

A Fundamental Limitation on Maximum Parameter Dimension for Accurate Estimation With Quantized Data

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Abstract—It is revealed that there is a link between the quantization approach employed and the dimension of the vector parameter which can be accurately estimated by a quantized estimation system. A critical quantity called inestimable dimension for quantized data (IDQD) is introduced, which does not depend on the quantization regions and the statistical models of the observations but instead depends only on the number of sensors and on the precision of the vector quantizers employed by the system. It is shown that the IDQD describes a quantization-induced fundamental limitation on the estimation capabilities of the system. To be specific, if the dimension of the desired vector parameter is larger than the IDQD of the quantized estimation system, then the Fisher information matrix for estimating the desired vector parameter is singular, and, moreover, there exist infinitely many nonidentifiable vector parameter points in the vector parameter space. Furthermore, it is shown that under some common assumptions on the statistical models of the observations and the quantization system, a smaller IDQD can be obtained, which can specify an even more limiting quantization induced fundamental limitation on the estimation capabilities of the system.

Index Terms—Distributed sensor parameter estimation, inestimable dimension for quantized data, singular Fisher information matrix, identifiability, quantization.

I. INTRODUCTION

BOLSTERED by recent technological advances in coding, digital wireless communications technology and digital devices, the employment of quantized data has become

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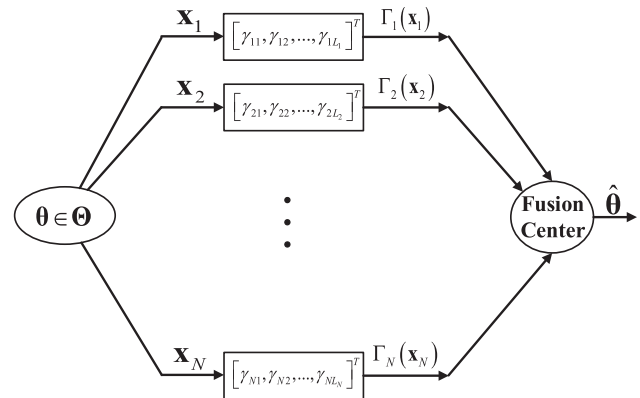


Fig. 1. Parameter estimation system with quantized data.

increasingly popular in many applications, such as sensor networking, the Internet of things, data-transmission systems and data-storage systems. Inspired by this tendency, parameter estimation utilizing quantized data has seen considerable interest in recent years, see [1]–[14] and references therein. In addition, the design of quantizers for parameter estimation has also been an active area of research recently [15]–[17]. For example, [17] considers the design of the optimal quantizers for parameter estimation systems using quantized data under the Bayesian criterion.

A parameter estimation system which employs quantized data is depicted by Fig. 1. The distribution of the observations $[\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]^T$ depends on an underlying vector parameter $\theta \in \Theta$. As shown in Fig. 1, for each j , a sequence of L_j vector quantizers denoted as $\Gamma_j \triangleq [\gamma_{j1}, \gamma_{j2}, \dots, \gamma_{jL_j}]^T$ is employed to convert the observation vector \mathbf{x}_j to digital data $\Gamma_j(\mathbf{x}_j)$, which is transmitted, without error, to the fusion center (FC). To be specific, as illustrated by Fig. 2, \mathbf{x}_j is partitioned into a sequence of L_j disjoint observation subvectors $\{\mathbf{x}_{jl}\}_{l=1}^{L_j}$ first, and then for each l , \mathbf{x}_{jl} is quantized to a scalar u_{jl} by the l -th vector quantizer γ_{jl} in the sequence Γ_j . The output of the sequence of vector quantizers $\Gamma_j(\mathbf{x}_j)$ is the vector $[u_{j1}, u_{j2}, \dots, u_{jL_j}]^T$ which gathers the quantized data from all vector quantizers $\{\gamma_{jl}\}$. After collecting the quantized data from all sequences of vector quantizers, the FC makes use of $\{\Gamma_j(\mathbf{x}_j)\}$ to estimate the value of the desired vector parameter θ .

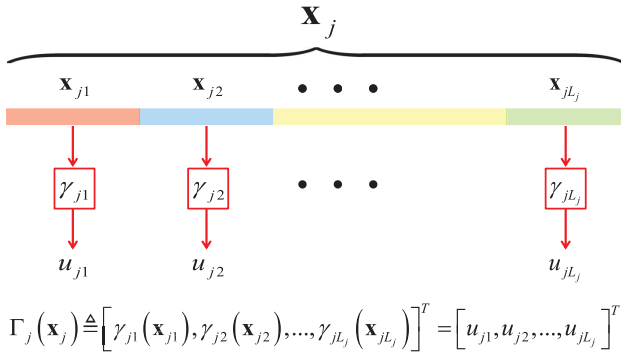


Fig. 2. The sequence of vector quantizers employed for \mathbf{x}_j .

In general, the output of a quantizer is a scalar, however, the output of Γ_j is a vector. To distinguish Γ_j from the commonly defined quantizers, we refer to the sequence of vector quantizers, Γ_j , as a superquantizer. It is worth mentioning that L_j can be any positive integer. The scenarios where $L_j > 1$ are widely considered in recent literature, see [5] for instance. For the scenario where $L_j = 1$, the superquantizer Γ_j is equivalent to a vector quantizer.

Our recent investigations into attacks on parameter estimation systems provide insight into some very effective attacks on systems utilizing quantized data [10]. Building on these ideas, this paper attempts to fully uncover the fundamental limitations on the estimation capabilities of the unattacked quantized estimation system shown in Fig. 1. In order to assess the estimation capabilities of the quantized estimation system shown in Fig. 1, two criteria are often adopted [18]. The first criterion is the information-regularity condition, which is defined as

Definition 1 (Information-Regularity Condition): The Fisher information matrix (FIM) for estimating the desired parameter is nonsingular.

The information-regularity condition guarantees the existence of the Cramer-Rao bound (CRB) for the desired parameter. Further, under mild additional conditions, it guarantees the estimation performance of an appropriate estimator can always be improved by an increase in the number of observations provided a suitably large set of observations is employed. Moreover, it can be shown that in most of the cases, if the FIM is singular, there is no unbiased estimator for the desired parameter with finite variance [19]. To this end, the information-regularity condition, which ensures the nonsingularity of the FIM for the desired parameter θ , is crucial in parameter estimation problems.

The second criterion to assess the estimation capabilities of the quantized estimation system is the identifiability condition, which is defined as

Definition 2 (Identifiability Condition): There exists no parameter value in the parameter space such that the conditional distribution of the data conditioned on the parameter is identical to that for some other parameter value in the parameter space.

The identifiability condition is sufficient to guarantee almost sure convergence of a class of estimators which includes the

maximum likelihood estimator as the sample size approaches infinity, given some reasonable conditions [18], [20]. Intuitively, if the identifiability condition fails for some parameter value, then there exists another parameter value which is just as likely as the true value based on the observations, and hence the desired parameter cannot be estimated consistently [21], [22]. Therefore, it is necessary that every parameter point in the parameter space Θ satisfies the identifiability condition for a meaningful estimation problem. In such cases, we say the vector parameter space is identifiable.

In general, the FIM nonsingularity and the identifiability of the vector parameter space are both determined by the statistical models of the observations, the value of the desired parameter, and the quantizer designs employed by the system which complicates the analysis of the estimation capabilities of a quantized estimation system. On the other hand, this paper provides a simple expression called the inestimable dimension for quantized data (IDQD) which describes a vector parameter dimension beyond which the FIM nonsingularity and the identifiability of the vector parameter space are both guaranteed to fail. These powerful results can be employed in preliminary design in many applications of quantized (digital) data. For example, in the most commonly studied case of real independent and identically distributed quantized data with a common binary quantizer at N sensors and K times, this paper showed that only a scalar parameter can be estimated efficiently for large N and K . Any vector parameter of interest with a dimension larger than 1 always brings about a singular FIM and nonidentifiable vector parameter space. This result has significant implications for big data problems which are attracting significant attention lately, where large dimension parameters frequently occur.

A. Summary of Results

1) For the general parameter estimation with quantized data system shown in Fig. 1, the impact of quantization with regard to the information-regularity condition is first studied. By exploring the structure of the FIM for estimating the desired vector parameter, it is shown that if the dimension of the desired vector parameter is larger than the IDQD, the FIM for estimating the desired vector parameter cannot be nonsingular for any statistical models of the observations, any value of the desired vector parameter, and any quantization regions. Hence, the IDQD specifies a quantization induced fundamental limitation on the estimation capabilities of the quantized estimation system with regard to the information-regularity condition, which limits the number of parameters which can be estimated by the quantized estimation system while maintaining a nonsingular FIM.

2) Next, we investigate the impact of quantization on the identifiability condition. It is shown that for any statistical models of the observations and any quantization regions employed by the system, if the dimension of the desired vector parameter is larger than the IDQD, then the vector parameter space is not identifiable, and moreover, there are infinitely many nonidentifiable vector parameter points in the vector parameter space. Thus, the IDQD indicates a quantization

induced fundamental limitation on the estimation capabilities of the quantized estimation system with regard to the identifiability condition.

3) We next show that there is no general equivalence between the just described quantization induced FIM singularity and the quantization induced nonidentifiability of the vector parameter space. In particular, there exist some cases where the necessary conditions for the existence of the FIM do not hold, but the quantization induced nonidentifiability of the vector parameter space can still be guaranteed. However, if the FIM exists, the condition that the dimension of the desired vector parameter is larger than the IDQD gives rise to both quantization-induced issues.

4) Some further investigations into quantization induced nonidentifiability are carried out. We show that in some cases where the dimension of the desired vector parameter is larger than the IDQD, every vector parameter point in the quantization induced nonidentifiable vector parameter space is nonidentifiable, while in some other cases, only some vector parameter points in the quantization induced nonidentifiable vector parameter space are identifiable. Thus the quantization induced FIM singularity does not necessarily determine the identifiability of the vector parameter point although it does determine the identifiability of the vector parameter space. Moreover, we show that the cardinality of a set of vector parameter points in the quantization induced nonidentifiable vector parameter space which are as likely as each other based on the observations can be as small as 1 and can also be as large as uncountably infinite.

5) Finally, as opposed to our previous general results, we consider scenarios where some commonly assumed specific assumptions on the statistical models of the observations are made. It is shown that under the assumptions, the fundamental limitation of the quantization system becomes more limiting. A smaller dimension of the vector parameter, called the refined IDQD (rIDQD), will guarantee the FIM singularity and the nonidentifiability of the vector parameter space.

B. Related Work

The information-regularity condition and the identifiability condition have been successfully applied in several engineering disciplines, including statistical inference, control theory, and array processing, see [19], [23]–[26] for examples. Previous work has illuminated an intimate link between the nonsingularity of the FIM and the local identifiability of the desired parameter [18], [19], [24]–[28]. Local identifiability implies identifiability in an open neighborhood of the true value of the desired parameter and is weaker than the identifiability discussed in this paper which is often called global identifiability. Rothenberg [27] shows that if the rank of the FIM is constant over some open neighborhood of the desired parameter, then the nonsingularity of the FIM is equivalent to the local identifiability of the desired parameter. For normal distributions, the work in [25] provides some other conditions which also guarantee the equivalence between the nonsingularity of the FIM and the local identifiability of the

desired parameter. However, it can be shown that quantization induced singularity of the FIM does not generally imply a lack of local identifiability.

