FIRST ORDER RIGIDITY OF NON-UNIFORM HIGHER RANK ARITHMETIC GROUPS

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In Memory of Daniel G. Mostow

ABSTRACT. If Γ is an irreducible non-uniform higher-rank characteristic zero arithmetic lattice (for example $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) and Λ is a finitely generated group that is elementarily equivalent to Γ, then Λ is isomorphic to Γ.

1. Introduction

In this article, we state and prove a new rigidity result for irreducible non-uniform higher-rank arithmetic lattices. This class includes the groups $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ and $\mathrm{SL}_n(\mathbb{Z}[1/p])$ for $n, p \geq 2$.

We recall the definitions. A lattice in a locally compact, second countable group $\mathbf{G}$ is a discrete subgroup $\Gamma \subset \mathbf{G}$ such that there is a fundamental domain with finite Haar measure for the translation action of Γ on $\mathbf{G}$. A lattice is called uniform if $\mathbf{G}/\Gamma$ is compact, and non-uniform otherwise. We say that Γ is irreducible if, for every non-compact normal subgroup $\mathbf{K} \lhd \mathbf{G}$, the closure of the image of Γ in $\mathbf{G}/\mathbf{K}$ is open.

In this paper, by a semisimple group we mean a locally compact group $\mathbf{G}$ of the form $\prod_{i=1}^{r} G_i(F_i)$, where F_i are local fields of characteristic zero and G_i are connected simple algebraic groups defined over F_i and $G_i(F_i)$ is non-compact for every $1 \leq i \leq r$. We say that a semisimple group $\mathbf{G}$ has higher-rank if $\sum \mathrm{rank}_{F_i} G_i \geq 2$ and has low-rank otherwise. A group which is an irreducible lattice in a semisimple group of higher-rank is called a higher-rank lattice. By Mostow's strong rigidity (see Theorem A on page 9 in [Mos]), a group cannot be an irreducible lattice in both a semisimple higher-rank group and a semisimple low-rank group, so being an irreducible lattice in a higher-rank group is a property of Γ. For example, $\mathrm{SL}_n(\mathbb{Z})$ is an irreducible non-uniform lattice in $\mathrm{SL}_n(\mathbb{R})$, $n \geq 2$; it is a higher-rank lattice if $n \geq 3$, while $\mathrm{SL}_n(\mathbb{Z}[1/p])$ is an irreducible non-uniform higher-rank lattice in $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$ for any $n, p \geq 2$.

Irreducible higher-rank lattices are very much related to arithmetic groups. We recall the construction of the latter. Let k be a number field with ring of integers O, let S be a finite set of places of k, containing all the archimedean ones, and let $O_S := \{x \in k \mid (\forall v \notin S)\, v(x) \geq 0\}$ be the ring of S-integers. Let G be a connected group scheme over O_S, and let G_k be the corresponding algebraic group over k. Assume that G_k is absolutely simple and simply connected. Any group which is abstractly commensurable to such $G(O_S)$ is called an arithmetic group. Borel and Harish-Chandra [BHC] proved that the image (under the diagonal embedding) of $G(O_S)$ in $\prod_{v\in S} G(K_v)$ is an irreducible lattice and so every arithmetic group is commensurable to an irreducible lattice in some semisimple group. Conversely, Margulis' Arithmeticity Theorem implies that any irreducible higher-rank lattice is commensurable to an arithmetic group. Note that even in the case $\mathbf{G} = \prod_{v\in S} G(K_v)$, the arithmetic group need not be $G(O_S)$ but can have the form $H(O_T)$ for a different algebraic group H and set of primes T.

Irreducible higher-rank lattices have many remarkable properties. For example, Margulis's Superrigidity Theorem roughly says that Γ (as abstract group) determines $\mathbf{G}$ and the embedding $\Gamma \hookrightarrow \mathbf{G}$ up to automorphisms of $\mathbf{G}$ (see Definition 4.1 for the accurate statement). Another amazing rigidity result for these groups is the following (for a quick formulation, we assume that Γ is non-uniform): If Λ is any finitely generated group which is quasi-isometric to Γ (i.e., the Cayley graph of Λ is quasi-isometric to that of Γ), then, up to finite index and finite normal subgroups, Γ and Λ are isomorphic, see [Far] and the reference therein.

The main goal of this paper is to show a new rigidity phenomenon for higher rank arithmetic groups. For the formulation, we need the following definitions:

Definition 1.1. *Two groups are said to be* elementarily equivalent *if every first order sentence in the language of groups that holds in one also holds in the other.*

Elementary equivalence is a fairly weak equivalence relation: every infinite group has an equivalent group of any infinite cardinality. From a group-theoretic perspective, it is reasonable to restrict the discussion to finitely generated groups. Luckily, characteristic zero arithmetic groups are always finitely generated (in fact, finitely presented).

Definition 1.2. *We say that a finitely generated group* Γ *is* first order rigid *(or quasi-axiomatizable or QA for short) if every finitely generated group that is elementarily equivalent to* Γ *is isomorphic to* Γ*.*

Remark 1.3. *The term quasi-axiomatizable was defined for the first time in* [Nie03] *and have been used in various papers in model theory (see* § *7 below). We prefer the term first order rigid to put Theorem 1.4 below in line with the various rigidity result for lattices in Lie groups described above.*

Finitely generated abelian groups are first order rigid. Nilpotent groups need not be first order rigid, but the elementary equivalence class of any finitely generated nilpotent group contains only finitely many finitely generated groups (see Remark 6.2). In general, elementary equivalence classes can be infinite. The celebrated work of Sela [Sel02] (see also [KM]) shows that all non-abelian free groups are elementarily equivalent, and are also equivalent to the fundamental groups of compact surfaces of genera at least two (free groups and fundamental groups of surfaces are all arithmetic groups, but not of higher-rank).

Our main result says that the situation for higher-rank arithmetic groups is very different. Recall that two groups are said to be abstractly commensurable if they contain isomorphic finite index subgroups.

Theorem 1.4. *Any group which is abstractly commensurable to an irreducible non-uniform higher-rank lattice is first order rigid.*

Remark 1.5. *First order rigidity is, in general, not preserved under abstract commensurability, see §6.*

Remark 1.6. (1) *Theorem 1.4 stands in a sharp contrast to lattices in low-rank groups:*

(a) By [Sel09]*, all torsion-free lattices in* $\mathrm{SL}_2(\mathbb{R})$ *are elementarily equivalent.*

(b) By [Sel09, Theorem 7.6]*, if* Γ *is torsion-free uniform lattice in a rank-one group, then* Γ *is elementarily equivalent to* $\Gamma * F_n$ *for all* $n \geq 1$*.*

2) *By* [Sel09, Proposition 7.1]*, two non-isomorphic uniform torsion-free lattices in rank-one groups other than* $\mathrm{SL}_2(\mathbb{R})$ *are never elementarily equivalent. We do not know what happens for non-uniform lattices.*

Remark 1.7. *As observed by the referee, the results extend to groups that are commensurable to products of irreducible non-uniform higher-rank arithmetic groups.*

The paper is organized as follows: Section 2 contains some preliminaries, including the crucial definitions of a prime group and the Brenner property. In the same section, we also show that $\mathrm{SL}_n(\mathbb{Z})$ is prime and has an element satisfying the Brenner property. In Section 3, we prove that a prime group with a finite center that has an element with the Brenner property is first-order rigid. This finishes the proof of rigidity for $\mathrm{SL}_n(\mathbb{Z})$. In Section 4 we show that superrigid arithmetic groups are prime and in Section 5 we show that irreducible higher-rank non-uniform lattices have elements with the Brenner property, finishing the proof of Theorem 1.4. In Section 6 we show that first-order rigidity is, in general, not preserved under commensurability. Finally, in Section 7 we discuss some related model theoretic and group theoretic properties.

This article is dedicated to the memory Daniel G. Mostow who is the founding father of modern rigidity. Dan was a role model and inspiration for us, professionally and personally.

