Theoretical and Mathematical Physics, **196**(3): 1241–1267 (2018)

INVERSE SCATTERING TRANSFORM FOR THE NONLOCAL REVERSE SPACE–TIME NONLINEAR SCHRÖDINGER EQUATION

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Nonlocal reverse space–time equations of the nonlinear Schrödinger (NLS) type were recently introduced. They were shown to be integrable infinite-dimensional dynamical systems, and the inverse scattering transform (IST) for rapidly decaying initial conditions was constructed. Here, we present the IST for the reverse space–time NLS equation with nonzero boundary conditions (NZBCs) at infinity. The NZBC problem is more complicated because the branching structure of the associated linear eigenfunctions is complicated. We analyze two cases, which correspond to two different values of the phase at infinity. We discuss special soliton solutions and find explicit one-soliton and two-soliton solutions. We also consider spatially dependent boundary conditions.

Keywords: inverse scattering transform, nonlocal RST NLS equation

DOI: 10.1134/S0040577918090015

1. Introduction

In 1965, Zabusky and Kruskal found that solitary wave solutions of the Korteweg–de Vries (KdV) equation [1] exhibited special interaction properties; they called these waves *solitons*. Motivated by this in 1967, for rapidly decaying initial data on the line, Gardner, Greene, Kruskal, and Miura connected the KdV equation to the linear Schrödinger equation and outlined a method for solving the Cauchy problem for the KdV equation using the inverse scattering [2]. Lax learned about these results and soon showed that the KdV and other equations could result from the compatibility condition of two linear operators; for the KdV equation, the linear Schrödinger equation was one of them [3].

In 1972, Zakharov and Shabat [4] found that another physically important equation, the nonlinear Schrödinger (NLS) equation, also had a Lax pair and could be solved (linearized) by inverse scattering.

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The research of M. J. Ablowitz was supported in part by the National Science Foundation (Grant No. DMS-1310200).

The research of Bao-Feng Feng was supported in part by the National Science Foundation (Grant No. DMS-1715991) and NSFC for Overseas Scholar Collaboration Research (No. 11728103).

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 196, No. 3, pp. 343–372, September, 2018.

The NLS equation, like the KdV equation, was known to arise universally [5]. Motivated by these results in 1973, Ablowitz, Kaup, Newell, and Segur (AKNS) [6] generalized the linear operators used by Zakharov and Shabat and showed that the NLS, sine-Gordon [7], modified-KdV, and KdV equations could be solved (linearized) by inverse scattering. Soon afterwards in 1974, AKNS [8] developed a general framework for finding integrable systems solvable by what they called the inverse scattering transform (IST). The method was associated with classes of equations (later called recursion operators) and was used to solve the initial value problem with rapidly decaying data on the line.

To solve a nonlinear equation using the IST, a nonlinear wave equation is first related [9]–[11] to a compatible linear scattering, or spectral, problem or Lax pair, denoted by $v_x = Xv$, and an associated linear time evolution equation, denoted by $v_t = Tv$. The operator X has a function (or functions) called a potential (or potentials). The operator X and the associated linear time evolution operator, denoted by T, are mutually compatible with the nonlinear wave equation that the potential satisfies. Here, we let $q(x,t)$ denote the potential (solution) of the nonlinear equation.

The idea in [8] was to consider the scattering problem

$$v_x = Xv = \begin{pmatrix} -ik & q(x,t) \\ r(x,t) & ik \end{pmatrix} v, \tag{1.1}$$

where $v(x,t) = (v_1(x,t), v_2(x,t))^{\mathrm{T}}$, k is a time-independent spectral parameter, and $q(x,t)$ and $r(x,t)$ are complex-valued functions of the real variables x and t. Associated with AKNS scattering problem (1.1) is the time evolution equation

$$v_t = Tv, \tag{1.2}$$

where the $2{\times}2$ matrix T is a function of $q(x,t)$, $r(x,t)$, and the spectral parameter k. Different matrices T yield different coupled partial differential equations for $q(x,t)$ and $r(x,t)$ from the compatibility condition $v_{xt} = v_{tx}$. Under a certain relation between $q(x,t)$ and $r(x,t)$ (also called symmetry reduction), the resulting system is compatible and leads to a single integrable evolution equation for $q(x,t)$ or $r(x,t)$. An important example is the NLS equation

$$iq_t = q_{xx} - 2\sigma|q|^2 q, \quad \sigma = \mp 1. \tag{1.3}$$

Many investigations of the IST treat initial-value problems with rapidly decaying data, for example, $q(x,t)$, $r(x,t) \to 0$ rapidly as $x \to \pm\infty$ (see [9]–[11]). But there has been keen interest in other NLS-type problems with nonzero boundary conditions (NZBCs). The first study using NZBCs was developed for the NLS equation [12]. The original method for solving the inverse problem for NZBCs used two Riemann surfaces associated with square-root branch points in the eigenfunctions (scattering data). An important improvement involved introducing a uniformization variable [13]. This transforms the inverse problem into a more standard inverse problem in the upper/lower half-planes in the new variable. Subsequently, several researchers studied the NLS equation and related problems in this manner (see [14]–[21]), thus substantially enhancing the applicability and range of the IST method.

New nonlocal symmetry reductions for the AKNS scattering problem were recently identified [22]. These include (here $\sigma = \mp 1$)

- $r(x,t) = \sigma q^*(x,-t)$,
- $r(x,t) = \sigma q^*(-x,t)$,
- $r(x,t) = \sigma q(-x,-t)$, and
- $r(x,t) = \sigma q^*(-x,-t)$.

Each of these symmetry reductions leads to new classes of nonlocal nonlinear integrable equations and new types of inverse problems. The IST with decaying data was constructed for many associated equations [22]. An important example is the PT-symmetric NLS equation [23], [24]. A nonlocal PT-symmetric Davey–Stewartson equation was also analyzed [25]. We recall that an evolution equation is said to be PT-symmetric if it is invariant under the combined action of the parity operator $P(x \to -x)$ and the time-reversal symmetry (complex conjugation) T. The IST with NZBCs has so far only been considered for the PT-symmetric case $r(x,t) = \sigma q^*(-x,t)$ [26], i.e., for the PT-symmetric NLS equation

$$iq_t = q_{xx} - 2\sigma q^2(x,t)q^*(-x,t), \quad \sigma = \mp 1. \tag{1.4}$$

In addition to PT-symmetric NLS equation (1.4), corresponding to the symmetry reduction $r(x,t) = \sigma q(-x,-t)$, the integrability of the nonlocal reverse space–time (RST) NLS equation

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q(-x,-t) \tag{1.5}$$

was also established, and the IST for rapidly decaying boundary conditions (BCs) was developed [22]. It is remarkable that this equation is such a simple modification of NLS equation (1.3). We mention two aspects of RST NLS equation (1.5). First, σ need not be only ± 1; it can be anywhere on the unit circle, i.e., $\sigma = e^{i\theta}$, $\theta \in \mathbb{R}$. Second, unlike the NLS equation or PT-symmetric NLS equation, the coefficient σ can be scaled away in this case. Namely, transforming (1.5) by $q \to (-1/\sigma)^{1/2} q$ allows considering only the case $\sigma = 1$ without loss of generality.

Although the IST under the symmetry reduction $r(x,t) = \sigma q(-x,-t)$ was analyzed for decaying data, the IST for the nonlocal RST NLS equation with NZBCs is still new and open. There are significant differences between this case and the PT-symmetric case considered in [26]. These nonlocal systems exhibit many interesting differences from their local counterparts.

The solution process via IST employs direct and inverse scattering. It can be briefly described as follows. From a suitable initial condition for the potential $q(x, t = 0)$, the data is transformed via the direct scattering problem X into the scattering data $S(k, t = 0)$. The associated operator T determines the time evolution of the scattering data $S(k,t)$ for any $t \neq 0$. Via inverse scattering, the scattering data at any time t is then used to reconstruct eigenfunctions from linear integral equations, and the solution of the nonlinear evolution equation $q(x,t)$ is recovered from this information.

While IST has been used to solve (linearize) the Cauchy initial-value problem with decaying and, in some cases, nonzero boundary values at infinity for numerous nonlinear equations, the method with NZBCs is more difficult because of more complex branching structure of the associated linear eigenfunctions.

Here, we analyze the direct and IST associated with the nonlocal RST NLS equation with the NZBCs

$$q(x,t) \to q_0 e^{i(2\sigma q_0^2 t + \theta_\pm)}, \quad x \to \pm\infty, \tag{1.6}$$

where $\theta_+ + \theta_- = 0$ and $\theta_+ + \theta_- = \pi$. We note that some properties of the NZBCs associated with RST NLS equation (1.5) can be obtained directly. If we assume that $q \to q_\pm(t)$ as $x \to \pm\infty$, then Eq. (1.5) yields

$$iq_{\pm,t}(t) = -2\sigma q_\pm^2(t) q_\mp(-t). \tag{1.7}$$

This implies that

$$q_+(t)q_-(-t) = C_0 = \text{const}, \tag{1.8}$$

and (1.7) is hence simplified to

$$iq_{\pm,t}(t) = -2\sigma C_0 q_\pm(t). \tag{1.9}$$

This equation has the solution

$$q_\pm(t) = |q_\pm(0)|e^{2i\sigma C_0 t}e^{i\theta_\pm}, \tag{1.10}$$

where $\theta_\pm$ are constant. Because $C_0 = |q_+(0)| \cdot |q_-(0)| \cdot e^{i(\theta_+ + \theta_-)}$, if $\theta_+ + \theta_- = 0$ or $\theta_+ + \theta_- = \pi$, then C_0 is real. Otherwise, it is complex, and the background either grows or decays exponentially as $|t| \to \infty$. Without loss of generality, we take $q_\pm(0) = q_0 = \text{const}$.

