initial data set satisfying the hypotheses of Theorem 1.2. Assume that  $M$  is spin. Then  $\text{ind}(\check{M}, h) = 0$  where  $h$  is the induced metric on  $\partial M$ , and  $\check{M}$  is the exterior of this MOTS with the asymptotically flat end compactified by a point.*

As a simple application, we have the following.

**PROPOSITION 4.3.** *Let  $(M, g, K)$  be an initial data set satisfying the hypotheses of Theorem 1.2, and assume  $M$  is spin. Let  $M'$  be a manifold which can be expressed as a connected sum,  $M' = M \# X$ , where  $X$  is a closed spin manifold. Suppose there exists initial data on  $M'$  satisfying the dominant energy condition and coinciding with  $(g, K)$  near the boundary. Then either  $\text{ind}(X) = 0$  or there exists a MOTS in the exterior of the boundary of  $M'$ .*

*Proof.* We have that  $\check{M}' = \check{M} \# X$ , where  $\check{M}$  is the one point compactification of  $M$ . The metrics on the boundaries from Theorem 1.2 are both conformal to a small perturbation of the boundary metric  $h$ . The proposition then follows from Proposition 4.2, together with the cobordism invariance of the index and the fact that a connected sum is cobordant to the disjoint union of the summands.  $\square$

For example, if, in the context of the Proposition 4.3,  $M$  is a four manifold and  $X$  is a K3 surface, then there must exist a MOTS in the exterior of  $M'$ .

We now make use of Gromov and Lawson's notion of enlargeability, as described in [19] and references therein.

**PROPOSITION 4.4.** *Let  $(M, g, K)$  be an initial data set satisfying the hypotheses of Theorem 1.2, and assume  $M$  is spin. Suppose  $M$  can be expressed as,  $M = N \# X$ , where  $X$  is a closed manifold. Then  $X$  is not enlargeable.*

*Proof.* Let  $\check{P}$  denote the double of  $\check{M} = \check{N} \# X$ .  $\check{P}$  is the connected sum of a closed spin manifold and an enlargeable manifold (namely  $X$  or its copy in the double). But this contradicts the fact, which is an immediate consequence of Theorem 1.2, that  $\check{P}$  admits a metric of positive scalar curvature.  $\square$

So, for example, as follows from results in [19], if  $(M, g, K)$  is as in Proposition 4.4, then  $X$  could not be a torus, or, more generally, a manifold that admits a metric of nonpositive sectional curvature.

This suggests a relationship between the results we have established here and a natural problem for the existence of metrics of positive scalar curvature.

**QUESTION 4.5.** *Let  $(M, g, K)$  be an initial data set satisfying the hypotheses of Theorem 1.2, with  $\partial M$  connected and spherical. Does the closed manifold  $\bar{M}$ , obtained by compactifying the end of  $M$  and attaching a standard ball to  $\partial M$ , admit a metric of positive scalar curvature?*

One should compare this with Problem 6.1 in [27]. We should note however that in this context we have

**PROPOSITION 4.6.** *Let  $(M, g, K)$  be an initial data set satisfying the hypotheses of Theorem 1.2, with  $\partial M$  connected and spherical and assume  $M$  is spin. Then the closed manifold  $\bar{M}$ , obtained by compactifying the end of  $M$  and attaching a standard ball to  $\partial M$ , is not enlargeable.*

*Proof.* Suppose  $\bar{M}$  were enlargeable then,  $\bar{M} \# \bar{M}$  would also be enlargeable by [19], but this is precisely the compactified double of  $\bar{M}$  which we have shown admits a metric of positive scalar curvature.  $\square$

**4.3. Minimal hypersurface obstructions.** For closed manifolds of dimension less than 8 one finds obstructions to positive scalar curvature by using the fact that area-minimizing hypersurfaces of a manifold with positive scalar curvature also allow positive scalar curvature metrics, see [32]. In this context, Schoen and Yau introduce, for each  $n$ , a class of  $n$ -manifolds  $\mathcal{C}_n$ , and prove that a closed orientable manifold  $M$  of dimension  $n$ ,  $3 \leq n \leq 7$ , having positive scalar curvature must belong to  $\mathcal{C}_n$ .

Similar classes of manifolds can be defined for compact manifolds  $M$  with mean convex boundary,  $H_{\partial M} \geq 0$  (with respect to the outward normal). Let  $\mathcal{C}'_3$  be the class of compact orientable 3-manifolds with (possibly empty) mean convex boundary such that for any finite covering manifold  $\tilde{M}$  of  $M$ ,  $\pi_1(\tilde{M})$  contains no subgroup isomorphic to the fundamental group of a compact surface of genus  $\geq 1$ . In general, we say that an  $n$ -dimensional,  $n \geq 4$ , compact orientable manifold  $M$  with (possibly empty) mean convex boundary is of class  $\mathcal{C}'_n$  if for any finite covering space of  $M$ , every nontrivial codimension one homology class can be represented by an embedded compact hypersurface of class  $\mathcal{C}'_{n-1}$ . Then, using the results in [20] for compact 3-manifolds with mean convex boundary, the proof of Theorem 1 in [32] is easily modified to show that a compact orientable manifold  $M$  with mean convex boundary of dimension  $n$ ,  $3 \leq n \leq 7$ , having positive scalar curvature must belong to  $\mathcal{C}'_n$ . This immediately yields the following.

