

PRODUCTS OF SYMMETRIC GROUP CHARACTERS

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ABSTRACT. In [OZ], the authors introduced a new basis of the ring of symmetric functions which evaluate to the irreducible characters of the symmetric group at roots of unity. The structure coefficients for this new basis are the stable Kronecker coefficients. In this paper we give combinatorial descriptions for several products. In addition, we identify some applications and instances where special cases of these products have occurred elsewhere in the mathematical literature.

1. INTRODUCTION

Schur functions, s_λ , form a fundamental basis for the ring of symmetric functions. One reason for this is that they specialize to the characters of polynomial representations of the general linear group, GL_n . The structure coefficients for the Schur functions are the Littlewood-Richardson coefficients, $c_{\lambda\mu}^\nu$,

$$s_\lambda s_\mu = \sum_{\nu \vdash |\lambda|+|\mu|} c_{\lambda\mu}^\nu s_\nu.$$

In [OZ], we introduced a new basis of symmetric functions, \tilde{s}_λ , that we believe in many ways is as fundamental as the basis of Schur functions since these functions specialize to the irreducible characters of the symmetric group, S_n . The structure coefficients for the \tilde{s} -basis,

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_{|\nu| \leq |\lambda|+|\mu|} \bar{g}_{\lambda\mu}^\nu \tilde{s}_\nu ,$$

are the stable Kronecker coefficients, $\bar{g}_{\lambda\mu}^\nu$, [BOR, Li1, Mur3, Val] and they can be seen to be a generalization of the Littlewood-Richardson coefficients in the sense that $c_{\lambda\mu}^\nu = \bar{g}_{\lambda\mu}^\nu$ when $|\nu| = |\lambda| + |\mu|$. The \tilde{s} -basis gives an elegant and natural symmetric function formulation for the stable Kronecker coefficients.

A polynomial irreducible representation of GL_n exists for each partition λ and the character of that representation is the Schur polynomial indexed by λ evaluated at the eigenvalues of the elements of GL_n . Since the subgroup of $n \times n$ permutation matrices is much smaller, each one of these polynomial irreducible GL_n representation will decompose into a direct sum of irreducible S_n representations. The general representation theory of GL_n and S_n laid out by Frobenius and Schur gives an explicit formula for the

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multiplicity of an S_n irreducible in a GL_n irreducible, but finding a positive, purely combinatorial expression for this multiplicity has been revisited many times in the literature [BK, Li2, Kin, Nis, ST, STW2] and is known as ‘the restriction problem.’ The irreducible character basis reformulates the restriction problem since, at least for n sufficiently large, an equivalent statement would be to give the change of basis coefficients between the Schur basis and the \tilde{s} -basis.

A notorious open problem is to find a positive combinatorial interpretation for Kronecker coefficients, $g_{\lambda\mu}^{\nu}$. These are the coefficients in the decomposition of the tensor product of two irreducible representations of S_n . The stable Kronecker coefficients are some of the Kronecker coefficients in the sense that for any three partitions α, β and γ , $\bar{g}_{\alpha,\beta}^{\gamma} = g_{(n-|\alpha|,\alpha)(n-|\beta|,\beta)}^{(n-|\gamma|,\gamma)}$ for $n \gg 0$. Many attempts have been made to find combinatorial interpretations and there are a large number of papers in the literature addressing several aspects of the Kronecker coefficients (for example, [BO, Bla, BOR, Man, PPV, Rem, RW, Ros, STW, Val]). Recent progress has lead to combinatorial interpretations of the special cases of $g_{(n-k,1^k)\mu}^{\lambda}$ [Bla, Liu, BL] and $\bar{g}_{(k)\mu}^{\lambda}$ [BO]. In terms of the \tilde{s} -basis, this could be translated into a combinatorial interpretation for the products of $\tilde{s}_k \tilde{s}_{\mu}$ and $\tilde{s}_{1^k} \tilde{s}_{\lambda}$, but the statement of Theorem 15 covers a much more general class of stable Kronecker coefficients than has previously been considered for Kronecker coefficients.

The \tilde{s} -expansion of a symmetric function specializing to an S_n -character corresponds to the decomposition of the corresponding representation into irreducible S_n characters. In this setting, products of symmetric functions correspond to tensor products of representations. In [OZ] we also introduced the \tilde{h}_{μ} symmetric functions which specialize to values of the trivial characters of a Young subgroup $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_{\ell(\mu)}} \subseteq S_n$ induced to the symmetric group S_n . The focus of this work are products involving the S_n -characters h_{μ} , \tilde{h}_{μ} and \tilde{s}_{μ} , where h_{μ} are the complete homogeneous functions.

The aim of this paper is to give combinatorial interpretations for the coefficients in the \tilde{s} -expansion for the products $\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_{\ell}} \tilde{s}_{\lambda}$, $\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell}} \tilde{s}_{\lambda}$, $\tilde{h}_{\mu} \tilde{s}_{\lambda}$, and $h_{\mu} \tilde{s}_{\lambda}$. Our combinatorial description for $h_{\mu} \tilde{s}_{\lambda}$ is in terms of column strict tableaux filled with multisets containing at most one barred entry and unbarred entries that satisfy a lattice condition by reading the barred entries with respect to the unbarred entries. When we look at the products $\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell}} \tilde{s}_{\lambda}$ we have the same objects but now we cannot repeat unbarred entries, that is, cells are filled with sets. And for the products $\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_{\ell}} \tilde{s}_{\lambda}$ we further require that only sets with more than one element can go on the first row of the tableau. Finally, for the products $\tilde{h}_{\mu} \tilde{s}_{\lambda}$ we have the conditions for $h_{\mu} \tilde{s}_{\lambda}$ and the additional condition that at most one barred and at most one unbarred entry can be in each cell. This provides a unified description using the same combinatorial objects for all these products.

For symmetric functions f and g , we say that $f \leq g$ if the coefficients of $g - f$ when expanded in the \tilde{s} -basis are non-negative. The combinatorial interpretations for the products described in this paper clearly illustrate the following inequalities.

$$\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_{\ell}} \tilde{s}_{\lambda} \leq \tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell}} \tilde{s}_{\lambda} \leq h_{\mu} \tilde{s}_{\lambda}$$

After some preliminaries we define a scalar product on symmetric functions for which the set of functions $\{\tilde{s}_\lambda\}_\lambda$ forms an orthonormal basis (Section 2.7). Our main results are the combinatorial rules for the products $\tilde{s}_{\mu_1}\tilde{s}_{\mu_2}\cdots\tilde{s}_{\mu_\ell}\tilde{s}_\lambda$, $\tilde{h}_{\mu_1}\tilde{h}_{\mu_2}\cdots\tilde{h}_{\mu_\ell}\tilde{s}_\lambda$ and $h_\mu\tilde{s}_\lambda$ (Theorem 13, 14 and 15) that appear in Section 3. We prove these formulas in Section 5 after also giving a combinatorial interpretation for the product $\tilde{h}_\mu\tilde{s}_\lambda$ in Section 4. In Section 6, we end the paper with a discussion of how our combinatorial interpretations give a unified way to view several different mathematical questions and constructions considered in the literature (e.g. the restriction problem of an irreducible GL_n module to S_n , quantum entanglements of q -bits, Grothendieck symmetric functions and dimensions of irreducible representations of the partition and quasi-partition algebras).

2. PRELIMINARIES

2.1. Partition and diagram notation. A partition of a non-negative integer n is a list of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ with $\lambda_i \geq \lambda_{i+1} > 0$ and $n = \sum_{i=1}^{\ell} \lambda_i$. We refer to n as the size of the partition (denoted $|\lambda| := n$ or $\lambda \vdash n$). The nonzero entries λ_i are called the parts of the partition. The number of nonzero parts $\ell = \ell(\lambda)$ is called the length of λ . We will often refer to the partition λ with the first part removed as $\bar{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_{\ell(\lambda)})$. Similarly, if a is an integer with $a \geq \lambda_1$, then (a, λ) represents the partition $(a, \lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$. The set of all partitions is denoted by Par , and $\text{Par}_{\leq n}$ will denote partitions of size less than or equal to n .

Partitions will be visually represented by their Young diagrams with n cells arranged in $\ell(\lambda)$ rows with λ_i cells in the i^{th} row. We will represent our partitions in French notation with the largest part of the partition on the bottom and the coordinates of the cells are (i, j) such that $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. The number of parts of the partition λ which are of size i will be denoted $m_i(\lambda) := |\{j : \lambda_j = i\}|$.

For any partition λ we define

$$z_\lambda = \prod_{i=1}^{\lambda_1} i^{m_i(\lambda)} m_i(\lambda)! .$$

The conjugate of a partition λ is denoted λ' and it is the partition with the i^{th} row equal to $\sum_{j \geq i} m_j(\lambda)$.

For two partitions λ and μ where $\ell(\lambda) \geq \ell(\mu)$ and $\lambda_i \geq \mu_i$ for all $1 \leq i \leq \ell(\mu)$. We define the skew partition λ/μ to be the cells of the partition λ which are not in the cells of μ . We say that λ/μ is a horizontal strip if $\lambda_{i+1} \leq \mu_i$ for $1 \leq i \leq \ell(\lambda) - 1$.

2.2. Multisets and multiset partitions. We will distinguish a multiset from a set using the notation $\{\!\{1^{a_1}, 2^{a_2}, \dots, m^{a_m}\}\!\}$ to indicate that the multiset contains a_i copies of i and the elements are considered without order. A multiset partition π of a multiset S is a multiset of non-empty multisets whose union (with multiplicities) is S . That is,

$$(1) \quad \pi = \{\!\{S_1, S_2, \dots, S_{\ell(\pi)}\}\!\}$$

with $S_1 \cup S_2 \cup \dots \cup S_{\ell(\pi)} = S$ (here the union is taken with multiplicities). We have denoted $\ell(\pi)$ as the number of parts of the multiset partition. If S is a multiset then we will indicate that π is a multiset partition of S with the notation $\pi \vdash S$.