In Sec. 3, we find the nonsingular dark one-soliton solution with $\sigma = 1$ and $\theta_+ + \theta_- = 0$

$$q(x,t) = \frac{q_0 e^{2iq_0^2 t}\left[e^{i\theta_+} \cdot e^{2q_0 x \sin\theta_+} + e^{-i\theta_+} \cdot e^{2q_0^2 t \sin(2\theta_+)}\right]}{e^{2q_0 x \sin\theta_+} + e^{2q_0^2 t \sin(2\theta_+)}}. \tag{1.11}$$

In Sec. 4, we show that there is no corresponding exponentially decaying one-soliton solution with $\sigma = 1$ and $\theta_+ + \theta_- = \pi$ because a single eigenvalue is found in the continuous spectrum. The simplest decaying pure reflectionless potential generates a two-soliton solution. There are nonsingular two-soliton solutions.

In Sec. 5, we show that there also exist solutions for the nonlocal RST NLS equation satisfying the spatially dependent BCs

$$q(x,t) \to q_0 e^{i(\alpha t + \beta x + \theta_\pm)} \quad \text{as } x \to \pm\infty, \tag{1.12}$$

where both α and β are real. We verify the Galilean invariance for the classical NLS equation.

We can also find novel types of solutions of the above equations that are singular along space–time lines (see [27]).

Finally, we emphasize that the IST for NZBCs associated with the novel nonlocal RST NLS equation leads to new solutions; the boundary-value problems studied here extend the technique and applicability of the IST method.

2. The RST NLS equation: Compatible linear system

Nonlocal RST NLS equation (1.5) is associated with the 2×2 compatible systems

$$v_x = Xv = \begin{pmatrix} -ik & q(x,t) \\ \sigma q(-x,-t) & ik \end{pmatrix} v, \tag{2.1}$$

$$v_t = Tv = \begin{pmatrix} 2ik^2 + i\sigma q(x,t)q(-x,-t) & -2kq(x,t) - iq_x(x,t) \\ -2\sigma k q(-x,-t) - \sigma i q_x(-x,-t) & -2ik^2 - i\sigma q(x,t)q(-x,-t) \end{pmatrix} v. \tag{2.2}$$

Such compatibility relations are well-known (see [8], [9]).

We see that as $x \to \pm\infty$, the eigenfunctions of the spatial scattering problem asymptotically satisfy

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} -ik & q_0 e^{i(\alpha t + \theta_\pm)} \\ \sigma q_0 e^{i(-\alpha t + \theta_\mp)} & ik \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.3}$$

where $\alpha = 2\sigma q_0^2$ (see Eq. (1.6)).

3. Nonlocal RST equation: $\boldsymbol{\sigma = 1}$ and $\boldsymbol{\theta_+ + \theta_- = 0}$

3.1. Direct scattering. In this section, we consider the NZBCs given above in (1.6) with $\sigma = 1$ and $\theta_+ + \theta_- = 0$. With this condition, Eq. (2.3) conveniently reduces to

$$\frac{\partial^2 v_j}{\partial x^2} = -(k^2 - q_0^2)v_j, \quad j = 1, 2. \tag{3.1}$$

Each of the two equations has two linearly independent solutions $e^{i\lambda x}$ and $e^{-i\lambda x}$ as $|x| \to \infty$, where $\lambda = \sqrt{k^2 - q_0^2}$. The variable k is then considered to belong to a Riemann surface $\mathbb{K}$ consisting of two sheets $\mathbb{C}_1$ and $\mathbb{C}_2$ with the complex plane cut along $(-\infty, -q_0] \cup [q_0, +\infty)$ and its edges glued such that $\lambda(k)$ is continuous through the cut. We introduce the local polar coordinates

$$k - q_0 = r_1 e^{i\theta_1}, \quad 0 \le \theta_1 < 2\pi, \qquad k + q_0 = r_2 e^{i\theta_2}, \quad -\pi \le \theta_2 < \pi, \tag{3.2}$$

where $r_1 = |k - q_0|$ and $r_2 = |k + q_0|$. The function $\lambda(k)$ then becomes single-valued on $\mathbb{K}$, i.e.,

$$\lambda(k) = \begin{cases} \lambda_1(k) = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}, & k \in \mathbb{C}_1, \\ \lambda_2(k) = -(r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}, & k \in \mathbb{C}_2. \end{cases} \tag{3.3}$$

Moreover, if $k \in \mathbb{C}_1$, then $\operatorname{Im}\lambda \ge 0$, and if $k \in \mathbb{C}_2$, then $\operatorname{Im}\lambda \le 0$. Hence, the variable λ is considered to belong to the complex plane consisting of the upper half-plane U_+ ($\operatorname{Im}\lambda \ge 0$) and the lower half-plane U_- ($\operatorname{Im}\lambda \le 0$) glued together along the real axis; the transition occurs at $\operatorname{Im}\lambda = 0$. The transformation $k \to \lambda$ maps $\mathbb{C}_1$ onto U_+, $\mathbb{C}_2$ onto U_-, the cut $(-\infty, -q_0] \cup [q_0, +\infty)$ onto the real axis, and the points $\pm q_0$ to 0 (see Figs. 1 and 2 on p. 6 in [26]).

3.2. Eigenfunctions. It is natural to introduce the eigenfunctions defined by the BCs

$$\phi(x,k) \sim w e^{-i\lambda x}, \qquad \bar{\phi}(x,k) \sim \overline{w} e^{i\lambda x} \tag{3.4}$$

as $x \to -\infty$ and

$$\psi(x,k) \sim v e^{i\lambda x}, \qquad \bar{\psi}(x,k) \sim \overline{v} e^{-i\lambda x} \tag{3.5}$$

as $x \to +\infty$. We substitute them in (2.3) and obtain

$$w = \begin{pmatrix} \lambda + k \\ iq_+ \end{pmatrix}, \qquad \overline{w} = \begin{pmatrix} -iq_- \\ \lambda + k \end{pmatrix}, \qquad v = \begin{pmatrix} -iq_+ \\ \lambda + k \end{pmatrix}, \qquad \overline{v} = \begin{pmatrix} \lambda + k \\ iq_- \end{pmatrix}, \tag{3.6}$$

which satisfy the BCs. These BCs reduce to the well-known BCs in the decaying case. We remark that the functions q_+ and q_- are in fact functions of time: $q_+ = q_+(-t)$ and $q_- = q_-(t)$ for w and $\overline{w}$ whose BCs are evaluated at $-\infty$, and $q_+ = q_+(t)$ and $q_- = q_-(-t)$ for v and $\overline{v}$ whose BCs are evaluated at $+\infty$. But for the purposes of this section, the precise time dependence is unimportant, and we therefore use a simpler notation and omit the explicit time dependence.

In the following analysis, it is convenient to consider functions with constant BCs. We define the bounded eigenfunctions as

$$\begin{aligned} M(x,k) &= e^{i\lambda x}\phi(x,k), \qquad & \overline{M}(x,k) &= e^{-i\lambda x}\bar{\phi}(x,k), \\ N(x,k) &= e^{-i\lambda x}\psi(x,k), \qquad & \overline{N}(x,k) &= e^{i\lambda x}\bar{\psi}(x,k). \end{aligned} \tag{3.7}$$

The eigenfunctions can be represented using the integral equations

$$\begin{aligned} M(x,k) &= \begin{pmatrix} \lambda + k \\ iq_+ \end{pmatrix} + \int_{-\infty}^{+\infty} G_-(x - x', k)((Q - Q_-)M)(x', k)\, dx', \\ \overline{M}(x,k) &= \begin{pmatrix} -iq_- \\ \lambda + k \end{pmatrix} + \int_{-\infty}^{+\infty} \overline{G}_-(x - x', k)((Q - Q_-)M)(x', k)\, dx', \\ N(x,k) &= \begin{pmatrix} -iq_+ \\ \lambda + k \end{pmatrix} + \int_{-\infty}^{+\infty} G_+(x - x', k)((Q - Q_+)M)(x', k)\, dx', \\ \overline{N}(x,k) &= \begin{pmatrix} \lambda + k \\ iq_- \end{pmatrix} + \int_{-\infty}^{+\infty} \overline{G}_+(x - x', k)((Q - Q_+)M)(x', k)\, dx'. \end{aligned} \tag{3.8}$$

Using the Fourier transform method, we obtain

$$
\begin{aligned}
G_-(x,k) &= \frac{\theta(x)}{2\lambda}[(1+e^{2i\lambda x})\lambda I - i(e^{2i\lambda x}-1)(ikJ+Q_-)],\\
\overline{G}_-(x,k) &= \frac{\theta(x)}{2\lambda}[(1+e^{-2i\lambda x})\lambda I + i(e^{-2i\lambda x}-1)(ikJ+Q_-)],\\
G_+(x,k) &= -\frac{\theta(-x)}{2\lambda}[(1+e^{-2i\lambda x})\lambda I + i(e^{-2i\lambda x}-1)(ikJ+Q_+)],\\
\overline{G}_+(x,k) &= -\frac{\theta(-x)}{2\lambda}[(1+e^{2i\lambda x})\lambda I - i(e^{2i\lambda x}-1)(ikJ+Q_+)],
\end{aligned}
\tag{3.9}
$$

where $\theta(x)$ is the Heaviside function, i.e., $\theta(x)=1$ if $x>0$ and $\theta(x)=0$ if $x<0$.

Definition 1. We say that $f \in L^1(\mathbb{R})$ if $\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty$, and we say that $f \in L^{1,N}(\mathbb{R})$ if $\int_{-\infty}^{+\infty} |f(x)|\cdot(1+|x|)^N\,dx < \infty$, where $N=1,2,\dots$ is a given positive integer.

We then have the following result.