**PROPOSITION 4.7.** *Let  $(M, g, K)$  be an  $n$ -dimensional initial data set,  $3 \leq n \leq 7$ , satisfying the hypotheses of Theorem 1.2, and assume  $M$  is orientable. Then the compactification  $\bar{M}$  belongs to the class  $\mathcal{C}'_n$ .*

## REFERENCES

- [1] A. ALAEE, H. K. KUNDURI, AND EDUARDO MARTÍNEZ P., *Notes on maximal slices of five-dimensional black holes*, Classical Quantum Gravity, 31:5 (2014), 055004, 20.
- [2] L. ANDERSSON, M. EICHMAIR, AND J. METZGER, *Jang's equation and its applications to marginally trapped surfaces*, Complex analysis and dynamical systems IV. Part 2, Contemp. Math., vol. 554, Amer. Math. Soc., Providence, RI, 2011, pp. 13–45.
- [3] L. ANDERSSON AND J. METZGER, *The area of horizons and the trapped region*, Comm. Math. Phys., 290:3 (2009), pp. 941–972.

- [4] R. BARTNIK, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math., 39:5 (1986), pp. 661–693.
- [5] P. T. CHRUSCIEL, *The geometry of black holes*, 2015, notes available at <http://homepage.univie.ac.at/piotr.chrusciel>.
- [6] P. T. CHRUSCIEL AND R. MAZZEO, *Initial Data Sets with Ends of Cylindrical Type: I. the Lichnerowicz Equation*, Annales Henri Poincaré (2014), pp. 1–36.
- [7] M. EICHMAIR, *The Plateau problem for marginally outer trapped surfaces*, J. Differential Geom., 83:3 (2009), pp. 551–583.
- [8] ———, *Existence, regularity, and properties of generalized apparent horizons*, Comm. Math. Phys., 294:3 (2010), pp. 745–760.
- [9] ———, *The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight*, Comm. Math. Phys., 319:3 (2013), pp. 575–593.
- [10] M. EICHMAIR, G. J. GALLOWAY, AND D. POLLACK, *Topological censorship from the initial data point of view*, J. Differential Geom., 95:3 (2013), pp. 389–405.
- [11] M. EICHMAIR, L.-H. HUANG, D. A. LEE, AND R. SCHOEN, *The spacetime positive mass theorem in dimensions less than eight*, 2011, <http://arxiv.org/abs/1110.2087v2>.
- [12] M. EICHMAIR AND J. METZGER, *Jenkins-Serrin type results for the Jang equation*, <http://arxiv.org/abs/1205.4301>.
- [13] R. EMPARAN AND H. S. REALL, *A rotating black ring solution in five dimensions*, Phys. Rev. Lett., 88:10 (2002), 101101, 4.
- [14] J. L. FRIEDMAN, K. SCHLEICH, AND D. M. WITT, *Topological censorship*, Phys. Rev. Lett., 71:10 (1993), pp. 1486–1489.
- [15] G. J. GALLOWAY, *On the topology of the domain of outer communication*, Classical Quantum Gravity, 12:10 (1995), pp. L99–L101.
- [16] ———, *Rigidity of marginally trapped surfaces and the topology of black holes*, Comm. Anal. Geom., 16:1 (2008), pp. 217–229.
- [17] G. J. GALLOWAY AND R. SCHOEN, *A generalization of Hawking’s black hole topology theorem to higher dimensions*, Comm. Math. Phys., 266:2 (2006), pp. 571–576.
- [18] D. GANNON, *Singularities in nonsimply connected space-times*, J. Mathematical Phys., 16:12 (1975), pp. 2364–2367.
- [19] M. GROMOV AND H. B. LAWSON, JR., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. (1983), no. 58, pp. 83–196 (1984).
- [20] J. HASS AND P. SCOTT, *The existence of least area surfaces in 3-manifolds*, Trans. Amer. Math. Soc., 310:1 (1988), pp. 87–114.
- [21] L.-H. HUANG, personal communication.
- [22] J. ISENBERG, R. MAZZEO, AND D. POLLACK, *On the topology of vacuum spacetimes*, Ann. Henri Poincaré, 4:2 (2003), pp. 369–383.
- [23] C. W. LEE, *A restriction on the topology of Cauchy surfaces in general relativity*, Comm. Math. Phys., 51:2 (1976), pp. 157–162.
- [24] V. G. MAZ’JA, *Sobolev spaces*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985, Translated from the Russian by T. O. Shaposhnikova.
- [25] J. METZGER, *Blowup of Jang’s equation at outermost marginally trapped surfaces*, Comm. Math. Phys., 294:1 (2010), pp. 61–72.
- [26] P. PIAZZA AND T. SCHICK, *Rho-classes, index theory and Stolz’ positive scalar curvature sequence*, Journal of Topology, 7:4 (2014), pp. 965–1004.
- [27] J. ROSENBERG AND S. STOLZ, *Metrics of positive scalar curvature and connections with surgery*, Surveys on surgery theory, Vol. 2, Ann. of Math. Stud., vol. 149, Princeton Univ. Press, Princeton, NJ, 2001, pp. 353–386.
- [28] T. SCHICK, *The topology of positive scalar curvature*, 2014, <http://arxiv.org/abs/1405.4220>.
- [29] K. SCHLEICH AND D. WITT, *Singularities from the topology and differentiable structure of asymptotically flat spacetimes*, 2010, <http://arxiv.org/abs/1006.2890>.
- [30] R. SCHOEN AND S.-T. YAU, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2), 110:1 (1979), pp. 127–142.
- [31] ———, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys., 65:1 (1979), pp. 45–76.
- [32] ———, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math., 28 (1979), no. 1-3, pp. 159–183.
- [33] ———, *Positivity of the total mass of a general space-time*, Phys. Rev. Lett., 43:20 (1979), pp. 1457–1459.
- [34] ———, *Proof of the positive mass theorem. II*, Comm. Math. Phys., 79:2 (1981), pp. 231–260.