2.3. Multiset valued tableaux. We now describe properties of the tableaux which are common to all the combinatorial interpretations presented in the following sections.

A *tableau* T is a map from the set of cells of the Young diagram to a set of labels. We say that a tableau is *column strict* (with respect to some order on the labels) if $T_{(i,j)} \leq T_{(i,j+1)}$ for all $1 \leq i \leq \ell(\lambda)$ and $1 \leq j < \lambda_i$ and if $T_{(i,j)} < T_{(i+1,j)}$ for all $1 \leq i < \ell(\lambda)$ and $1 \leq j \leq \lambda_{i+1}$. We are interested in column strict tableaux with labels being multisets, thus we need to define the order on multisets that we will use throughout the paper. We assume that $\bar{1} < \bar{2} < \dots < 1 < 2 < \dots$. Let $M = \{\{m_1 \leq m_2 \leq \dots \leq m_r\}\}$ and $N = \{\{n_1 \leq n_2 \leq \dots \leq n_s\}\}$ be two multisets with elements from the set $\{\bar{1}, \bar{2}, \dots, 1, 2, \dots\}$ and at most one barred entry. We define $M < N$ if $m_{r-i} = n_{s-i}$ for $0 \leq i \leq k$ and $m_{r-k-1} < n_{s-k-1}$ or $r-i = 0$ and $s-i > 0$. This is the *reverse lexicographic order (reverse lex)* on the entries of the multisets. For example, $\{\{\bar{5}, 1, 1, 1, 2, 2, 3, 4\}\} < \{\{\bar{2}, 1, 1, 2, 3, 3, 4\}\}$.

The *content* of a tableau T is defined as the multiset which contains $a_i^{m_i}$ where $a_i \in \{\bar{1}, \bar{2}, \dots, 1, 2, \dots\}$ occurs m_i times in T . The cells of the tableaux will be filled with non-empty multisets with elements chosen from the set $\{\bar{1}, \bar{2}, \bar{3}, \dots, 1, 2, 3, \dots\}$ such that each multiset contains at most one barred entry. Thus, the content of the tableau is $\{\{\bar{1}^{\alpha_1}, \bar{2}^{\alpha_2}, \dots, \bar{\ell}^{\alpha_\ell}, 1^{\beta_1}, 2^{\beta_2}, \dots, k^{\beta_k}\}\}$ for some weak compositions α and β .

The *shape* of a tableau T is the sequence obtained by reading the lengths of each row in T . We denote by $sh(T)$ the shape of T . All of our tableaux will be of shape $(r, \gamma)/(\gamma_1)$ for a partition γ and some integer $r \geq \gamma_1$.

Definition 1. Let γ be a partition, α and β be compositions. Then the set $\mathcal{MCT}_\gamma(\alpha, \beta)$ contains tableaux T that are column strict with respect to the reverse lex order, have shape $(r, \gamma)/(\gamma_1)$, and content $\{\{\bar{1}^{\alpha_1}, \bar{2}^{\alpha_2}, \dots, \bar{\ell}^{\alpha_\ell}, 1^{\beta_1}, 2^{\beta_2}, \dots, k^{\beta_k}\}\}$ with at most one barred entry in each cell. Further, we require that cells in the first row cannot be filled with multisets containing only barred entries.

In our examples of tableaux, to save space, the $\{\}$ are dropped from the labels of the entries.

Example 2. The following tableau is in $\mathcal{MCT}_{(3,3,3,2,1)}((6,4,2), (12,4))$

| | |
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| $T =$ | $\begin{array}{ c c c } \hline \bar{1}12 & & \\ \hline \bar{1}2 & \bar{3}12 & \\ \hline 11 & \bar{1}11 & \bar{2}2 \\ \hline \bar{2} & \bar{2}1 & \bar{1}11 \\ \hline \bar{1} & \bar{1}1 & \bar{2}1 \\ \hline & & \bar{3}1 \\ \hline \end{array}$ |
|-------|--|

Then $sh(T) = (4, 3, 3, 3, 2, 1)/(3)$ and $\overline{sh(T)} = (3, 3, 3, 2, 1)$.

For any tableau T , let $read(T)$ be the word of entries in the tableau from bottom row to top row and from right to left in the rows. If S is a multiset of non-barred entries, then let $read(T|_S)$ represent the reading word of the barred entries in the cells with $\{\bar{j}, S\}$ as a label and let $read(T|_-)$ represent the reading word of the cells which have only a barred entry, i.e., when $S = \emptyset$. The multisets with no barred entries do not contribute to the reading word.

In the tableau of Example 2, we have $read(T|_-) = \bar{12}$, $read(T|_1) = \bar{3212}$, $read(T|_{1^2}) = \bar{11}$, $read(T|_2) = \bar{21}$, $read(T|_{12}) = \bar{31}$. We remark that we have omitted the brackets in the indexing sets to unclutter the notation.

We indicate the concatenation of several words of the form $read(T|_S)$ by placing dots at the end of each word $read(T|_S)$. In our examples, consecutive dots will indicate that some words are empty. For example, $read(T|_-)read(T|_1)read(T|_{1^2})read(T|_2)read(T|_{12}) = \bar{12}.\bar{3212}.\bar{11}.\bar{21}.\bar{31}$ in the tableau of Example 2.

Let w be a word of content $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$. For any subword u of w , let $n_i(u)$ be the number of occurrences of i in u . A word w is called *lattice* if for all subwords u and v such that $w = uv$, $n_i(u) \geq n_{i+1}(u)$ for all $1 \leq i \leq \ell$. We use this definition to define a lattice multiset tableau.

Definition 3. If $T \in \mathcal{MCT}_\gamma(\lambda, \mu)$ for some partitions λ, μ and γ , then let $S_1 < S_2 < \dots < S_d$ be the multisets of the unbarred entries that appear in T (ignoring the barred entries). We say that T is a *lattice tableau* if the word

$$read(T|_-)read(T|_{S_1})read(T|_{S_2}) \cdots read(T|_{S_d})$$

is lattice.

Example 4. Consider the two tableaux

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| $\bar{1}$ | $\bar{11}$ | $\bar{21}$ | $\bar{22}$ |
| | | | |

and

| | | | |
|-----------|------------|------------|------------|
| $\bar{1}$ | $\bar{21}$ | $\bar{21}$ | $\bar{12}$ |
| | | | |

which are both elements of $\mathcal{MCT}_{(4)}((2, 2), (2, 1))$. The tableau on the left is lattice because $read(T|_-)read(T|_1)read(T|_2) = \bar{1}.\bar{21}.\bar{2}$; however, the tableau on the right is not lattice since $read(T|_-)read(T|_1)read(T|_2) = \bar{1}.\bar{22}.\bar{1}$.

Remark 5. The results that we use from [OZ] depend on choosing and fixing a total order on multisets. For the application in this paper, the total order that we have chosen at the beginning of this section is reverse lexicographic because it most easily captures the reading order that is compatible with our applications of the Littlewood-Richardson rule.

2.4. Symmetric function notation. Define the Hopf algebra of symmetric functions to be the polynomial ring

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \dots] = \mathbb{Q}[p_1, p_2, p_3, \dots]$$

where the generators are related by the equation

$$h_m = \sum_{\lambda \vdash m} \frac{p_\lambda}{z_\lambda}.$$

Then, for a partition λ , the complete and power sum symmetric functions are defined by

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}} \quad p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}.$$

Since Λ is a polynomial ring, it has as linear bases $\{h_\lambda\}_{\lambda \in \text{Par}}$ and $\{p_\lambda\}_{\lambda \in \text{Par}}$, where Par denotes the set of all partitions.

A definition of the Schur basis, $\{s_\lambda\}_{\lambda \in \text{Par}}$, is by the formula

$$h_\mu = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu} s_\lambda,$$

where $K_{\lambda\mu}$ is the Kostka coefficient and it is equal to the number of column strict tableau of shape λ and content μ .

The ring of symmetric functions is endowed with a scalar product for which the power sum symmetric functions are orthogonal, $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ where $\delta_{\lambda\lambda} = 1$ and $\delta_{\lambda\mu} = 0$ if $\lambda \neq \mu$, and the Schur basis is self dual, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. A useful application of the scalar product and the dual basis is to give an expansion of one symmetric function in one of these bases since for any symmetric function f ,

$$(2) \quad f = \sum_{\lambda \in \text{Par}} \langle f, s_\lambda \rangle s_\lambda = \sum_{\lambda \in \text{Par}} \langle f, p_\lambda \rangle \frac{p_\lambda}{z_\lambda}.$$

The structure coefficients of Schur functions are known as the Littlewood-Richardson coefficients, denoted by $c_{\lambda\mu}^\nu$,

$$(3) \quad s_\lambda s_\mu = \sum_{\nu \vdash |\lambda| + |\mu|} c_{\lambda\mu}^\nu s_\nu,$$

and in general, the coefficient of s_λ in the product $s_{\nu(1)} s_{\nu(2)} \cdots s_{\nu(\ell)}$ is denoted $c_{\nu(1)\nu(2)\dots\nu(\ell)}^\lambda$. Combinatorial interpretations for the coefficients $c_{\lambda\mu}^\nu$ are given in Section 2.5.