Theorem 1. *Let the elements of $Q-Q_\pm$ belong to $L^1(\mathbb{R})$. Then for each $x\in\mathbb{R}$, the eigenfunctions $M(x,k)$ and $N(x,k)$ are continuous for $k\in\overline{\mathbb{C}}_1\setminus\{\pm q_0\}$ and analytic for $k\in\mathbb{C}_1$, and $\overline{M}(x,k)$ and $\overline{N}(x,k)$ are continuous for $k\in\overline{\mathbb{C}}_2\setminus\{\pm q_0\}$ and analytic for $k\in\mathbb{C}_2$. In addition, if the elements of $Q-Q_\pm$ belong to $L^{1,2}(\mathbb{R})$, then for each $x\in\mathbb{R}$, the eigenfunctions $M(x,k)$ and $N(x,k)$ are continuous for $k\in\overline{\mathbb{C}}_1$ and analytic for $k\in\mathbb{C}_1$, and $\overline{M}(x,k)$ and $\overline{N}(x,k)$ are continuous for $k\in\overline{\mathbb{C}}_2$ and analytic for $k\in\mathbb{C}_2$. Here, $\overline{\mathbb{C}}_j$, $j=1,2$, are the closures of $\mathbb{C}_j$, $j=1,2$.*

The proof uses Neumann series with $\operatorname{Im}\lambda\ge 0$ or $\operatorname{Im}\lambda>0$ appropriately; it is similar to the proof in [26].

Remark 1. Similarly to Theorem 3.2 in [26], we can rewrite the Green's functions in terms of the projectors. For example,

$$
M(x,k) = \begin{pmatrix}\lambda+k\\ -iq_+^*\end{pmatrix} + \int_{-\infty}^{x} [P_{-i\lambda}^- + e^{2i\lambda(x-x')}P_{i\lambda}^-]((Q-Q_-)M)(x',k)\,dx', \tag{3.10}
$$

where

$$
P_{-i\lambda}^{\pm}(k) = \frac{1}{2\lambda}\begin{pmatrix}\lambda+k & iq_\pm\\ iq_\mp & \lambda-k\end{pmatrix}, \qquad P_{i\lambda}^{\pm}(k) = \frac{1}{2\lambda}\begin{pmatrix}\lambda-k & -iq_\pm\\ -iq_\mp & \lambda+k\end{pmatrix}. \tag{3.11}
$$

To extend the continuity property tot $k=\pm q_0$, we rewrite $P_{-i\lambda}^- + e^{2i\lambda(x-x')}P_{i\lambda}^-$ as $I_2 + [e^{2i\lambda(x-x')}-1]P_{i\lambda}^-$ and use the estimate

$$
\begin{aligned}
&\|I_2 + [e^{2i\lambda(x-x')}-1]P_{i\lambda}^-\| \le 1 + 2|x-x'|\cdot|k| \le\\
&\qquad\le \max\{1,2|k|\}(1+|x|)(1+|x'|) \le \max\{1,2|k|\}(1+|x|)^2,
\end{aligned}
\tag{3.12}
$$

and the elements of $Q-Q_\pm$ belonging to $L^{1,2}(\mathbb{R})$ are therefore required.

3.3. Scattering data. The two eigenfunctions ϕ, $\bar{\phi}$, ψ, $\bar{\psi}$ are linearly independent. Indeed, if $u(x,k) = (u_1(x,k), u_2(x,k))^{\mathrm{T}}$ and $v(x,k) = (v_1(x,k), v_2(x,k))^{\mathrm{T}}$ are any two solutions of (2.1) for $\sigma = 1$, then we have

$$\frac{d}{dx} W(u,v) = 0, \tag{3.13}$$

where the Wronskian $W(u,v)$ of u and v is given by $W(u,v) = u_1 v_2 - u_2 v_1$. It follows from asymptotic formulas (3.4) and (3.5) that

$$W(\phi, \bar{\phi}) = \lim_{x \to -\infty} W(\phi(x,k), \bar{\phi}(x,k)) = 2\lambda(\lambda + k), \tag{3.14}$$

$$W(\psi, \bar{\psi}) = \lim_{x \to +\infty} W(\psi(x,k), \bar{\psi}(x,k)) = -2\lambda(\lambda + k), \tag{3.15}$$

which proves that the functions $\phi(x,k)$ and $\bar{\phi}(x,k)$ are linearly independent, as are ψ and $\bar{\psi}$, with the only exception being the branch points $\pm q_0$. Hence, we can write $\phi(x,k)$ and $\bar{\phi}(x,k)$ as linear combinations of $\psi(x,k)$ and $\bar{\psi}(x,k)$, or vice versa. Therefore, the relations

$$\phi(x,k) = b(k)\psi(x,k) + a(k)\bar{\psi}(x,k), \tag{3.16}$$

$$\bar{\phi}(x,k) = \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k) \tag{3.17}$$

hold for any k such that all four eigenfunctions exist. Combining (3.14) and (3.15), we can deduce that the scattering data satisfy the characterization equation

$$a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1. \tag{3.18}$$

The scattering data can be represented in terms of Wronskians of the eigenfunctions, i.e.,

$$\begin{aligned}
a(k) &= \frac{W(\phi(x,k), \psi(x,k))}{W(\bar{\psi}(x,k), \psi(x,k))} = \frac{W(\phi(x,k), \psi(x,k))}{2\lambda(\lambda + k)}, \\
\bar{a}(k) &= -\frac{W(\bar{\phi}(x,k), \bar{\psi}(x,k))}{W(\bar{\psi}(x,k), \psi(x,k))} = -\frac{W(\bar{\phi}(x,k), \bar{\psi}(x,k))}{2\lambda(\lambda + k)}, \\
b(k) &= -\frac{W(\phi(x,k), \bar{\psi}(x,k))}{W(\bar{\psi}(x,k), \psi(x,k))} = -\frac{W(\phi(x,k), \bar{\psi}(x,k))}{2\lambda(\lambda + k)}, \\
\bar{b}(k) &= \frac{W(\bar{\phi}(x,k), \psi(x,k))}{W(\bar{\psi}(x,k), \psi(x,k))} = \frac{W(\bar{\phi}(x,k), \psi(x,k))}{2\lambda(\lambda + k)}.
\end{aligned} \tag{3.19}$$

We then obtain the following theorem from the analytic behavior of the eigenfunctions.

Theorem 2. *Let the elements of $Q - Q_\pm$ belong to $L^1(\mathbb{R})$. Then $a(k)$ is continuous for $k \in \overline{\mathbb{C}}_1 \setminus \{\pm q_0\}$ and analytic for $k \in \mathbb{C}_1$, and $\bar{a}(k)$ is continuous for $k \in \overline{\mathbb{C}}_2 \setminus \{\pm q_0\}$ and analytic for $k \in \mathbb{C}_2$. Moreover, $b(k)$ and $\bar{b}(k)$ are continuous in $k \in (-\infty, -q_0) \cup (q_0, +\infty)$. In addition, if the elements of $Q - Q_\pm$ belong to $L^{1,2}(\mathbb{R})$, then $a(k)\lambda(k)$ is continuous for $k \in \overline{\mathbb{C}}_1$ and analytic for $k \in \mathbb{C}_1$, and $\bar{a}(k)\lambda(k)$ is continuous for $k \in \overline{\mathbb{C}}_2$ and analytic for $k \in \mathbb{C}_2$. Moreover, $b(k)\lambda(k)$ and $\bar{b}(k)\lambda(k)$ are continuous for $k \in \mathbb{R}$. If the elements of $Q - Q_\pm$ do not increase faster than e^{-ax^2}, where a is a positive real number, then $a(k)\lambda(k)$, $\bar{a}(k)\lambda(k)$, $b(k)\lambda(k)$, and $\bar{b}(k)\lambda(k)$ are analytic for $k \in \mathbb{K}$.*

The proof of Theorem 2 follows from the Wronskian relations (also see [26]).

We note that (3.16) and (3.17) can be written as

$$\mu(x,k) = \rho(k)e^{2i\lambda x}N(x,k) + \overline{N}(x,k), \qquad \bar{\mu}(x,k) = N(x,k) + \bar{\rho}(k)e^{-2i\lambda x}\overline{N}(x,k), \tag{3.20}$$

where $\mu(x,k) = M(x,k)a^{-1}(k)$, $\bar{\mu}(x,k) = \overline{M}(x,k)\bar{a}^{-1}(k)$, $\rho(k) = b(k)a^{-1}(k)$, and $\bar{\rho}(k) = \bar{b}(k)\bar{a}^{-1}(k)$. We introduce the 2×2 matrices

$$m_+(x,k) = (\mu(x,k), N(x,k)), \qquad m_-(x,k) = (\overline{N}(x,k), \bar{\mu}(x,k)), \tag{3.21}$$

which are respectively meromorphic in $\mathbb{C}_1$ and $\mathbb{C}_2$. Hence, we can write the Riemann–Hilbert problem or "jump" conditions in the k plane as

$$m_+(x,k) - m_-(x,k) = m_-(x,k)\begin{pmatrix} -\rho(k)\bar{\rho}(k) & -\bar{\rho}(k)e^{-2i\lambda x} \\ \rho(k)e^{2i\lambda x} & 0 \end{pmatrix} \tag{3.22}$$

on the contour Σ: $k \in (-\infty, -q_0] \cup [q_0, +\infty)$. We recall that $\operatorname{Im}\lambda > 0$ on $\mathbb{C}_1$, $\operatorname{Im}\lambda < 0$ on $\mathbb{C}_2$, and $\operatorname{Im}\lambda = 0$ on Σ.

3.4. Symmetry reductions. The symmetry in the potential induces a symmetry between the eigenfunctions. Indeed, if $v(x,k) = (v_1(x,k), v_2(x,k))^{\mathrm{T}}$ solves (2.1) for $\sigma = 1$, then $(v_2(-x,k), -v_1(-x,k))^{\mathrm{T}}$ also solves (2.1) for $\sigma = 1$. Taking BCs (3.6) into account, we can obtain

$$\psi(x,k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\phi(-x,k), \qquad \bar{\psi}(x,k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\bar{\phi}(-x,k). \tag{3.23}$$

Using (3.7), we can obtain the symmetry relations for the eigenfunctions, i.e.,

$$N(x,k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}M(-x,k), \qquad \overline{N}(x,k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{M}(-x,k). \tag{3.24}$$

From the Wronskian representations for the scattering data and the above symmetry relations, we obtain

$$\bar{b}(k) = b(k). \tag{3.25}$$

3.5. Uniformization coordinates. Before discussing the zeros of the scattering data and solving the inverse problem, we introduce a uniformization variable z defined by the conformal map

$$z = z(k) = k + \lambda(k), \tag{3.26}$$

where $\lambda = \sqrt{k^2 - q_0^2}$ and the inverse map is given by $k = k(z) = (z + q_0^2/z)/2$. Then $\lambda(z) = (z - q_0^2/z)/2$. We observe that

1. the upper sheet $\mathbb{C}_1$ and the lower sheet $\mathbb{C}_2$ of the Riemann surface $\mathbb{K}$ are mapped onto the respective upper half-plane $\mathbb{C}^+$ and lower half-plane $\mathbb{C}^-$ of the complex-z plane,
2. the cut $(-\infty, -q_0] \cup [q_0, +\infty)$ on the Riemann surface is mapped onto the real-z axis, and
3. the segments $[-q_0, q_0]$ on $\mathbb{C}_1$ and $\mathbb{C}_2$ are mapped onto the respective upper and lower semicircles of radius q_0 centered at the origin of the complex-z plane.