The relationship between the identifiability and the dimension of the vector parameter to be estimated has been studied in the area of array processing for a particular class of multivariate Gaussian distributed signal models [25]. For the particular class of models considered in [25], the task of examining the identifiability can be simplified to examining whether different values of the parameters give rise to different values of the covariance matrix. However, there are major differences between the work in [25] and that in this paper. First and foremost, the array processing models considered in [25] do not employ quantization which is the focus of our work. Moreover, we don't make any assumption on the model of the received signals, and our results hold for arbitrary statistical models of the observations, arbitrary value of the desired vector parameter, and arbitrary quantizer designs.

For the parameter estimation system illustrated in Fig. 1 with $N = 2$ and $L_1 = L_2 = 1$, a related but different problem has been considered in [29]. Zhang and Berger [29] show that under some assumptions, as the number of observations at each sensor increases to infinity, there exists an asymptotically unbiased estimator employing some quantizers with quantization rates no greater than some prescribed values, which can achieve a bounded variance. It is worth mentioning that for a given quantization rate, the number of quantization levels increases exponentially with respect to the number of observations at each sensor. Thus, the work in [29] essentially focuses on the case where the number of quantization levels employed is sufficiently large. In contrast, we are primarily interested in the case where the number of quantization levels employed is fixed and independent of the number of observations at each sensor, and we show that the quantization with fixed number of quantization levels induces a fundamental limitation on the dimension of the vector parameter which can be accurately estimated by the quantization estimation system.

C. Notation and Organization

In this paper, bold upper case letters and bold lower case letters are used to represent matrices and column vectors respectively. The symbol $\mathbf{1}$ stands for the all-one column vector, and $\mathbf{0}$ for the all-zero column vector. Let $[\mathbf{A}]_{i,j}$ denote the element in the i -th row and j -th column of the matrix \mathbf{A} . The l -th element of a vector $\boldsymbol{\theta}$ is denoted by θ_l . $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{A} \succeq \mathbf{0}$ imply that the matrix \mathbf{A} is positive definite and positive semidefinite respectively. For any set \mathcal{S} , $|\mathcal{S}|$ represents the number of elements in the set \mathcal{S} . For any given L , \mathbb{R}^L denotes the set of all L -tuples real numbers. The rank and expectation operators are denoted by $\text{rank}(\cdot)$ and $\mathbb{E}(\cdot)$ respectively.

The remainder of the paper is organized as follows. A general quantized estimation system and its IDQD are introduced in *Section II*. In *Section III*, the impact of quantization on the information-regularity condition is investigated. *Section IV* studies the impact of quantization on the identifiability condition. The specialization of the results in *Section III* and *Section IV* to cases with some commonly assumed assumptions

is considered in *Section V*. Finally, *Section VII* provides our conclusions.

II. QUANTIZED PARAMETER ESTIMATION SYSTEM MODEL AND INESTIMABLE DIMENSION FOR QUANTIZED DATA

Consider an N -sensor system as shown in Fig. 1 where the j -th sensor¹ produces a K -dimensional vector \mathbf{x}_j . The statistical description of \mathbf{x}_j depends on a D_θ -dimensional vector parameter $\theta \in \Theta \subset \mathbb{R}^{D_\theta}$ that we wish to estimate. As a generalization² to the standard quantized parameter estimation system, partition the observation vector into L_j parts as

$$\mathbf{x}_j = [\mathbf{x}_{j1}^T, \mathbf{x}_{j2}^T, \dots, \mathbf{x}_{jL_j}^T]^T. \quad (1)$$

Next the l -th observation subvector \mathbf{x}_{jl} of \mathbf{x}_j is converted to the quantized value u_{jl} by employing the vector quantizer γ_{jl} using³

$$u_{jl} \triangleq \gamma_{jl}(\mathbf{x}_{jl}) = \sum_{r=1}^{R_{jl}} r \mathbb{1}\{\mathbf{x}_{jl} \in I_{jl}^{(r)}\}, \quad (2)$$

where $\mathbb{1}\{\mathbf{x}_{jl} \in I_{jl}^{(r)}\} = 1$ if $\mathbf{x}_{jl} \in I_{jl}^{(r)}$ and otherwise it is zero. Thus γ_{jl} is an R_{jl} -level vector quantizer with given quantization regions $\{I_{jl}^{(r)}\}_{r=1}^{R_{jl}}$ which are disjoint and cover the domain of γ_{jl} . Next, we collect all the quantized data corresponding to \mathbf{x}_j , into \mathbf{u}_j which we call the superquantized vector, such that

$$\begin{aligned} \mathbf{u}_j &= [u_{j1}, u_{j2}, \dots, u_{jL_j}]^T \\ &\triangleq \Gamma_j(\mathbf{x}_j) \\ &= [\gamma_{j1}(\mathbf{x}_{j1}), \gamma_{j2}(\mathbf{x}_{j2}), \dots, \gamma_{jL_j}(\mathbf{x}_{jL_j})]^T. \end{aligned} \quad (3)$$

We assume for simplicity that the quantities \mathbf{u}_j , $j = 1, 2, \dots, N$ are transmitted without error to the FC to be used for estimating θ .

Without loss of generality, we can assume that the observation vectors $\{\mathbf{x}_j\}_{j=1}^N$ are independent, but the elements of \mathbf{x}_j are not necessarily independent for each j . This is because the scenarios where $\{\mathbf{x}_j\}_{j=1}^N$ are not independent can be considered as a special case of the system which only consists of 1 sensor, since the observation vectors $\{\mathbf{x}_j\}_{j=1}^N$ at different sensors are just a partition of the overall observation vector $[\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]^T$. We assume that \mathbf{x}_j follows a statistical model $(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)$ for each j , where \mathcal{X}_j is some set endowed with a σ -algebra \mathcal{F}_j . The probability measure \mathcal{P}_j^θ of \mathbf{x}_j belongs to a family of probability measures $\{\mathcal{P}_j^\theta : \theta \in \Theta\}$ on $(\mathcal{X}_j, \mathcal{F}_j)$ indexed by a D_θ -dimensional vector parameter θ lying in a set $\Theta \subset \mathbb{R}^{D_\theta}$. The superquantizer $\Gamma_j : (\mathcal{X}_j, \mathcal{F}_j) \rightarrow (\mathbb{R}^{L_j}, \mathcal{B}_j)$ is a measurable function with

¹It should be noted that if the data comes from something other than a sensor, the results still apply.

²The case of performing L_j scalar quantizations at each sensor, is common, as is pure vector quantization with $L_j = 1$, for example.

³It is worth mentioning that in this paper, we use R_{jl} to denote the number of quantization levels of the vector quantizer γ_{jl} , while the notation R is usually reserved for the quantization rate in the literature on vector quantization.

respect to \mathcal{F}_j and \mathcal{B}_j for all j , where \mathcal{B}_j is the Borel algebra on \mathbb{R}^{L_j} .

Before proceeding, we define a critical quantity, which is called the inestimable dimension for quantized data (IDQD) of the quantized estimation system.

Definition 3 (Inestimable Dimension for Quantized Data): The IDQD $\lambda(N, \{R_{jl}\})$ of the quantized estimation system described above is defined as

$$\lambda(N, \{R_{jl}\}) \triangleq \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N. \quad (4)$$

It is seen from (4) that the defined IDQD $\lambda(N, \{R_{jl}\})$ of the quantized estimation system does not depend on the value of the desired vector parameter θ , the statistical models $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$ and the quantization regions $\{I_{jl}^{(r)}\}$, but is only determined by the number N of sensors and the numbers $\{R_{jl}\}$ of the quantization levels employed at the sensors.

It is worth mentioning that the definition of the IDQD in (4) implies that the IDQD has some scaling properties. To be specific, for the case with $N = 1$, $L = 1$ and $R_{jl} = R$, we can obtain $\lambda(1, \{R\}) = R - 1$. Similarly, for the case with $L_1 = L_2 = \dots = L_N = 1$ and $R_{jl} = R$ for all j and l , we have $\lambda(N, \{R\}) = N(R - 1)$, which illustrates a linear effect of increasing the number of sensors. Likewise, we also can obtain

$$N \left(R_{\min}^{L_{\min}} - 1 \right) \leq \lambda(N, \{R_{jl}\}) \leq N \left(R_{\max}^{L_{\max}} - 1 \right), \quad (5)$$

where $R_{\min} \triangleq \min_{j,l} R_{jl}$, $R_{\max} \triangleq \max_{j,l} R_{jl}$, $L_{\min} \triangleq \min_l L_l$, and $L_{\max} \triangleq \max_l L_l$. This clearly shows the scaling property of the IDQD with respect to N , $\{L_j\}$ and $\{R_{jl}\}$.

In the following, we will show that there is a close link between the IDQD and the estimation capabilities of the quantized estimation system in terms of the information-regularity condition and the identifiability condition.

III. IDQD AND INFORMATION-REGULARITY CONDITION

In this section, we first formulate the FIM for estimating θ , and then based on the expression of the FIM, we show that the IDQD of the quantized estimation system describes a fundamental limitation of the quantized estimation system with respect to the information-regularity condition.

Let \mathcal{S}_j denote the set of all possible outcomes of the j -th superquantizer Γ_j

$$\mathcal{S}_j = \left\{ \mathbf{s}_1^{(j)}, \mathbf{s}_2^{(j)}, \dots, \mathbf{s}_{|\mathcal{S}_j|}^{(j)} \right\}. \quad (6)$$

It is clear that the size of \mathcal{S}_j can be written as

$$|\mathcal{S}_j| = \prod_{l=1}^{L_j} R_{jl}. \quad (7)$$

Let \mathbf{u} denote a vector containing all the quantized data $\{\mathbf{u}_j\}$ received at the FC

$$\mathbf{u} \triangleq [\mathbf{u}_1^T, \mathbf{u}_2^T, \dots, \mathbf{u}_N^T]^T. \quad (8)$$

For any given quantized data \mathbf{u} received at the FC, the log-likelihood function $L(\boldsymbol{\theta})$ can be expressed as⁴

$$\begin{aligned} L(\boldsymbol{\theta}) &\triangleq \ln \Pr(\mathbf{u} | \boldsymbol{\theta}) \\ &= \ln \prod_{j=1}^N \Pr(\mathbf{u}_j | \boldsymbol{\theta}) \\ &= \sum_{j=1}^N \sum_{\mathbf{s} \in \mathcal{S}_j} \mathbb{1}\{\mathbf{u}_j = \mathbf{s}\} \ln q_j^{(\mathbf{s})}(\boldsymbol{\theta}) \end{aligned} \quad (9)$$

where $\forall j$, $q_j^{(\mathbf{s})}(\boldsymbol{\theta})$ is defined as

$$q_j^{(\mathbf{s})}(\boldsymbol{\theta}) \triangleq \mathcal{P}_j^\theta(\Gamma_j(\mathbf{x}_j) = \mathbf{s}) \quad (10)$$

for any given vector $\mathbf{s} \in \mathcal{S}_j$.

Define the following assumptions.

Assumption 1: We assume the estimation problem has already been formulated with the smallest possible dimension of the vector parameter $\boldsymbol{\theta}$ using all information available. Thus, there are no additional known constraints between the elements of $\boldsymbol{\theta}$ that imply we can estimate a vector parameter with a smaller dimension and then use the constraints to find $\boldsymbol{\theta}$. Further, no additional side or prior information is available involving the elements of $\boldsymbol{\theta}$.

Assumption 2: For the set Θ in \mathbb{R}^{D_θ} , the interior of Θ is not empty.

Assumption 3: For all j and \mathbf{s} , $q_j^{(\mathbf{s})}(\boldsymbol{\theta})$ in (10) is twice differentiable with respect to $\boldsymbol{\theta}$ for all $\boldsymbol{\theta} \in \Theta$.