The Kronecker product is a second product on symmetric functions defined such that $\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda}$. The Kronecker coefficients, $g_{\lambda\mu}^\nu$, are the structure coefficients with respect to the Schur basis as in the coefficients of the equation

$$(4) \quad s_\lambda * s_\mu = \sum_{\nu} g_{\lambda\mu}^\nu s_\nu.$$

We assume that the reader is familiar with basic facts about the Kronecker coefficients including the basic stability properties [BOR, Mur2, Mur3, Val] and that $\langle f * g, h \rangle = \langle f, g * h \rangle$. We will use the notation $\bar{g}_{\lambda\mu}^\nu := g_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)}^{(n-|\nu|, \nu)}$ for the stable limit of these coefficients if n is sufficiently large.

2.5. Littlewood-Richardson rules. Our combinatorial interpretations are built upon tableau constructions of the Littlewood-Richardson rule. We will need the following two forms of the Littlewood-Richardson rule that we will use in the proofs below.

For a tableau T of content $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$ (potentially skew shape), we say that T is lattice if $\text{read}(T)$ is lattice (see Section 2.3 for the definition of $\text{read}(T)$ and lattice word). The other form of the Littlewood-Richardson rule uses jeu de Taquin and we will assume

that the reader is familiar with the relevant definitions to use these forms of the statement of the Littlewood-Richardson rule.

Proposition 6. ([Ful], *Exe. 1, Sect. 5.2, pp. 66*) *The Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ is equal to the number of pairs of tableaux (S, T) where S is of shape λ and T is of shape μ such that $\text{read}(S)\text{read}(T)$ is a lattice word of content $\{1^{\nu_1}, 2^{\nu_2}, \dots, \ell^{\nu_{\ell(\nu)}}\}$.*

We will represent these pairs (S, T) graphically by placing S below and to the right of T so that $\text{read}(S)\text{read}(T)$ is the extension of the reading of the skew tableau.

Example 7. The coefficient $c_{(2,1),(3,1,1)}^{(4,2,1,1)} = 2$ is equal to the number of the following tableaux

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where the reading words of these tableaux are 11123214 and 11123412 respectively. If we change the roles of $(2, 1)$ and $(3, 1, 1)$,

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where the reading words of these tableaux are 11221134 and 11231124 respectively.

Proposition 8. ([Ful], *Corollary 1-4, [Sta], Fomin's Appendix, Theorem A1.3.1*) *The Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ is equal to the number of column strict tableaux of shape ν/μ which are jeu de Taquin equivalent to a particular (arbitrary) column strict tableau of shape λ .*

Example 9. The coefficient $c_{(2,1),(3,1,1)}^{(4,2,1,1)} = 2$ is equal to the number of the following tableaux:

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or by interchanging the roles of $(2, 1)$ and $(3, 1, 1)$:

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2.6. Characters as symmetric functions. Every partition λ with $\ell(\lambda) \leq n$ indexes an irreducible polynomial module of $GL_n(\mathbb{C})$ with character at an element with eigenvalues x_1, x_2, \dots, x_n equal to $s_\lambda[X_n]$ where $X_n = x_1 + x_2 + \dots + x_n$. The square brackets after a symmetric function indicate plethystic notation where $p_k[E(x_1, x_2, \dots, x_n)] = E(x_1^k, x_2^k, \dots, x_n^k)$ for an expression $E(x_1, x_2, \dots, x_n)$.

Now the symmetric group, S_n , as permutation matrices, is contained in $GL_n(\mathbb{C})$. In [OZ], we introduced two new bases of symmetric functions $\{\tilde{s}_\lambda\}_{\lambda \in \text{Par}}$ and $\{\tilde{h}_\lambda\}_{\lambda \in \text{Par}}$ that have the property that they specialize to characters of the symmetric group when evaluated at the eigenvalues of permutation matrices. In particular, the functions \tilde{s}_λ specialize to irreducible S_n characters. In what follows, we cite some results from [OZ] that we will need in this paper.

The \tilde{s}_λ and \tilde{h}_λ are defined via change of basis formulae. For a set partition π of length $\ell(\pi)$, we define $\tilde{m}(\pi)$ to be the partition of $\ell(\pi)$ consisting of the multiplicities of the multisets occurring in π , for example, $\tilde{m}(\{\{\{1, 1, 2\}, \{1, 1, 2\}, \{1, 4\}\}\}) = (2, 1)$.

Definition 10. Fix a partition μ and a positive integer $n \geq 2|\mu|$,

$$(5) \quad h_\mu = \sum_{\pi \vdash \{\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell(\mu)^{\mu_{\ell(\mu)}}\}\}} \tilde{h}_{\tilde{m}(\pi)}$$

$$(6) \quad \tilde{h}_\mu = \sum_{\lambda \in \text{Par}_{\leq |\mu|}} K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)} \tilde{s}_\lambda = \sum_{\nu \vdash |\mu|} \sum_{\lambda} K_{\nu \mu} \tilde{s}_\lambda$$

where the $K_{\lambda \mu}$ are the Koskta coefficients and the inner sum on the right is over all partitions λ such that ν/λ is a horizontal strip.

Theorem 11. (Equations (6)–(10) of [OZ]) For a partition λ , the symmetric functions \tilde{s}_λ and \tilde{h}_λ have the property that for a positive integer $n \geq |\lambda| + \lambda_1$ and $\mu \vdash n$,

$$(7) \quad \tilde{s}_\lambda[\Xi_\mu] = \langle s_{(n-|\lambda|, \lambda)}, p_\mu \rangle = \chi^{(n-|\lambda|, \lambda)}(\mu) \quad \text{and} \quad \tilde{h}_\lambda[\Xi_\mu] = \langle h_{(n-|\lambda|, \lambda)}, p_\mu \rangle .$$

where for a symmetric function f , $f[\Xi_\mu]$ is the evaluation of the symmetric polynomial $f[X_{|\mu|}]$ at the eigenvalues of a permutation matrix with cycle structure μ .

Note that this implies that the characters \tilde{s}_λ and \tilde{h}_λ are the characters of representations of $S_n \subseteq GL_n(\mathbb{C})$ in the same way that the Schur functions are the characters of the irreducible polynomial representations of $GL_n(\mathbb{C})$. They have the reduced Kronecker coefficients as structure coefficients (see Theorem 4 of [OZ]) so that

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_{\nu: |\nu| \leq |\lambda| + |\mu|} \bar{g}_{\lambda \mu}^\nu \tilde{s}_\nu .$$

2.7. A scalar product on characters. The usual scalar product on characters can be used to define a scalar product on symmetric functions. Choose an n sufficiently large (any

n satisfying $n \geq 2 \max(\deg(f), \deg(g))$ will do), then define

$$(8) \quad \langle f, g \rangle_{\circledast} = \sum_{\nu \vdash n} \frac{f[\Xi_{\nu}]g[\Xi_{\nu}]}{z_{\nu}} = \frac{1}{n!} \sum_{\sigma \in S_n} f[\Xi_{cyc(\sigma)}]g[\Xi_{cyc(\sigma)}].$$

where $cyc(\sigma)$ is a partition representing the cycle structure of $\sigma \in S_n$. We use the \circledast -symbol as a subscript of the right angle bracket to differentiate this scalar product from the usual scalar product where $\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda\mu}$. One should recognize this scalar product as a translation of the usual scalar product on characters of a finite group when the group is S_n .

At first glance, there should be some concern about the dependence of the right hand side of this expression on the value n , but quite surprisingly this expression is independent of n as long as it is sufficiently large. This is not an obvious property, but it follows from the fact that the \tilde{s}_{λ} elements form a basis of the symmetric functions, linearity of the scalar product, and the following proposition.

Proposition 12. *For all partitions λ and μ ,*

$$(9) \quad \langle \tilde{s}_{\lambda}, \tilde{s}_{\mu} \rangle_{\circledast} = \delta_{\lambda\mu}.$$

Proof. Let n be greater than or equal to $2 \max(|\lambda|, |\mu|)$, then $(n - |\lambda|, \lambda)$ and $(n - |\mu|, \mu)$ are partitions and we calculate

$$(10) \quad \langle \tilde{s}_{\lambda}, \tilde{s}_{\mu} \rangle_{\circledast} = \sum_{\nu \vdash n} \frac{\tilde{s}_{\lambda}[\Xi_{\nu}] \tilde{s}_{\mu}[\Xi_{\nu}]}{z_{\nu}}$$

$$(11) \quad = \sum_{\nu \vdash n} \frac{\chi^{(n-|\lambda|, \lambda)}(\nu) \chi^{(n-|\mu|, \mu)}(\nu)}{z_{\nu}} = \delta_{\lambda\mu}.$$

This last equality follows by the character relations. \square

This scalar product allows us to characterize the \tilde{s}_{λ} symmetric functions as the orthonormal basis with respect to the scalar product $\langle \cdot, \cdot \rangle_{\circledast}$. This basis is triangular with respect to the Schur basis (\tilde{s}_{λ} is equal to s_{λ} plus terms of lower degree) and hence it may be calculated using Gram-Schmidt orthonormalization with respect to this scalar product.

3. COMBINATORIAL RULES FOR THE PRODUCTS $h_{\mu} \tilde{s}_{\lambda}$, $\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_k} \tilde{s}_{\lambda}$ AND $\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_k} \tilde{s}_{\lambda}$

In this section we state the combinatorial rules for the products, the proofs are in Section 5. The combinatorial interpretations we state here will follow from a formula for $\tilde{h}_{\mu} \tilde{s}_{\lambda}$ (Theorem 17 and Corollary 23) proved in Section 4. We state the following theorems in terms of increasing number of restrictions.