It follows from Theorem 1 that the eigenfunctions $M(x,z)$ and $N(x,z)$ are analytic in the upper half-plane, i.e., $z \in \mathbb{C}^+$, and $\overline{M}(x,z)$ and $\overline{N}(x,z)$ are analytic in the lower half-plane, i.e., $z \in \mathbb{C}^-$. Moreover, according to Theorem 2, $a(z)$ is analytic in the upper half-plane ($z \in \mathbb{C}^+$) and $\bar{a}(z)$ is analytic in the lower half-plane ($z \in \mathbb{C}^-$).

3.6. Symmetries via uniformization coordinates. From the eigenfunction symmetries above, we obtain

$$\psi(x,z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \phi(-x,z), \qquad \bar{\psi}(x,z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\phi}(-x,z). \tag{3.27}$$

Further, $(k,\lambda) \to (k,-\lambda)$ as $z \to q_0^2/z$. Hence,

$$\phi\left(x, \frac{q_0^2}{z}\right) = \frac{q_0^2/z}{-iq_-} \bar{\phi}(x,z), \qquad \psi\left(x, \frac{q_0^2}{z}\right) = \frac{-iq_+}{z} \bar{\psi}(x,z), \quad \operatorname{Im} z < 0. \tag{3.28}$$

Similarly, we can obtain

$$\begin{gathered} N(x,z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M(-x,z), \qquad \overline{N}(x,z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{M}(-x,z), \\ \bar{b}(z) = b(z), \qquad a\left(\frac{q_0^2}{z}\right) = e^{2i\theta_+} \bar{a}(z), \quad \operatorname{Im} z < 0, \qquad b\left(\frac{q_0^2}{z}\right) = -\bar{b}(z). \end{gathered} \tag{3.29}$$

3.7. Asymptotic behavior of eigenfunctions and scattering data. To solve the inverse problem, we must determine the asymptotic behavior of the eigenfunctions and scattering data both as $z \to \infty$ and as $z \to 0$. From the integral equations (in terms of Green's functions), we obtain

$$\begin{gathered} \overline{N}(x,z) \sim \begin{pmatrix} z\dfrac{q(x)}{q_+} \\ iq_- \end{pmatrix}, \quad z \to 0, \\ a(z) = \begin{cases} 1, & z \to \infty, \\ e^{2i\theta_+}, & z \to 0, \end{cases} \qquad \bar{a}(z) = \begin{cases} 1, & z \to \infty, \\ e^{-2i\theta_+}, & z \to 0, \end{cases} \end{gathered} \tag{3.30}$$

$$\lim_{z\to\infty} zb(z) = 0, \qquad \lim_{z\to 0} \frac{b(z)}{z^2} = 0. \tag{3.31}$$

3.8. Riemann–Hilbert problem via uniformization coordinates.

3.8.1. Left scattering problem. To take the behavior of the eigenfunctions into account, we can write the "jump" conditions at the real-z axis as

$$\begin{aligned} \frac{M(x,z)}{za(z)} - \frac{\overline{N}(x,z)}{z} &= \rho(z) e^{i(z-q_0^2/z)x} \frac{N(x,z)}{z}, \\ \frac{\overline{M}(x,z)}{z\bar{a}(z)} - \frac{N(x,z)}{z} &= \bar{\rho}(z) e^{-i(z-q_0^2/z)x} \frac{\overline{N}(x,z)}{z}. \end{aligned} \tag{3.32}$$

and the functions are hence bounded at infinity, although they have an additional pole at $z = 0$. We note that $M(x,z)/a(z)$ as a function of z is defined in the upper half-plane $\mathbb{C}^+$, where it by assumption has simple poles z_j, i.e., $a(z_j) = 0$, and that $\overline{M}(x,z)/\bar{a}(z)$ is defined in the lower half-plane $\mathbb{C}^-$, where it has simple poles $\bar{z}_j$, i.e., $\bar{a}(\bar{z}_j) = 0$. Indeed, this is similar to the case of the NLS equation where $a(z)$ and $\bar{a}(z)$ can have multiple zeros and/or zeros on the real axis [8]. The notion of "proper zeros" omits these nongeneric possibilities. At zeros of a and $\bar{a}$, we have

$$M(x,z_j) = b(z_j) e^{i(z_j - q_0^2/z_j)x} N(x,z_j), \qquad \overline{M}(x,\bar{z}_j) = \bar{b}(\bar{z}_j) e^{-i(\bar{z}_j - q_0^2/\bar{z}_j)x} \overline{N}(x,\bar{z}_j). \tag{3.33}$$

Subtracting the values at infinity, the induced pole at the origin, and the poles (assumed to be simple) in the respective upper and lower half-planes at $a(z_j) = 0$, $j = 1, 2, \ldots, J$, and $\bar{a}(\bar{z}_j)$, $j = 1, 2, \ldots, \bar{J}$ (we later see that $J = \bar{J}$), yields

$$\left[\frac{M(x,z)}{za(z)} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} 0 \\ iq_- \end{pmatrix} - \sum_{j=1}^{J} \frac{M(x,z_j)}{(z-z_j)z_j a'(z_j)}\right] -$$

$$- \left[\frac{\overline{N}(x,z)}{z} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} 0 \\ iq_- \end{pmatrix} - \sum_{j=1}^{J} \frac{b(z_j)e^{i(z_j - q_0^2/z_j)x} N(x,z_j)}{(z-z_j)z_j a'(z_j)}\right] =$$

$$= \rho(z) e^{i(z-q_0^2/z)x} \frac{N(x,z)}{z}, \tag{3.34}$$

$$\left[\frac{\overline{M}(x,z)}{z\bar{a}(z)} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} -iq_+ \\ 0 \end{pmatrix} - \sum_{j=1}^{\bar{J}} \frac{\overline{M}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j a'(\bar{z}_j)}\right] -$$

$$- \left[\frac{N(x,z)}{z} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} -iq_+ \\ 0 \end{pmatrix} - \sum_{j=1}^{\bar{J}} \frac{\bar{b}(\bar{z}_j)e^{-i(\bar{z}_j - q_0^2/\bar{z}_j)x} \overline{N}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j \bar{a}'(\bar{z}_j)}\right] =$$

$$= \bar{\rho}(z) e^{-i(z-q_0^2/z)x} \frac{\overline{N}(x,z)}{z}. \tag{3.35}$$

We now introduce the projection operators

$$P_\pm(f)(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - (z \pm i0)}\, d\xi, \tag{3.36}$$

which are well defined for any function $f(\xi)$ that is integrable on the real axis. If $f_\pm(\xi)$ is analytic in the upper and lower z plane and decays at large ξ, then

$$P_\pm(f_\pm)(z) = \pm f_\pm(z), \qquad P_\mp(f_\pm)(z) = 0. \tag{3.37}$$

Applying P_- to (3.34) and P_+ to (3.35), we can obtain

$$\overline{N}(x,z) = \begin{pmatrix} z \\ iq_- \end{pmatrix} + \sum_{j=1}^{J} \frac{z b(z_j) e^{i(z_j - q_0^2/z_j)x} N(x,z_j)}{(z-z_j)z_j a'(z_j)} +$$

$$+ \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{\rho(\xi)}{\xi(\xi - z)} e^{i(\xi - q_0^2/\xi)x} N(x,\xi)\, d\xi, \tag{3.38}$$

$$N(x,z) = \begin{pmatrix} -iq_+ \\ z \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{z \bar{b}(\bar{z}_j) e^{-i(\bar{z}_j - q_0^2/\bar{z}_j)x} \overline{N}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j \bar{a}'(\bar{z}_j)} -$$

$$- \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{\bar{\rho}(\xi)}{\xi(\xi - z)} e^{-i(\xi - q_0^2/\xi)x} \overline{N}(x,\xi)\, d\xi. \tag{3.39}$$

Because the symmetries are between eigenfunctions defined at both $\pm\infty$, we proceed to obtain the inverse scattering integral equations defined from the right end.

