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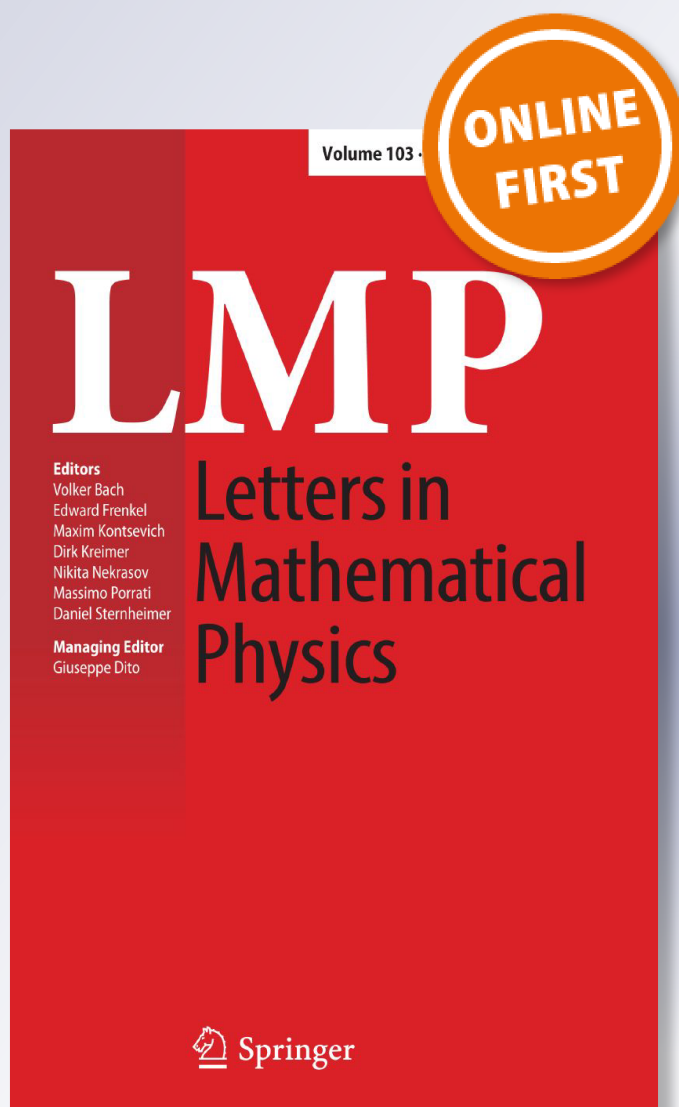
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# Rigidity in vacuum under conformal symmetry

Gregory J. Galloway<sup>1</sup> · Carlos Vega<sup>2</sup>

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**Abstract** Motivated in part by Eardley et al. (Commun Math Phys 106(1):137–158, 1986), in this note we obtain a rigidity result for globally hyperbolic vacuum spacetimes in arbitrary dimension that admit a timelike conformal Killing vector field. Specifically, we show that if  $M$  is a Ricci flat, timelike geodesically complete spacetime with compact Cauchy surfaces that admits a timelike conformal Killing field  $X$ , then  $M$  must split as a metric product, and  $X$  must be Killing. This gives a partial proof of the Bartnik splitting conjecture in the vacuum setting.

**Keywords** Lorentzian rigidity · Vacuum equations · Conformal symmetry

**Mathematics Subject Classification** 53C50 · 83C75

## 1 Introduction

The classical Hawking–Penrose singularity theorems form a cornerstone in the global theory of spacetime geometry and general relativity. These theorems guarantee the existence of incomplete causal geodesics, (i.e., ‘singularities’), in large, generic classes of spacetimes satisfying natural energy conditions.

The singularity theorems can be viewed as Lorentzian analogs to Riemannian Ricci comparison theorems like Myers’ theorem and rely on *strict* curvature conditions. In the early 1980s, S.-T. Yau put forth the question of the *rigidity* of the singularity theorems and posed a Lorentzian analog to the Cheeger–Gromoll splitting theorem in

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his famous problem section [18], in 1982. This was settled in a series of papers by the end of the decade [5, 8, 15], with the basic version of the result (due to Eschenburg) as follows:

**Theorem 1.1** (Lorentzian splitting theorem) *Let  $M$  be a globally hyperbolic, time-like geodesically complete spacetime, satisfying the timelike convergence condition,  $\text{Ric}(X, X) \geq 0$ , for all timelike  $X$ . If  $M$  admits a timelike line (i.e., an inextendible globally maximizing timelike geodesic), then  $M$  splits as an isometric product*

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h) \quad (1.1)$$

where  $\Sigma^n$  is a smooth, geodesically complete, spacelike (Cauchy) hypersurface, with induced metric  $h$ .