Note that Assumption 1, Assumption 2 and Assumption 3 are called regularity conditions and are commonly adopted in the signal processing literature [30].

To gain insights into whether the information-regularity condition is satisfied, we first explore the FIM $\mathbf{J}(\boldsymbol{\theta})$ for estimating $\boldsymbol{\theta}$. Under Assumption 2 and Assumption 3, the (l, m) -th element of the FIM $\mathbf{J}(\boldsymbol{\theta})$ is defined as [30]

$$[\mathbf{J}(\boldsymbol{\theta})]_{l,m} \triangleq -\mathbb{E} \left\{ \frac{\partial^2 L(\boldsymbol{\theta})}{\partial \theta_l \partial \theta_m} \right\}, \quad (11)$$

where θ_l and θ_m denote the l -th and m -th elements of $\boldsymbol{\theta}$ respectively. Hence, by employing (9), $\mathbf{J}(\boldsymbol{\theta})$ can be expressed as

$$\mathbf{J}(\boldsymbol{\theta}) = \sum_{j=1}^N \sum_{\mathbf{s} \in \mathcal{S}_j} \frac{1}{q_j^{(\mathbf{s})}(\boldsymbol{\theta})} \frac{\partial q_j^{(\mathbf{s})}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left[\frac{\partial q_j^{(\mathbf{s})}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]^T. \quad (12)$$

By employing (12), we can obtain the following theorem with regard to the singularity of the FIM.

Theorem 1: Under Assumption 1, Assumption 2 and Assumption 3, for any given $\boldsymbol{\theta}$, any quantization regions $\{I_{jl}^{(r)}\}$ and any statistical models $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$, the FIM $\mathbf{J}(\boldsymbol{\theta})$ described in (12) is singular, if the dimension D_θ of the vector parameter $\boldsymbol{\theta}$ is greater than the IDQD, i.e.,

$$D_\theta > \lambda(N, \{R_{jl}\}), \quad (13)$$

where $\lambda(N, \{R_{jl}\})$ is defined in (4).

Proof: Refer to Appendix A. ■

⁴Note that if $q_j^{(\mathbf{s})}(\boldsymbol{\theta}) = 0$ for some j and \mathbf{s} , then the corresponding summand in (9) should be eliminated in computing (9).

Remark 1: Further discussion on Assumption 1 can found in Section VI.

Theorem 1 reveals a fundamental limitation when utilizing quantized data for estimating a vector parameter, and sheds light on the preliminary design of a quantized estimation system. To be specific, the quantization and sensing approach employed should guarantee that the IDQD of the quantized estimation system, $\lambda(N, \{R_{jl}\})$, is larger than or equal to the dimension of the vector parameter of interest. Otherwise, the FIM for estimating the vector parameter of interest is singular definitely singular for any $\boldsymbol{\theta}$, $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$. Since this fundamental limitation is quantization induced, we refer to this singularity of the FIM which is caused by the condition $D_\theta > \lambda(N, \{R_{jl}\})$ as quantization induced singularity. In order to alleviate this undesirable outcome, it is seen from the definition of $\lambda(N, \{R_{jl}\})$ in (4) that one can employ finer quantizers $\{\gamma_{jl}\}$ with larger $\{R_{jl}\}$ or add more sensors in the system. However, enlarging $\{R_{jl}\}$ can significantly increase the data rate from each sensor to the FC, and adding more sensors increases the cost of quantized estimation system.

If the dimension of the vector parameter of interest is smaller than the IDQD $\lambda(N, \{R_{jl}\})$, it is still possible that the FIM for estimating the vector parameter of interest is singular for some $\boldsymbol{\theta}$, $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$. However, in this case the singularity depends on $\boldsymbol{\theta}$, $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$. Hence, the condition $D_\theta > \lambda(N, \{R_{jl}\})$ is not generally necessary for guaranteeing the singularity of the FIM.

IV. IDQD AND IDENTIFIABILITY CONDITION

In this section, we study the relationship between the identifiability of the vector parameter space Θ and the IDQD of the quantized estimation system defined in (4). The definitions of identifiability which are employed in this paper are first described. Then, we show that the IDQD of the quantized estimation system describes a fundamental limitation of the quantized estimation system with respect to the identifiability of the vector parameter space.

A. IDQD and Identifiability of the Vector Parameter Space

In order to characterize the impact of the quantization on the identifiability condition, we first formally give the following definitions with regard to the identifiability.

Definition 4 (Observational Equivalence [27]): Two distinct vector parameter points in Θ , $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$, are said to be observationally equivalent if $\Pr(\mathbf{u} | \boldsymbol{\theta}) = \Pr(\mathbf{u} | \boldsymbol{\theta}')$ for all possible \mathbf{u} in (8).

Definition 5 (Identifiable Vector Parameter Point [27]): The vector parameter point $\boldsymbol{\theta} \in \Theta$ is called identifiable, if there is no other $\boldsymbol{\theta}' \in \Theta \setminus \{\boldsymbol{\theta}\}$ which is observationally equivalent to $\boldsymbol{\theta}$.

Definition 6 (Identifiable Vector Parameter Space): The vector parameter space Θ is considered identifiable, if every vector parameter point $\boldsymbol{\theta} \in \Theta$ is identifiable.

It is worth pointing out that in some literature, if a parameter is said to be identifiable, it means that the parameter space Θ is identifiable, for instance, see [21], [31]. In general, for a meaningful estimation problem, it is necessary that the parameter space Θ is identifiable.

Let $\mathcal{A} \triangleq \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{D_{\mathbf{u}}}\}$ denote the set of all possible realizations of \mathbf{u} in (8), where the number of all possible realizations is

$$D_{\mathbf{u}} = \prod_{j=1}^N \prod_{l=1}^{L_j} R_{jl}. \quad (14)$$

According to *Definition 4* and *Definition 6*, the vector parameter space Θ is identifiable if and only if there are no distinct vector parameter points θ_1 and θ_2 in Θ such that $\Pr(\mathbf{u}|\theta_1) = \Pr(\mathbf{u}|\theta_2)$ for all $\mathbf{u} \in \mathcal{A}$. In other words, the vector parameter space Θ is identifiable if and only if the mapping

$$\begin{aligned} \varphi_{\mathbf{u}} : \Theta &\longrightarrow \mathbb{R}^{D_{\mathbf{u}}} \\ \theta &\longmapsto [\Pr(\mathbf{a}_1|\theta), \Pr(\mathbf{a}_2|\theta), \dots, \Pr(\mathbf{a}_{D_{\mathbf{u}}|\theta})]^T \end{aligned} \quad (15)$$

is injective. Thus, we can examine the injectivity of the mapping $\varphi_{\mathbf{u}}$ in (15) to investigate the identifiability condition. On the other hand, the following lemmas simplify this investigation.

Let us define a $(\sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N)$ -dimensional vector $\Psi(\theta)$

$$\Psi(\theta) \triangleq [\psi_1(\theta)^T, \psi_2(\theta)^T, \dots, \psi_N(\theta)^T]^T, \quad (16)$$

where for each j , $\psi_j(\theta)$ is defined as

$$\psi_j(\theta) \triangleq \left[q_j^{(s_1^{(j)})}(\theta), q_j^{(s_2^{(j)})}(\theta), \dots, q_j^{(s_{|\mathcal{S}_j|-1}^{(j)})}(\theta) \right]^T, \quad (17)$$

$q_j^{(s)}(\theta)$ is defined in (10), and $s_i^{(j)}$ is defined in (6) for all $i = 1, 2, \dots, |\mathcal{S}_j| - 1$.

Lemma 1: The mapping $\varphi_{\mathbf{u}}$ in (15) is injective if and only if the mapping

$$\begin{aligned} \Psi : \Theta &\longrightarrow \mathbb{R}^{\sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N} \\ \theta &\longmapsto \Psi(\theta) \end{aligned} \quad (18)$$

is injective. Therefore, a necessary and sufficient condition under which the vector parameter space Θ is identifiable is that the mapping Ψ in (18) is injective. Moreover, the dimension of the vector $\Psi(\theta)$ in (16) is strictly smaller than that of $\varphi_{\mathbf{u}}(\theta)$ in (15) for any given N and $\{R_{jl}\}$.

Proof: Refer to Appendix B. ■

As *Lemma 1* demonstrates, the identifiability of the vector parameter space Θ can also be determined by the injectivity of the mapping Ψ in (18). To this end, we only need to inspect the injectivity of the mapping Ψ in (18) to investigate the identifiability of the vector parameter space. What's more, it is seen that the dimension of the vector $\Psi(\theta)$ in (16) is precisely the IDQD of the quantized estimation system $\lambda(N, \{R_{jl}\})$, which is shown to be strictly smaller than the dimension of $\varphi_{\mathbf{u}}(\theta)$ in (15) for any given N and $\{R_{jl}\}$. In the following, we will show that because of the smaller dimension of $\Psi(\theta)$, inspecting the injectivity of the mapping Ψ in (18) is easier than inspecting the injectivity of the mapping $\varphi_{\mathbf{u}}$ in (15) under the condition that $D_{\theta} > \lambda(N, \{R_{jl}\})$.

Before proceeding, let us first introduce a helpful result in algebraic topology.

Lemma 2 (Invariance of Domain Theorem [32]): If \mathcal{U} is an open subset of \mathbb{R}^n and $f : \mathcal{U} \rightarrow \mathbb{R}^n$ is an injective continuous mapping, then $\mathcal{V} \triangleq f(\mathcal{U})$ is open in \mathbb{R}^n , and f is a homeomorphism between \mathcal{U} and \mathcal{V} .

The proof of *Lemma 2* can be found in [32]. Next, we make the following assumption throughout this section, which is weaker than *Assumption 3*.

Assumption 4: For all j and all \mathbf{s} , $q_j^{(\mathbf{s})}(\theta)$ in (10) is continuous with respect to θ .

By employing *Lemma 1* and *Lemma 2*, we provide the following theorem with regard to the identifiability of the vector parameter space.

Theorem 2: Under *Assumption 1*, *Assumption 2* and *Assumption 4*, for any given quantization regions $\{I_{jl}^{(r)}\}$ and statistical models $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^{\theta})\}$, if the dimension of the desired vector parameter θ is larger than the IDQD of the quantized estimation system, i.e.,

$$D_{\theta} > \lambda(N, \{R_{jl}\}), \quad (19)$$

then the vector parameter space Θ is not identifiable. Moreover, for any open subset $\mathcal{U} \subset \Theta$ in $\mathbb{R}^{D_{\theta}}$, there are infinitely many vector parameter points in \mathcal{U} which are not identifiable.

Proof: Refer to Appendix C. ■

Theorem 2 demonstrates that under *Assumption 1*, *Assumption 2* and *Assumption 4*, for any given $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^{\theta})\}$, the condition $D_{\theta} > \lambda(N, \{R_{jl}\})$ is sufficient to guarantee that there exist infinitely many nonidentifiable vector parameter points in the vector parameter space Θ . Hence, the vector parameter space Θ is not identifiable. This nonidentifiability of the vector parameter space is also quantization induced, and doesn't depend on the statistical models of the observations and the design of the quantization regions. We refer to the nonidentifiability of the vector parameter space which is caused by the condition $D_{\theta} > \lambda(N, \{R_{jl}\})$ as quantization induced nonidentifiability. However, it is worth mentioning that the condition $D_{\theta} \leq \lambda(N, \{R_{jl}\})$ cannot guarantee the identifiability of the vector parameter space, which is determined by the vector parameter space Θ , the quantizer designs $\{I_{jl}^{(r)}\}$, and the statistical models $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^{\theta})\}$.