3.8.2. Right scattering problem. The right scattering problem can be written as

$$\psi(x,z)=\alpha(z)\bar{\phi}(x,z)+\beta(z)\phi(x,z), \qquad \bar{\psi}(x,z)=\bar{\alpha}(z)\phi(x,z)+\bar{\beta}(z)\bar{\phi}(x,z), \tag{3.40}$$

where $\alpha(z)$, $\bar{\alpha}(z)$, $\beta(z)$, and $\bar{\beta}(z)$ are the right scattering data. Moreover, we can obtain the right scattering data and left scattering data satisfying the relations

$$\bar{\alpha}(z)=\bar{a}(z), \qquad \alpha(z)=a(z), \qquad \bar{\beta}(z)=-b(z), \qquad \beta(z)=-\bar{b}(z). \tag{3.41}$$

Thence, by the symmetry relations for the scattering data, we have

$$\begin{aligned}\frac{N(x,z)}{za(z)}-\frac{\overline{M}(x,z)}{z}&=\rho^*(-z^*)e^{-i(z-q_0^2/z)x}\frac{M(x,z)}{z},\\ \frac{\overline{N}(x,z)}{z\bar{a}(z)}-\frac{M(x,z)}{z}&=\bar{\rho}^*(-z^*)e^{i(z-q_0^2/z)x}\frac{\overline{M}(x,z)}{z},\end{aligned} \tag{3.42}$$

and the functions are hence bounded at infinity, although they have an additional pole at $z=0$. We note that $N(x,z)/a(z)$ as a function of z is defined in the upper half-plane $\mathbb{C}^+$, where (by assumption) it has simple poles z_j, i.e., $a(z_j)=0$, and that $\overline{N}(x,z)/\bar{a}(z)$ is defined in the lower half-plane $\mathbb{C}^-$, where it has simple poles $\bar{z}_j$, i.e., $\bar{a}(\bar{z}_j)=0$. At the zeros of a and $\bar{a}$,

$$\begin{aligned}N(x,z_j)&=-\bar{b}(z_j)M(x,z_j)e^{-i(z_j-q_0^2/z_j)x},\\ \overline{N}(x,\bar{z}_j)&=-b(\bar{z}_j)\overline{M}(x,\bar{z}_j)e^{i(\bar{z}_j-q_0^2/\bar{z}_j)x}.\end{aligned} \tag{3.43}$$

As before, subtracting the values at infinity, the induced pole at the origin, and the poles (assumed to be simple) in the respective upper and lower half-planes at $a(z_j)=0$, $j=1,2,\ldots,J$, and $\bar{a}(\bar{z}_j)$, $j=1,2,\ldots,\bar{J}$, then yields

$$\begin{aligned}&\left[\frac{N(x,z)}{za(z)}-\begin{pmatrix}0\\1\end{pmatrix}-\frac{1}{z}\begin{pmatrix}-iq_-\\0\end{pmatrix}-\sum_{j=1}^{J}\frac{N(x,z_j)}{(z-z_j)z_ja'(z_j)}\right]-\\ &\quad-\left[\frac{\overline{M}(x,z)}{z}-\begin{pmatrix}0\\1\end{pmatrix}-\frac{1}{z}\begin{pmatrix}-iq_-\\0\end{pmatrix}-\sum_{j=1}^{J}\frac{-\bar{b}(z_j)M(x,z_j)e^{-i(z_j-q_0^2/z_j)x}}{(z-z_j)z_ja'(z_j)}\right]=\\ &\qquad=\rho^*(-z^*)e^{-i(z-q_0^2/z)x}\frac{M(x,z)}{z},\end{aligned} \tag{3.44}$$

$$\begin{aligned}&\left[\frac{\overline{N}(x,z)}{z\bar{a}(z)}-\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{z}\begin{pmatrix}0\\iq_+\end{pmatrix}-\sum_{j=1}^{\bar{J}}\frac{\overline{N}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j\bar{a}'(\bar{z}_j)}\right]-\\ &\quad-\left[\frac{M(x,z)}{z}-\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{z}\begin{pmatrix}0\\iq_+\end{pmatrix}-\sum_{j=1}^{\bar{J}}\frac{-b(\bar{z}_j)\overline{M}(x,\bar{z}_j)e^{i(\bar{z}_j-q_0^2/\bar{z}_j)x}}{(z-\bar{z}_j)\bar{z}_j\bar{a}'(\bar{z}_j)}\right]=\\ &\qquad=\bar{\rho}^*(-z^*)e^{i(z-q_0^2/z)x}\frac{\overline{M}(x,z)}{z}.\end{aligned} \tag{3.45}$$

Applying P_- to (3.44) and P_+ to (3.45), we can obtain

$$
\begin{aligned}
\overline{M}(x,z) &= \begin{pmatrix} -iq_- \\ z \end{pmatrix} + \sum_{j=1}^{J} \frac{-z\bar{b}(z_j)M(x,z_j)e^{-i(z_j-q_0^2/z_j)x}}{(z-z_j)z_j a'(z_j)} + \\
&+ \frac{z}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho^*(-\xi)}{\xi(\xi-z)} e^{-i(\xi-q_0^2/\xi)x} M(x,\xi)\,d\xi, \\
M(x,z) &= \begin{pmatrix} z \\ iq_+ \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{-zb(\bar{z}_j)\overline{M}(x,\bar{z}_j)e^{i(\bar{z}_j-q_0^2/\bar{z}_j)x}}{(z-\bar{z}_j)\bar{z}_j\bar{a}'(\bar{z}_j)} - \\
&- \frac{z}{2\pi i}\int_{-\infty}^{+\infty} \frac{\bar{\rho}^*(-\xi)}{\xi(\xi-z)} e^{i(\xi-q_0^2/\xi)x}\overline{M}(x,\xi)\,d\xi.
\end{aligned}
\tag{3.46}
$$

3.9. Recovering the potentials. We use asymptotic formulas to reconstruct the potential. For example, we obtain $\overline{N}_1(x,z)/z \sim q(x)/q_+$ as $z \to 0$ from Eq. (3.30). From (3.38), we can obtain

$$
\begin{aligned}
\frac{\overline{N}_1(x,z)}{z} &\sim 1 + \sum_{j=1}^{J} \frac{b(z_j)e^{i(z_j-q_0^2/z_j)x}}{-z_j^2 a'(z_j)} N_1(x,z_j) + \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\xi)}{\xi^2} e^{i(\xi-q_0^2/\xi)x} N_1(x,\xi)\,d\xi
\end{aligned}
\tag{3.47}
$$

as $z \to 0$. Hence,

$$
\begin{aligned}
q(x) = q_+\Bigg[1 &+ \sum_{j=1}^{J} \frac{b(z_j)e^{i(z_j-q_0^2/z_j)x}}{-z_j^2 a'(z_j)} N_1(x,z_j) + \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\xi)}{\xi^2} e^{i(\xi-q_0^2/\xi)x} N_1(x,\xi)\,d\xi\Bigg].
\end{aligned}
\tag{3.48}
$$

3.10. Closing the system. From $a(q_0^2/z) = e^{2i\theta_+}\bar{a}(z)$, we obtain $J = \bar{J}$. Combining integral equations (3.38) and (3.39) previously obtained, we obtain

$$
\begin{aligned}
&\begin{pmatrix} N_1(x,z) \\ N_2(x,z) \end{pmatrix} = \begin{pmatrix} -iq_+ \\ z \end{pmatrix} + \sum_{j=1}^{J} \frac{z\bar{b}(\bar{z}_j)e^{-i(\bar{z}_j-q_0^2/\bar{z}_j)x}}{(z-\bar{z}_j)\bar{z}_j\bar{a}'(\bar{z}_j)} \times \\
&\times \begin{pmatrix} \bar{z}_j + \sum_{l=1}^{J} \frac{\bar{z}_j b(z_l)e^{i(z_l-q_0^2/z_l)x}}{(\bar{z}_j-z_l)z_l a'(z_l)} N_1(x,z_l) + \frac{\bar{z}_j}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\xi)}{\xi(\xi-\bar{z}_j)} e^{i(\xi-q_0^2/\xi)x} N_1(x,\xi)\,d\xi \\ iq_- + \sum_{l=1}^{J} \frac{\bar{z}_j b(z_l)e^{i(z_l-q_0^2/z_l)x}}{(\bar{z}_j-z_l)z_l a'(z_l)} N_2(x,z_l) + \frac{\bar{z}_j}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\xi)}{\xi(\xi-\bar{z}_j)} e^{i(\xi-q_0^2/\xi)x} N_2(x,\xi)\,d\xi \end{pmatrix} - \\
&- \frac{z}{2\pi i}\int_{-\infty}^{+\infty} \frac{\bar{\rho}(\xi)\,d\xi}{\xi(\xi-z)} e^{-i(\xi-q_0^2/\xi)x} \times \\
&\times \begin{pmatrix} \xi + \sum_{l=1}^{J} \frac{\xi b(z_l)e^{i(z_l-q_0^2/z_l)x}}{(\xi-z_l)z_l a'(z_l)} N_1(x,z_l) + \frac{\xi}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\eta)}{\eta(\eta-\xi)} e^{i(\eta-q_0^2/\eta)x} N_1(x,\eta)\,d\eta \\ iq_- + \sum_{l=1}^{J} \frac{\xi b(z_l)e^{i(z_l-q_0^2/z_l)x}}{(\xi-z_l)z_l a'(z_l)} N_2(x,z_l) + \frac{\xi}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\eta)}{\eta(\eta-\xi)} e^{i(\eta-q_0^2/\eta)x} N_2(x,\eta)\,d\eta \end{pmatrix}.
\end{aligned}
\tag{3.49}
$$

Potential (3.48) can be reconstructed from the solution of integral equation (3.49).

We can obtain an analogous equation for $\overline{M}(x,z)$ [27].

3.11. Trace formula. We have shown that $a(z)$ and $\bar{a}(z)$ are analytic in the respective upper and lower z planes. As mentioned above, we assume that $a(z)$ has simple zeros, which we call z_j. From the symmetry relation $a(q_0^2/z) = e^{2i\theta_+}\bar{a}(z)$, we can deduce that $\bar{a}(z)$ has simple zeros q_0^2/z_j. We set

$$\gamma(z) = a(z)\prod_{j=1}^{J}\frac{z - q_0^2/z_j}{z - z_j}, \qquad \overline{\gamma}(z) = \bar{a}(z)\prod_{j=1}^{J}\frac{z - z_j}{z - q_0^2/z_j}. \tag{3.50}$$

Then $\gamma(z)$ and $\overline{\gamma}(z)$ are analytic in the respective upper and lower z planes and have no zeros in their respective half-planes. We can obtain

$$\begin{aligned} \log\gamma(z) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\gamma(\xi)}{\xi - z}\,d\xi, \qquad \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\overline{\gamma}(\xi)}{\xi - z}\,d\xi = 0, \quad \operatorname{Im} z > 0,\\ \log\overline{\gamma}(z) &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\overline{\gamma}(\xi)}{\xi - z}\,d\xi, \qquad \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\gamma(\xi)}{\xi - z}\,d\xi = 0, \quad \operatorname{Im} z < 0. \end{aligned} \tag{3.51}$$