Acknowledgement: The authors are grateful to Zlil Sela for fruitful conversations and insights which improved the original proof. We are also thankful to Goulnara Arzhantseva, Andre Nies, Andrei Rapinchuk and Tyakal Nanjundiah Venkataramana for pointing out to us several background references. Finally, we thank the anonymous referees for their careful reading and many suggestions. The first author was partially support by NSF grant no. DMS-1303205 and BSF grant no. 2012247. The second author was partially support by ERC, NSF and BSF. The third author was partially supported by ISF grant no. 662/15 and BSF grant no. 2014099.

2. Preliminaries

The following is a theorem of Malcev.

Proposition 2.1 ([Mal]). *If Λ is a group that is elementarily equivalent to a linear group, then Λ is linear. If, in addition, Λ is finitely generated, then Λ is residually finite.*

Definition 2.2. *A homomorphism $f : \Gamma \to \Lambda$ is called an* elementary embedding *if, for every first order formula $\phi(\vec{x})$ with n free variables and every $\vec{a} \in \Gamma^n$, the statement $\phi(\vec{a})$ holds in Γ if and only if $\phi(f(\vec{a}))$ holds in Λ.*

Definition 2.3. *We say that a group Γ is* prime *if, for every group Λ that is elementary equivalent to Γ, there is an elementary embedding $\Gamma \hookrightarrow \Lambda$.*

The following is proved by Oger and Sabbagh:

Theorem 2.4 ([OS]). *Let Γ be a finitely generated group. The following are equivalent:*

(1) Γ is prime.
(2) There is a generating tuple $\vec{g} \in \Gamma^n$ and a formula $\phi(\vec{x})$ such that, for any n-tuple $\vec{h} \in \Gamma^n$, the statement $\phi(\vec{h})$ holds in Γ if and only if $\vec{h}$ is in the $\mathrm{Aut}(\Gamma)$ orbit of $\vec{g}$.

Example 2.5. $\mathrm{SL}_n(\mathbb{Z})$*, $n \geq 3$, is prime: We use the following consequence of superrigidity: Any endomorphism of $\mathrm{SL}_n(\mathbb{Z})$ is either trivial or an automorphism.*

Fix a finite presentation $\langle g_1, \ldots, g_a \mid r_1, \ldots, r_b \rangle$ of $\mathrm{SL}_n(\mathbb{Z})$, and let $\phi(x_1, \ldots, x_a)$ be the formula

$$\phi(\vec{x}) = (x_1 \neq 1) \wedge \bigwedge_{j=1}^{b} (r_j(\vec{x}) = 1).$$

If $\vec{h} \in (\mathrm{SL}_n(\mathbb{Z}))^a$ and $\phi(\vec{h})$ holds, then the map $g_i \mapsto h_i$ extends to a non-trivial endomorphism of Γ, so it must be an automorphism, so $\vec{h}$ is a generating tuple.

Notation 2.6. *For a set $S \subset \Gamma$ and $n \geq 1$, let $[S]^n = \{g_1 \cdots g_n \mid g_i \in S \cup \{1\}\}$.*

Definition 2.7. *We say that an element $b \in \Gamma$ has the* Brenner Property *if there exists a constant $D \geq 1$ for which the following statement hold:*

For every $h \in \Gamma$, if $|[h^\Gamma \cup (h^{-1})^\Gamma]^D| > D$ then $[h^\Gamma \cup (h^{-1})^\Gamma]^D \cap Z(C_\Gamma(b)) \neq \{1\}$.

Remark 2.8. *Let $h \in \Gamma \smallsetminus Z(\Gamma)$ and denote $S = h^\Gamma \cup (h^{-1})^\Gamma$. We claim that if $|[S]^D| \leq D$ for some $D \geq 1$ then $[S]^D$ is the normal subgroup of Γ generated by h, and in particular $[S]^D = [S]^C$ for every $C \geq D$. Indeed, since $|S| > 1$, there is a natural number $k < D$ such that $[S]^k = [S]^{k+1}$. By induction, $[S]^k = [S]^\ell$ for every $\ell > k$ and hence $[S]^D = [S]^k$ is a group.*

Note that the claim implies that if $h \in \Gamma$ is not contained in any finite normal subgroup then $|[S]^D| > D$ for every $D \geq 1$.

Let $e_{i,j} \in \mathrm{SL}_n(\mathbb{Z})$ be the elementary matrix with 1s on the diagonal and entry (i, j) and zero elsewhere.

Lemma 2.9 ([Bre]). *Let $n \geq 3$. Then $e_{1,n}$ has the Brenner Property in $\mathrm{SL}_n(\mathbb{Z})$.*

Proof. Denote $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Since the center of G is finite it is enough to show that there is a constant C such that for every $h \in \Gamma \setminus Z(\Gamma)$, $[h^\Gamma \cup (h^{-1})^\Gamma]^C \cap Z(C_\Gamma(e_{1,n})) \neq \{1\}$.

Let $h \in \Gamma \backslash Z(\Gamma)$ and define $S := h^\Gamma \cup (h^{-1})^\Gamma$. For every $k \geq 1$, $[S]^k$ is a symmetric normal subset. Thus, if $t \in [S]^k$ and $q \in \Gamma$ then $t^\Gamma \cup (t^{-1})^\Gamma \subseteq [S]^k$ and $[t, q] := tqt^{-1}q^{-1} \in [S]^{2k}$. For a matrix $t \in \mathrm{SL}_n(\mathbb{Q})$ let $V_t := \{v \in \mathbb{Q}^n \mid tv = v\}$. As $\mathrm{SL}_n(\mathbb{Z}) = \langle e_{i,j} \mid 1 \leq i \neq j \leq n\rangle$, there exists $1 \leq r \neq s \leq n$ such that $1 \neq h^* := [e_{r,s}, h] \in [S]^2$. Since $\dim(V_{e_{r,s}}) = \dim(V_{he_{r,s}^{-1}h^{-1}}) = n-1$, we get $n-2 \leq \dim(V_{h^*}) \leq n-1$. By the structure theorem of finitely generated abelian groups, there exist $0 \neq A \in M_{n-2}(\mathbb{Z})$ and $B \in \mathrm{SL}_2(\mathbb{Z})$ such that h is conjugate in $\mathrm{SL}_n(\mathbb{Z})$ to

$$h^{**} = \begin{pmatrix} I_{n-2} & A \\ 0 & B \end{pmatrix} \in [S]^2.$$

By considering the cases $B = \pm I_2$ and $B \neq \pm I_2$ separately, it is easy to see that there exist $0 \neq A', A'' \in M_{n-2}(\mathbb{Z})$ and $B' \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$1 \neq h^{***} := \left[\begin{pmatrix} I_{n-2} & A \\ 0 & B \end{pmatrix}, \begin{pmatrix} I_{n-2} & A' \\ 0 & B' \end{pmatrix}\right] = \begin{pmatrix} I_{n-2} & A'' \\ 0 & I_2 \end{pmatrix} \in [S]^4.$$

If h^{***} differs from the identity matrix only in the last column then h^{***} is conjugate to $e_{1,n}^k$ for some $k \neq 0$. Otherwise, $h^{****} := [h^{**}, e_{n-1,n}] \in [S]^8$ is a non-identity matrix which differs from the identity matrix only in the last column. □

3. Primeness and Brenner property imply first order rigidity

Recall that the FC-center of a group G is the collection of all finite conjugacy classes. The FC-center of G is a characteristic subgroup which is denoted by $\mathrm{FC}(G)$.

Proposition 3.1. *Let Γ be a finitely generated group with finite FC-center. Suppose that Λ is a finitely generated group, $i : \Gamma \to \Lambda$ is an elementary embedding, and $b \in \Gamma$. Then*

(1) $i(FC(\Gamma)) = FC(\Lambda)$. In particular, any non-trivial finite normal subgroup of Λ is contained in $i(\Gamma)$.
(2) $i(Z(C_\Gamma(b))) = Z(C_\Lambda(i(b))) \cap i(\Gamma)$.
(3) If Λ is finitely presented or Γ is linear then, for every $t \in \Lambda \smallsetminus \{1\}$, there is $\Delta_t \lhd \Lambda$ such that $\Lambda = \Delta_t \rtimes i(\Gamma)$ and $t \notin \Delta_t$. In particular, $Z(C_\Lambda(i(b)))$ is the direct sum of $Z(C_\Lambda(i(b))) \cap \Delta_t$ and $i(Z(C_\Gamma(b)))$.