B. Remarks on Quantization Induced Nonidentifiable Vector Parameter Space

A particular note of interest is that *Assumption 4* employed in *Theorem 2* is much weaker than *Assumption 3* employed in *Theorem 1*. The continuity of $q_j^{(\mathbf{s})}(\theta)$ assumed in *Assumption 4* is not enough to guarantee the existence of the FIM. Thus, in some cases where the FIM for estimating the desired vector parameter doesn't exist, the quantization induced nonidentifiability of the vector parameter space can still be guaranteed by *Theorem 2* under the condition that $D_{\theta} > \lambda(N, \{R_{jl}\})$. Hence, in general, there is no equivalence between the quantization induced singularity of the FIM and the quantization induced nonidentifiability of the vector parameter space. However, if both *Assumption 2* and

Assumption 3 hold, the condition that $D_{\theta} > \lambda(N, \{R_{jl}\})$ ensures both singularity of the FIM and nonidentifiability of the vector parameter space.

According to *Theorem 2*, we know that under *Assumption 2* and *Assumption 4*, for any given $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^{\theta})\}$, if the dimension of the desired vector parameter θ is larger than the IDQD of the quantized estimation system, then there exist infinitely many nonidentifiable vector parameter points in Θ . However, there still remain two interesting questions which are not answered by *Theorem 2*. First, although *Theorem 2* shows that under the condition $D_{\theta} > \lambda(N, \{R_{jl}\})$, there are infinitely many nonidentifiable vector parameter points in Θ , whether every vector parameter point in Θ is nonidentifiable or not is still unknown. Second, under the condition $D_{\theta} > \lambda(N, \{R_{jl}\})$, what is the cardinality of a given set of observationally equivalent points in the nonidentifiable vector parameter space?

In this subsection, we employ examples to show that in some cases, every vector parameter point in the vector parameter space is not identifiable, while in other cases, there exist some vector parameter points which are identifiable. Moreover, the examples illustrate that under the condition $D_{\theta} > \lambda(N, \{R_{jl}\})$, the cardinality of a set of observationally equivalent points can be very different for various cases. It can be as small as 1 and can also be as large as uncountably infinite.

1) *Every Vector Parameter Point in the Nonidentifiable Vector Parameter Space is Nonidentifiable and Every Set of Observationally Equivalent Points is Uncountable.*

Example 1: Consider a quantized estimation system with $N = 1$, $L_j = 1$, and the dimension of \mathbf{x}_{jl} is 1 for all l . In this case the single sensor makes a scalar observation which we denote as x for simplicity. The Gaussian assumed probability density function (pdf) of x is

$$f(x|\theta) = \frac{1}{\sqrt{2\pi\beta}} e^{-\frac{(x-a)^2}{2\beta}}, \quad (20)$$

where the unknown vector parameter is $\theta \triangleq [\alpha, \beta]^T$. The vector parameter space is

$$\Theta = \{[\alpha, \beta]^T : \alpha \in \mathbb{R}, \beta \geq 0\}. \quad (21)$$

It is clear that the interior of Θ is not empty. We assume that the sensor employs a binary quantizer to convert x to $u \in \{1, 2\}$ by using the nonempty quantization regions

$$I^{(1)} = (a, b) \text{ and } I^{(2)} = \mathbb{R} \setminus I^{(1)}, \quad (22)$$

for some a and b with $-\infty \leq a < b \leq \infty$. This quantizer model is common and widely considered in recent literature, for instance, see [8], [33]. By the definition of the IDQD in (4), we can obtain

$$\lambda(N, \{R_{jl}\}) = \lambda(1, \{2\}) = 1 < 2 = D_{\theta}, \quad (23)$$

and hence, by *Theorem 2*, the vector parameter space Θ is not identifiable.

Proposition 1: For the quantized estimation system described in *Example 1*, if the sensor doesn't employ quantization, then every vector parameter point $\theta \in \Theta$ is identifiable. Since the vector parameter space Θ

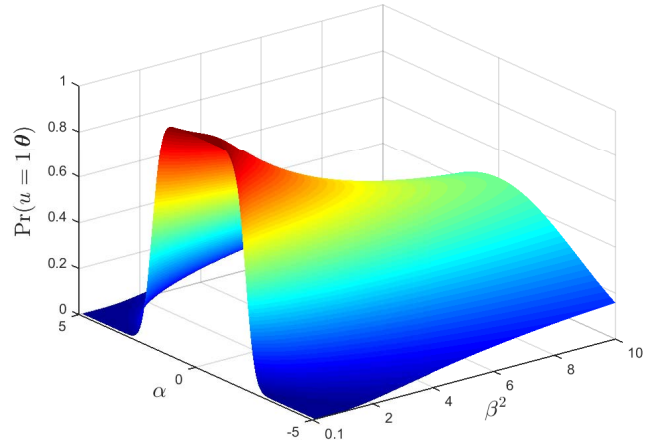


Fig. 3. $\Pr(u = 1|\theta)$ versus θ for the case where $I^{(1)} = (-2, 2)$ and $I^{(2)} = \mathbb{R} \setminus I^{(1)}$.

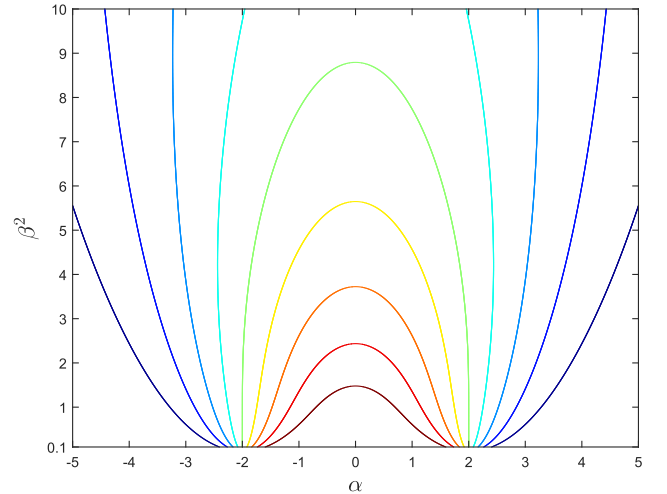


Fig. 4. Contours of $\Pr(u = 1|\theta)$ for the case where $I^{(1)} = (-2, 2)$ and $I^{(2)} = \mathbb{R} \setminus I^{(1)}$.

is not identifiable when the quantizer is employed, the nonidentifiability of the vector parameter space is indeed quantization induced. Furthermore, if the sensor employs the quantizer with any given $I^{(1)}$ and $I^{(2)}$ with the forms in (22), every vector parameter point in Θ is not identifiable, and moreover, for any vector parameter point $\theta \in \Theta$, the set of vector parameter points which are observationally equivalent to θ is uncountable.

Proof: Refer to Appendix D. ■

As *Proposition 1* demonstrates, under the condition $D_{\theta} > \lambda(N, \{R_{jl}\})$, there exist some cases where for any vector parameter point $\theta \in \Theta$, the set of vector parameter points which are observationally equivalent to θ is uncountable, and hence, every vector parameter point in Θ is not identifiable.

To corroborate the theoretic analysis, we present some numerical results which illustrate the identifiability of the vector parameter points in Θ . Fig. 3 depicts the relationship between $\Pr(u = 1|\theta)$ and θ for a particular case where $I^{(1)} = (-2, 2)$ and $I^{(2)} = \mathbb{R} \setminus I^{(1)}$, and Fig. 4 shows the contour of $\Pr(u = 1|\theta)$ for the same case. Since $\Pr(u = 2|\theta) = 1 - \Pr(u = 1|\theta)$, we know that for a given θ in Θ ,

if $\Pr(u = 1 | \theta') = \Pr(u = 1 | \theta)$ for some other $\theta' \in \Theta \setminus \{\theta\}$, then by *Definition 4*, θ' is observationally equivalent to θ , and hence, θ is not identifiable. Therefore, every contour curve in Fig. 4 illustrates a set of observationally equivalent vector parameter points. Moreover, it is easy to see from Fig. 3 that every vector parameter point is not identifiable.

2) *Existence of Identifiable Vector Parameter Point in the Nonidentifiable Vector Parameter Space.*

Example 2: Consider a quantized estimation system with $N = 1$ and $K = 2$, where the observation $\mathbf{x} = [x_1, x_2]^T$ follows the distribution⁵ $\mathcal{N}(\theta, \mathbf{I})$ with unknown vector parameter $\theta \triangleq [\theta_1, \theta_2]^T$. The vector parameter space Θ is \mathbb{R}^2 which is open. We assume that the sensor employs a binary vector quantizer to convert \mathbf{x} to $u \in \{1, 2\}$ by using the nonempty quantization regions

$$I^{(1)} = (a_1, b_1) \times (a_2, b_2) \text{ and } I^{(2)} = \mathbb{R}^2 \setminus I^{(1)}, \quad (24)$$

for some a_1, a_2, b_1 and b_2 , where $-\infty \leq a_1 < b_1 \leq \infty$ and $-\infty \leq a_2 < b_2 \leq \infty$.

From the definition of the IDQD in (4), we can obtain

$$\lambda(N, \{R_{jl}\}) = \lambda(1, \{2\}) = 1 < 2 = D_\theta, \quad (25)$$

and hence, by *Theorem 2*, the vector parameter space Θ is not identifiable.

Proposition 2: For the quantized estimation system described in Example 2 and for any given $I^{(1)}$ and $I^{(2)}$ with the forms in (24), there exists an identifiable vector parameter point in Θ .

Proof: Refer to Appendix E. ■

As illustrated by *Proposition 2*, under the condition $D_\theta > \lambda(N, \{R_{jl}\})$, there exist some cases where the quantization induced nonidentifiable vector parameter space Θ contains some identifiable vector parameter points. According to *Definition 5*, for any identifiable vector parameter point θ , the set of points which are observationally equivalent to θ consists of only one point, that is, θ . Thus, under the condition $D_\theta > \lambda(N, \{R_{jl}\})$, the cardinality of some observationally equivalent set in nonidentifiable vector parameter space Θ can be as small as 1 in some cases, since it is possible that some vector parameter point in Θ is identifiable. Furthermore, it is worth mentioning that under the condition $D_\theta > \lambda(N, \{R_{jl}\})$, the FIM evaluated at any vector parameter point in Θ is singular for any case. Hence, as *Proposition 2* demonstrates, the singularity of the FIM does not necessarily contradict the identifiability of the vector parameter point.

Some numerical results for a particular case where $a_1 = a_2 = -1$ and $b_1 = b_2 = 1$ are provided in support of the theoretical analysis. Fig. 5 depicts the value of $\Pr(u = 1 | \theta)$ for each vector parameter point in Θ , and Fig. 6 illustrates the contour of $\Pr(u = 1 | \theta)$. Since $\Pr(u = 2 | \theta) = 1 - \Pr(u = 1 | \theta)$, according to *Definition 5*, it is clear that if θ is not identifiable, then there exists some other $\theta' \in \Theta \setminus \{\theta\}$ such that $\Pr(u = 1 | \theta) = \Pr(u = 1 | \theta')$. Fig. 5 shows that $\Pr(u = 1 | \theta)$ achieves its unique global maximum at $\theta = \mathbf{0}$, which demonstrates that $\theta = \mathbf{0}$ is an identifiable vector

⁵ $\mathcal{N}(\theta, \mathbf{I})$ denotes a multivariate Gaussian distribution with mean vector θ and covariance matrix \mathbf{I} , where \mathbf{I} is the 2-by-2 identity matrix.