Adding or subtracting these equations in the corresponding half-plane, we obtain

$$\begin{aligned} \log\gamma(z) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\gamma(\xi)\overline{\gamma}(\xi)}{\xi - z}\,d\xi, \qquad \operatorname{Im} z > 0,\\ \log\overline{\gamma}(z) &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\gamma(\xi)\overline{\gamma}(\xi)}{\xi - z}\,d\xi, \quad \operatorname{Im} z < 0. \end{aligned} \tag{3.52}$$

We note that $\gamma(z)\overline{\gamma}(z) = a(z)\bar{a}(z)$, and from the unitarity condition $a(z)\bar{a}(z) - b(z)\bar{b}(z) = 1$ and the symmetry $b(z) = \bar{b}(z)$, we obtain

$$\begin{aligned} \log a(z) &= \log\left(\prod_{j=1}^{J}\frac{z - z_j}{z - q_0^2/z_j}\right) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log(1 + b^2(\xi))}{\xi - z}\,d\xi, \quad \operatorname{Im} z > 0,\\ \log \bar{a}(z) &= \log\left(\prod_{j=1}^{J}\frac{z - q_0^2/z_j}{z - z_j}\right) - \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log(1 + b^2(\xi))}{\xi - z}\,d\xi, \quad \operatorname{Im} z < 0. \end{aligned} \tag{3.53}$$

Hence, we can reconstruct $a(k)$ and $\bar{a}(k)$ in terms of the eigenvalues (zeros) and only one function $b(k)$. Because $a(z) \sim e^{2i\theta_+}$ as $z \to 0$, from the trace formula when $b(\xi) = 0$ on the real axis, we obtain the constraint on the reflectionless potentials

$$\prod_{j=1}^{J} z_j^2 = q_0^{2J} e^{2i\theta_+}. \tag{3.54}$$

3.12. Discrete scattering data and their symmetries. To find reflectionless potentials (solitons), we must calculate the relevant scattering data: $b(z_j)$ and $\bar{b}(\bar{z}_j)$, $a'(z_j)$ and $\bar{a}'(z_j)$, $j = 1, 2, \ldots, J$. The latter functions can be calculated via the trace formulas. We therefore concentrate on the former. Because

$$\begin{aligned} &N_1(x,z) = -M_2(-x,z), \qquad N_2(x,z) = M_1(-x,z),\\ &M_1(x,z_j) = b(z_j)e^{i(z_j - q_0^2/z_j)x}N_1(x,z_j),\\ &M_2(x,z_j) = b(z_j)e^{i(z_j - q_0^2/z_j)x}N_2(x,z_j), \end{aligned} \tag{3.55}$$

we have

$$N_1(x,z_j) = -b(z_j)e^{-i(z_j-q_0^2/z_j)x}N_2(-x,z_j), \tag{3.56}$$

$$N_2(x,z_j) = b(z_j)e^{-(z_j-q_0^2/z_j)x}N_1(-x,z_j). \tag{3.57}$$

Rewriting (3.57), we obtain

$$N_2(-x,z_j) = b(z_j)e^{(z_j-q_0^2/z_j)x}N_1(x,z_j). \tag{3.58}$$

Combining (3.58) with (3.56), we can deduce the symmetry condition on the discrete data $b(z_j)$

$$-b^2(z_j) = 1. \tag{3.59}$$

A similar analysis shows that $\bar{b}(\bar{z}_j)$ satisfies an analogous equation $-\bar{b}^2(\bar{z}_j) = 1$, i.e.,

$$b(z_j) = \pm i, \qquad \bar{b}(\bar{z}_j) = \pm i. \tag{3.60}$$

By the symmetry relation $\bar{b}(z) = b(z)$, we have $\bar{b}(z_j) = b(z_j)$ and $b(\bar{z}_j) = \bar{b}(\bar{z}_j)$.

For $J = 1$, assuming that $0 < \theta_+ < \pi$, we have $z_1 = q_0e^{i\theta_+}$. By the trace formula with $b(\xi) = 0$ on the real axis, we obtain

$$\begin{aligned} a'(z_1) &= a'(q_0e^{i\theta_+}) = \frac{1}{q_0(e^{i\theta_+} - e^{-i\theta_+})}, \\ \bar{a}'(\bar{z}_1) &= a'(q_0e^{-i\theta_+}) = \frac{1}{q_0(e^{-i\theta_+} - e^{i\theta_+})}. \end{aligned} \tag{3.61}$$

Moreover, from the symmetry relation $b(q_0^2/z) = -\bar{b}(z)$, we obtain

$$\bar{b}(q_0e^{-i\theta_+}) = -b(q_0e^{i\theta_+}). \tag{3.62}$$

For convenience, we write $b(q_0e^{i\theta_+}) = \delta i$, and then $\bar{b}(q_0e^{-i\theta_+}) = -\delta i$, where $\delta = \pm 1$.

3.13. Time evolution. Because

$$v_t = Tv = \begin{pmatrix} 2ik^2 + i\sigma q_0^2 & -2kq_0e^{i(2\sigma q_0^2 t+\theta_\pm)} \\ -2\sigma kq_0e^{i(2\sigma q_0^2 t+\theta_\pm)} & -2ik^2 + i\sigma q_0^2 \end{pmatrix} v := \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v, \tag{3.63}$$

where $\sigma = 1$, we find that both $a(z,t)$ and $\bar{a}(z,t)$ are independent of time and

$$\begin{aligned} b(z;t) &= b(z;0)\exp\left\{-2i\left[q_0^2 + \frac{1}{2}\left(z^2 - \frac{q_0^4}{z^2}\right)\right]t\right\}, \\ \bar{b}(z;t) &= \bar{b}(z;0)\exp\left\{2i\left[q_0^2 + \frac{1}{2}\left(z^2 - \frac{q_0^4}{z^2}\right)\right]t\right\}. \end{aligned} \tag{3.64}$$

Hence,

$$\begin{aligned} b(q_0e^{i\theta_+};t) &= \delta i\exp\{-2iq_0^2[1 + i\sin(2\theta_+)]t\}, \\ \bar{b}(q_0e^{-i\theta_+};t) &= -\delta i\exp\{2iq_0^2[1 - i\sin(2\theta_+)]t\}. \end{aligned} \tag{3.65}$$

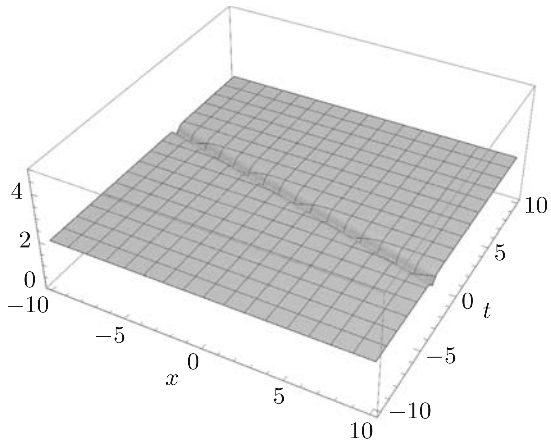

Fig. 1. The amplitude of the one-soliton solution $q(x,t)$ plotted as a function of x and t with the parameters $\sigma = 1$, $\delta = 1$, $\theta_+ = \pi/3$, and $q_0 = 2$.

3.14. Pure one-soliton solution. With $J = 1$ and $b(\xi, t) = 0$ (reflectionless potential), from (3.49) by solving the corresponding Eq. (3.48) (which is a linear system), we obtain

$$q(x,t) = q_0 e^{i(2q_0^2 t+\theta_+)}\left[1 - \frac{2i\sin\theta_+ b(q_0e^{i\theta_+};t)e^{-2q_0x\sin\theta_+}}{q_0e^{2i\theta_+}} N_1(x, q_0e^{i\theta_+};t)\right], \tag{3.66}$$

where

$$N_1(x, q_0e^{i\theta_+}) = \frac{-iq_0e^{i(2q_0^2t+\theta_+)} - q_0e^{i\theta_+}e^{-2q_0x\sin\theta_+}\bar{b}(q_0e^{-i\theta_+};t)}{1 - e^{-4q_0x\sin\theta_+}b(q_0e^{i\theta_+};t)\bar{b}(q_0e^{-i\theta_+};t)}. \tag{3.67}$$

Solving the corresponding discrete system (3.49) with $J = 1$, we find that there is a nonsingular pure one-soliton solution only with $\delta = 1$, which is given by

$$q(x,t) = \frac{1}{2}q_0e^{i(2q_0^2t+\theta_+)}[(1 + e^{-2i\theta_+}) + (1 - e^{-2i\theta_+})\tanh(q_0x\sin\theta_+ - q_0^2t\sin(2\theta_+))]. \tag{3.68}$$

We display a typical nonsingular one-soliton solution in Fig. 1.

4. Nonlocal RST NLS equation: $\sigma = 1$ with $\theta_+ + \theta_- = \pi$

4.1. Direct scattering. In this section, we consider the NZBCs given in (1.6) above with $\sigma = 1$ and $\theta_+ + \theta_- = \pi$. Under this condition, Eq. (2.3) conveniently reduces to

$$\frac{\partial^2 v_j}{\partial x^2} = -(k^2 + q_0^2)v_j, \quad j = 1, 2. \tag{4.1}$$

Each of the two equations has two linearly independent solutions $e^{i\lambda x}$ and $e^{-i\lambda x}$ as $|x| \to \infty$, where we introduce the local polar coordinates

$$k - iq_0 = r_1e^{i\theta_1}, \quad -\frac{\pi}{2} \le \theta_1 < \frac{3\pi}{2}, \qquad k + iq_0 = r_2e^{i\theta_2}, \quad -\frac{\pi}{2} \le \theta_2 < \frac{3\pi}{2}, \tag{4.2}$$

with $r_1 = |k - iq_0|$ and $r_2 = |k + iq_0|$. We can write $\lambda(k) = (r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2+im\pi}$ with $m = 0, 1$ and the respective sheets I ($\mathbb{K}_1$) and II ($\mathbb{K}_2$). The variable k is then considered to belong to a Riemann surface $\mathbb{K}$ consisting of sheets I and II, both coinciding with the complex plane cut along $\Sigma := [-iq_0, iq_0]$ with its edges glued such that $\lambda(k)$ is continuous through the cut. Along the real k axis, we have $\lambda(k) = \pm\mathrm{sgn}(k)\sqrt{k^2+q_0^2}$, where the plus and minus signs apply on the respective sheets I and II of the Riemann surface and where the square root sign denotes the principal branch of the real-valued square root function. We let $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the open upper and lower complex half-planes and $\mathbb{K}^+$ and $\mathbb{K}^-$ denote the open upper and lower complex half-planes cut along Σ. Then λ provides one-to-one correspondences between the sets

1. $k \in \mathbb{K}^+ = \mathbb{C}^+ \setminus (0, iq_0]$ and $\lambda \in \mathbb{C}^+$,
2. $k \in \partial\mathbb{K}^+ = \mathbb{R} \cup \{is - 0^+ : 0 < s < q_0\} \cup \{iq_0\} \cup \{is + 0^+ : 0 < s < q_0\}$ and $\lambda \in \mathbb{R}$,
3. $k \in \mathbb{K}^- = \mathbb{C}^- \setminus [-iq_0, 0)$ and $\lambda \in \mathbb{C}^-$, and
4. $k \in \partial\mathbb{K}^- = \mathbb{R} \cup \{is - 0^+ : -q_0 < s < 0\} \cup \{-iq_0\} \cup \{is + 0^+ : -q_0 < s < 0\}$ and $\lambda \in \mathbb{R}$.