Proof. Identify Γ with its image in Λ. Let $\vec{g}$ be a generating n-tuple of Γ. We first show (1). Let $\mathrm{FC}(\Gamma) = \{a_1, \ldots, a_m\}$. Let $\psi(x)$ be the formula saying that the conjugacy class of x has at most m elements. Since $\psi(a_i)$ holds in Γ, it holds in Λ, so $\mathrm{FC}(\Gamma) \subset \mathrm{FC}(\Lambda)$. To show the converse, for every natural number k, let $\varphi_k(x_1, \ldots, x_m)$ be the formula saying that, for any x different from $x_1, \ldots, x_m$, the conjugacy class of x has more than k elements. Since $\varphi(a_1, \ldots, a_m)$ holds in Γ, it holds in Λ, showing $\mathrm{FC}(\Gamma) \supset \mathrm{FC}(\Lambda)$.

For every word $w(\vec{x})$, let $\nu_w(\vec{x}, y)$ be the first order formula saying that $w(\vec{x})$ is in the center of the centralizer of y. If $\nu_w(\vec{g}, b)$ holds in Γ, then it holds in Δ. This implies (2).

Finally we prove (3). In order to prove the first part of (3) it is enough to find an epimorphism $\varphi_t : \Lambda \to \Gamma$ whose restriction to Γ is the identity map and such that $\varphi_t(t) \neq 1$ (since, in this case, $\Lambda = \ker(\varphi_t) \rtimes \Gamma$). Let $\langle y_1, \ldots, y_m \mid r_1, \ldots \rangle$ be a presentation for Λ, and let $\vec{h} := (h_1, \ldots, h_m)$ be a generating m-tuple of Λ corresponding to this presentation. If Λ is finitely presented then there are only finitely many relations $r_1, \ldots, r_s$ and for every m matrices $l_1, \ldots, l_m \in \mathrm{GL}_d(F)$ which satisfy these relations, the map $h_i \mapsto l_i$ extends to a homomorphism from Λ to Γ . If Γ is linear, Hilbert's basis theorem implies that there exists a number s such that any m matrices $l_1, \ldots, l_m \in \mathrm{GL}_d(F)$ satisfying the relations $r_1, \ldots, r_s$ also satisfy the rest of the relations r_i, and, in particular, the map $h_i \mapsto l_i$ extends to a homomorphism. Let $w_1(\vec{x}), \ldots, w_n(\vec{x})$ be words such that $g_i = w_i(\vec{h})$ for every $1 \leq i \leq n$, and let $u(\vec{x})$ be a word such that $t = u(\vec{h})$. Let $\eta(y_1, \ldots, y_m, x_1, \ldots, x_n)$ be the first order formula which is the conjunction of

(1) $\vec{y}$ satisfies $r_1, \ldots, r_s$.
(2) $\bigwedge_{1 \leq i \leq n} w_i(\vec{y}) = x_i$.
(3) $u(\vec{y}) \neq 1$.

The tuple $\vec{h}$ is a testament that the formula $(\exists \vec{y})\eta(\vec{y}, \vec{g})$ holds in Λ. Hence, this formula also holds in Γ. Let $\vec{k} \in \Gamma^m$ be such that $\eta(\vec{k}, \vec{g})$ holds. By the first part of η, the map $h_i \mapsto k_i$ extends to a homomorphism $\varphi_t : \Lambda \to \Gamma$. By the second part of the definition of η, $\varphi_t(g_i) = \varphi_t(w_i(\vec{h})) = w_i(\vec{k}) = g_i$ for every $1 \leq i \leq n$, so the restriction of φ_t to Γ is the identity map. By the third part of the definition of η, $\varphi_t(t) = \varphi_t(u(h)) = u(\vec{k}) \neq 1$. The second part of (3) follows from the first part and (2).

□

Remark 3.2. *In this paper, the requirement that $t \notin \Delta_t$ in part (3) of Proposition 3.1 does not play any role. This requirement becomes important when dealing with the positive characteristic case and it is included here for future reference.*

Theorem 3.3. *Let Γ be a finitely generated group. Assume that*

(1) Γ is linear.
(2) $\mathrm{FC}(\Gamma)$ is finite.
(3) Γ is prime.
(4) b has the Brenner property in Γ and the Prufer rank of $Z(C_\Gamma(b))$ is finite (this means that there is a natural number k such that every finitely generated subgroup of $Z(C_\Gamma(b))$ is generated by k elements.)

Then Γ is first order rigid.

Proof. Let Λ be a finitely generated group that is elementarily equivalent to Γ. Since Γ is prime, there is an elementary embedding $i : \Gamma \to \Lambda$. As before, we identify Γ with $i(\Gamma)$. By Proposition 3.1(3) (with any non-trivial t), there is a subgroup $\Delta \triangleleft \Lambda$ such that $\Lambda = \Delta \rtimes \Gamma$. We will show that $\Delta = 1$.

If there is a non-trivial element in Δ, the Brenner property of b, Remark 2.8, part (1) of Proposition 3.1 and the normality of Δ imply that there is a non-trivial element in $\Delta \cap Z(C_\Lambda(b))$. Hence, it is enough to prove that $\Delta \cap Z(C_\Lambda(b)) = \{1\}$.

For an abelian group Φ, denote the set of m-powers in Φ by $P_m(\Phi)$, and note that this is a subgroup. Since every finitely generated subgroup of $Z(C_\Gamma(b))$ is generated by at most k elements, the group $Z(C_\Gamma(b))/P_m(Z(C_\Gamma(b)))$ is finite. Let d_m be its size. There is a first order formula $\nu_m(x)$ that says that the quotient of the center of the centralizer of x by the collection of m-th powers of the center of the centralizer of x has size d_m. Since $\nu_m(b)$ holds in Γ, it also holds in Λ. Hence, $|Z(C_\Lambda(b))/P_m(Z(C_\Lambda(b)))| = |Z(C_\Gamma(b))/P_m(Z(C_\Gamma(b)))|$. Proposition 3.1(3) implies that, for every m, $Z(C_\Lambda(b)) \cap \Delta = P_m(Z(C_\Lambda(b)) \cap \Delta)$. Hence, $Z(C_\Lambda(b)) \cap \Delta$ is divisible. By Proposition 2.1, Λ is linear. Since Λ is finitely generated, it is residually finite. It follows that $Z(C_\Lambda(b)) \cap \Delta$ is a divisible and residually finite group, a contradiction. ☐

Combining Theorem 3.3, Lemma 2.9, Example 2.5, and noting that $Z(C_{\mathrm{SL}_n(\mathbb{Z})}(e_{1,n}))$ is the cyclic group generated by $e_{1,n}$, we get

Corollary 3.4. *If $n \geq 3$, $\mathrm{SL}_n(\mathbb{Z})$ is first order rigid.*

4. Superrigid lattices are prime

In this section, we prove that superrigid lattices are prime. Recall our notation that $G, H, \ldots$ denote algebraic groups and $\mathbf{G}, \mathbf{H}, \ldots$ denote locally compact groups.

Definition 4.1. *A subgroup Γ of a locally compact group $\mathbf{G}$ is called* superrigid *if, for any simple adjoint algebraic group H defined over a local field L, any homomorphism from Γ to $H(L)$, whose image is unbounded and Zariski dense, extends to a homomorphism from $\mathbf{G}$ to $H(L)$.*

There are many examples of superrigid subgroups:

Example 4.2.
1. *By Theorem (2) in page 2 of* [Mar]*, irreducible lattices in higher-rank semisimple groups are superrigid.*
2. *By* [Cor] *and* [GS]*, lattices in* $\mathrm{Sp}(n,1)$ *and in* $F_4^{(-20)}$ *are superrigid.*
3. *In* [BL] *there were given examples of groups which are superrigid but not lattices.*

Recall that a semisimple group is a locally compact group $\mathbf{G} = \prod_{i=1}^r G_i(F_i)$, where F_i are local fields of characteristic zero and G_i are connected simple algebraic groups defined over F_i and $G_i(F_i)$ is non-compact for every $1 \leq i \leq r$. The purpose of this section is to prove the following:

Theorem 4.3. *Let Φ be a group which is abstractly commensurable to an irreducible lattice in higher-rank (characeterstic zero) semisimple group. Then Φ is prime.*

Some preparation is needed for the proof of Theorem 4.3 which is given below.