Theorem 13. *Let λ and γ be partitions and α a composition, then the coefficient of \tilde{s}_{γ} in $h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} \tilde{s}_{\lambda}$ is equal to the number of $T \in \mathcal{MCT}_{\gamma}(\lambda, \alpha)$ such that T is a lattice tableau.*

We next state the combinatorial rule for the product of an irreducible character basis element and $\tilde{h}_{\mu_1}\tilde{h}_{\mu_2}\cdots\tilde{h}_{\mu_{\ell(\mu)}}$ and leave the proof for Section 5.2. The difference between the previous combinatorial description is that the labeled entries must be sets rather than multisets.

Theorem 14. *Let λ and γ be partitions and α a composition, then the coefficient of \tilde{s}_γ in $\tilde{h}_{\alpha_1}\tilde{h}_{\alpha_2}\cdots\tilde{h}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda$ is equal to the number of $T \in \mathcal{MCT}_\gamma(\lambda, \alpha)$ such that the entries of the tableaux are sets (no repeated entries) and T is a lattice tableau.*

The expansion of the expression $\tilde{s}_{\mu_1}\tilde{s}_{\mu_2}\cdots\tilde{s}_{\mu_{\ell(\mu)}}\tilde{s}_\lambda$ has a combinatorial interpretation where there is only one extra condition that sets consisting of a single un-barred entry may not appear in the first row (and we already have the condition that sets consisting of a single barred entry may not occur in the first row as part of the definition of $\mathcal{MCT}_\gamma(\lambda, \alpha)$).

Theorem 15. *Let λ and γ be partitions and α a composition, then the coefficient of \tilde{s}_γ in $\tilde{s}_{\alpha_1}\tilde{s}_{\alpha_2}\cdots\tilde{s}_{\alpha_{\ell(\alpha)}}\tilde{s}_\lambda$ is equal to the number of $T \in \mathcal{MCT}_\gamma(\lambda, \alpha)$ such that the entries of the tableaux are sets (no repeated entries), T is a lattice tableau, and only labels of sets of size greater than 1 are allowed in the first row.*

Example 16. The coefficient of \tilde{s}_4 in $h_{21}\tilde{s}_{22}$ is equal to 8. The same coefficient in $\tilde{h}_2\tilde{h}_1\tilde{s}_{22}$ is equal to 7 and the coefficient of \tilde{s}_4 in $\tilde{s}_2\tilde{s}_1\tilde{s}_{22}$ is equal to 5. These expressions are represented by the following 8 tableaux (equal to the coefficient of \tilde{s}_4 in $h_{21}\tilde{s}_{22}$) such that the first 7 represent the coefficient of \tilde{s}_4 in $\tilde{h}_2\tilde{h}_1\tilde{s}_{22}$ and the first 5 represent the coefficient of \tilde{s}_4 in $\tilde{s}_2\tilde{s}_1\tilde{s}_{22}$.

| | | | |
|--|---|---|---|
| $\begin{array}{ c c c c } \hline \bar{1} & \bar{1}1 & \bar{2}1 & \bar{2}2 \\ \hline \end{array}$ | $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & \bar{2}1 & 2 \\ \hline & & & \bar{2}1 \\ \hline \end{array}$ | $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & 1 & \bar{2}1 \\ \hline & & & \bar{2}2 \\ \hline \end{array}$ | $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & 1 & \bar{2}2 \\ \hline & & & \bar{2}1 \\ \hline \end{array}$ |
| $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}12 \\ \hline & & & \\ \hline \end{array}$ | $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}2 \\ \hline & & & 1 \\ \hline \end{array}$ | $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}1 \\ \hline & & & 2 \\ \hline \end{array}$ | $\begin{array}{ c c c c } \hline \bar{1} & \bar{1} & \bar{2}11 & \bar{2}2 \\ \hline & & & \\ \hline \end{array}$ |

We will build our proof of Theorem 15 from Theorem 17 which is an expression for the product $\tilde{h}_\mu\tilde{s}_\lambda$. We will first complete the statements of the expressions and then follow with the proofs.

4. A COMBINATORIAL RULE FOR THE PRODUCT OF $\tilde{h}_\mu\tilde{s}_\lambda$

In developing a combinatorial interpretation for $\tilde{h}_\mu\tilde{s}_\lambda$ we will start by enumerating sequences of tableaux with certain restrictions and show that they are in bijection with multiset tableaux.

Theorem 17. *For partitions λ and γ and a composition α let $\mu = \text{sort}(\alpha)$, $r = \ell(\lambda)$ and $\ell = \ell(\mu)$. The coefficient of \tilde{s}_γ in $\tilde{h}_\mu\tilde{s}_\lambda$ is equal to the number of $T \in \mathcal{MCT}_\gamma(\lambda, \alpha)$ whose entries are sets with at most one non-barred entry (and at most one barred entry) and such that T is a lattice tableau.*

Notice that the difference between the combinatorial interpretation in Theorem 17 and the one that appears in Theorem 14 is that here we require that there is at most one barred entry and at most one unbarred entry in each set.

Example 18. The coefficient of \tilde{s}_4 in $\tilde{h}_{21}\tilde{s}_{22}$ is equal to 6 since the tableaux described by the theorem are

| | | | | | |
|---|---|-------------------------------------|-------------------------------------|-------------------------------------|---|
| $\bar{1} \bar{1} \bar{2} \bar{1} \bar{2} \bar{2}$ | $\bar{1} \bar{1} \bar{2} \bar{1} \bar{2} \bar{1}$ | $\bar{1} \bar{1} \bar{2} \bar{1} 2$ | $\bar{1} \bar{1} 1 \bar{2} \bar{1}$ | $\bar{1} \bar{1} 1 \bar{2} \bar{2}$ | $\bar{1} \bar{1} \bar{2} \bar{1} \bar{2} \bar{2}$ |
| | | 2 | $\bar{2} \bar{1}$ | $\bar{2} \bar{2}$ | 1 |

Their respective reading words are $\bar{1}.\bar{2}.\bar{1}.\bar{2}$, $\bar{1}.\bar{1}.\bar{2}.\bar{2}.$, $\bar{1}.\bar{1}.\bar{2}.\bar{1}.$, $\bar{1}.\bar{1}.\bar{2}.\bar{2}$, $\bar{1}.\bar{1}.\bar{2}.\bar{2}$, $\bar{1}.\bar{1}.\bar{2}.\bar{2}$, where the periods indicate the transitions of the reading words.

The coefficient of \tilde{s}_γ in $\tilde{h}_\mu \tilde{s}_\lambda$ is equal to the scalar product $\langle \tilde{h}_\mu \tilde{s}_\lambda, \tilde{s}_\gamma \rangle_{\circledast}$ and we apply Example 23 of Section I.7 of [Mac] to compute it. Using equation (8) we have that

$$\begin{aligned}
 \langle \tilde{h}_\mu \tilde{s}_\lambda, \tilde{s}_\gamma \rangle_{\circledast} &= \sum_{\nu} \frac{\tilde{h}_\mu[\Xi_\nu] \tilde{s}_\lambda[\Xi_\nu] \tilde{s}_\gamma[\Xi_\nu]}{z_\nu} \\
 &= \sum_{\nu} \langle h_{(n-|\mu|, \mu)}, p_\nu \rangle \langle s_{(n-|\lambda|, \lambda)}, p_\nu \rangle \langle s_{(n-|\gamma|, \gamma)}, p_\nu \rangle \frac{1}{z_\nu} \\
 &= \langle h_{(n-|\mu|, \mu)} * s_{(n-|\lambda|, \lambda)}, s_{(n-|\gamma|, \gamma)} \rangle \\
 (12) \quad &= \sum_{\tau^{(0)} \vdash n-|\mu|} \sum_{\tau^{(1)} \vdash \mu_1} \cdots \sum_{\tau^{(\ell)} \vdash \mu_\ell} c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)} c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\lambda|, \lambda)}.
 \end{aligned}$$

Definition 19. Assume that $\lambda, \gamma, \tau^{(i)}$ for $0 \leq i \leq \ell$ are partitions such that $|\lambda| = |\gamma| = \sum_{i=0}^{\ell} |\tau^{(i)}|$. Define $B_{\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(\ell)}}^{\gamma, \lambda}$ to be the set of sequences of tableaux of the form

$$(13) \quad (T_0, T_1, T_2, \dots, T_\ell)$$

satisfying

- T_0 is the super-standard tableau of shape $\tau^{(0)}$.
- T_i is a skew-shape column-strict tableau with the outer shape of T_{i-1} as the inner shape of T_i .
- T_i is jeu de Taquin equivalent to a tableau of shape $\tau^{(i)}$.
- The word $read(T_0)read(T_1)\cdots read(T_\ell)$ has content $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_{\ell(\lambda)}}\}$ and is lattice.
- γ is the outer shape of T_ℓ .

Example 20. Consider the following example with $\gamma = (5, 4)$, $\lambda = (5, 2, 2)$ and $|\tau^{(0)}| = 6$, $|\tau^{(1)}| = 2$ and $|\tau^{(2)}| = 1$.

For the partitions $(\tau^{(0)}, \tau^{(1)}, \tau^{(2)}) = ((5, 1), (2), (1))$, we have that $|B_{(5,1)(2)(1)}^{(5,4)(5,2,2)}| = 1$ and contains:

$$\left(\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 2 & 3 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline \end{array} \right)$$

For $(\tau^{(0)}, \tau^{(1)}, \tau^{(2)}) = ((4, 2), (2), (1))$, we have that $|B_{(4,2)(2)(1)}^{(5,4)(5,2,2)}| = 4$ and contains:

$$\left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 3 & 3 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 3 \\ \hline & & 3 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 1 \\ \hline & & 3 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 1 \\ \hline & & 3 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 3 \\ \hline & & 3 \\ \hline \end{array} \right)$$

For $(\tau^{(0)}, \tau^{(1)}, \tau^{(2)}) = ((4, 2), (1, 1), (1))$, we have that $|B_{(4,2)(1,1)(1)}^{(5,4)(5,2,2)}| = 1$ and contains:

$$\left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 3 \\ \hline & & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline & 3 \\ \hline & & 3 \\ \hline \end{array} \right)$$

We will use the two versions of the Littlewood-Richardson rule in Propositions 6 and 8 to show the following result.