For basic background in Lorentzian geometry and causal theory used here and below, we refer the reader to the references [14, 16].

Despite the resolution of Theorem 1.1, however, the result did not ultimately settle the original motivating rigidity question. A concrete formulation of this was posed by Bartnik in 1988 as follows:

**Conjecture 1.2** (Bartnik splitting conjecture) *Let  $M$  be a spacetime with compact Cauchy surfaces, which satisfies  $\text{Ric}(X, X) \geq 0$  for all timelike  $X$ . If  $M$  is timelike geodesically complete, then  $M$  splits as in (1.1), (with  $\Sigma$  compact).*

Thus, according to the conjecture, only under quite exceptional circumstances can  $M$  fail to be singular, i.e., timelike geodesically incomplete. The conjecture has been established under various auxiliary conditions; see, for example, [1, 6, 7, 9–11]. It was proven under the stronger sectional curvature condition in [4], using [13].

Conjecture 1.2 is most simply illustrated by the special case of a warped product  $(M^{n+1}, g) = (I \times \Sigma^n, -d\tau^2 + \phi^2(\tau)\tilde{h})$ , where  $(\Sigma^n, \tilde{h})$  is a compact Riemannian manifold,  $I \subset \mathbb{R}$  an open interval, and  $\phi : I \rightarrow (0, \infty)$  a smooth, positive function. For such a spacetime, the timelike convergence condition forces  $\phi'' \leq 0$ . But then if  $M$  is timelike complete, we must have  $I = \mathbb{R}$ , which forces  $\phi = c$  to be constant. (Then,  $h := c^2\tilde{h}$  is the induced metric on  $\Sigma$ , and hence,  $M$  splits as above.)

While the warped product case is trivial, one may ask what happens when this is ‘weakened’ to the assumption of a timelike conformal symmetry, i.e., the existence of a timelike conformal Killing field. By the latter, we mean a timelike vector field  $X$  such that  $\mathcal{L}_X g = 2\sigma g$ , where  $\mathcal{L}$  is the Lie derivative and  $\sigma : M \rightarrow \mathbb{R}$  is smooth. For example, a warped product as above has timelike conformal Killing vector field  $X = \phi(\tau)\partial_\tau$ , with conformal factor  $\sigma = \phi'(\tau)$ . In [3], various results are established showing the ‘rigidity’ imposed by the existence of conformal symmetries on solutions  $(M^4, g)$  of the Einstein equations. Theorem 3 in [3], for example, shows that a vacuum solution with a proper conformal symmetry must be one of a few special types. The proof makes special use of the dimension  $3 + 1$ .

Interestingly, Conjecture 1.2 remains open even in the vacuum setting,  $\text{Ric} \equiv 0$ . Indeed, we are not aware of any prior results in this direction. The main result established here is the following:



**Theorem 1.3** *Let  $n \geq 2$  and suppose that  $(M^{n+1}, g)$  is a Ricci flat, timelike geodesically complete spacetime, with compact Cauchy surfaces. If  $M$  admits a timelike conformal Killing field  $X$ , then  $M$  splits isometrically as*

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h) \quad (1.2)$$

*with the (Riemannian) fiber  $(\Sigma^n, h)$  compact and Ricci flat, and  $X$  is in fact Killing.*

## 2 Preliminary results

Recall that a smooth vector field  $X$  on a semi-Riemannian manifold  $(M, g)$  is *Killing* if  $g$  is invariant under the flow of  $X$ , i.e., if  $\mathcal{L}_X g = 0$ , where  $\mathcal{L}$  is the Lie derivative. More generally, by a *conformal Killing field* on  $(M, g)$ , we mean a smooth vector field  $X$ , such that  $\mathcal{L}_X g = 2\sigma g$ , for some smooth function  $\sigma : M \rightarrow \mathbb{R}$ . In the special case that  $\sigma$  is a constant,  $X$  is called a *homothetic Killing field*.

We begin with the following standard observation. (The notation is suggestive for our applications below, but note that we are not assuming that  $X$  is timelike or that  $M$  is Lorentzian.)