Definition 4.4. *Let $f : \mathbf{G} \to \mathbf{H}$ be a homomorphism between two locally compact groups. We say that f is* locally measure preserving *if there is a neighborhood $1 \in U \subset \mathbf{G}$ such that $f|_U : U \to f(U)$ is a measure preserving homeomorphism.*

Note that, if $f : \mathbf{G} \to \mathbf{H}$ is locally measure preserving and the restriction of f to $\Omega \subset \mathbf{G}$ is one-to-one, then $f|_\Omega$ is measure preserving.

Example 4.5. *If G, H are semisimple algebraic groups defined over a local field K and $f : G \to H$ is a central isogeny (i.e., f is surjective and $\ker(f)$ is a finite subgroup of the center of G) with invertible derivative, then, up to normalization of the Haar measures by constants, the map $f : G(K) \to H(K)$ is locally measure preserving.*

Suppose that $f : \mathbf{G} \to \mathbf{H}$ is locally measure preserving and onto, and let $\Lambda \subset \mathbf{G}$ be a discrete subgroup such that $\Lambda \supset \mathrm{Ker}(f)$. If $\Omega \subset \mathbf{G}$ is a fundamental domain for Λ in $\mathbf{G}$, then $f(\Omega)$ is a fundamental domain for $f(\Lambda)$ and $f|_\Omega : \Omega \to f(\Omega)$ is one-to-one. It follows that the covolume of Λ in $\mathbf{G}$ is equal to the covolume of $f(\Lambda)$ in $\mathbf{H}$.

We will use a theorem of Borel and Tits. In the following statement, if G is an algebraic group over a field F, we denote by G^+ the subgroup of $G(F)$ generated by the subgroups $U(F)$, where U ranges over the unipotent radicals of parabolic subgroups of G. If $\beta : F \to F'$ is a homomorphism of fields, we denote the base change of G by β by ${}^\beta G$.

Theorem 4.6 ([BT73], Theorem A)**.** *Let F, F' be fields. Let G and G' be absolutely simple connected algebraic groups over F and F' respectively. Assume that G' is adjoint and G^+ is Zariski dense in G. Let $f : G(F) \to G'(F')$ be a homomorphism with Zariski-dense image. Then there is a field homomorphism $\beta : F \to F'$, an F'-isogeny with invertible derivative $\phi : {}^\beta G \to G'$, and a homomorphism $\gamma : G(F) \to Z(G'(F'))$ such that $f(g) = \gamma(g)\phi(\beta(g))$.*

Remark 4.7. *By the solution to the Knesser–Tits conjecture, if F is either a local or a global field and G is F-isotropic, then G^+ is Zariski dense in G. This implies also that $Z(G(F)) = Z(G)$. We will only apply the theorem under the assumption that F and F' are either local or global, so the condition on G^+ is always satisfied and $Z(G'(F')) = Z(G') = 1$.*

Theorem 4.8. *Let G be a connected absolutely simple group over a number field k, let S be a finite set of valuations, containing all archimedean ones. Assume that every finite index subgroup of $G(O_S)$ is superrigid in $\prod_{v \in S} G(k_v)$. Let Γ be a finite index subgroup of $G(O_S)$ and let $\rho : \Gamma \to G(O_S)$ be a homomorphism with infinite image. Then*

(1) $\ker(\rho)$ *is finite.*
(2) If ρ is injective, then $[G(O_S) : \Gamma] = [G(O_S) : \rho(\Gamma)]$. In particular, if $\rho(\Gamma) \subset \Gamma$, then ρ is an automorphism.

Proof. Note first that it suffices to prove (1) and (2) for some finite index subgroup of Γ, so we can replace Γ with a finite index subgroup whenever it is needed.

Step 1 Since $Z(G(O_S))$ is finite, by passing to a finite index subgroup of Γ we may assume that $\Gamma \cap Z(G(O_S)) = 1$. We can also assume that, for any $v \in S$, Γ is unbounded in the valuation v (otherwise, after passing to a finite index subgroup, $\Gamma \subset G(O_{S \smallsetminus \{v\}})$). Denote $\mathbf{G} = \prod_{v \in S} G(k_v)$, and let $\delta : G(k) \to \mathbf{G}$ be the diagonal embedding.

Step 2 Let $H = \overline{\rho(\Gamma)}^Z$ be the Zariski closure of the image of Γ and let H^0 be the connected component of identity. Since the image of ρ is infinite, H^0 is not trivial. Replace Γ with $\Gamma \cap H^0$ (and still call it Γ). There is $v \in S$ and a non-trivial adjoint k-factor $q : H^0 \to K$ such that $q(\rho(\Gamma))$ is unbounded in the valuation v.

Proof: Assume the contrary. Let $q : H^0 \to K$ be an adjoint factor defined over k, and choose a k-embedding $K \hookrightarrow \mathrm{GL}_n$. Since q is defined over k, the group $q \circ \rho(\Gamma)$ is commensurable to a subgroup of $K(k) \cap \mathrm{GL}_n(O_S)$, so it is discrete in $\prod_{v \in S} K(k_v)$. Being pre-compact, $q \circ \rho(\Gamma)$ is finite. Since $q \circ \rho(\Gamma)$ is also Zariski dense in the connected group K, it follows that K is trivial. Since this holds for every adjoint factor K, H^0 is solvable. Thus, H^0 has an infinite abelianization, so Γ has a finite index subgroup with an infinite abelianization, a contradiction to superrigidity.

Step 3 $G = H = H^0$ and $K = G^{ad}$. I.e., $\rho(\Gamma)$ is Zariski dense.

Proof: Since $q \circ \rho(\Lambda)$ is Zariski dense in K and unbounded in $K(k_v)$, Superrigidity implies that $q \circ \rho$ extends to a map $f_v : \mathbf{G} \to K(k_v)$, so, in particular, there is $w \in S$ and a non-trivial map $f_{w,v} : G(k_w) \to K(k_v)$. By Borel–Tits, there is a field homomorphism $\beta = \beta_{w,v} : k_w \to k_v$ and a non-trivial algebraic homomorphism $\phi : {}^{\beta}G \to K$ such that $f_{w,v}$ is the composition of $\beta : G(k_w) \to {}^{\beta}G(k_v)$ and ϕ. Since ${}^{\beta}G$ is simple, $\dim K \geq \dim {}^{\beta}G = \dim G \geq \dim H^0 \geq \dim K$, so H^0 is open in G. Since G is connected, we get that $G = H^0 = H$ and $K = G^{ad}$.

Step 4 $\ker(\rho)$ is finite.

Proof: In the last step we showed that $f_{w,v}(G(k_w))$ is Zariski dense in K. Since K has trivial center, the image under f_v of any other factor of $\mathbf{G}$ is trivial. It follows that f_v is the composition of the projection $\mathbf{G} \to G(k_w)$ and $f_{w,v}$. Hence, ρ is the composition of the embedding $\Gamma \to G(k_w)$ and $f_{w,v}$. By Borel–Tits, $f_{w,v}$ is a composition of a field homomorphism, which is necessarily injective, and a non-trivial central isogeny, so the kernel of $f_{w,v}$ is finite.

Step 5 Let $f_v : \mathbf{G} \to G^{ad}(k_v)$ be the map constructed in Step 3. Then $f_v(\delta(G(k))) \subset G^{ad}(k)$.

Proof: Denote the algebraic closure of k_w by $\overline{k_w}$. Let $g \in G(k)$, and assume that $f_v(\delta(g)) \in G^{ad}(k_w) \smallsetminus G^{ad}(k) \subset G^{ad}(\overline{k_w}) \smallsetminus G^{ad}(k)$. Then there is a field automorphism $\sigma \in \mathrm{Gal}(\overline{k_w}/k)$ such that $\sigma(f_v(\delta(g))) \neq f_v(\delta(g))$. Denote the conjugation by an element h by c_h. Since $g \in G(k)$, there is a finite index subgroup $\Lambda \subset \Gamma$ such that $c_g(\Lambda) \subset \Gamma$. It follows that $c_{f_v(\delta(g))}(f_v(\delta(\Lambda))) \subset f_v(\delta(\Gamma)) = q \circ \rho(\Gamma) \subset G^{ad}(k)$. We get that, for each $h \in f_v(\delta(\Lambda))$, $c_{f_v(\delta(g))}(h) = c_{\sigma(f_v(\delta(g)))}(h)$, meaning that $\sigma(f_v(\delta(g)))\,(f_v(\delta(g)))^{-1} \neq 1$ commutes with $f_v(\delta(\Lambda))$. Since $f_v(\delta(\Lambda))$ has finite index in $f_v(\delta(\Gamma))$, it is Zariski dense. Since G^{ad} has trivial center, we get a contradiction.