Proposition 21. *Let λ, γ , and $\tau^{(i)}$, $0 \leq i \leq \ell$, be partitions satisfying $|\lambda| = |\gamma| = \sum_{i=0}^{\ell} |\tau^{(i)}|$, then*

$$|B_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell)}}^{\gamma, \lambda}| = c_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell)}}^{\gamma} c_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell)}}^{\lambda}.$$

Proof. The enumeration will follow by induction on ℓ . First, we note that

$$c_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell)}}^{\gamma} = \sum_{\tau^{(0)} \subseteq \gamma^{(1)} \subseteq \dots \subseteq \gamma^{(\ell-1)} \subseteq \gamma} c_{\tau^{(0)}\tau^{(1)}}^{\gamma^{(1)}} c_{\gamma^{(1)}\tau^{(2)}}^{\gamma^{(2)}} \cdots c_{\gamma^{(\ell-1)}\tau^{(\ell)}}^{\gamma^{(\ell)}}$$

where the sum is over chains of partitions $\gamma^{(i)}$ for $1 \leq i \leq \ell - 1$ such that $|\tau^{(i)}| = |\gamma^{(i)}| - |\gamma^{(i-1)}|$ with the convention that $\gamma^{(0)} = \tau^{(0)}$ and $\gamma^{(\ell)} = \gamma$. This follows by the associativity of products of Schur functions since both sides of this equation are the coefficient of s_{γ} in $s_{\tau^{(0)}} s_{\tau^{(1)}} \cdots s_{\tau^{(\ell)}}$. For the induction argument we only need that

$$c_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell)}}^{\gamma} = \sum_{\zeta \subseteq \gamma} c_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell-1)}}^{\zeta} c_{\zeta\tau^{(\ell)}}^{\gamma}.$$

Now for $\ell = 0$ we are taking as a base case the definition that $B_{\tau}^{\gamma, \lambda}$ is empty unless $\lambda = \gamma = \tau$ and in this case, $B_{\tau}^{\tau\tau} = \{(T)\}$ where T is the superstandard tableau of shape τ . The base case of the coefficient is the coefficient $c_{\tau}^{\tau} = 1$.

Now assume that the proposition is true for tableaux sequences of length $\ell - 1$. We observe that the number of sequences in $B_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell-1)}}^{\gamma, \lambda}$ (see Definition 19) is the same as the number of pairs

$$((T_1, T_2, \dots, T_{\ell-1}), T_{\ell})$$

where $(T_1, T_2, \dots, T_{\ell-1}) \in B_{\tau^{(0)}\tau^{(1)}\dots\tau^{(\ell-1)}}^{\zeta, \nu}$ for some partitions $\nu \subseteq \lambda$ and $\zeta \subseteq \gamma$ and the last tableau T_{ℓ} has the following properties

- (1) $\tau^{(\ell)}$ is the straight shape of the tableau obtained by applying jeu de Taquin to T_ℓ .
- (2) $1^{\nu_1} 2^{\nu_2} \cdots \ell(\nu)^{\nu_{\ell(\nu)}} \text{read}(T_\ell)$ is a lattice word of content λ .
- (3) T_ℓ is of shape γ/ζ .

Let S_ℓ be the straight shape tableau obtained by applying jeu de Taquin to T_ℓ , then $1^{\nu_1} 2^{\nu_2} \cdots \ell(\nu)^{\nu_{\ell(\nu)}} \text{read}(T_\ell)$ is lattice if and only if $1^{\nu_1} 2^{\nu_2} \cdots \ell(\nu)^{\nu_{\ell(\nu)}} \text{read}(S_\ell)$ is lattice. The number of T_ℓ satisfying (1), (2) and (3) is equal to the number of pairs (T_ℓ, S_ℓ) satisfying:

- (1') S_ℓ is equal to the straight shape tableau obtained by applying jeu de Taquin to T_ℓ .
- (2') S_ℓ is of shape $\tau^{(\ell)}$.
- (3') $1^{\nu_1} 2^{\nu_2} \cdots \ell(\nu)^{\nu_{\ell(\nu)}} \text{read}(S_\ell)$ is a lattice word of content λ .
- (4') T_ℓ is of shape γ/ζ .

By Proposition 6 there are $c_{\tau^{(\ell)}, \nu}^\lambda$ possible S_ℓ satisfying (2') and (3'). If we fix a straight shape S_ℓ , then Proposition 8 says that the number of T_ℓ satisfying (1') and (4') is equal to $c_{\tau^{(\ell)}, \zeta}^\gamma$. Therefore we have $c_{\tau^{(\ell)}, \nu}^\lambda c_{\tau^{(\ell)}, \zeta}^\gamma$ pairs (T_ℓ, S_ℓ) .

By the inductive hypothesis $|B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell-1)}}^{\zeta, \nu}| = c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell-1)}}^\zeta c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell-1)}}^\nu$, then by the addition and multiplication principles we have

$$\begin{aligned} |B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{\gamma, \lambda}| &= \sum_{\zeta \subseteq \gamma} \sum_{\nu \subseteq \lambda} c_{\tau^{(\ell)}, \zeta}^\gamma c_{\tau^{(0)}, \nu}^\lambda c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell-1)}}^\zeta c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell-1)}}^\nu \\ &= c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^\gamma c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^\lambda. \end{aligned}$$

The proposition follows by induction. \square

We are now ready to prove the combinatorial interpretation of Theorem 17 for $\tilde{h}_\mu \tilde{s}_\lambda$.

Proof. (of Theorem 17) Fix a positive integer n which is sufficiently large. The combinatorial interpretation for $B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)(n-|\lambda|, \lambda)}$ in this proof will depend on n , but the cardinality of the set is independent of n as long as the number of 1's in T_0 is greater than γ_1 for all sequences $T^* \in B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)(n-|\lambda|, \lambda)}$.

In order to show that this combinatorial interpretation is correct we will show that the number of tableaux in the statement of the theorem is equal to Equation (12). That is, we will show that the number of tableaux described in Theorem 17 is equal to

$$\langle \tilde{h}_\mu \tilde{s}_\lambda, \tilde{s}_\gamma \rangle_{\text{@}} = \sum_{\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(\ell)}} c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)} c_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\lambda|, \lambda)} = \sum_{\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(\ell)}} |B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)(n-|\lambda|, \lambda)}|$$

by Proposition 21. That is, we will show that there is a bijection between the set of tableaux described in the theorem and the set

$$(14) \quad B_\alpha^{\gamma, \lambda} := \biguplus_{\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(\ell)}} B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)(n-|\lambda|, \lambda)},$$

where the union is over all partitions $\tau^{(0)} \vdash n - |\mu|$ and $\tau^{(i)} \vdash \alpha_i$ and the set $B_{\tau^{(0)} \tau^{(1)} \cdots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)(n-|\lambda|, \lambda)}$ is defined in Definition 19.

As an example to follow along, we note that the tableaux in Example 18 are in bijection with the elements of $B_{(2,1)}^{(5,4),(5,2,2)}$ in Example 20. The elements in both examples are given in order so that they correspond when we apply the bijection described below.

Let $T \in \mathcal{MCT}_\gamma(\lambda, \alpha)$ have set entries with at most one barred and at most one non-barred entry and such that T is a lattice tableau. The tableau T is a skew tableau of shape $(r, \gamma)/(\gamma_1)$ and we think of the cells in the inner shape (γ_1) as filled with empty sets. First pad T with $n - |T|$ additional empty cells in the first row so that it is of shape $(n - |\gamma|, \gamma)$ to make a tableau T' . Next, let $T'|_-$ be the skew tableau consisting of the cells of T' with no un-barred entries (including the empty cells) and $T'|_i$ for $1 \leq i \leq \ell$ be the skew tableau consisting of the cells of the form $\{i\}$ or $\{\bar{j}, i\}$. In $T'|_-$ replace empty cells with 1 and \bar{j} with $j + 1$ and in $T'|_i$ replace $\{i\}$ with a 1 and $\{\bar{j}, i\}$ with a $j + 1$. Then set

$$T^* = (T'|_-, T'|_1, \dots, T'|_\ell),$$

that is $T_0 = T'|_-$ and $T_i = T'|_i$ for $1 \leq i \leq \ell$.

To ensure this correspondence is clear, we have for example (with $n = 12$), (note that we padded T_0 with $n - |T| = 2$ extra cells)

$$\begin{array}{c|cc} \bar{2} & \bar{3}2 \\ \hline \bar{1} & \bar{1}1 & \bar{2}2 \\ \hline & \bar{1}1 & \bar{3}2 \end{array} \longrightarrow T_0 = \begin{array}{c|cc} 3 \\ \hline 2 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array}, \quad T_1 = \begin{array}{c|cc} & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 2 \end{array}, \quad T_2 = \begin{array}{c|cc} & 4 \\ \hline & 3 \\ \hline 1 & 1 & 1 & 1 & 1 & 4 \end{array}.$$

To establish that $\text{read}(T_0)\text{read}(T_1) \cdots \text{read}(T_\ell)$ is a lattice word we note that if $w = \text{read}(T|_-)$ with \bar{j} replaced by $j + 1$ then since no cells labeled with $\{\bar{j}\}$ are allowed in the first row of T , then $\text{read}(T_0) = 1^{n-|T|+\gamma_1} w$. The fact that $\text{read}(T|_-)\text{read}(T|_1) \cdots \text{read}(T|_\ell)$ is a lattice word implies that $\text{read}(T_0)\text{read}(T_1) \cdots \text{read}(T_\ell)$ is also lattice word since the number of 2's in $\text{read}(T_0)\text{read}(T_1) \cdots \text{read}(T_\ell)$ will be less than the $n - |T| + \gamma_1$ 1's at the beginning of the word.