**Lemma 2.1** *Let  $M = (M^{n+1}, g)$  be a semi-Riemannian manifold, and suppose that  $X$  is a conformal Killing field, with  $\mathcal{L}_X g = 2\sigma g$ . Then, in any local coordinates  $\{t = x^0, x^1, \dots, x^n\}$  with  $X = \partial_t$ , we have:*

$$g(t, x^1, \dots, x^n) = e^{2f(t, x^1, \dots, x^n)} \sum_{i,j=0}^n G_{ij}(x^1, \dots, x^n) dx^i \otimes dx^j$$

where  $f(t, x^1, \dots, x^n) := \int_0^t \sigma(s, x^1, \dots, x^n) ds$ . In particular, note that  $\partial_t f = \sigma$ .

*Proof* Because  $\mathcal{L}_X(\partial_i) = [\partial_t, \partial_i] = 0$ , we have  $2\sigma g_{ij} = (2\sigma g)(\partial_i, \partial_j) = (\mathcal{L}_X g)(\partial_i, \partial_j) = X[g(\partial_i, \partial_j)] = Xg_{ij} = \partial_t g_{ij}$ . Hence, for any indices  $i, j \in \{0, 1, 2, \dots, n\}$ , we have  $\partial_t g_{ij} = 2\sigma g_{ij}$ . Both sides are functions of  $(t = x^0, x^1, \dots, x^n)$ , but holding  $x^i$  constant for all  $i \geq 1$ , we have a first-order linear equation in the single variable  $t$ . Using the integrating factor  $\mu = e^{-2f}$  gives  $g_{ij}(t, x^1, \dots, x^n) = e^{2f(t, x^1, \dots, x^n)} G_{ij}(x^1, \dots, x^n)$ .  $\square$

We note that for a warped product spacetime metric  $g = -d\tau^2 + \phi^2(\tau)\tilde{h}$ , the above result holds globally. For example, letting  $t := \int_c^\tau 1/\phi(s) ds$ , then  $dt = d\tau/\phi(\tau)$ , and

$$g = -d\tau^2 + \phi^2(\tau)\tilde{h} = \phi^2(\tau(t))(-dt^2 + \tilde{h}) = e^{2f(t)}(-dt^2 + \tilde{h})$$

Indeed, this is precisely the form of the metric induced as in Lemma 2.1 by the conformal Killing field  $X = \phi(\tau)\partial_\tau = \partial_t$ , with  $df/dt = d\phi/d\tau = \sigma$ .

We shall make use of the following:

**Lemma 2.2** *Let  $M$  be a semi-Riemannian manifold, and let  $X$  be a conformal Killing field, with  $\mathcal{L}_X g = 2\sigma g$ . Then, we have:*

- (1)  $g(\nabla_Y X, Y) = \sigma g(Y, Y)$ , for all smooth vector fields  $Y$ .
- (2) Let  $\gamma = \gamma(s)$  be any affinely parameterized geodesic, and set  $C := g(\gamma'(s), \gamma'(s))$ . Then along the geodesic  $\gamma = \gamma(s)$ , we have:

$$\frac{d}{ds} g(X, \gamma') = \sigma(\gamma(s))C \quad (2.1)$$

*Proof* (1) follows from the standard formula:

$$(\mathcal{L}_X g)(V, W) = g(\nabla_V X, W) + g(\nabla_W X, V)$$

To prove (2), note that for any curve  $\gamma$ , we have:

$$\gamma' g(X, \gamma') = g(\nabla_{\gamma'} X, \gamma') + g(X, \nabla_{\gamma'} \gamma')$$

If  $\gamma$  is a geodesic, the last term vanishes, and the result follows from (1).  $\square$

The following lemma is the key analytic result needed to prove Theorem 1.3.

**Lemma 2.3** *For  $n \geq 2$ , suppose that  $(M^{n+1}, g)$  is a semi-Riemannian manifold, with  $\text{Ric}_g = \lambda g$ , for some real number  $\lambda \in \mathbb{R}$ , and suppose that  $X$  is a nowhere vanishing conformal Killing field, with  $\mathcal{L}_X g = 2\sigma g$ . Then, with  $\Delta_g \sigma = \text{tr}(\text{Hess}_g(\sigma))$ , we have:*

$$\text{Hess}_g(\sigma) = -\left(\frac{\Delta_g \sigma + 2\lambda \sigma}{n-1}\right)g \quad (2.2)$$

which after tracing gives:

$$\Delta_g \sigma = -\lambda \left(\frac{n+1}{n}\right)\sigma \quad (2.3)$$

*Proof* For the convenience of the reader, we provide an outline of the proof, which is a lengthy computation. (See also the proof of Theorem 3 in [3], which treats  $\lambda = 0$ , citing [17] for the relevant formula in this case.)