Moreover, $\lambda^+(k)$ and $\lambda^-(k)$ denote the boundary values taken by $\lambda(k)$ for $k \in \Sigma$ from the right and left edges of the cut with $\lambda^\pm(k) = \pm\sqrt{q_0^2 - |k|^2}$, $k = is \pm 0^+$, $|s| < q_0$ on the right and left edge of the cut (see Figs. 5 and 6 on p. 30 in [26]).

4.2. Eigenfunctions. As in Sec. 3, we introduce the eigenfunctions $\phi(x,k)$, $\bar\phi(x,k)$, $\psi(x,k)$, and $\bar\psi(x,k)$ defined by their BCs. We substitute them in (2.3) with $\sigma = 1$ and from (3.5) and (3.6) obtain

$$w = \begin{pmatrix} \lambda + k \\ iq_+ \end{pmatrix}, \qquad \overline{w} = \begin{pmatrix} -iq_- \\ \lambda + k \end{pmatrix}, \qquad v = \begin{pmatrix} -iq_+ \\ \lambda + k \end{pmatrix}, \qquad \overline{v} = \begin{pmatrix} \lambda + k \\ iq_- \end{pmatrix}, \tag{4.3}$$

where $\lambda = \sqrt{k^2 + q_0^2}$. We also consider functions with constant BCs and define the same bounded eigenfunctions $M(x,k)$, $N(x,k)$, $\overline{M}(x,k)$, and $\overline{N}(x,k)$ defined as in (3.7) with the new definition of λ.

Moreover, the bounded eigenfunctions can be represented using integral equations with the same formulas as (3.8) but with the different definition of λ given above.

Using similar methods as in the preceding case ($\theta_+ + \theta_- = 0$), we obtain the following result.

Theorem 3. *Let the elements of $Q - Q_\pm$ belong to $L^{1,1}(\mathbb{R})$. Then for each $x \in \mathbb{R}$, the eigenfunctions $M(x,k)$ and $N(x,k)$ are continuous for $k \in \overline{\mathbb{K}^+} \cup \partial\overline{\mathbb{K}^-}$ and analytic for $k \in \mathbb{K}^+$, and $\overline{M}(x,k)$ and $\overline{N}(x,k)$ are continuous for $k \in \overline{\mathbb{K}^-} \cup \partial\overline{\mathbb{K}^+}$ and analytic for $k \in \mathbb{K}^-$.*

The proof uses Neumann series and is similar to the proof in [26].

4.2.1. Scattering data. We have

$$\phi(x,k) = b(k)\psi(x,k) + a(k)\bar\psi(x,k), \qquad \bar\phi(x,k) = \bar a(k)\psi(x,k) + \bar b(k)\bar\psi(x,k) \tag{4.4}$$

for any k such that all four eigenfunctions exist. Moreover,

$$a(k)\bar a(k) - b(k)\bar b(k) = 1, \tag{4.5}$$

where the formulas for $a(k)$, $\bar a(k)$, $b(k)$, and $\bar b(k)$ are given in (3.19) but the definition of λ given in this section differs from the case in Sec. 3.

If $k \in (-iq_0, iq_0)$, then the above scattering data and eigenfunctions are defined using the corresponding values on the right and left edges of the cut and are labeled with superscripts $\pm$ as clarified below. Explicitly, for $k \in (-iq_0, iq_0)$, we have

$$\begin{aligned} a^\pm(k) &= \frac{W(\phi^\pm(x,k), \psi^\pm(x,k))}{2\lambda^\pm(\lambda^\pm + k)}, \qquad & \bar a^\pm(k) &= -\frac{W(\bar\phi^\pm(x,k), \bar\psi^\pm(x,k))}{2\lambda^\pm(\lambda^\pm + k)}, \\ b^\pm(k) &= -\frac{W(\phi^\pm(x,k), \bar\psi^\pm(x,k))}{2\lambda^\pm(\lambda^\pm + k)}, \qquad & \bar b^\pm(k) &= \frac{W(\bar\phi^\pm(x,k), \psi^\pm(x,k))}{2\lambda^\pm(\lambda^\pm + k)}. \end{aligned} \tag{4.6}$$

We then obtain the following theorem from the analytic behavior of the eigenfunctions.

Theorem 4. *Let the elements of $Q - Q_\pm$ belong to $L^{1,1}(\mathbb{R})$. Then $a(k)$ is continuous for $k \in \overline{\mathbb{K}^+} \cup \partial\overline{\mathbb{K}^-} \setminus \{\pm iq_0\}$ and analytic for $k \in \mathbb{K}^+$, and $\bar{a}(k)$ is continuous for $k \in \overline{\mathbb{K}^-} \cup \partial\overline{\mathbb{K}^+} \setminus \{\pm iq_0\}$ and analytic for $k \in \mathbb{K}^-$. Moreover, $b(k)$ and $\bar{b}(k)$ are continuous in $k \in \mathbb{R} \cup (-iq_0, iq_0)$. In addition, if the elements of $Q - Q_\pm$ do not increase faster than e^{-ax^2}, where a is a positive real number, then $a(k)\lambda(k)$, $\bar{a}(k)\lambda(k)$, $b(k)\lambda(k)$, and $\bar{b}(k)\lambda(k)$ are analytic for $k \in \mathbb{K}$.*

The proof uses Neumann series and is similar to the proof in [26].

4.3. Symmetry reductions. The symmetry in the potential induces a symmetry between the eigenfunctions. Indeed, if $v(x,k) = (v_1(x,k), v_2(x,k))^{\mathrm{T}}$ solves (2.1) for $\sigma = 1$, then $(v_2(-x,k), -v_1(-x,k))^{\mathrm{T}}$ also solves (2.1) for $\sigma = 1$. Taking the BCs into account, we obtain

$$\psi(x,k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \phi(-x,k), \qquad \bar{\psi}(x,k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\phi}(-x,k). \tag{4.7}$$

We can similarly obtain the symmetry relations of the eigenfunctions, i.e.,

$$N(x,k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M(-x,k), \qquad \overline{N}(x,k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{M}(-x,k). \tag{4.8}$$

We obtain $\bar{b}(k) = b(k)$ from the Wronskian representations for the scattering data and the above symmetry relations.

4.4. Uniformization coordinates. Similarly, we introduce a uniformization variable z defined by the conformal map

$$z = z(k) = k + \lambda(k), \tag{4.9}$$

where $\lambda = \sqrt{k^2 + q_0^2}$ and the inverse map is given by $k = k(z) = (z - q_0^2/z)/2$. Then $\lambda(z) = (z + q_0^2/z)/2$. We let C_0 be the circle of radius q_0 centered at the origin in z plane. We note the following:

1. The branch cut on either sheet is mapped onto C_0. In particular, $z(\pm iq_0) = \pm iq_0$ from either sheet, $z(0_{\mathrm{I}}^{\pm}) = \pm q_0$ and $z(0_{\mathrm{II}}^{\pm}) = \mp q_0$.

2. $\mathbb{K}_1$ is mapped onto the exterior of C_0, and $\mathbb{K}_2$ is mapped onto the interior of C_0. In particular, $z(\infty_{\mathrm{I}}) = \infty$ and $z(\infty_{\mathrm{II}}) = 0$. The first and second quadrants of $\mathbb{K}_1$ are mapped into the respective first and second quadrants outside C_0; the first and second quadrants of $\mathbb{K}_2$ are mapped into the respective second and first quadrants inside C_0; $z_{\mathrm{I}} z_{\mathrm{II}} = q_0^2$.

3. The regions in the k plane such that $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ are respectively mapped onto $D^+ = \{z \in \mathbb{C} : (|z|^2 - q_0^2) \cdot \operatorname{Im} z > 0\}$ and $D^- = \{z \in \mathbb{C} : (|z|^2 - q_0^2) \cdot \operatorname{Im} z < 0\}$ (see Fig. 11 on p. 36 in [26]).

We then find that the eigenfunctions M and N are analytic for $z \in D^+$ and the eigenfunctions $\overline{M}$ and $\overline{N}$ are analytic for $z \in D^-$.