Step 6 Let $f = f_v \circ \delta : G(k) \to G^{ad}(k)$. By Borel–Tits, there is a field endomorphism $\alpha : k \to k$ and a central isogeny $\psi : {}^{\alpha}G \to G^{ad}$ such that $f = \psi \circ \alpha$. Since the characteristic of k is zero, α is an automorphism. The automorphism α defines a bijection on the set of valuations of k by $(\alpha(w))(x) = w(\alpha^{-1}(x))$. We claim that $\alpha(S) = S$.

Proof: Note that by our assumption $w \in S$ iff $G(O_S)$ is unbounded in $G(k_v)$. Let $w \in S$. By our assumptions, $G(O_S)$ is unbounded in the w valuation, so $\alpha(G(O_S))$ is unbounded in the $\alpha(w)$ valuation. Since ψ is a central isogeny, this implies that $\psi(\alpha(G(O_S)))$ is unbounded in the $\alpha(w)$ valuation. Since $\psi(\alpha(G(O_S)))$ is commensurable to $q(\rho(\Gamma))$ and $q(\rho(\Gamma)) \cap G^{ad}(O_S)$ has finite index in $q(\rho(\Gamma))$, it follows that $G^{ad}(O_S)$ is unbounded in the $\alpha(w)$ valuation, so $\alpha(w) \in S$.

Step 7 Let $q : G \to G^{ad}$ be the quotient by the center from before. Let

$$\mathbf{q} : \mathbf{G} = \prod_{w \in S} G(k_w) \to \prod_{w \in S} G^{ad}(k_w) = \prod_{w \in S} G(k_w)/Z(G(k_w)) = \mathbf{G}/Z(\mathbf{G})$$

be the map induced by q. Then the composition $\Gamma \xrightarrow{\rho} G(O_S) \xrightarrow{\delta} \mathbf{G} \xrightarrow{\mathbf{q}} \mathbf{G}/Z(\mathbf{G})$ extends to a locally measure preserving map $\mathbf{h} : \mathbf{G} \to \mathbf{G}/Z(\mathbf{G})$ (i.e., $\mathbf{h} \circ \delta = \mathbf{q} \circ \delta \circ \rho$), whose kernel is $Z(\mathbf{G})$.

Proof: For every $w \in S$, the map $f : G(k) \to G^{ad}(k)$ is uniformly continuous if we put the w-topology on $G(k)$ and the $\alpha(w)$-topology on $G^{ad}(k)$, so it extends to a continuous map $h_w : G(k_w) \to G^{ad}(k_{\alpha(w)})$. Let $\mathbf{h} : \mathbf{G} \to \prod_{w \in S} G^{ad}(k_w) = \mathbf{G}/Z(\mathbf{G})$ be the product map. Each h_w is a composition of an isomorphism and a central isogeny, so it is locally measure preserving. It is easy to see that $\mathbf{h}$ extends $\mathbf{q} \circ \delta \circ \rho$.

Step 8 We have $[G(O_S) : \Gamma] = [G(O_S) : \rho(\Gamma)]$.

Proof: By Step 7, $\mathbf{q}(\delta(\rho(\Gamma))) = \mathbf{h}(\delta(\Gamma))$. We have

$$\operatorname{covol}_{\mathbf{G}/Z(\mathbf{G})}(\mathbf{h}(\delta(\Gamma))) = \operatorname{covol}_{\mathbf{G}/Z(\mathbf{G})}(\mathbf{h}(\delta(\Gamma) Z(\mathbf{G}))) = \operatorname{covol}_{\mathbf{G}}(\delta(\Gamma) Z(\mathbf{G})) =$$
$$= \operatorname{covol}_{\mathbf{G}}(\delta(\Gamma)) \cdot [\delta(\Gamma) Z(\mathbf{G}) : \delta(\Gamma)] = \operatorname{covol}_{\mathbf{G}}(\delta(\Gamma)) \cdot |Z(\mathbf{G})|,$$

where the first equality is because the kernel of $\mathbf{h}$ is $Z(\mathbf{G})$, the second is because $\mathbf{h}$ is locally measure preserving, the third is clear, and the forth is because $Z(\mathbf{G}) \cap \delta(\Gamma) \subset \delta(Z(\Gamma)) = 1$. The same proof shows that

$$\operatorname{covol}_{\mathbf{G}/Z(\mathbf{G})}(\mathbf{q}(\delta(\rho(\Gamma)))) = \operatorname{covol}_{\mathbf{G}}(\delta(\rho(\Gamma))) \cdot |Z(\mathbf{G})|,$$

which implies the claim.

This complete the proof Theorem 4.8. □

Corollary 4.9. *Let Φ be a group which is abstractly commensurable to an irreducible non-uniform higher-rank lattice. Let $\rho : \Phi \to \Phi$ be an endomorphism with an infinite image. Then:*

(1) $\ker \rho$ *is finite.*

(2) If ρ is injective then ρ is an automorphism.

Proof. Margulis' Arithmeticity Theorem implies that there exist connected absolutely simple group G defined over a number field k, a finite set of valuations S which contains all archimedean ones such that $G(K_v)$ is unbounded for every $v \in S$, and a finite index subgroup Γ of $G(O_S)$ such that Γ is isomorphic to a finite index subgroup of Φ. Margulis' superrigidity theorem implies that all finite index subgroups of $G(O_S)$ are superrigid in $\prod_{v\in S} G(k_v)$. We identify Γ with its image in Φ. There exists a finite index subgroup Γ_1 of Γ such that $\rho(\Gamma_1) \leq \Gamma$.

Since Γ_1 has a finite index in Φ then $\rho(\Gamma_1)$ is infinite. Part (1) of Theorem 4.8 implies that $\ker\rho \cap \Gamma_1$ is finite. Since Γ_1 has a finite index in Φ, $\ker\rho$ is also finite.

Assume that ρ is injective. Part (2) of Theorem 4.8 implies that

$$[G(O_S) : \Gamma][\Gamma : \Gamma_1] = [G(O_S) : \Gamma_1] = [G(O_S) : \rho(\Gamma_1)] = [G(O_S) : \Gamma][\Gamma : \rho(\Gamma_1)].$$

Thus, $[\Gamma : \Gamma_1] = [\Gamma : \rho(\Gamma_1)]$ and $[\Phi : \Gamma_1] = [\Phi : \Gamma][\Gamma : \Gamma_1] = [\Phi : \Gamma][\Gamma : \rho(\Gamma_1)] = [\Phi : \rho(\Gamma_1)]$. Since ρ is injective, $[\rho(\Phi) : \rho(\Gamma_1)] = [\Phi : \Gamma_1] = [\Phi : \rho(\Gamma_1)]$ and ρ is surjective. □

Lemma 4.10. *Let Φ be a group which is abstractly commensurable to an irreducible non-uniform higher-rank lattice. Then*

(1) $\mathrm{FC}(\Phi)$ *is finite.*
(2) Φ *is finitely presented.*
(3) There exists a constant N such that every finite subgroup of Φ has a normal abelian subgroup of index at most N.
(4) Φ *contains a non-abelian free subgroup.*

Proof. Φ has a finite index subgroup Γ which is a lattice in a group of the form $\prod G_i(F_i)$. By passing to a finite index subgroup, we may assume that G_i are connected. It follows that the projection of Γ to each G_i is Zariski dense. Since conjugacy classes in $G_i(F_i)$ are either central or infinite, (1) follows.

For part (2), recall that lattices in semisimple groups (of characteristic zero) are finitely presented and that finite presentability is preserved under abstract commensurability. Part (3) is just Jordan's theorem about finite linear group of characteristic zero. Part (4) follows from Tit's alternative [Tit]. □

We can now prove Theorem 4.3.