The sequence T^* is an element in $B_\alpha^{\gamma, \lambda}$ and we can determine the sequence of partitions $\tau^{(i)}$ from T^* by setting $\tau^{(i)}$ to be the shape of $T'|_i$ once it is brought to straight shape using jeu de Taquin. This gives us a sequence of partitions such that $T^* \in B_{\tau^{(0)}\tau^{(1)} \dots \tau^{(\ell)}}^{(n-|\gamma|, \gamma)(n-|\lambda|, \lambda)}$ by Definition 19.

Moreover, every $(T_0, T_1, \dots, T_\ell) \in B_\alpha^{\gamma, \lambda}$ corresponds to a set tableau T of shape $(n - |\gamma|, \gamma)$ by reversing the bijection and replacing 1 with a blank cell in T_0 and a $j + 1$ with a $\{\bar{j}\}$, and in T_i replacing a 1 with a cell labeled with an $\{i\}$ and a $j + 1$ with $\{\bar{j}, i\}$ and overlaying the resulting skew tableaux of shape $(n - |\gamma|, \gamma)$. The result may have too many blank cells, but they can be deleted until there are precisely γ_1 .

Notice that $\text{read}(T|_-)\text{read}(T|_1) \cdots \text{read}(T|_\ell)$ is equal to the word formed by starting with $\text{read}(T_0)\text{read}(T_1) \cdots \text{read}(T_\ell)$ and then deleting the 1's and replacing $j + 1$ with \bar{j} . Thus, if $\text{read}(T_0)\text{read}(T_1) \cdots \text{read}(T_\ell)$ is a lattice word then $\text{read}(T|_-)\text{read}(T|_1) \cdots \text{read}(T|_\ell)$ will also be a lattice word. \square

To prove the other combinatorial interpretations presented in this paper, we need Corollary 23, (stated below) where we establish a combinatorial interpretation for the product $\tilde{h}_{\tilde{m}(\pi)}\tilde{s}_\lambda$ where π is a multiset partition.

Fix π , a multiset partition of $\{\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_\ell}\}\}$, and let $S_1 < S_2 < \dots < S_d$ be the (distinct) parts of the multiset partition ordered in reverse lex order and let $\alpha(\pi) = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be the composition of $\ell(\pi)$ of length d such that α_i is the multiplicity of S_i in π . The symbol $\alpha(\pi)$ is defined so that $\text{sort}(\alpha(\pi)) = \tilde{m}(\pi)$.

For partitions λ and γ , we define $\mathcal{MCT}'_\gamma(\lambda, \pi)$ to be the set of tableaux $T \in \mathcal{MCT}_\gamma(\lambda, \mu)$ such that the labels of the cells satisfy the following two conditions

- the multiset of multisets of unbarred entries (ignoring the barred entries and those that don't have any unbarred entries) is equal to the multiset partition π ;
- T is a lattice tableau.

Example 22. In the following tableau, if we ignore barred entries there are three boxes with unbarred entries. The distinct multisets of unbarred entries are $S_1 = \{\{1\}\}$ and $S_2 = \{\{1, 2\}\}$ (occurs twice).

| | | | |
|---|---|----|-----|
| 1 | 1 | 21 | 212 |
| | | | 12 |

Since $\text{read}(T|_-)\text{read}(T|_{S_1})\text{read}(T|_{S_2}) = \overline{11}\overline{2}\overline{2}$ is lattice, then this tableau is an element of $\mathcal{MCT}'_{(4)}((2, 2), \{\{\{1\}\}, \{\{1, 2\}\}, \{\{1, 2\}\}\})$.

Corollary 23. For λ a partition and $\pi \vdash \{\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_\ell}\}\}$,

$$(15) \quad \tilde{h}_{\tilde{m}(\pi)}\tilde{s}_\lambda = \sum_{\gamma} \sum_{T \in \mathcal{MCT}'_\gamma(\lambda, \pi)} \tilde{s}_\gamma$$

where the sum is over all partitions γ such that $|\gamma| \leq \ell(\pi) + |\lambda|$.

Proof. By Theorem 17, to demonstrate this result, we need to establish a bijection for each partition γ between the set of $T \in \mathcal{MCT}_\gamma(\lambda, \alpha(\pi))$ and $T' \in \mathcal{MCT}'_\gamma(\lambda, \pi)$ such that

$$\text{read}(T'|_-)\text{read}(T'|_{S_1}) \dots \text{read}(T'|_{S_d}) = \text{read}(T|_-)\text{read}(T|_1) \dots \text{read}(T|_d).$$

We take the $S_1 < S_2 < \dots < S_d$ to be the (distinct) parts of the multiset partition ordered in reverse lex order. The bijection between these sets is to replace the unbarred label i in T with the set S_i for each $1 \leq i \leq d$ to obtain T' . The inverse bijection is to replace the occurrences of the set S_i in T' with the label i to obtain the corresponding $T \in \mathcal{MCT}_{\overline{sh(T')}}(\lambda, \alpha(\pi))$.

Hence we have by Theorem 17 that

$$(16) \quad \tilde{h}_{\tilde{m}(\pi)}\tilde{s}_\lambda = \sum_{\gamma: |\gamma| \leq |\tilde{m}(\pi)| + |\lambda|} \sum_{T \in \mathcal{MCT}_\gamma(\lambda, \alpha(\pi))} \tilde{s}_\gamma = \sum_{\gamma: |\gamma| \leq |\tilde{m}(\pi)| + |\lambda|} \sum_{T \in \mathcal{MCT}'_\gamma(\lambda, \pi)} \tilde{s}_\gamma.$$

□

Example 24. We provide an example to clarify the notation in the bijection of the proof of Corollary 23. Let $\mu = (5, 1)$ and then $\pi = \{\{1\}, \{1, 1\}, \{1, 1\}, \{2\}\}$ is a multiset partition of $\{1^5, 2\}$ and

$$T = \begin{array}{|c|c|c|} \hline \bar{2} & \bar{1}2 & \bar{1}3 \\ \hline \bar{1} & \bar{1} & 1 \\ \hline & & \bar{2}2 \\ \hline \end{array}$$

is an element of $\mathcal{MCT}_{(3,3)}((4, 2), (1, 2, 1))$. Since the multisets of π are ordered $\{1\} < \{1, 1\} < \{2\}$, the corresponding T' is equal to

$$T' = \begin{array}{|c|c|c|} \hline \bar{2} & \bar{1}11 & \bar{1}2 \\ \hline \bar{1} & \bar{1} & 1 \\ \hline & & \bar{2}11 \\ \hline \end{array}$$

and this is an element of $\mathcal{MCT}'_{(3,3)}((4, 2), \pi) \subseteq \mathcal{MCT}_{(3,3)}((4, 2), (5, 1))$. We also have $\text{read}(T|_-)\text{read}(T|_1)\text{read}(T|_2)\text{read}(T|_3) = \text{read}(T'|_-)\text{read}(T'|_{S_1})\text{read}(T'|_{S_2})\text{read}(T'|_{S_3}) = \bar{1}12.. \bar{2}1.. \bar{1}$, where $S_1 = \{1\}$, $S_2 = \{1, 1\}$ and $S_3 = \{2\}$.

5. PROOFS OF THEOREMS 13, 14 AND 15

5.1. A proof of Theorem 13. Recall that Theorem 13 gives a combinatorial expression for the coefficients in the expansion of $h_\mu \tilde{s}_\lambda$ in the irreducible character basis, \tilde{s}_λ . We will rely on Corollary 23 and Equation (5).

Proof. (Theorem 13) The symmetric group S_ℓ acts on the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \dots, \ell^{\alpha_\ell}\}$ as well as multiset partitions $\pi \vdash \{1^{\alpha_1}, 2^{\alpha_2}, \dots, \ell^{\alpha_\ell}\}$ by permuting the values of the multiset and multiset partitions. It is not hard to see that $\tilde{m}(\sigma(\pi)) = \tilde{m}(\pi)$ for all $\sigma \in S_\ell$. This implies that the obvious analogue of Equation (5) holds for compositions α , that is,

$$h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell} = \sum_{\pi \vdash \{1^{\alpha_1} 2^{\alpha_2} \dots \ell^{\alpha_\ell}\}} \tilde{h}_{\tilde{m}(\pi)}.$$

Notice that by Corollary 23. We have,

$$\begin{aligned} h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell} \tilde{s}_\lambda &= \sum_{\pi \vdash \{1^{\alpha_1} 2^{\alpha_2} \dots \ell^{\alpha_\ell}\}} \tilde{h}_{\tilde{m}(\pi)} \tilde{s}_\lambda \\ &= \sum_{\pi \vdash \{1^{\alpha_1} 2^{\alpha_2} \dots \ell^{\alpha_\ell}\}} \sum_{\gamma} \sum_{T \in \mathcal{MCT}'_\gamma(\lambda, \pi)} \tilde{s}_\gamma \end{aligned}$$

where the sum is over all partitions γ such that $|\gamma| \leq |\lambda| + \ell(\pi)$.

This establishes exactly the conditions of the statement of Theorem 13 where the labels of the tableaux have labels whose entries have at most one barred entry and a multiset of non-barred entries. And if we read the barred entries using the reverse lex order on the distinct multisets consisting of the non-barred entries the reading word is lattice. \square

5.2. A proof of Theorem 14. We next establish a combinatorial interpretation for the terms in $\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_k} \tilde{s}_\lambda$.