Fix local coordinates  $\{t = x^0, x^1, \dots, x^n\}$ , and  $f$  and  $G$ , as in Lemma 2.1, with  $X = \partial_t$ . Hence, in the neighborhood  $U$  covered by the chart,  $g$  is conformal to a metric whose components are independent of  $t$ , that is, on  $U$  we have  $g = e^{2f} G$ , with  $G = G(x^1, \dots, x^n)$ . Using the formula for Ricci under conformal change in [2], we have:

$$\text{Ric}_g = \text{Ric}_G - (n-1) \left( H_G^f - df \otimes df \right) - \left( \Delta_G f + (n-1) \|df\|_G^2 \right) G \quad (2.4)$$

where  $\|df\|_G^2 = G(\nabla_G f, \nabla_G f)$ ,  $H_G^f = \text{Hess}_G(f)$  and  $\Delta_G f = \text{tr}(H_G^f)$ . (Note: the sign convention in [2] is  $\Delta_G f := -\text{tr}(H_G^f)$ .) Applying the Einstein condition, we obtain:

$$\left(\frac{\lambda}{n-1}\right)g = \frac{\text{Ric}_G}{n-1} - H_G^f + df \otimes df - \|df\|_G^2 G - \left(\frac{\Delta_G f}{n-1}\right)G. \quad (2.5)$$

We now take the Lie derivative of (2.5) with respect to  $X = \partial_t$ . First note that  $\mathcal{L}_X(\text{Ric}_G) = \mathcal{L}_{\partial_t}(\text{Ric}_G)$  vanishes, since the coefficients  $(\text{Ric}_G)_{ij}$ , which depend only on  $G_{ij}$  and its derivatives, are independent of  $t$ . Thus, taking the Lie derivative of (2.5) gives:

$$\left(\frac{2\lambda\sigma}{n-1}\right)g = -\mathcal{L}_{\partial_t}H_G^f + \mathcal{L}_{\partial_t}(df \otimes df) - \mathcal{L}_{\partial_t}\left(\|df\|_G^2 G\right) - \mathcal{L}_{\partial_t}\left(\frac{\Delta_G f}{n-1}G\right) \quad (2.6)$$

One may now proceed to compute the four Lie derivatives in (2.6), using, where appropriate, the fact that the  $G_{ij}$ 's, and quantities defined in terms of the  $G_{ij}$ 's, have vanishing  $t$ -derivative, and  $\partial_t f = \sigma$ . One obtains,

$$\begin{aligned} \mathcal{L}_{\partial_t}H_G^f &= H_G^\sigma, \quad \mathcal{L}_{\partial_t}[(\Delta_G f)G] = (\Delta_G \sigma)G \\ \mathcal{L}_{\partial_t}(df \otimes df) &= d\sigma \otimes df + df \otimes d\sigma, \quad \partial_t\|df\|_G^2 = 2G(\nabla_G \sigma, \nabla_G f). \end{aligned}$$

Substituting these into (2.6) gives:

$$\left(\frac{2\lambda\sigma}{n-1}\right)g = -H_G^\sigma + d\sigma \otimes df + df \otimes d\sigma - 2G(\nabla_G \sigma, \nabla_G f)G - \left(\frac{\Delta_G \sigma}{n-1}\right)G \quad (2.7)$$

We now translate all the  $G$ -terms in (2.7) back to the metric  $g$ . First note that:

$$G(\nabla_G \sigma, \nabla_G f)G = g(\nabla_g \sigma, \nabla_g f)g$$

By standard formulas, we have for the Hessian and Laplacian,

$$\begin{aligned} H_G^\sigma &= H_g^\sigma + d\sigma \otimes df + df \otimes d\sigma - g(\nabla_g \sigma, \nabla_g f)g \\ \Delta_G \sigma &= e^{2f}\left(\Delta_g \sigma - (n-1)g(\nabla_g \sigma, \nabla_g f)\right) \end{aligned}$$

By plugging these pieces into (2.7), after some simple manipulations we arrive at (2.2) and (2.3).  $\square$

The proof of Theorem 1.3 eventually reduces to the static case. We will then make use of the following curve lifting result.