Proof of Theorem 4.3. By Theorem 2.4, we need to show that there is a generating set $g_1, \dots, g_n$ and a formula $\phi(x_1, \dots, x_n)$ such that, for any tuple $(h_1, \dots, h_n) \in \Phi^n$, if $\phi(\vec{h})$ holds, then there is an automorphism of Φ sending g_i to h_i for every $1 \leq i \leq n$.

We are going to use freely the facts mentioned in Lemma 4.10. Find a generating tuple $g_1, \dots g_n$ and let $r_1, \dots, r_a$ be the corresponding defining relations. Let $w_1, \dots, w_b$ be words such that $\{w_1(\vec{g}), \dots, w_b(\vec{g})\}$ is the set of non-trivial elements in the maximal finite normal subgroup of Φ. Let N be the constant defined in part (4) of Lemma 4.10. Since Φ contains a non-abelian free subgroup, there are words u_1, u_2 such that $[u_1(\vec{g})^N, u_2(\vec{g})^N] \neq 1$. Let $\phi(x_1, \dots, x_n)$ be the formula

$$\big([u_1(\vec{x})^N, u_2(\vec{x})^N] \neq 1\big) \wedge \left(\bigwedge_{j=1}^{a} r_j(\vec{x}) = 1\right) \wedge \left(\bigwedge_{i \leq b} w_i(\vec{x}) \neq 1)\right).$$

Assume that $h_1, \dots, h_n \in \Phi$ and $\phi(\vec{h})$ holds. There exists an endomorphism $\rho : \Gamma \to \Gamma$ which sends g_i to h_i for every $1 \leq i \leq n$. Since $[u_1(\vec{h})^N, u_2(\vec{h})^N] \neq 1$, the images of the form $\rho(u_i(\vec{g})) = u_i(\vec{h})$ are not contained in any finite subgroup of Γ. Hence, the image of ρ is infinite.

Part (1) of Corollary 4.9 implies that $\ker(\rho)$ is finite, and hence contained in $\{1\} \cup \{w_1(\vec{g}), \dots, w_b(\vec{g})\}$. By the definition of ϕ, we get that ρ is one-to-one. Part (2) of Corollary 4.9 implies that ρ is an automorphism, confirming the required condition. □

Remark 4.11. *The converse of Theorem 4.3, namely, that a prime lattice is superrigid, is false: by* [Sel09], *torsion-free cocompact lattices in $SO(n,1)$, $n \geq 3$ are prime. It is well known that these lattices are not necessarily (and probably never) superrigid.*

Remark 4.12. *Prime groups need not be first-order rigid. For example, any cocompact lattice in* $\mathrm{Sp}(n,1)$ *satisfies the assumptions of Theorem 4.8 (and hence prime) but is not first order rigid by Theorem 7.6 of* [Sel09].

Remark 4.13. *The crucial property needed in the proof of Theorem 4.3 above is the property stated in Corollary 4.9: Every injective endomorphism of* Γ *is an automorphism (*Γ *is said to be co-hopfian). This property does not hold for positive characteristic higher rank lattices. For example, if F be a finite a field and $n \geq 3$ then* $\mathrm{SL}_n(F[t])$ *is supperrigid but it has many proper subgroups that are isomorphic to itself, e.g.,* $\mathrm{SL}_n(F[t^m])$ *for every $m \geq 2$. Nevertheless, we can prove that* $\mathrm{SL}_n(F[t])$ *and all its finite index subgroups are prime and first order rigid.*

Remark 4.14. *We thank the referee for the following remark. Suppose that Γ is finitely generated, linear, just-infinite, and co-hopfian, and suppose that Λ is universally equivalent to Γ. An argument similar to the proof of Theorem 4.3 shows that there is a monomorphism $i : \Gamma \to \Lambda$. An argument similar to the proof of Proposition 3.1 shows that there is a morphism $j : \Lambda \to \Gamma$ such that $j \circ i$ is injective. It follows that $\Lambda = \Gamma \ltimes N$, for some group N.*

In particular, if Γ_1 and Γ_2 are non-isomorphic finitely generated, just-infinite, co-hopfian, and linear groups, then their universal theories are different.

5. Brenner property for higher-rank groups

Theorem 5.1. *Let k be a global field and let S be a finite set of valuations, containing all archimedean ones. Let G be a simple algebraic group over k which is k-isotropic and has S-rank at least 2. Let P be a maximal proper k-parabolic subgroup, and let Γ be a finite index subgroup of $G(O_S)$. There is a constant C such that, for any non-central $\gamma \in \Gamma$,* *the set* $[\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^C$ *contains a finite index subgroup of* $U \cap \Gamma$ *where* U *is the unipotent radical of* P.

Proof. The claim essentially appears in the proof of Theorem 2.1 of [Rag]. We sketch the argument. Fix a maximal k-split torus S contained in P. There is a simple k-root α and an ordering of the simple k-roots such that P is the parabolic corresponding to α and the positive roots. By [BT65, §5], there is $w \in N_G(S)(k)$ that switches the positive and negative roots. The image of w in the Weyl group is of order 2. This means that $w^2 \in C_G(S)$, so $P^{w^2} = P$ (because $C_G(S) \subset P$). In particular, $P \cap P^w$ is w-invariant. Let $\alpha' = -w(\alpha)$ (so α' is positive), let P' be the (maximal) parabolic corresponding to α', and let U' be its unipotent radical. By [BT65, Theorem 5.15], the map $(u, b) \overset{\varphi}{\mapsto} uwb$ is a k-isomorphism between $U' \times P$ and an open dense set in G. By definition, φ^{-1} is also defined over k, so $uwb \in G(k)$ implies that $u \in U'(k)$ and $b \in P(k)$.

We first claim that there is a constant C_1 such that $[\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{C_1}$ is Zariski dense. Note that $[\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^2$ contains a Zariski-dense subset of $\gamma^G \cdot (\gamma^{-1})^G$ and the later contains the identity. The conjugacy class γ^G is irreducible and has positive dimension, and, hence, so is $[\gamma^G \cdot (\gamma^{-1})^G]^n$, for all $n \geq 1$. Note also that $n \mapsto \dim[\gamma^G \cdot (\gamma^{-1})^G]^n$ is non-decreasing. If $\dim[\gamma^G \cdot (\gamma^{-1})^G]^n = \dim[\gamma^G \cdot (\gamma^{-1})^G]^{n+1}$, it follows that the Zariski closures of $[\gamma^G \cdot (\gamma^{-1})^G]^n$ and $[\gamma^G \cdot (\gamma^{-1})^G]^{n+1}$ coincide. Therefore, the Zariski closure of $[\gamma^G \cdot (\gamma^{-1})^G]^n$ is a normal subgroup of G, so it must be G.

Let $u \in U'$ and $b \in P$ such that $uwb \in [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{C_1}$ and suppose that $x \in \Gamma \cap P \cap P^w$ satisfies $[x, u] \in \Gamma$. We consider the effect of conjugating by x and by $[x, u] := xux^{-1}u^{-1}$ on the Bruhat decomposition of uwb:

$$(xux^{-1})w(x^w bx^{-1}) = xux^{-1}xwbx^{-1} = x(uwb)x^{-1} \in [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{C_1}$$

and

$$(xux^{-1})w(buxu^{-1}x^{-1}) = [x, u](uwb)[x, u]^{-1} \in [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{C_1}.$$

Taking the quotient,

$$xb^{-1}(x^{-1})^w buxu^{-1}x^{-1} \in [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{2C_1}$$

as $[\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{2C_1}$ is closed to conjugation by elements of Γ,

$$b^{-1}(x^{-1})^w buxu^{-1} \in [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{2C_1}. \tag{1}$$

Note that U' is generated by rational positive roots so it is contained in P and in particular $u \in P$. Since $P \cap P^w$ is w-invariant, our assumptions on x imply that every term in equation (1) is in P. Thus, the element in (1) is also contained in P. Let

$$A = \{(u, b, x) \in U' \times P \times (P \cap P^w) \mid uwb \in \Gamma, [x, u] \in \Gamma, x \in \Gamma\}$$

and let $f : U' \times P \times (P \cap P^w) \to P$ be the function

$$f(u, b, x) = b^{-1}(x^{-1})^w buxu^{-1}.$$

We just showed that $f(A) \subset [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{2C_1}$. Let M be the connected component of the Zariski closure of $(P \cap P^w)(O_S)$. We claim that A is Zariski dense in $U' \times P \times M$.