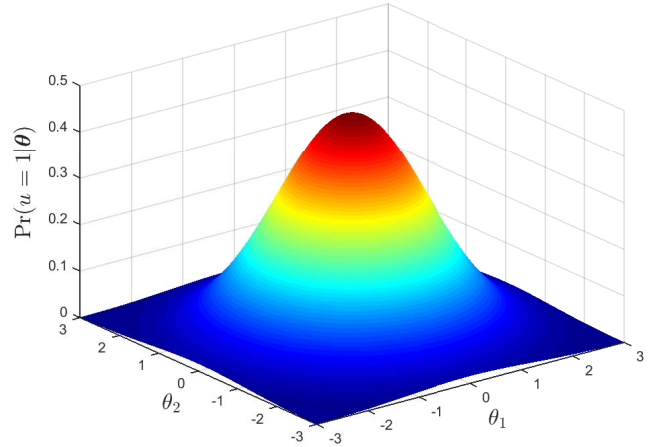


Fig. 5. $\Pr(u = 1 | \theta)$ versus θ for the case where $a_1 = a_2 = -1$ and $b_1 = b_2 = 1$.

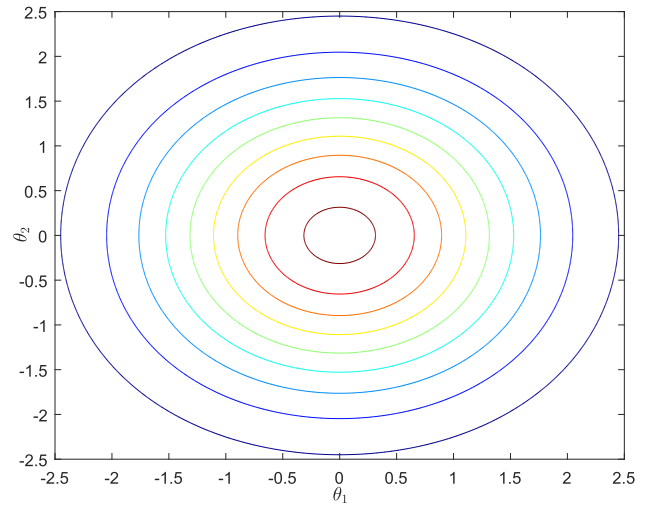


Fig. 6. Contours of $\Pr(u = 1 | \theta)$ for the case where $a_1 = a_2 = -1$ and $b_1 = b_2 = 1$.

parameter point in this particular case. It is seen from Fig. 6 that every set of observationally equivalent points forms a circle with the center at $\theta = \mathbf{0}$. Hence, except the set of points which are observationally equivalent to $\theta = \mathbf{0}$, every other set of observationally equivalent points is uncountable.

V. IDQD WITH ADDITIONAL ASSUMPTIONS

In *Section III* and *Section IV*, we make no assumptions about the quantizers and the statistical model of the observations at each sensor. Hence, *Theorem 1* and *Theorem 2* apply to any case with any $\{\gamma_{jl}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$. However, in general, the sufficient condition $D_\theta > \lambda(N, \{R_{jl}\})$ employed in *Theorem 1* and *Theorem 2* for guaranteeing the FIM singularity and the nonidentifiability of the vector parameter space is not strictly necessary. Hence, weaker conditions which still imply singularity and nonidentifiability when some additional assumptions are valid are of interest. In fact, under some common assumptions, we will show we can obtain a smaller IDQD compared to the results given in *Theorem 1* and *Theorem 2* which better describes the limitations imposed by quantization under these assumptions.

A. Existence of Identical Sensor Observation Statistical Models and Identical Superquantizers

In this subsection, we consider the scenarios where the following assumption is valid.

Assumption 5: The statistical models of the observation vectors at some different sensors are known to be the same for all θ such that the number of different statistical models at all the N sensors is $P < N$.

Collect all the sensors indices that employ the p -th statistical model in the group \mathcal{G}_p such that

$$\{1, 2, \dots, N\} = \bigcup_{p=1}^P \mathcal{G}_p, \text{ and } \mathcal{G}_p \cap \mathcal{G}_{p'} = \emptyset, \forall p \neq p'. \quad (26)$$

For the sake of notational simplicity, we use $(\hat{\mathcal{X}}_p, \hat{\mathcal{F}}_p, \hat{\mathcal{P}}_p^\theta)$ to denote the statistical model for any \mathbf{x}_j with $j \in \mathcal{G}_p$.

In general, it is possible that some different sensors employ an identical superquantizer⁶ to convert its observation vector to digital data. Each \mathcal{G}_p can be further divided into M_p disjoint nonempty subgroups $\{\mathcal{G}_p^{(m)}\}_{m=1}^{M_p}$ of sensors that use different superquantizers

$$\mathcal{G}_p = \bigcup_{m=1}^{M_p} \mathcal{G}_p^{(m)}, \text{ and } \mathcal{G}_p^{(m)} \cap \mathcal{G}_p^{(m')} = \emptyset, \forall m \neq m'. \quad (27)$$

For simplicity, we use

$$\hat{\Gamma}_p^{(m)} \triangleq \left[\hat{\gamma}_{p1}^{(m)}, \hat{\gamma}_{p2}^{(m)}, \dots, \hat{\gamma}_{pL_p^{(m)}}^{(m)} \right]^T \quad (28)$$

to denote the superquantizer employed by the sensors in $\mathcal{G}_p^{(m)}$, where $L_p^{(m)}$ is the number of vector quantizers in $\hat{\Gamma}_p^{(m)}$.

Moreover, we use $\hat{R}_{pl}^{(m)}$ and $\{\hat{I}_{mpl}^{(r)}\}_{r=1}^{\hat{R}_{pl}^{(m)}}$ to respectively denote the number of quantization levels of $\hat{\gamma}_{pl}^{(m)}$ and the quantization regions of $\hat{\gamma}_{pl}^{(m)}$ for each m, p and l .

Thus, under *Assumption 5*, if j and j' are contained in $\mathcal{G}_p^{(m)}$ for some p and m , then for any θ and any outcome \mathbf{s} of the superquantizer $\hat{\Gamma}_p^{(m)}$,

$$q_j^{(\mathbf{s})}(\theta) = q_{j'}^{(\mathbf{s})}(\theta) = \hat{\mathcal{P}}_p^\theta(\hat{\Gamma}_p^{(m)}(\mathbf{x}_j) = \mathbf{s}). \quad (29)$$

By employing (29) and similar arguments as those in *Section III* and *Section IV*, the following theorem can be obtained.

Theorem 3: Under *Assumption 1*, *Assumption 2*, *Assumption 3* and *Assumption 5*, for any given θ , any quantization regions $\{\hat{I}_{mpl}^{(r)}\}$ and any statistical models $\{(\hat{\mathcal{X}}_p, \hat{\mathcal{F}}_p, \hat{\mathcal{P}}_p^\theta)\}$, if the dimension D_θ of the vector parameter θ is greater than $\sum_{p=1}^P \sum_{m=1}^{M_p} (\prod_{l=1}^{L_p^{(m)}} \hat{R}_{pl}^{(m)} - 1)$, i.e.,

$$D_\theta > \lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\}) \\ \triangleq \sum_{p=1}^P \sum_{m=1}^{M_p} \left(\prod_{l=1}^{L_p^{(m)}} \hat{R}_{pl}^{(m)} - 1 \right), \quad (30)$$

⁶Here the order is important, thus an identical superquantizer uses the same vector quantizers in the same order.

then the FIM for estimating θ is singular. Furthermore, under *Assumption 1*, *Assumption 2*, *Assumption 4* and *Assumption 5*, for any given $\{\hat{I}_{mpl}^{(r)}\}$ and $\{(\hat{\mathcal{X}}_p, \hat{\mathcal{F}}_p, \hat{\mathcal{P}}_p^\theta)\}$, if (30) holds, then the vector parameter space Θ is not identifiable. Moreover, for any open subset $\mathcal{U} \subset \Theta$ in \mathbb{R}^{D_θ} , there are infinitely many vector parameter points in \mathcal{U} which are not identifiable.

The proof of *Theorem 3* is omitted, since it is similar to the proofs of *Theorem 1* and *Theorem 2* after properly accounting for the impact of *Assumption 5*, which effectively reduces the IDQD. At least in terms of the FIM singularity, the fact that the IDQD is reduced seems reasonable since the statistically identical models assumed in *Assumption 5* leads to identical terms in the sum in (12) which leads to a smaller dimension of θ at which the FIM must be singular. The impact of *Assumption 5* on identifiability can be similarly justified.

By the definition of $\lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\})$ in (30), we know that the critical quantity $\lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\})$ does not depend on $\{\hat{I}_{mpl}^{(r)}\}$ and $\{(\hat{\mathcal{X}}_p, \hat{\mathcal{F}}_p, \hat{\mathcal{P}}_p^\theta)\}$, but is only determined by the number of groups $\{\mathcal{G}_p\}$, the number of subgroups $\{\mathcal{G}_p^{(m)}\}$ and the precision of the vector quantizers employed by the system.

What's more, we can obtain the relationship between the IDQD $\lambda(N, \{R_{jl}\})$ in (4) and the quantity $\lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\})$ in (30) as

$$\begin{aligned} \lambda(N, \{R_{jl}\}) &= \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N \\ &= \sum_{p=1}^P \sum_{m=1}^{M_p} \sum_{j \in \mathcal{G}_p^{(m)}} \left(\prod_{l=1}^{L_p^{(m)}} \hat{R}_{pl}^{(m)} - 1 \right) \\ &= \sum_{p=1}^P \sum_{m=1}^{M_p} |\mathcal{G}_p^{(m)}| \left(\prod_{l=1}^{L_p^{(m)}} \hat{R}_{pl}^{(m)} - 1 \right) \\ &\geq \sum_{p=1}^P \sum_{m=1}^{M_p} \left(\prod_{l=1}^{L_p^{(m)}} \hat{R}_{pl}^{(m)} - 1 \right) \\ &= \lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\}) \end{aligned} \quad (31)$$

where the inequality in (31) is a consequence of the fact that $|\mathcal{G}_p^{(m)}| \geq 1$ for all p and all m . Thus from (32), rather than utilizing the IDQD $\lambda(N, \{R_{jl}\})$ in (4), it would be better to employ the critical quantity $\lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\})$ in (30) to specify the fundamental limitation of the quantized estimation system under *Assumption 5* given the conditions of *Theorem 3* apply. To this end, the critical quantity $\lambda_{\text{ISM}}(\{\mathcal{G}_p^{(m)}\}, \{\hat{R}_{pl}^{(m)}\})$ in (30) is referred to as the *refined inestimable dimension for quantized data* (rIDQD) for the quantized estimation system under *Assumption 5*.

Additionally, (31) implies that in order to reduce the severity of the fundamental limitation of the quantized estimation system, for any set of sensors whose observation vectors obey the same statistical model, we should employ distinct superquantizers at each of the sensors in this set, so that we can achieve $|\mathcal{G}_p^{(m)}| = 1$ for all p and all m . Otherwise, the

quantization induced fundamental limitation becomes more limiting implying the FIM singularity and the nonidentifiability of the vector parameter space for an even smaller vector parameter dimension.

B. Independent Observation Subvectors

In this subsection, we make the following assumption.