**Lemma 2.4** *Let  $M$  be a globally hyperbolic spacetime, with smooth spacelike Cauchy surface  $S$ . If  $M$  admits a complete timelike Killing field  $X$ , then every spatial curve in  $S$  lifts (along the integral curves of  $X$ ) to a timelike curve in  $M$ .*

*Proof* Because  $X$  is complete, we have a diffeomorphic splitting  $M \approx \mathbb{R} \times S$ , given by flowing along the integral curves of  $X$ . By reparameterizing if necessary, we may suppose that each integral curve  $\gamma(t)$  of  $X$  meets  $S$  at  $t = 0$ .

We will now prepare a convenient collection of coordinate patches on  $M^{n+1}$ . First note that, choosing any local coordinates  $\{x^1, \dots, x^n\}$  on  $S$ , then  $\{t = x^0, x^1, \dots, x^n\}$  give local coordinates on  $M$ , and by Lemma 2.1 we have:

$$g(t, x^1, \dots, x^n) = \sum_{i,j=0}^n G_{ij}(x^1, \dots, x^n) dx^i \otimes dx^j \quad (2.8)$$

For  $p \in S$ , let  $U_p$  be a neighborhood of  $p$  in  $S$ , with local coordinates  $\{x^1, \dots, x^n\}$ . Let  $V_p$  be a smaller neighborhood, with  $p \in V_p \subset\subset U_p$ . Hence,  $\{t = x^0, x^1, \dots, x^n\}$  give local coordinates on  $\mathbb{R} \times V_p$ , on which  $g$  has coordinate representation (2.8). Because  $g$  is continuous, and  $V_p \subset\subset U_p$ , and because the component functions  $G_{ij}$  are independent of  $t$ , the  $G_{ij}$ 's are bounded on  $\mathbb{R} \times V_p$ . Moreover, because  $G_{00}$  is negative on  $\mathbb{R} \times V_p$ , we have

$$m_p := \min\{-G_{00}(z) : z \in \mathbb{R} \times V_p\} > 0 \quad (2.9)$$

Now fix any spatial curve  $\beta : [0, \ell] \rightarrow S$ . Since the image of  $\beta$  is compact, we can find finitely many points  $\{p_1, \dots, p_N\}$  such that  $\text{Im}(\beta) \subset (V_{p_1} \cup \dots \cup V_{p_N})$ . Hence,  $\beta$  breaks into finitely many subsegments, with each contained in a single patch. Provided that we are able to lift to any desired 'initial height' or 'starting time,' it thus suffices to assume that  $\beta$  lies in a single chart as above, say,  $\text{Im}(\beta) \subset V_p$ . Hence, we have  $\beta(u) = (\beta_1(u), \dots, \beta_n(u))$ , for  $u \in [0, \ell]$ . For  $0 < s \leq 1$ , consider  $\beta_s(u) = \beta(su)$ , for  $u \in [0, \ell/s]$ . Consider the simple lift up to the starting time  $t = t_0$ , given by  $\alpha_s(t) = (t + t_0, \beta(st))$ . Then, for  $t \in [0, \ell/s]$ ,

$$\begin{aligned} g(\alpha'_s(t), \alpha'_s(t)) &= G_{00}(\beta(st)) + 2sG_{0i}(\beta(st))\beta'_i(st) + s^2G_{ij}(\beta(st))\beta'_i(st)\beta'_j(st) \\ &\leq -m_p + 2sG_{0i}(\beta(st))\beta'_i(st) + s^2G_{ij}(\beta(st))\beta'_i(st)\beta'_j(st) \end{aligned}$$

where we sum over all repeated indices, with  $i, j \in \{1, \dots, n\}$ . Since  $-m_p$  is strictly negative and everything else is bounded, we can find an  $s$  small enough so that this last quantity is negative, and hence so that  $\alpha_s(t)$  is timelike, for all  $t \in [0, \ell/s]$ .  $\square$

### 3 Proof of the splitting result

*Proof of Theorem 1.3* Applying Lemma 2.3 with  $\lambda = 0$ , we see that  $\nabla\sigma$  is parallel. Fix any smooth, spacelike Cauchy surface,  $S$ . By compactness,  $\sigma|_S$  attains a maximum at some  $p \in S$ , and thus,  $(\nabla\sigma)_p$  is normal to  $S$ . Thus, either  $(\nabla\sigma)_p$  is timelike or zero. But since  $\nabla\sigma$  is parallel, then either  $\nabla\sigma$  is everywhere timelike, or it vanishes identically. That is, either  $\nabla\sigma$  is an everywhere timelike vector field, or  $\sigma$  is constant. The proof will show that, in fact,  $\sigma$  must be zero, but we proceed by considering each case below.