Proof. (Theorem 14) Proposition 3.4 of [OZ2] allows us to expand the product $\tilde{h}_\lambda \tilde{h}_\mu$ in terms of the induced character basis elements, i.e., \tilde{h} -basis. In particular, a product of the form $\tilde{h}_{\alpha_1} \tilde{h}_{\alpha_2} \cdots \tilde{h}_{\alpha_k}$ is equal to the sum over all set partitions of a multiset $\{\!\{1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k}\}\!}$ (that is, multiset partitions where the sets in the partition do not contain repeated elements), hence we have the expansion

$$(17) \quad \tilde{h}_{\alpha_1} \tilde{h}_{\alpha_2} \cdots \tilde{h}_{\alpha_k} = \sum_{\pi} \tilde{h}_{\tilde{m}(\pi)}$$

where the sum is over all set partitions π of the multiset $\{\!\{1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k}\}\!}$ and $\tilde{m}(\pi)$ is the partition of the number $\ell(\pi)$ consisting of multiplicities of each multiset occurring in π .

It follows from Corollary 23 that

$$(18) \quad \tilde{h}_{\alpha_1} \tilde{h}_{\alpha_2} \cdots \tilde{h}_{\alpha_k} \tilde{s}_\lambda = \sum_{\pi} \sum_{\gamma} \sum_{T \in \mathcal{MCT}'_\gamma(\lambda, \pi)} \tilde{s}_\gamma$$

where the first sum is over set partitions π of $\{\!\{1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k}\}\!}$.

For a fixed γ , there is one term in this sum for each set valued tableau in $\mathcal{MCT}_\gamma(\lambda, \alpha)$ where the entries are sets and with the condition that if $S_1 < S_2 < \dots < S_d$ are the sets of the unbarred entries that appear (ignoring the barred entries), then the word

$$\text{read}(T|_-) \text{read}(T|_{S_1}) \text{read}(T|_{S_2}) \cdots \text{read}(T|_{S_d})$$

is lattice because T is in $\mathcal{MCT}'_\gamma(\lambda, \pi)$. \square

5.3. A proof of Theorem 15. Building on Theorem 14 we can develop a combinatorial expression for the coefficients in the expansion of $\tilde{s}_{\alpha_1} \tilde{s}_{\alpha_2} \cdots \tilde{s}_{\alpha_k} \tilde{s}_\lambda$ in the irreducible character basis. We note that $\tilde{h}_k = \tilde{s}_k + \tilde{s}_{k-1} + \cdots + \tilde{s}_0$ by Equation (6) and so by induction we have that $\tilde{s}_0 = \tilde{h}_0 = 1$, $\tilde{s}_1 = \tilde{h}_1 - \tilde{h}_0$ and $\tilde{s}_k = \tilde{h}_k - \tilde{h}_{k-1}$ for $k > 1$.

Proof. (Theorem 15) We replace \tilde{s}_k by $\tilde{h}_k - \tilde{h}_{k-1}$ for each term \tilde{s}_{α_i} and expand.

$$(19) \quad \begin{aligned} \tilde{s}_{\alpha_1} \tilde{s}_{\alpha_2} \cdots \tilde{s}_{\alpha_k} \tilde{s}_\lambda &= (\tilde{h}_{\alpha_1} - \tilde{h}_{\alpha_1-1})(\tilde{h}_{\alpha_2} - \tilde{h}_{\alpha_2-1}) \cdots (\tilde{h}_{\alpha_k} - \tilde{h}_{\alpha_k-1}) \tilde{s}_\lambda \\ &= \sum_{S \subseteq [k]} (-1)^{|S|} \tilde{h}_{\alpha_1 - \chi(1 \in S)} \tilde{h}_{\alpha_2 - \chi(2 \in S)} \cdots \tilde{h}_{\alpha_k - \chi(k \in S)} \tilde{s}_\lambda \end{aligned}$$

By Equation (17), we have that Equation (19) is equal to the following sum which we can expand using Corollary 23,

$$(20) \quad = \sum_{S \subseteq [k]} \sum_{\pi} (-1)^{|S|} \tilde{h}_{\tilde{m}(\pi)} \tilde{s}_\lambda = \sum_{S \subseteq [k]} \sum_{\pi} \sum_{\gamma} \sum_{T \in \mathcal{MCT}'_\gamma(\lambda, \pi)} (-1)^{|S|} \tilde{s}_\gamma.$$

where the second sum is over all set partitions π of $\{\!\{1^{\alpha_1 - \chi(1 \in S)}, 2^{\alpha_2 - \chi(2 \in S)}, \dots, k^{\alpha_k - \chi(k \in S)}\}\!}$ and the third sum is over all partitions γ such that $|\gamma| \leq |\lambda| + \ell(\pi)$.

Now for each subset $S \subseteq [k]$, we can define an injective map

$$\phi_S : \biguplus_{\pi} \biguplus_{\gamma} \mathcal{MCT}'_{\gamma}(\lambda, \pi) \rightarrow \biguplus_{\pi'} \biguplus_{\gamma} \mathcal{MCT}'_{\gamma}(\lambda, \pi')$$

where on the left we have the union over set partitions π of the multiset

$$\{1^{\alpha_1 - \chi(1 \in S)}, 2^{\alpha_2 - \chi(2 \in S)}, \dots, k^{\alpha_k - \chi(k \in S)}\}$$

while on the right the union is over set partitions π' of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k}\}$. For an element $T \in \mathcal{MCT}'_{\gamma}(\lambda, \pi)$ then $\phi_S(T)$ is equal to the tableau where a cell with a label of $\{i\}$ for each $i \in S$ is added to the first row. Note that $\overline{sh(T)} = \overline{sh(\phi_S(T))} = \gamma$ and the reading words of the barred entries do not change. Therefore, ϕ_S maps the set $\mathcal{MCT}'_{\gamma}(\lambda, \pi)$ to $\mathcal{MCT}'_{\gamma}(\lambda, \pi \uplus \{\{i\} : i \in S\})$.

We note that ϕ_S is an injective map and $T \in \mathcal{MCT}'_{\gamma}(\lambda, \alpha)$ is in the image of ϕ_S if and only if there are cells with a label of $\{i\}$ where $i \in S$ in the first row. Let $a(T)$ be the set of i such that $\{i\}$ is a label in the first row of a $T \in \mathcal{MCT}'_{\gamma}(\lambda, \alpha)$. By applying this injection, we can interchange the sum over subsets of $S \subseteq [k]$ in Equation (20) so that we first sum over all multiset tableaux and for each fixed tableau, T , there is one term for each subset S of the set $a(T)$. We write this as

$$(21) \quad = \sum_{\pi'} \sum_{\gamma} \sum_{T \in \mathcal{MCT}'_{\gamma}(\lambda, \pi')} \sum_{S \subseteq a(T)} (-1)^{|S|} \tilde{s}_{\gamma} .$$

Since the summand \tilde{s}_{γ} is independent of the sum over subsets S , the terms with $a(T)$ not equal to the empty set will sum to zero. Therefore Equation (21) is equal to the expression

$$= \sum_T \tilde{s}_{\overline{sh(T)}}$$

where this sum is over multiset tableaux $T \in \mathcal{MCT}'_{\gamma}(\lambda, \alpha)$ for some partition γ such that

- the labels of un-barred entries (ignoring the barred entries) are sets, that is, they have no repeated entries;
- T is a lattice tableau;
- the set $a(T)$ is empty, that is, there are no labels of size less than 2 in the first row.

These are equivalent to the conditions stated in Theorem 15. \square

6. CONCLUDING REMARKS AND APPLICATIONS

There is a formula for the stable Kronecker coefficients in terms of Littlewood-Richardson coefficients and Kronecker coefficients due to Littlewood [Li2]. It says that

$$(22) \quad \bar{g}_{\alpha\beta}^{\gamma} = \sum g_{\delta\epsilon\zeta} c_{\delta\sigma\tau}^{\alpha} c_{\epsilon\rho\tau}^{\beta} c_{\zeta\rho\sigma}^{\gamma}$$

where the sum is over all partitions $\delta, \epsilon, \zeta, \rho, \tau$ and σ . This equation has been rediscovered and reproved in the literature (see for example [BDO, Bri, BK, ST, Thi, STW, Wag]).

In the case when γ is a single row or column then this formula simplifies to an equation only involving Littlewood-Richardson coefficients. Our combinatorial interpretation for the product $\tilde{s}_k \tilde{s}_{\lambda}$ in Theorem 15 reduces to Littlewood's formula in that case.

Our main goal in developing the results in this paper is to provide a combinatorial answer to two fundamental open problems in representation theory, a formula for $\bar{g}_{\lambda\mu\nu}$ and the multiplicity of an irreducible symmetric group representation in an irreducible polynomial GL_n representation. These correspond to the coefficients of \tilde{s}_ν in the expressions $\tilde{s}_\mu \tilde{s}_\lambda$ and s_μ (respectively). The inequalities discussed in the introduction may be extended to the following diagram.

$$\begin{array}{ccc} \tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_\ell} \tilde{s}_\lambda & \leq & \tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_\ell} \tilde{s}_\lambda & \leq & h_\mu \tilde{s}_\lambda \\ | \vee & & | \vee & & | \vee \\ \tilde{s}_\mu \tilde{s}_\lambda & \leq & \tilde{h}_\mu \tilde{s}_\lambda & \leq & s_\mu \tilde{s}_\lambda \end{array}$$

We therefore believe that if such a combinatorial formula exists then it can be formulated in terms of the tableaux introduced in this paper.