Assumption 6: All the partitioned observation subvectors $\{\mathbf{x}_{jl}\}$ are known to be independent. We denote the statistical model of \mathbf{x}_{jl} by $(\mathcal{X}_{jl}, \mathcal{F}_{jl}, \mathcal{P}_{jl}^\theta)$ for each j and l .

Note that *Assumption 6* is commonly assumed in recent literature on parameter estimation with quantized data, see [5] for example. It is clear that the quantized estimation system under *Assumption 6* is a special case of the general quantized estimation system described in *Section II*. However, as stated previously we show we can find a smaller IDQD under *Assumption 6* that better describes the limitations imposed by using quantized data.

It is clear that under *Assumption 6*, for any θ and any outcome $\mathbf{s} = [s_1, s_2, \dots, s_{L_j}]^T$ of the superquantizer Γ_j , we have

$$\begin{aligned} q_j^{(\mathbf{s})}(\theta) &= \mathcal{P}_j^\theta(\Gamma_j(\mathbf{x}_j) = \mathbf{s}) \\ &= \prod_{l=1}^{L_j} \mathcal{P}_{jl}^\theta(\gamma_{jl}(\mathbf{x}_{jl}) = s_l). \end{aligned} \quad (33)$$

Thus, under *Assumption 6*, we can obtain the following theorem by employing (33) and similar arguments as those in *Section III* and *Section IV*.

Theorem 4: Under *Assumption 1*, *Assumption 2*, *Assumption 3* and *Assumption 6*, for any given θ , any quantization regions $\{I_{jl}^{(r)}\}$ and any statistical models $\{(\mathcal{X}_{jl}, \mathcal{F}_{jl}, \mathcal{P}_{jl}^\theta)\}$, if the dimension D_θ of the vector parameter θ is greater than $\sum_{j=1}^N \sum_{l=1}^{L_j} R_{jl} - \sum_{j=1}^N L_j$, i.e.,

$$D_\theta > \lambda_{\text{Indep}}(N, \{R_{jl}\}) \triangleq \sum_{j=1}^N \sum_{l=1}^{L_j} R_{jl} - \sum_{j=1}^N L_j, \quad (34)$$

then the FIM for estimating θ is singular. Furthermore, under *Assumption 1*, *Assumption 2*, *Assumption 4* and *Assumption 6*, for any given $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_{jl}, \mathcal{F}_{jl}, \mathcal{P}_{jl}^\theta)\}$, if (34) holds, then the vector parameter space Θ is not identifiable. Moreover, for any open subset $\mathcal{U} \subset \Theta$ in \mathbb{R}^{D_θ} , there are infinitely many vector parameter points in \mathcal{U} which are not identifiable.

Theorem 4 can be justified as we now explain. Under *Assumption 6*, all the partitioned observation subvectors $\{\mathbf{x}_{jl}\}$ are known to be independent. Hence, without any impact, for each j , we can view the j -th sensor as L_j effective ‘‘sensors’’ where the observation vector of the l -th effective ‘‘sensor’’ is \mathbf{x}_{jl} and the l -th effective ‘‘sensor’’ employs the vector quantizer γ_{jl} to quantize its observation vector \mathbf{x}_{jl} . As a result, under *Assumption 6*, the original N -sensor system where the j -th sensor employs the superquantizer Γ_j for each j , is equivalent to a $(\sum_{j=1}^N L_j)$ -sensor system where each sensor just employs a vector quantizer. We use a pair of indices $\{(j, l)\}_{j=1, l=1}^{N, L_j}$ to index the sensors in the $(\sum_{j=1}^N L_j)$ -sensor system, and the

number of quantization levels of the vector quantizer employed at the (j, l) -th sensor is R_{jl} . Therefore, by the equivalence between these two sensor systems and by replacing N by $(\sum_{j=1}^N L_j)$ and replacing $\sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl}$ by $\sum_{j=1}^N \sum_{l=1}^{L_j} R_{jl}$ (the new sum over all sensors of the number of quantization levels at each sensor) in the formula in (4), it follows that

$$\lambda_{\text{Indep}}(N, \{R_{jl}\}) = \sum_{j=1}^N \sum_{l=1}^{L_j} R_{jl} - \sum_{j=1}^N L_j, \quad (35)$$

which justifies *Theorem 4*.

Noting that $R_{jl} \geq 1$ for all j and l , and by employing the fact that for any positive integer L_j , if $a_i \geq 0$ for all $i = 1, 2, \dots, L_j$, then $\prod_{i=1}^{L_j} (1 + a_i) \geq 1 + \sum_{i=1}^{L_j} a_i$, we can obtain

$$\begin{aligned} \lambda(N, \{R_{jl}\}) &= \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N \\ &= \sum_{j=1}^N \left\{ \prod_{l=1}^{L_j} [1 + (R_{jl} - 1)] - 1 \right\} \\ &\geq \sum_{j=1}^N \left\{ 1 + \sum_{l=1}^{L_j} (R_{jl} - 1) - 1 \right\} \\ &= \lambda_{\text{Indep}}(N, \{R_{jl}\}). \end{aligned} \quad (36)$$

Thus, it is seen from (36) that the critical quantity $\lambda_{\text{Indep}}(N, \{R_{jl}\})$ in (34) allows us to guarantee the FIM is singular and the vector parameter space is nonidentifiable at a smaller dimension of θ . We refer to $\lambda_{\text{Indep}}(N, \{R_{jl}\})$ in (34) as the rIDQD for the quantized estimation system under *Assumption 6*.

What’s more, it is seen from (34) that the rIDQD $\lambda_{\text{Indep}}(N, \{R_{jl}\})$ under *Assumption 6* is precisely the number of quantization levels employed by the effective sensor system minus the number of effective sensors. Hence, the rIDQD $\lambda_{\text{Indep}}(N, \{R_{jl}\})$ under *Assumption 6* does not depend on $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_{jl}, \mathcal{F}_{jl}, \mathcal{P}_{jl}^\theta)\}$.

C. Independent Observation Subvectors, Identical Sensor Observation Statistical Models and Identical Vector Quantizers

In this subsection, we consider the following assumption which is stronger than *Assumption 6*.

Assumption 7: All the partitioned observation subvectors $\{\mathbf{x}_{jl}\}$ are known to be independent, and moreover, some partitioned observation subvectors \mathbf{x}_{jl} are known to obey the same statistical model such that $W < \sum_{j=1}^N L_j$ distinct statistical models of the partitioned observation subvectors $\{\mathbf{x}_{jl}\}$ exist.

Collect all the partitioned observation subvector indices that employ the w -th statistical model in the group \mathcal{A}_w . For notational simplicity, we use $(\tilde{\mathcal{X}}_w, \tilde{\mathcal{F}}_w, \tilde{\mathcal{P}}_w^\theta)$ to denote the statistical model for any \mathbf{x}_{jl} with its index contained in \mathcal{A}_w .

Moreover, each group \mathcal{A}_w can be divided into T_w disjoint nonempty subgroups $\{\mathcal{A}_w^{(t)}\}_{t=1}^{T_w}$ of partitioned observation

which yields

$$\mathbf{A} = \frac{1}{\sqrt{D_\theta}} [1, 1, \dots, 1]^T \quad (46)$$

by employing (43). As a result, we know

$$\mathbf{A}^T \mathbf{J}(\boldsymbol{\theta}) \mathbf{A} = \frac{1}{D_\theta} \sum_{l,m} [\mathbf{J}(\boldsymbol{\theta})]_{l,m} \neq 0, \quad (47)$$

which implies that the information regularity is guaranteed.

B. Additional Information on the Prior Distribution of $\boldsymbol{\theta}$

Consider the case where we know that the desired vector parameter $\boldsymbol{\theta}$ is a random vector with some distribution $f(\boldsymbol{\theta})$. The corresponding posterior Fisher information matrix \mathbf{J}_B can be expressed as [35]

$$\begin{aligned} \mathbf{J}_B &\triangleq \mathbb{E}_{\mathbf{u}, \boldsymbol{\theta}} \left\{ \frac{\partial \ln \Pr(\mathbf{u}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln \Pr(\mathbf{u}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \right\} \\ &= \mathbb{E}_{\mathbf{u}, \boldsymbol{\theta}} \left\{ \frac{\partial \ln \Pr(\mathbf{u} | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln \Pr(\mathbf{u} | \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \right\} \\ &\quad + \mathbb{E}_{\boldsymbol{\theta}} \left\{ \frac{\partial \ln f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \right\} \\ &= \mathbb{E}_{\boldsymbol{\theta}} \{ \mathbf{J}(\boldsymbol{\theta}) \} + \mathbb{E}_{\boldsymbol{\theta}} \left\{ \frac{\partial \ln f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \right\}, \quad (48) \end{aligned}$$

where $\mathbf{J}(\boldsymbol{\theta})$ is given by (12). From *Theorem 1*, we know that if $D_\theta > \lambda(N, \{R_{jl}\})$, then $\mathbf{J}(\boldsymbol{\theta})$ is singular. However, if the distribution $f(\boldsymbol{\theta})$ gives rise to

$$\mathbb{E}_{\boldsymbol{\theta}} \left\{ \frac{\partial \ln f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \right\} > 0, \quad (49)$$

then \mathbf{J}_B is always nonsingular. Thus, this additional information $f(\boldsymbol{\theta})$ ensures information regularity, and the inverse of \mathbf{J}_B provides a lower bound on the covariance matrix of the estimation error.

VII. CONCLUSION

In this paper, we investigate the impact of quantization on the estimation capabilities with respect to the information-regularity condition and the identifiability condition. A critical quantity, called IDQD, is introduced, which describes a fundamental limitation of using quantized data. To be specific, under the condition that the dimension of the desired vector parameter is larger than the IDQD, the FIM for estimating the desired vector parameter is singular for any value of the desired vector parameter, any quantization regions, and any statistical models of the observations. Furthermore, it is shown that under the same condition, the vector parameter space is not identifiable, and moreover, there are infinitely many nonidentifiable vector parameter points in the vector parameter space. It is worth mentioning that there is no general equivalence between the quantization induced FIM singularity and the quantization induced nonidentifiability of the vector parameter space. Further, in the quantization induced non-identifiable vector parameter space, every vector parameter

point is nonidentifiable in some cases, while in some other cases, there exist some identifiable vector parameter points. Thus the quantization induced FIM singularity does not necessarily determine the identifiability of the vector parameter point although it does determine the identifiability of the vector parameter space. Moreover, the cardinality of a set of observationally equivalent points in the quantization induced nonidentifiable vector parameter space can be as small as 1 and can also be as large as uncountably infinite. In addition, some commonly assumed specific assumptions on the statistical models of the observations are considered in this paper. It is shown that under these assumptions, a refined IDQD becomes smaller than the standard IDQD, implying the FIM singularity and the nonidentifiability of the vector parameter space can be guaranteed for an even smaller vector parameter dimension.

APPENDIX A PROOF OF THEOREM 1

By employing (12), the rank of $\mathbf{J}(\boldsymbol{\theta})$ is upper bounded by $\text{rank}(\mathbf{J}(\boldsymbol{\theta}))$

$$\begin{aligned} &= \text{rank} \left(\sum_{j=1}^N \sum_{s \in \mathcal{S}_j} \frac{1}{q_j^{(s)}(\boldsymbol{\theta})} \frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left[\frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]^T \right) \\ &\leq \sum_{j=1}^N \text{rank} \left(\sum_{s \in \mathcal{S}_j} \frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left[\frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]^T \right). \quad (50) \end{aligned}$$

Noticing that

$$\sum_{s \in \mathcal{S}_j} q_j^{(s)}(\boldsymbol{\theta}) = 1, \quad \forall j, \quad (51)$$

we can obtain that

$$\sum_{s \in \mathcal{S}_j} \frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{0}, \quad \forall j, \quad (52)$$

and therefore,

$$\begin{aligned} &\text{rank} \left(\sum_{s \in \mathcal{S}_j} \frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left[\frac{\partial q_j^{(s)}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]^T \right) \\ &\leq |\mathcal{S}_j| - 1 \\ &= \prod_{l=1}^{L_j} R_{jl} - 1, \quad \forall j, \quad (53) \end{aligned}$$

where (53) follows from (7).