*Case 1* Suppose first that  $\nabla\sigma$  is everywhere timelike. By reversing the time orientation of  $M$  if necessary, we may suppose that  $\nabla\sigma$  is future pointing. Since  $\nabla\sigma$  is parallel, its integral curves are timelike geodesics and hence complete by assumption. Moreover, the quantity  $g(\nabla\sigma, \nabla\sigma)$  is constant, which by a rescaling can be taken to be  $-1$ . The condition  $g(\nabla\sigma, \nabla\sigma) = -1$  then forces the integral curves of  $\nabla\sigma$  to be maximal (and hence to be timelike lines). To see this, fix one such integral curve,  $\gamma$ . Since  $\nabla\sigma$  is future

pointing,  $\gamma$  is a future-directed, unit-speed timelike geodesic. Without loss of generality, suppose that  $\alpha : [a, b] \rightarrow M$  is another future timelike curve from  $\alpha(a) = \gamma(0)$  to  $\alpha(b) = \gamma(\ell)$ . Since  $-1 = g(\nabla\sigma, \nabla\sigma) = g(\nabla\sigma, \gamma') = (\sigma \circ \gamma)'$ , note that along  $\gamma$  we have  $\sigma(\gamma(u)) = -u + \sigma(\gamma(0))$ . Also, because  $\nabla\sigma$  is future pointing, we have  $g(\alpha', \nabla\sigma) < 0$ . Define the ‘spacelike part’ of  $\alpha'$  by  $N := \alpha' + g(\alpha', \nabla\sigma)\nabla\sigma$ . It follows that  $N$  is a vector field on  $\alpha$  with  $g(N, N) \geq 0$  and  $g(N, \nabla\sigma) = 0$ , and  $\alpha' = -g(\alpha', \nabla\sigma)\nabla\sigma + N$ . Then,  $|\alpha'| = [-g(\alpha', \alpha')^{1/2}] = [g(\alpha', \nabla\sigma)^2 - g(N, N)]^{1/2} \leq -g(\alpha', \nabla\sigma) = -\alpha'(\sigma) = -\frac{d}{ds}(\sigma(\alpha(s)))$ . Integrating this gives  $L(\alpha) = \int_a^b |\alpha'| ds \leq \sigma(\alpha(a)) - \sigma(\alpha(b)) = \sigma(\gamma(0)) - \sigma(\gamma(\ell)) = \ell = L(\gamma|_{[0, \ell]})$ . This shows that the (arbitrary) subsegment  $\gamma|_{[0, \ell]}$  is maximal, and thus,  $\gamma$  is a timelike line. (A local version of this basic maximality argument appears, for example, in Proposition 34 in Chapter 5 of [16].)

It now follows that  $M$  splits as a product, as in (1.1), with compact, totally geodesic spacelike slices  $\{t\} \times \Sigma$ . Since  $\nabla\sigma$  is parallel, it follows from a standard maximum principle argument that each level set of  $\sigma$  must coincide with a slice in the splitting. Fix any nonzero level set  $\{\sigma = k\}$ ,  $k \neq 0$ . Since  $\{\sigma = k\}$  is a slice in the product, it is totally geodesic and compact and hence admits a closed spacelike geodesic,  $\gamma$ . But then (2.1) in Lemma 2.2 leads to a contradiction as we traverse a full circuit of  $\gamma$ .

*Case 2* We have shown that  $\sigma$  must in fact be constant, i.e., that  $X$  is in fact a homothetic Killing field,  $\mathcal{L}_X g = 2cg$ , for some constant  $c$ . We now claim that  $c = 0$  and  $X$  is Killing. To see this, let  $\alpha : \mathbb{R} \rightarrow M$  be a complete unit-speed timelike geodesic. Then, we have  $-c = cg(\alpha', \alpha') = g(\nabla_{\alpha'} X, \alpha') = \alpha'(g(X, \alpha'))$ . This implies  $g(X, \alpha'(s)) = -cs + d$ . If  $c \neq 0$ , then  $g(X, \alpha'(d/c)) = 0$  gives a contradiction, since both  $\alpha'$  and  $X$  are timelike.