Indeed, the collection of (u,b) satisfying $uwb \in \Gamma$ is Zariski dense in $U' \times P$, so it is enough to show that, for every $u \in U'(k)$, the collection of x's satisfying $[x,u] \in \Gamma$ contains a finite index subgroup of $M(O_S)$. After passing to a finite index subgroup, we can assume that Γ is normal in $G(O_S)$. Consider the polynomial function $x \mapsto [x,u]$. It has k-rational coefficients and maps 1 to 1. It follows that there is an ideal $\mathfrak{a}$ of O_S such that, if $x \in G(O_S)(\mathfrak{a})$, then $[x,u] \in G(O_S)$. Consider the map $c : M(O_S)(\mathfrak{a}) \cap \Gamma \to G(O_S)/\Gamma$ defined by $c(x) = [x,u]\Gamma$. Since $[xy,u] = xyuy^{-1}x^{-1}u^{-1} = x[y,u]x^{-1}[x,u]$, it follows that c is a homomorphism. Every element x in $\ker(c)$ (which has finite index in $M(O_S)(\mathfrak{a}) \cap \Gamma$ and hence in $M(O_S)$) satisfies $[x,u] \in \Gamma$, which is what we wanted to prove.

It follows that, in the notation above, $f(A)$ is Zariski dense in $f(U' \times P \times M)$. Hence, the Zariski closure of $[\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{2C_1} \cap P$ contains $f(U' \times P \times M)$. Denoting the Levi subgroup of P by L, [Rag, Lemma 2.8] says that the group generated by $f(U' \times P \times M)$ contains the identity component of the Zariski-closure $\overline{L(O_S)}^Z$ of $L(O_S)$.

Let U^i, $i = 1, \dots, N$ be the ascending central series of U. Each U^i/U^{i+1} is a vector space on which P acts by conjugation. If $v \in (U^i \cap \Gamma)/(U^{i+1} \cap \Gamma)$ and $z \in f(A)$, then $(\mathrm{Ad}(z) - 1)v = [v,z] \in [\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{4C_1}$. We will use the following simple lemma:

Lemma 5.2. *Let k be a global field, O its ring of integers, and S a finite set of valuations. For $h_1, \dots, h_t \in \mathrm{GL}_n(O_S)$ generating a subgroup H, the following are equivalent:*

(1) $\mathrm{span}\left\{(h-1)k^n \mid h \in \overline{H}^Z\right\} = k^n$.
(2) $\mathrm{span}\{(h-1)k^n \mid h \in H\} = k^n$.
(3) *There is no H-invariant linear functional on k^n.*
(4) $\mathrm{span}\{(h_i - 1)k^n \mid 1 \le i \le t\} = k^n$.
(5) $(h_1 - 1)O_S^n + \cdots + (h_t - 1)O_S^n$ *has a finite index in* O_S^n.

By [Rag, Claim 2.11], $(\overline{L(O_S)}^Z)^0$ acting on U^i/U^{i+1} satisfies condition (1). Since the Zariski closure of $\langle f(A)\rangle$ contains $(\overline{L(O_S)}^Z)^0$, the action of $\langle f(A)\rangle$ also satisfies this condition. It follows that there are finitely many elements $h_1, \dots, h_t \in f(A)$ that satisfy the claim of the lemma. In particular, $[h_1, (U^i \cap \Gamma)/(U^{i+1} \cap \Gamma)] + \ldots + [h_t, (U^i \cap \Gamma)/(U^{i+1} \cap \Gamma)]$ has finite index in $(U^i \cap \Gamma)/(U^{i+1} \cap \Gamma)$. By induction, it follows that $[\gamma^\Gamma \cup (\gamma^{-1})^\Gamma]^{4tNC_1} \cap U$ has finite index in $U(O_S)$ (and, hence, in $U(O_S) \cap \Gamma$). □

Corollary 5.3. *Let Φ be a group which is abstractly commensurable to an irreducible non-uniform higher-rank lattice. There exits $g \in \Phi$ which has the Brenner Property .*

Proof. Margulis' Arithmeticity Theorem implies that Φ has a finite index subgroup Γ which satisfies the assumptions of Theorem 5.1. We claim that any element $g \in \Gamma \cap U$ of infinite order has the Brenner Property where U is as in the statement of Theorem 5.1. Indeed, let $D = 2ABC$ where $A := [\Phi : \Gamma]$, $B = |Z(\Gamma)|$ and C is the constant defined in Theorem 5.1. Let $h \in \Phi$ and assume that $|[h^\Phi \cup {h^{-1}}^\Phi]^D| > D$. Remark 2.8 implies that $|[h^\Phi \cup {h^{-1}}^\Phi]^{AB}| > AB$ so $[h^\Phi \cup {h^{-1}}^\Phi]^{2AB}$ contains a non-central element of Γ. The definition of C implies that $[h^\Gamma \cup (h^{-1})^\Gamma]^{2ABC}$ contains a finite index subgroup of $\langle g\rangle$. Since g has an infinite order, $[h^\Gamma \cup (h^{-1})^\Gamma]^{2ABC} \cap \langle g\rangle \neq \{1\}$. □

Lemma 5.4. *Let Φ be a group which is abstractly commensurable to an irreducible higher-rank lattice. There exists a constant D such that any finitely generated abelian subgroup of Φ is generated by at most D elements.*

Proof. Selberg's lemma implies that any finitely generated linear group of characteristic zero has a torsion free finite index subgroup. Thus, Φ has a torsion free finite index subgroup which is an irreducible higher-rank lattice. It is known [Ser] that such lattices have a finite cohomological dimension. Let C be the cohomological dimension of Γ. Every finitely generated subgroup of Γ has cohomological dimension at most C. The cohomological dimension of $\mathbb{Z}^n$ is n so the rank of every finitely generated abelian subgroup of Γ is at most C. Thus, the minimal number of generators of any abelian subgroup of Φ is at most $[\Phi : \Gamma]C$. □

We can now prove Theorem 1.4:

Proof of Theorem 1.4. Theorem 4.3, Lemma 4.10, Corollary 5.3 and Lemma 5.4 show that Φ satisfies the conditions of Theorem 3.3. Hence Φ is first-order rigid. □

6. First-order rigidity is not commensurability invariant

The goal of this section is to show that first order rigidity is not preserved by finite index subgroups nor by finite extensions. The key ingredient is the following theorem of Oger:

Theorem 6.1 ([Oge91, Oge96])**.** *Let G and H be finitely generated finite-by-nilpotent groups. Then G and H are elementarily equivalent if and only if $G \times \mathbb{Z}$ and $H \times \mathbb{Z}$ are isomorphic.*

Remark 6.2. *The following theorems imply that elementary equivalence classes of finitely generated nilpotent groups are finite:*

(1) Baumslag [Bau] *proved that if A, B, C and D are finitely generated group such that $A \times B \cong C \times D$ and B and D have the same finite quotients then A and C have the same finite quotients.*
(2) Pickel [Pic] *proved that if G is a nilpotent group then the collection of isomorphism classes of nilpotent groups with the same finite quotients as G is finite.*

We start by giving an example of a first order rigid group which has a finite extension that is not first order rigid. The example follows Baumslag's construction [Bau] of non-isomorphic finitely generated groups with the same finite quotients. Every finitely generated abelian group is first order rigid. In particular, the infinite cyclic group $\mathbb{Z}$ is first order rigid. For every coprime $n, m \in \mathbb{N}^+$, let C_n be the cyclic group of order n and let $\rho_m : \mathbb{Z} \to \mathrm{Aut}(C_n)$ be the homomorphism defined by $\rho_m(1) := \alpha_m$ where α_m is the automorphism of C_n which sends each element to its m-th power. Define $G_6 := C_{25} \rtimes_{\rho_6} \mathbb{Z}$ and $G_{11} := C_{25} \rtimes_{\rho_{11}} \mathbb{Z}$. Note that $\mathbb{Z}$ is isomorphic to a finite index subgroup of G_6, so it is enough to prove that G_6 is not first order rigid.