By employing (50) and (53), we can bound the rank of $\mathbf{J}(\boldsymbol{\theta})$ above by

$$\begin{aligned} \text{rank}(\mathbf{J}(\boldsymbol{\theta})) &\leq \sum_{j=1}^N \left(\prod_{l=1}^{L_j} R_{jl} - 1 \right) \\ &= \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N. \quad (54) \end{aligned}$$

Thus, noting that the size of $\mathbf{J}(\boldsymbol{\theta})$ is D_θ -by- D_θ , if $D_\theta > \lambda(N, \{R_{jl}\}) = \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N$, $\mathbf{J}(\boldsymbol{\theta})$ is singular for any given $\boldsymbol{\theta}$, $\{I_{jl}^{(r)}\}$ and $\{(\mathcal{X}_j, \mathcal{F}_j, \mathcal{P}_j^\theta)\}$.

APPENDIX B
PROOF OF LEMMA 1

First, consider a mapping $\bar{\Psi}$

$$\begin{aligned} \bar{\Psi} : \Theta &\longrightarrow \mathbb{R}^{\sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl}} \\ \theta &\longmapsto \bar{\Psi}(\theta), \end{aligned} \quad (55)$$

where the $(\sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl})$ -dimensional vector $\bar{\Psi}(\theta)$ is defined as

$$\bar{\Psi}(\theta) \triangleq [\bar{\psi}_1(\theta)^T, \bar{\psi}_2(\theta)^T, \dots, \bar{\psi}_N(\theta)^T]^T, \quad (56)$$

and for each j , the $|\mathcal{S}_j|$ -dimensional vector $\bar{\psi}_j(\theta)$ is defined as

$$\bar{\psi}_j(\theta) \triangleq \left[q_j^{(s_1^{(j)})}(\theta), q_j^{(s_2^{(j)})}(\theta), \dots, q_j^{(s_{|\mathcal{S}_j|}^{(j)})}(\theta) \right]^T, \quad (57)$$

$q_j^{(s)}(\theta)$ is defined in (10), and $s_i^{(j)}$ is defined in (6) for all $i = 1, 2, \dots, |\mathcal{S}_j|$.

We first show that the mapping $\varphi_{\mathbf{u}}$ in (15) is not injective if and only if the mapping $\bar{\Psi}$ in (55) is not injective, and hence, the injectivity of the mapping $\bar{\Psi}$ in (55) is the same as that of the mapping $\varphi_{\mathbf{u}}$ in (15).

Suppose the mapping $\bar{\Psi}$ in (55) is not injective. Then, there exist two distinct $\theta_1, \theta_2 \in \Theta$ such that $\bar{\Psi}(\theta_1) = \bar{\Psi}(\theta_2)$.

Noting that

$$\Pr(\mathbf{u}|\theta) = \prod_{j=1}^N \prod_{s \in \mathcal{S}_j} [q_j^{(s)}(\theta)]^{\mathbb{1}\{\mathbf{u}_j=s\}}, \quad (58)$$

it is clear that $\Pr(\mathbf{u}|\theta_1) = \Pr(\mathbf{u}|\theta_2)$ for all $\mathbf{u} \in \mathcal{A}$, if there exist two distinct $\theta_1, \theta_2 \in \Theta$ such that $\bar{\Psi}(\theta_1) = \bar{\Psi}(\theta_2)$. Thus, the mapping $\varphi_{\mathbf{u}}$ in (15) is not injective.

On the other hand, suppose the mapping $\varphi_{\mathbf{u}}$ in (15) is not injective. Then, there exist two distinct $\theta_1, \theta_2 \in \Theta$ such that

$$\Pr(\mathbf{u}|\theta_1) = \Pr(\mathbf{u}|\theta_2), \quad \forall \mathbf{u} \in \mathcal{A}, \quad (59)$$

where $\Pr(\mathbf{u}|\theta)$ is defined in (58).

Note that for each i , we have

$$\sum_{s \in \mathcal{S}_i} q_i^{(s)}(\theta_1) = 1. \quad (60)$$

Hence, for each i , there exists some $\mathbf{h}_i \in \mathcal{S}_i$ such that

$$q_i^{(\mathbf{h}_i)}(\theta_1) \neq 0. \quad (61)$$

For any given j and any given $\mathbf{s} \in \mathcal{S}_j$, we are going to show that $q_j^{(\mathbf{s})}(\theta_1) = q_j^{(\mathbf{s})}(\theta_2)$.

Consider a realization of \mathbf{u} that

$$\mathbf{u} = [\mathbf{h}_1^T, \mathbf{h}_2^T, \dots, \mathbf{h}_{j-1}^T, \mathbf{s}^T, \mathbf{h}_{j+1}^T, \dots, \mathbf{h}_N^T]^T. \quad (62)$$

By employing (58) and (59), we have

$$q_j^{(\mathbf{s})}(\theta_1) \prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_1) = q_j^{(\mathbf{s})}(\theta_2) \prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_2), \quad (63)$$

By (61), we know that $\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_1) \neq 0$, and therefore, from (63), we can obtain

$$q_j^{(\mathbf{s})}(\theta_1) = q_j^{(\mathbf{s})}(\theta_2) \frac{\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_2)}{\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_1)}. \quad (64)$$

Furthermore, by noting that $\sum_{s \in \mathcal{S}_j} q_j^{(\mathbf{s})}(\theta_2) = 1$ and employing (60) and (64), we can obtain

$$\begin{aligned} \frac{\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_2)}{\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_1)} &= \frac{\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_2)}{\prod_{i \neq j} q_i^{(\mathbf{h}_i)}(\theta_1)} \sum_{s \in \mathcal{S}_j} q_j^{(\mathbf{s})}(\theta_2) \\ &= \sum_{s \in \mathcal{S}_j} q_j^{(\mathbf{s})}(\theta_1) \\ &= 1, \end{aligned} \quad (65)$$

which implies

$$q_j^{(\mathbf{s})}(\theta_1) = q_j^{(\mathbf{s})}(\theta_2). \quad (66)$$

Therefore, by the definitions of $\bar{\Psi}(\theta)$ and $\bar{\psi}_j(\theta)$ in (56) and (57), we know that

$$\bar{\Psi}(\theta_1) = \bar{\Psi}(\theta_2), \quad (67)$$

and hence, the mapping $\bar{\Psi}$ in (55) is not injective. As a result, we know that the mapping $\varphi_{\mathbf{u}}$ in (15) is not injective if and only if the mapping $\bar{\Psi}$ in (55) is not injective, which implies that the injectivity of the mapping $\bar{\Psi}$ in (55) is the same as that of the mapping $\varphi_{\mathbf{u}}$ in (15).

Furthermore, for any given j , by the definitions of $\psi_j(\theta)$ and $\bar{\psi}_j(\theta)$ in (17) and (57) respectively, and noticing that $\sum_{s \in \mathcal{S}_j} q_j^{(\mathbf{s})}(\theta) = 1$ for all j , we can express $\bar{\psi}_j(\theta)$ as

$$\bar{\psi}_j(\theta) = [\psi_j(\theta)^T, 1 - \mathbf{1}^T \psi_j(\theta)]^T. \quad (68)$$

It is clear that if $\bar{\psi}_j(\theta)$ is not injective, then $\psi_j(\theta)$ is not injective. On the other hand, if $\psi_j(\theta)$ is not injective, then there exist two distinct $\theta_1, \theta_2 \in \Theta$ such that $\psi_j(\theta_1) = \psi_j(\theta_2)$, and hence $1 - \mathbf{1}^T \psi_j(\theta_1) = 1 - \mathbf{1}^T \psi_j(\theta_2)$. Consequently, we have $\bar{\psi}_j(\theta_1) = \bar{\psi}_j(\theta_2)$ by (68), which implies that $\bar{\psi}_j(\theta)$ is not injective. Therefore, the injectivity of $\bar{\psi}_j(\theta)$ is the same as that of $\psi_j(\theta)$ for all j , which implies that $\bar{\Psi}(\theta)$ is injective if and only if $\Psi(\theta)$ is injective. Since we have proven that the mapping $\varphi_{\mathbf{u}}$ in (15) is not injective if and only if the mapping $\bar{\Psi}$ in (55) is not injective, we know that the mapping $\varphi_{\mathbf{u}}$ in (15) is injective if and only if the mapping Ψ in (18) is injective.

In order to show that the dimension of the vector $\Psi(\theta)$ in (16) is strictly smaller than that of $\varphi_{\mathbf{u}}(\theta)$ in (15) for any given N and $\{R_{jl}\}$, it suffices to show that

$$D_{\mathbf{u}} = \prod_{j=1}^N \prod_{l=1}^{L_j} R_{jl} > \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N, \quad (69)$$

for any given N and $\{R_{jl}\}$.

Since N is the number of sensors and R_{jl} denotes the number of quantization levels of the quantizer γ_{jl} for each j and l , we know that $N \geq 1$ and $R_{jl} \geq 1$ for all j and all l . Hence, we can obtain that

$$\prod_{l=1}^{L_j} R_{jl} \geq 1, \quad \forall j. \quad (70)$$

Furthermore, notice that if $x \geq 0$ and $y \geq 0$, then we have the following inequality

$$(1+x)(1+y) = 1+x+y+xy \geq 1+x+y. \quad (71)$$

Therefore, by induction, we can obtain that if $x_i \geq 0$ for all $i = 1, 2, \dots, N$, then

$$\prod_{i=1}^N (1+x_i) \geq 1 + \sum_{i=1}^N x_i. \quad (72)$$

By employing (70) and (72), we can obtain

$$\begin{aligned} \prod_{j=1}^N \prod_{l=1}^{L_j} R_{jl} &= \prod_{j=1}^N \left[\left(\prod_{l=1}^{L_j} R_{jl} - 1 \right) + 1 \right] \\ &\geq 1 + \sum_{j=1}^N \left(\prod_{l=1}^{L_j} R_{jl} - 1 \right) \\ &> \sum_{j=1}^N \prod_{l=1}^{L_j} R_{jl} - N. \end{aligned} \quad (73)$$

This completes the proof.

APPENDIX C PROOF OF THEOREM 2

Under *Assumption 2*, the interior of Θ is not empty. Thus, there exists a subset \mathcal{U} of Θ which is open in \mathbb{R}^{D_θ} .

Define a D_θ -dimensional vector $\hat{\Psi}(\theta)$

$$\hat{\Psi}(\theta) \triangleq \left[\Psi(\theta)^T, \mathbf{0}^T \right]^T, \quad (74)$$

where $\Psi(\theta)$ is defined in (16), and the dimension of the all-zero vector in (74) is $D_\theta - \lambda(N, \{R_{jl}\})$.

Under *Assumption 4*, for all j and all \mathbf{s} , $q_j^{(s)}(\theta)$ is a continuous function with respect to θ . Hence, by (16), (17) and (74), we know that the restriction $\hat{\Psi}|_{\mathcal{U}}$ of the mapping $\hat{\Psi}$ to \mathcal{U}

$$\begin{aligned} \hat{\Psi}|_{\mathcal{U}} : \mathcal{U} &\longrightarrow \mathbb{R}^{D_\theta} \\ \theta &\longmapsto \hat{\Psi}(\theta) \end{aligned} \quad (75)$$

is continuous with respect to θ .