Hence  $X$  is Killing. The assumption of timelike completeness implies, in fact, that  $X$  is a complete Killing vector field; cf. [12, Lemma 1]. Without loss of generality, we may assume that  $X$  is future pointing. For  $a \in S$ , we can think of the integral curve  $L_a$  of  $X$  as the ‘spatial location’ corresponding to  $a \in S$ , flowing in time. Since  $X$  is complete and Killing, it follows from Lemma 2.4 that any spatial curve in  $S$  lifts, along the integral curves of  $X$ , to a timelike curve in  $M$ . It follows that for each  $a, b \in S$ , there is some finite time  $t \in (0, \infty)$  for which  $L_b(t) \in I^+(a)$ . For  $a, b \in S$ , define the shortest such ‘commute time’ from  $a$  to  $b$ , (really from  $a$  to  $L_b$ ) by:

$$C_a(b) := \inf\{t : L_b(t) \in I^+(a)\} \quad (3.1)$$

Letting  $C(a, b) = C_a(b)$ , we claim that  $C : S \times S \rightarrow [0, \infty)$  is upper semicontinuous. Suppose so for the moment. Since  $S$  is compact, it then follows that  $C$  is bounded, that is, there is a ‘maximum commute time’  $\tau$ , such that  $L_b(\tau) \in I^+(a)$ , for all  $a, b \in S$ . Let  $\gamma : [0, \infty) \rightarrow M$  be any future timelike unit-speed  $S$ -ray. (Hence,  $\gamma$  is future inextendible, with  $d(S, \gamma(s)) = s$  for all  $s \geq 0$ .) Note that the integral curves of  $X$  give a diffeomorphic splitting  $M \approx \mathbb{R} \times S$ . Since the slab  $[0, \tau] \times S$  is compact,  $\gamma$  must meet the slice  $\{\tau\} \times S$ , at some point  $\gamma(s_0) = (\tau, a_0) = L_{a_0}(\tau)$ . It follows that  $S \subset I^-(\gamma(s_0)) \subset I^-(\gamma)$ . But then by Theorem A in [6],  $M$  contains a timelike line. Thus, again, by the Lorentzian splitting theorem,  $M$  splits.

It remains to show that ‘commute time’  $C : S \times S \rightarrow [0, \infty)$  is upper semicontinuous. Fix  $a, b \in S$ , and  $\epsilon > 0$ . Let  $t_0 := C_a(b)$ , and set  $t_1 := t_0 + \epsilon/2$ . Hence,  $L_b(t_1) \in I^+(a)$ . Letting  $\pi_S : M \rightarrow S$  be the standard projection,  $\pi_S(L_x(t)) = x$ , define  $U_a := \pi_S(I^-(L_b(t_1)))$ . Hence,  $U_a$  is an open neighborhood of  $a$  in  $S$ , such that  $L_b(t_1) \in I^+(x)$ , for all  $x \in U_a$ . Letting  $t_2 := t_0 + \epsilon$  and  $S_{t_2} := \{t_2\} \times S$ , define  $V_b := \pi_S(I^+(L_b(t_1)) \cap S_{t_2})$ . Hence,  $V_b$  is a neighborhood of  $b$  in  $S$  such that  $L_y(t_2) \in I^+(L_b(t_1))$ , for all  $y \in V_b$ . But then, for all  $(x, y) \in U_a \times V_b \subset S \times S$ , we have  $L_y(t_2) \in I^+(L_b(t_1)) \subset I^+(x)$ , i.e.,  $L_y(t_0 + \epsilon) \in I^+(x)$ . In other words, for all  $(x, y) \in U_a \times V_b$ , we have  $C_x(y) \leq t_0 + \epsilon = C_a(b) + \epsilon$ .

We have now shown that  $M$  splits as in (1.2), for some smooth, compact space-like Cauchy hypersurface  $\Sigma^n$ . That this, with its induced Riemannian metric  $h$ , has  $\text{Ric}_h = 0$  follows, for example, from the warped product curvature formulas in [16], with  $\text{Ric}_g = 0$  and  $f = 1$ .

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