Proposition 6.3. *G_6 is not first order rigid.*

Proof. For every $i \in \{6, 11\}$, the set $\mathrm{T}(G_i)$ of torsion elements of G_i is a subgroup of G_i which is isomorphic to C_{25}. If $g \in G$ and $g\,\mathrm{T}(G_i)$ generates $G_i/\,\mathrm{T}(G_i) \simeq C$ then the conjugation action of g on $\mathrm{T}(G_i)$ induces either α_6 or α_{21} if $i = 6$ and α_{11} or α_{16} if $i = 11$. In particular, G_6 and G_{11} are not isomorphic. On the other hand the map $\psi : G_{11} \times \mathbb{Z} \to G_6 \times \mathbb{Z}$ defined by $\psi(((r, s), t)) = ((r, 2s + 5t), s + 2t)$ is an isomorphism. Theorem 6.1 implies that G_6 is not first order rigid. □

Our next goal it to show that G_6 has a finite extension which is first order rigid.

Lemma 6.4. *Let M be a finite group such that all the automorphisms of M are inner and define $H_1 := M \times \mathbb{Z}$. Then H_1 is first order rigid.*

Proof. Since H_1 is finite-by-nilpotent every group which is elementarily equivalent to H_1 is finite-by-nilpotent. Thus, Theorem 6.1 implies that it is enough to show that if $H_1 \times \mathbb{Z} \cong H_2 \times \mathbb{Z}$ then $H_1 \cong H_2$. Choose an isomorphism $\iota : H_1 \times \mathbb{Z} \to H_2 \times \mathbb{Z}$. Note that M and thus $\iota(M)$ are the torsion subgroups of $H_1 \times \mathbb{Z}$ and $H_2 \times \mathbb{Z}$. In particular, $\iota(M) \leq H_2$. Thus,

$$\mathbb{Z} \times \mathbb{Z} \cong (H_1/M) \times \mathbb{Z} \cong (H_1 \times \mathbb{Z})/M \cong (H_2 \times \mathbb{Z})/\iota(M) = (H_2/\iota(M)) \times \mathbb{Z}.$$

The structure theorem of finitely generated abelian groups implies that $\mathbb{Z} \cong H_1/M \cong H_2/\iota(M)$. Thus, $H_2 \cong M \rtimes_\delta \mathbb{Z}$ for some homomorphism $\delta : \mathbb{Z} \to \mathrm{Aut}(M)$. Since all the automorphisms of M are inner, $H_2 \cong M \times \mathbb{Z} \cong H_1$. □

Proposition 6.5. *G_6 embeds as a finite index subgroup of a first order rigid group.*

Proof. Embed $C_{25} \rtimes \mathrm{Aut}(C_{25})$ in the symmetric group S_n for some $n \geq 7$ and recall that all the automorphism of S_n are inner for $n \neq 6$. Every automorphism of C_{25} is the restriction of an inner automorphism of $C_{25} \rtimes \mathrm{Aut}(C_{25})$, in particular, every automorphism of C_{25} is the restriction of an inner automorphism of S_n. Define $H := S_n \rtimes_\gamma \mathbb{Z}$ where $\gamma : \mathbb{Z} \to \mathrm{Aut}(S_n)$ is a homomorphism for which $\gamma(1) \in \mathrm{Aut}(S_n)$ is an automorphism which preserves C_{25} and acts on it as α_6. Then, G_6 can be identified as a finite index subgroup of H. Since all the automorphisms of S_n are inner, $H \cong S_n \times \mathbb{Z}$. Lemma 6.4 implies that H is first order rigid. □

7. Discussion and further questions

There are other model theoretic notions that are related to quasi axiomatizability. In [Nie03], Nies gave the following definition:

Definition 7.1. *A finitely generated group Γ is called quasi-finitely axiomatizable (or QFA for short) if there exists a first order sentence ψ such that every finitely generated group which satisfies ψ is isomorphic to Γ.*

It is clear that a QFA group is also QA. On the other hand, finitely generated infinite abelian groups are QA but not QFA. Nies [Nie03] proved that the free step-2 nilpotent group of rank 2 is QFA. Oger and Sabbagh [OS] proved that a finitely generated nilpotent group Γ is QFA if and only if it is prime if and only if $Z(\Gamma) \subseteq \Delta(\Gamma)$ where $\Delta(\Gamma)$ is the isolator of the commutator subgroup of Γ,

$$\Delta(\Gamma) := \{g \mid \exists n > 0 \text{ such that } g^n \in [\Gamma, \Gamma]\}.$$

Lasserre [Las13] gave a similar characterization of QFA for polycyclic groups. Khelif [Khe] proved that a finitely generated group which is bi-interpretable with the ring $\mathbb{Z}$ is prime and QFA. Lasserre [Las14] used this to prove that the Thompson groups T and F are QFA. It is interesting to understand the connections between the properties QA, QFA and prime. Nies [Nie07] showed that there are $2^{\aleph_0}$ finitely generated prime groups. Since there are only countably many QFA groups, not every finitely generated prime group is QFA. An explicit example of a finitely generated prime group which is not even QA was given by Houcine [Hou] (see also Remark 4.12 above). To the best of our knowledge, it is unknown whether there exist prime groups which are QA but not QFA. In a sequel to this article we will show that many non-uniform higher-rank arithmetic groups are in fact QFA and we believe that all of them are. We will also show that many uniform higher-rank arithmetic groups are first order rigid. At the moment, We do not have a single example of a uniform higher-rank arithmetic group for which we know whether it is QFA or not.

Question 7.2. *Does there exist a uniform higher-rank arithmetic group which is QFA?*

A positive answer to Question 7.2 will give the first example (to the best of our knowledge) of a QFA group which does not have a non-abelian solvable subgroup.

All the higher-rank arithmetic groups for which we know to prove first order rigidity have the congruence subgroup property. In fact, the proof of first order rigidity relies on arguments which are used in the proof of the congruence subgroup property. For example, the fact that non-uniform higher-rank arithmetic groups have elements with the Brenner property follows from Raghunathan's proof of the congruence subgroup property. However, it should be emphasized that Raghunathan's proof yields more than what is actually needed in order to prove the congruence subgroup property. With the notations of Theorem 5.1, in order to prove first order rigidity we need the existence of the constant C, while if one only wants to prove the congruence subgroup property, it is enough to show that the normal subgroup generated by every non-central element $\gamma \in \Gamma$ contains a finite index subgroup of $U \cap \Gamma$. We are wondering if one can use the congruence subgroup property directly in order to prove first order rigidity.

Question 7.3. *Let Γ be an arithmetic group which has the congruence subgroup property. Does Γ must be first order rigid?*

Note that the congruence subgroup property implies superrigidity (see [Rag]).

It is known that all lattices in $\mathrm{Sp}(n, 1)(\mathbb{R})$ are superrigid and arithmetic. It follows from Sela's results that the uniform ones are never first order rigid (see remark 4.12 above), and it is a wide open question whether these lattices have the congruence subgroup property

(Serre conjectured that rank-1 arithmetic groups cannot have the congruence subgroup property). A positive answer to Question 7.3 will show that these uniform lattices do not have the congruence subgroup property and will give an indication that Serre's conjecture is true. It is worth mentioning that if these groups do have the congruence subgroup property, then an argument of the second author ([Lub, 4.2]) shows that the quotient of such a group with respect to the normal subgroup generated by a random element is a hyperbolic group which is not residually finite. It is a major open question whether such hyperbolic groups exist.

While we do not know if superrigidiy together with the congruence subgroup property imply first order rigidity, we do know that superrigidiy together with bounded generation imply first order rigidity (the proof will appear in a sequel to this paper). It is known that many non-uniform higher-rank arithmetic groups are boundedly generated (see [Tav], [ER], and [MRS] for the state of the art), but it is a wide open problem whether uniform higher-rank arithmetic groups are boundedly generated. To the best of our knowledge, there is not even a single example of a uniform higher-rank arithmetic where the answer to this question is known. As all uniform higher-rank arithmetic groups are supperrigid and many of them are indeed first order rigid, this leads to the following question:

Question 7.4. *Are all uniform higher-rank arithmetic groups first order rigid?*

This question is especially interesting in the cases where the congruence subgroup property is not known.

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