It is clear that \mathcal{U} is an open set in \mathbb{R}^{D_θ} , but by the definition of $\hat{\Psi}(\theta)$ in (74), $\hat{\Psi}|_{\mathcal{U}}(\mathcal{U})$ is not open in \mathbb{R}^{D_θ} . Thus, by *Lemma 2*, the mapping $\hat{\Psi}|_{\mathcal{U}}$ is not injective. As a result, by *Lemma 1*, the vector parameter space Θ is not identifiable.

What's more, according to *Definition 5* and *Definition 6*, the nonidentifiability of the vector parameter space implies that we can find two distinct nonidentifiable points $\theta_1 \in \mathcal{U} \subset \Theta$ and $\theta_2 \in \mathcal{U} \subset \Theta$ which are observationally equivalent to each other. Note that the set $\mathcal{U}^* \triangleq \mathcal{U} \setminus \{\theta_1, \theta_2\} \subset \mathcal{U}$ is

also an open subset of Θ in \mathbb{R}^{D_θ} . Therefore, by the same argument, the restriction $\hat{\Psi}|_{\mathcal{U}^*}$ of the mapping $\hat{\Psi}$ to \mathcal{U}^* is also not injective, and hence, there also exist two distinct points $\theta_1^* \in \mathcal{U}^* \subset \mathcal{U} \subset \Theta$ and $\theta_2^* \in \mathcal{U}^* \subset \mathcal{U} \subset \Theta$ which are not identifiable. Thus, by induction, there are infinitely many vector parameter points in $\mathcal{U} \subset \Theta$ which are not identifiable. This completes the proof.

APPENDIX D PROOF OF PROPOSITION 1

If the observation x is not quantized, suppose there exist two distinct vector parameter points $\theta_1 = [\alpha_1, \beta_1]^T$ and $\theta_2 = [\alpha_2, \beta_2]^T$ which give rise to $f(x|\theta_1) = f(x|\theta_2)$ for all $x \in \mathbb{R}$, then we can obtain

$$\begin{aligned} \ln \frac{f(x|\theta_1)}{f(x|\theta_2)} &= \frac{\beta_1 - \beta_2}{2\beta_1\beta_2} x^2 + \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \right) x + \frac{1}{2} \left(\frac{\alpha_2^2}{\beta_2} - \frac{\alpha_1^2}{\beta_1} + \ln \frac{\beta_2}{\beta_1} \right) \\ &= 0, \quad \forall x \in \mathbb{R}. \end{aligned} \quad (76)$$

The fundamental theorem of algebra demonstrates that (76) holds if and only if

$$\begin{cases} \beta_1 - \beta_2 = 0 \\ \frac{2\beta_1\beta_2}{\alpha_1} - \frac{\alpha_2}{\beta_2} = 0 \\ \frac{\alpha_2^2}{\beta_2} - \frac{\alpha_1^2}{\beta_1} + \ln \frac{\beta_2}{\beta_1} = 0, \end{cases} \quad (77)$$

which implies $\theta_1 = [\alpha_1, \beta_1^2]^T = [\alpha_2, \beta_2^2]^T = \theta_2$, and hence, we reach a contradiction. Thus, without quantization, every vector parameter point $\theta \in \Theta$ is identifiable.

Now, consider the case where the binary quantizer in (22) is employed at the sensor.

For any given vector parameter point $\theta_0 \triangleq [\alpha_0, \beta_0]^T \in \Theta$, let $\theta_\rho \triangleq [\alpha_\rho, \rho\beta_0]^T$ denote a vector parameter point in Θ for some $\rho \in (0, 1)$ and some α_ρ . We will show that for any $\rho \in (0, 1)$, there exists an α_ρ such that θ_ρ is observationally equivalent to θ_0 .

Define a function $g(\alpha, \beta)$ as

$$g(\alpha, \beta) \triangleq \Pr(u = 1|\theta) = \int_a^b \frac{1}{\sqrt{2\pi\beta}} e^{-\frac{(x-\alpha)^2}{2\beta}} dx. \quad (78)$$

Since $\Pr(u = 2|\theta) = 1 - \Pr(u = 1|\theta) = 1 - g(\alpha, \beta)$, it is clear that if $g(\alpha_\rho, \rho\beta_0) = g(\alpha_0, \beta_0)$, then by *Definition 4*, θ_ρ are observationally equivalent to θ_0 , and hence, θ_0 is not identifiable.

Since $I^{(1)}$ and $I^{(2)}$ are both nonempty sets, a and b cannot be both unbounded. Without loss of generality, we assume $-\infty \leq a < b < \infty$. The case where $-\infty < a < b \leq \infty$ can be proved in a similar way. By (78) and noting $b < \infty$, we can obtain that for any given α_0 and β_0 ,

$$\begin{aligned} \lim_{a \rightarrow \infty} g(\alpha, \rho\beta_0) &= \lim_{a \rightarrow \infty} \int_{\frac{a-\alpha}{\sqrt{\rho\beta}}}^{\frac{b-\alpha}{\sqrt{\rho\beta}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= 0 < g(\alpha_0, \beta_0). \end{aligned} \quad (79)$$

In the following, we will consider the case where $a = -\infty$ and the case where $a > -\infty$ respectively. We will show that for both cases, there exists some $\theta_\rho = [\alpha_\rho, \rho\beta_0]^T$ such that $g(\alpha_\rho, \rho\beta_0) = g(\alpha_0, \beta_0)$.

Suppose $a = -\infty$, then for any given α_0 and β_0 ,

$$\lim_{\alpha \rightarrow -\infty} g(\alpha, \rho\beta_0) = \lim_{\alpha \rightarrow -\infty} \int_{-\infty}^{\frac{b-\alpha}{\sqrt{\rho\beta}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 > g(\alpha_0, \beta_0), \quad (80)$$

Therefore, by (79) and (80), and noticing that $g(\alpha, \beta)$ is a continuous function for all $\theta = [\alpha, \beta^2]^T \in \Theta$, we know that there exists an $\alpha_\rho \in (-\infty, \infty)$ such that

$$g(\alpha_\rho, \rho\beta_0) = g(\alpha_0, \beta_0) \quad (81)$$

for any given ρ by employing *Intermediate Value Theorem*.

Suppose $a > -\infty$. Noticing that for any given β , the equation

$$\frac{\partial}{\partial \alpha} g(\alpha, \beta) = \frac{1}{\sqrt{2\pi\beta}} \left[e^{-\frac{(a-\alpha)^2}{2\beta}} - e^{-\frac{(b-\alpha)^2}{2\beta}} \right] = 0 \quad (82)$$

only admits one solution

$$\alpha = \frac{1}{2}(a+b) \in (-\infty, \infty). \quad (83)$$

Moreover, since

$$\frac{\partial^2}{\partial \alpha^2} g(\alpha, \beta) \Big|_{\alpha=\frac{1}{2}(a+b)} = -\frac{b-a}{\sqrt{2\pi\beta^3}} e^{-\frac{(b-a)^2}{8\beta}} < 0, \quad (84)$$

we know that $\alpha = \frac{1}{2}(a+b)$ maximizes the function $g(\alpha, \beta)$ for any given β . Hence,

$$g\left(\frac{1}{2}(a+b), \beta_0\right) \geq g(\alpha_0, \beta_0). \quad (85)$$

Furthermore, note that

$$\frac{\partial}{\partial \beta} g\left(\frac{1}{2}(a+b), \beta\right) = -\frac{b-a}{2\sqrt{2\pi\beta^3}} e^{-\frac{(b-a)^2}{8\beta}} < 0, \quad (86)$$

which yields that $g(\frac{1}{2}(a+b), \beta)$ is a strictly decreasing function with respect to β . As a result, by employing (85) and (86), we can obtain

$$g\left(\frac{1}{2}(a+b), \rho\beta_0\right) > g\left(\frac{1}{2}(a+b), \beta_0\right) \geq g(\alpha_0, \beta_0), \quad (87)$$

since $\rho \in (0, 1)$. Thus, by (79) and (87), and by employing *Intermediate Value Theorem*, we know that there exists an $\alpha_\rho \in (\frac{1}{2}(a+b), \infty)$ such that

$$g(\alpha_\rho, \rho\beta_0) = g(\alpha_0, \beta_0), \quad (88)$$

since $g(\alpha, \beta)$ is continuous.

By (81) and (88), we know that no matter what a and b are, for any given $\theta_0 = [\alpha_0, \beta_0]^T \in \Theta$ and for any $\rho \in (0, 1)$, there exists some $\theta_\rho = [\alpha_\rho, \rho\beta_0]^T$ such that $g(\alpha_\rho, \rho\beta_0) = g(\alpha_0, \beta_0)$. Hence, every vector parameter point in Θ is not identifiable. Moreover, since the set $(0, 1)$ is an uncountable set, for any vector parameter point $\theta_0 \in \Theta$, the set of vector parameter points which are observationally equivalent to θ_0 is uncountable. This completes the proof.

APPENDIX E

PROOF OF PROPOSITION 2

Define a function $g(\theta)$ as

$$\begin{aligned} g(\theta) &\triangleq \ln \Pr(u = 1 | \theta) \\ &= \ln \int_{a_1}^{b_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-\theta_1)^2}{2}} dx_1 \\ &\quad + \ln \int_{a_2}^{b_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_2-\theta_2)^2}{2}} dx_2. \end{aligned} \quad (89)$$

First, we claim that if a function $f(x, y)$ is a log-concave function of two arguments $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, then for any convex set $C \subseteq \mathbb{R}^n$, the function $\int_C f(x, y) dy$ is also log-concave. The proof of this claim can be found in Theorem 6 of [36]. Note that (a_1, b_1) and (a_2, b_2) are convex sets, and $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is a log-concave function. Hence, we know that $g(\theta)$ is concave. By employing (89), we can obtain

$$\frac{d}{d\theta} g(\theta) = \begin{bmatrix} \frac{e^{-\frac{1}{2}(b_1-\theta_1)^2} - e^{-\frac{1}{2}(a_1-\theta_1)^2}}{\int_{a_1}^{b_1} \frac{1}{2\pi} e^{-\frac{(x_1-\theta_1)^2}{2}} dx_1} \\ \frac{e^{-\frac{1}{2}(b_2-\theta_2)^2} - e^{-\frac{1}{2}(a_2-\theta_2)^2}}{\int_{a_2}^{b_2} \frac{1}{2\pi} e^{-\frac{(x_2-\theta_2)^2}{2}} dx_2} \end{bmatrix}, \quad (90)$$

and moreover, by setting $\frac{d}{d\theta} g(\theta) = \mathbf{0}$, we obtain only one solution

$$\theta^* = \left[\frac{b_1 - a_1}{2}, \frac{b_2 - a_2}{2} \right]^T. \quad (91)$$

Thus, $g(\theta)$ achieves the unique globally maximum at θ^* , since $g(\theta)$ is concave. Furthermore, since $\Pr(u = 2 | \theta) = 1 - \Pr(u = 1 | \theta)$, it is clear that if there exists a vector parameter point θ such that $g(\theta) \neq g(\theta')$ for all $\theta' \in \Theta \setminus \{\theta\}$, then by *Definition 5*, θ is identifiable. As a result, θ^* is an identifiable vector parameter point in Θ . This completes the proof.

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