6.1. Multiset tableaux and paths in a Bratteli diagrams. The coefficient of \tilde{s}_λ in the symmetric function expression $(\tilde{h}_1)^r$ is equal to the dimension of an irreducible representation of the partition algebras [HR, Mar1, Mar2] indexed by the partition λ . Previously in the literature, these dimensions have mainly been enumerated in terms of paths in a Bratteli diagram and these are sometimes referred to in the literature as vascillating tableaux [HL, MR]. A consequence of our results is that it is much more natural to enumerate these in terms of set valued tableaux since the combinatorial interpretation more clearly reflects the generalization from standard tableaux for dimensions of symmetric group modules to set valued tableaux for dimensions of partition algebra modules.

Similarly, the dimensions of irreducible quasi-partition modules [DO] are equal to the coefficients of \tilde{s}_λ in the symmetric function $(\tilde{s}_1)^r$, these dimensions were first computed in [CG] and the objects they enumerate correspond to paths in the Bratteli diagram of the quasi-partition algebra, these paths were called Kronecker tableaux.

In a recent paper, Benkart, Halverson and Harman [BHH] gave formulas for dimensions of irreducible partition algebra modules in terms of Stirling numbers and Kostka numbers reflecting this connection with set valued tableaux. Moreover, in a recent paper Benkart and Halverson [BH, Section 3.3.B] provide a bijection between vascillating tableaux and set valued tableaux.

6.2. Bases of symmetric functions, set tableaux and Grothendieck symmetric functions. A consequence of the formulae that we present in this paper is that the families of symmetric functions $\{\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell(\mu)}}\}_\mu$ and $\{\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_{\ell(\mu)}}\}_\mu$ are two non-homogenous multiplicative bases of the ring of symmetric functions. As special case of Theorem 14 and Theorem 15 we have a combinatorial interpretations for the transition coefficients between these bases and the \tilde{s} -basis.

There are several ways of describing the S_n -module with character $\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell(\mu)}}$. It is a module of dimension

$$\binom{n}{\mu_1} \binom{n}{\mu_2} \cdots \binom{n}{\mu_{\ell(\mu)}}$$

and the symmetric function $\tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_{\ell(\mu)}}$ is the character of the symmetric group module

$$\mathbb{S}^{(n-\mu_1, \mu_1)} \otimes \mathbb{S}^{(n-\mu_2, \mu_2)} \otimes \cdots \otimes \mathbb{S}^{(n-\mu_{\ell(\mu)}, \mu_{\ell(\mu)})}$$

where \mathbb{S}^{λ} is an irreducible symmetric group module indexed by a partition λ and the symmetric group acts diagonally on the tensors.

For a partition $\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$,

$$\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell(\mu)}} = \sum_{\nu} a_{\mu, \nu} \tilde{h}_{\nu}$$

where $a_{\mu, \nu}$ is equal to the number of set valued partitions whose multiplicity of the sets is equal to ν and whose content is equal to $\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_{\ell(\mu)}}\}$.

And as a consequence of Equation (6)

$$\tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \cdots \tilde{h}_{\mu_{\ell(\mu)}} = \sum_{\nu} b_{\mu, \nu} \tilde{s}_{\nu}$$

where $b_{\mu, \nu}$ is equal to the number of set valued tableaux of shape $(r, \nu) / (\nu_1)$ and content $\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_{\ell(\mu)}}\}$.

This family of symmetric functions has close connections (but they are not exactly the same) to (skew) Grothendieck symmetric functions. The coefficient of a monomial symmetric function m_{μ} in a Grothendieck symmetric function indexed by the partition λ or the skew tableaux λ / ν (see for instance [Buch] for a combinatorial formula for a monomial expansion of symmetric functions $G_{\lambda / \mu}$ in terms of set valued tableaux) is equal to $(-1)^{|\mu| - |\lambda|}$ times the number of set valued tableaux of shape λ and content μ .

This connection demonstrates how these symmetric functions are closely related but the fact that the first row can be of different sizes means that (for example) the tableaux $\boxed{12}$ and $\boxed{1} \boxed{2}$ contribute a 2 to the coefficient of $\tilde{s}_{()} \tilde{h}_1 \tilde{h}_1$ while the same tableaux contribute a term of $-m_{11}$ in G_1 and a term of $+m_{11}$ in G_2 .

6.3. Quantum entanglement. Multiple products of Kronecker coefficients have been considered in the literature with relation to quantum information theory. A measure of quantum entanglement of qubits can be reformulated as the calculation of repeated Kronecker products of the symmetric functions $s_{d,d}$ or $h_{d,d}$ [GWZ1, GWZ2, GMWX, LT1, LT2, Wal1, Wal2] and our combinatorial interpretation can be useful in this application.

In particular, the approach taken in [GMWX] to calculating the measurement of qubits is to compute the multiplicities in the Kronecker products of $h_{d,d}$ and $h_{d+1,d-1}$ and express the k -fold Kronecker product $\underbrace{s_{d,d} * s_{d,d} * \cdots * s_{d,d}}_{k \text{ times}}$ in terms of these expressions.

A consequence of Theorem 14 is that the coefficient of $\tilde{s}_{()}$ in the expression $(\tilde{h}_d)^a (\tilde{h}_{d-1})^{k-a}$ is equal to the number of set valued tableau of shape (r) where the content is equal to $\{1^d, 2^d, \dots, a^d, (a+1)^{d-1}, \dots, k^{d-1}\}$. This leads us to the following result for the repeated Kronecker product of complete homogeneous functions of the form $h_{d,d}$ and $h_{d+1,d-1}$.

Proposition 25. *The coefficient of the Schur function s_{2d} in the Kronecker product*

$$\underbrace{h_{d,d} * h_{d,d} * \cdots * h_{d,d}}_{a \text{ times}} * \underbrace{h_{d+1,d-1} * h_{d+1,d-1} * \cdots * h_{d+1,d-1}}_{k-a \text{ times}}$$

is equal to the number of set valued tableaux of shape (r) where $r \leq 2d$ and the content is equal to $\{\{1^d, 2^d, \dots, a^d, (a+1)^{d-1}, \dots, k^{d-1}\}\}$.

6.4. A relationship between the irreducible character basis and plethysm. Littlewood [Li2] proved an expression for the multiplicity of an irreducible S_n module indexed by the partition $(n - |\lambda|, \lambda)$ in an irreducible polynomial GL_n module indexed by the partition ν . Scharf and Thibon [ST] proved this formula using modern notation and symmetric function techniques to show explicitly that it was equal to

$$\langle s_\nu, s_{(n-|\lambda|, \lambda)}[1 + h_1 + h_2 + h_3 + \cdots] \rangle .$$

Here we are using $f[g]$ to represent the plethysm of two symmetric functions (see [Mac, ST]).

Since this multiplicity is also equal to the coefficient of an irreducible character basis element \tilde{s}_λ in the Schur function s_ν , then we have (again, for n sufficiently large) that

$$s_\nu = \sum_{\lambda} \langle s_\nu, s_{(n-|\lambda|, \lambda)}[1 + h_1 + h_2 + h_3 + \cdots] \rangle \tilde{s}_\lambda$$

and hence we can extend this symmetric function expression linearly and we conclude that for any symmetric function f ,

$$(23) \quad f = \sum_{\lambda} \langle f, s_{(n-|\lambda|, \lambda)}[1 + h_1 + h_2 + h_3 + \cdots] \rangle \tilde{s}_\lambda .$$

6.5. Characterizations by Pieri rules. One reason to focus on the Pieri rule of a symmetric function is that it provides a way of defining or characterizing a basis.

Definition 26. The family of symmetric functions $\{\tilde{s}_\lambda\}_\lambda$ may be defined recursively as the unique set of symmetric functions satisfying $\tilde{s}_{()} = 1$ and

$$\tilde{s}_\lambda = h_{\lambda_1} \tilde{s}_{\bar{\lambda}} - \sum_T \tilde{s}_{\overline{sh(T)}}$$

where the sum is over all $T \in \mathcal{MCT}(\bar{\lambda}, (k))$ with $\overline{sh(T)} \neq \lambda$ and such that T is a lattice tableau.

Example 27. We use this definition to calculate the expansion of \tilde{s}_λ in the complete homogeneous basis for $|\lambda| \leq 3$. For example,

- $\tilde{s}_1 = h_1 - \tilde{s}_{()}$ $= h_1 - 1$.
- $\tilde{s}_2 = h_2 - (2\tilde{s}_1 + 2\tilde{s}_{()}) = h_2 - 2h_1$.
- $\tilde{s}_{11} = h_1 \tilde{s}_1 - (\tilde{s}_2 + 2\tilde{s}_1 + \tilde{s}_{()}) = h_{11} - h_2 - h_1 + 1$.
- $\tilde{s}_3 = h_3 - (\tilde{s}_{11} + 2\tilde{s}_2 + 4\tilde{s}_1 + 3\tilde{s}_{()}) = h_3 - h_2 - h_{11} + h_1$.
- $\tilde{s}_{21} = h_2 \tilde{s}_1 - (\tilde{s}_3 + 3\tilde{s}_2 + 3\tilde{s}_{11} + 5\tilde{s}_1 + 2\tilde{s}_{()}) = h_{21} - h_3 - 2h_{11} + 3h_1$.
- $\tilde{s}_{111} = h_1 \tilde{s}_{11} - (\tilde{s}_{21} + \tilde{s}_2 + 2\tilde{s}_{11} + \tilde{s}_1) = h_{111} - 2h_{21} + h_3 + h_2 - h_{11} + h_1 - h_{()}$.

We can similarly define $\{\tilde{s}_\lambda\}_\lambda$ through the Pieri rules that are given in Theorem 14 and Theorem 15.

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