4.5. Symmetries via uniformization coordinates. From the above eigenfunction symmetry relations, we obtain

$$\psi(x,z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \phi(-x,z), \qquad \bar{\psi}(x,z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\phi}(-x,z). \tag{4.10}$$

Further, if $z \to -q_0^2/z$, then $(k, \lambda) \to (k, -\lambda)$. Hence,

$$\phi\left(x, -\frac{q_0^2}{z}\right) = \frac{q_0^2/z}{iq_-}\bar{\phi}(x,z), \qquad \psi\left(x, -\frac{q_0^2}{z}\right) = \frac{-iq_+}{z}\bar{\psi}(x,z), \quad z \in D^-. \tag{4.11}$$

Similarly, we obtain

$$N(x,z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M(-x,z), \qquad \overline{N}(x,z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{M}(-x,z), \tag{4.12}$$

$$\bar{b}(z) = b(z), \qquad a\left(-\frac{q_0^2}{z}\right) = -e^{2i\theta_+}\bar{a}(z), \quad z \in D^-, \qquad b\left(-\frac{q_0^2}{z}\right) = -\bar{b}(z). \tag{4.13}$$

4.6. Asymptotic behavior of eigenfunctions and scattering data. To solve the inverse problem, we must determine the asymptotic behavior of eigenfunctions and scattering data both as $z \to \infty$ in $\mathbb{K}_1$ and as $z \to 0$ in $\mathbb{K}_2$. We have

$$\begin{gathered} \overline{N}(x,z) \sim \begin{pmatrix} z\dfrac{q(x)}{q_+} \\ iq_- \end{pmatrix}, \quad z \to 0, \\ a(z) = \begin{cases} 1, & z \to \infty, \\ -e^{2i\theta_+}, & z \to 0, \end{cases} \qquad \bar{a}(z) = \begin{cases} 1, & z \to \infty, \\ -e^{-2i\theta_+}, & z \to 0, \end{cases} \\ \lim_{z\to\infty} zb(z) = 0, \qquad \lim_{z\to 0} \frac{b(z)}{z^2} = 0. \end{gathered} \tag{4.14}$$

4.7. Riemann–Hilbert problem via uniformization coordinates.

4.7.1. Left scattering problem. To take the behavior of the eigenfunctions into account, we can write the "jump" conditions at Σ, where

$$\begin{aligned} \Sigma := (-\infty, -q_0) \cup (q_0, +\infty) \cup \overrightarrow{(q_0, -q_0)} \cup \{q_0 e^{i\theta}, \pi \le \theta \le 2\pi\}_{\text{clockwise, upper circle}} \cup \\ \cup \{q_0 e^{i\theta}, -\pi \le \theta \le 0\}_{\text{counterclockwise, lower circle}}, \end{aligned}$$

as

$$\begin{aligned} \frac{M(x,z)}{za(z)} - \frac{\overline{N}(x,z)}{z} &= \rho(z)e^{i(z+q_0^2/z)x}\frac{N(x,z)}{z}, \\ \frac{\overline{M}(x,z)}{z\bar{a}(z)} - \frac{N(x,z)}{z} &= \bar{\rho}(z)e^{-i(z+q_0^2/z)x}\frac{\overline{N}(x,z)}{z}, \end{aligned} \tag{4.15}$$

and the functions are hence bounded at infinity, although they have an additional pole at $z = 0$. We note that $M(x,z)/a(z)$ as a function of z is defined in D^+, where (by assumption) it has simple poles z_j, i.e., $a(z_j) = 0$, and $\overline{M}(x,z)/\bar{a}(z)$, is defined in D^-, where it has simple poles $\bar{z}_j$, i.e., $\bar{a}(\bar{z}_j) = 0$. It follows that

$$\begin{aligned} M(x,z_j) &= b(z_j)e^{i(z_j+q_0^2/z_j)x}N(x,z_j), \\ \overline{M}(x,\bar{z}_j) &= \bar{b}(\bar{z}_j)e^{-i(\bar{z}_j+q_0^2/\bar{z}_j)x}\overline{N}(x,\bar{z}_j). \end{aligned} \tag{4.16}$$

Subtracting the values at infinity, the induced pole at the origin, and the poles assumed to be simple in D^+ and D^- respectively at $a(z_j) = 0$, $j = 1, 2, \ldots, J$, and $\bar{a}(\bar{z}_j)$, $j = 1, 2, \ldots, \bar{J}$, yields

$$\begin{aligned}
&\left[\frac{M(x,z)}{za(z)} - \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{z}\begin{pmatrix}0\\iq_-\end{pmatrix} - \sum_{j=1}^{J}\frac{M(x,z_j)}{(z-z_j)z_j a'(z_j)}\right] - \\
&\qquad - \left[\frac{\overline{N}(x,z)}{z} - \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{z}\begin{pmatrix}0\\iq_-\end{pmatrix} - \sum_{j=1}^{J}\frac{b(z_j)e^{i(z_j+q_0^2/z_j)x}N(x,z_j)}{(z-z_j)z_j a'(z_j)}\right] = \\
&\qquad\qquad = \rho(z)e^{i(z+q_0^2/z)x}\frac{N(x,z)}{z},
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&\left[\frac{\overline{M}(x,z)}{z\bar{a}(z)} - \begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{z}\begin{pmatrix}-iq_+\\0\end{pmatrix} - \sum_{j=1}^{\bar{J}}\frac{\overline{M}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j a'(\bar{z}_j)}\right] - \\
&\qquad - \left[\frac{N(x,z)}{z} - \begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{z}\begin{pmatrix}-iq_+\\0\end{pmatrix} - \sum_{j=1}^{\bar{J}}\frac{\bar{b}(\bar{z}_j)e^{-i(\bar{z}_j+q_0^2/\bar{z}_j)x}\overline{N}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j \bar{a}'(\bar{z}_j)}\right] = \\
&\qquad\qquad = \bar{\rho}(z)e^{-i(z+q_0^2/z)x}\frac{\overline{N}(x,z)}{z}.
\end{aligned} \tag{4.18}$$

We now introduce the projection operators

$$P_\pm(f)(z) = \frac{1}{2\pi i}\int_\Sigma \frac{f(\xi)}{\xi - (z \pm i0)}\, d\xi, \tag{4.19}$$

where z is in the $\pm$ regions and

$$\begin{aligned}
\Sigma := (-\infty, -q_0) \cup (q_0, +\infty) \cup \overrightarrow{(q_0, -q_0)} \cup \{q_0 e^{i\theta}, \pi \le \theta \le 2\pi\}_{\text{clockwise, upper circle}} \cup \\
\cup \{q_0 e^{i\theta}, -\pi \le \theta \le 0\}_{\text{counterclockwise, lower circle}}.
\end{aligned}$$

If $f_\pm(\xi)$ is analytic in $D^\pm$ and decays at large ξ, then

$$P_\pm(f_\pm)(z) = \pm f_\pm(z), \qquad P_\mp(f_\pm)(z) = 0. \tag{4.20}$$

Applying P_- to (4.17) and P_+ to (4.18), we obtain

$$\begin{aligned}
\overline{N}(x,z) &= \begin{pmatrix}z\\ & iq_-\end{pmatrix} + \sum_{j=1}^{J}\frac{zb(z_j)e^{i(z_j+q_0^2/z_j)x}N(x,z_j)}{(z-z_j)z_j a'(z_j)} + \\
&\quad + \frac{z}{2\pi i}\int_\Sigma \frac{\rho(\xi)}{\xi(\xi-z)}e^{i(\xi+q_0^2/\xi)x}N(x,\xi)\,d\xi, \\
N(x,z) &= \begin{pmatrix}-iq_+\\ z\end{pmatrix} + \sum_{j=1}^{\bar{J}}\frac{z\bar{b}(\bar{z}_j)e^{-i(\bar{z}_j+q_0^2/\bar{z}_j)x}\overline{N}(x,\bar{z}_j)}{(z-\bar{z}_j)\bar{z}_j \bar{a}'(\bar{z}_j)} - \\
&\quad - \frac{z}{2\pi i}\int_\Sigma \frac{\bar{\rho}(\xi)}{\xi(\xi-z)}e^{-i(\xi+q_0^2/\xi)x}\overline{N}(x,\xi)\,d\xi.
\end{aligned} \tag{4.21}$$

We can similarly solve the right scattering problem, which is

$$\begin{aligned}\overline{M}(x,z) = &\begin{pmatrix}-iq_-\\ z\end{pmatrix} + \sum_{j=1}^{J}\frac{-z\bar{b}(z_j)M(x,z_j)e^{-i(z_j+q_0^2/z_j)x}}{(z-z_j)z_ja'(z_j)} + \\ &+ \frac{z}{2\pi i}\int_\Sigma \frac{\rho^*(-\xi^*)}{\xi(\xi-z)}e^{-i(\xi+q_0^2/\xi)x}M(x,\xi)\,d\xi,\\ M(x,z) = &\begin{pmatrix}z\\ iq_+\end{pmatrix} + \sum_{j=1}^{\bar{J}}\frac{-zb(\bar{z}_j)\overline{M}(x,\bar{z}_j)e^{i(\bar{z}_j+q_0^2/\bar{z}_j)x}}{(z-\bar{z}_j)\bar{z}_j\bar{a}'(\bar{z}_j)} - \\ &- \frac{z}{2\pi i}\int_\Sigma \frac{\bar{\rho}^*(-\xi^*)}{\xi(\xi-z)}e^{i(\xi+q_0^2/\xi)x}\overline{M}(x,\xi)\,d\xi.\end{aligned} \tag{4.22}$$

4.8. Recovery of the potentials. We note that as $z \to 0$, $\overline{N}_1(x,z)/z \sim q(x)/q_+$,

$$\frac{\overline{N}_1(x,z)}{z} \sim 1 + \sum_{j=1}^{J}\frac{b(z_j)e^{i(z_j+q_0^2/z_j)x}}{-z_j^2a'(z_j)}N_1(x,z_j) + \frac{1}{2\pi i}\int_\Sigma \frac{\rho(\xi)}{\xi^2}e^{i(\xi+q_0^2/\xi)x}N_1(x,\xi)\,d\xi, \tag{4.23}$$

and we have

$$\begin{aligned}q(x) =q_+\Bigg[1 + &\sum_{j=1}^{J}\frac{b(z_j)e^{i(z_j+q_0^2/z_j)x}}{-z_j^2a'(z_j)}N_1(x,z_j) + \\ &+ \frac{1}{2\pi i}\int_\Sigma \frac{\rho(\xi)}{\xi^2}e^{i(\xi+q_0^2/\xi)x}N_1(x,\xi)\,d\xi\Bigg].\end{aligned} \tag{4.24}$$

4.9. Closing the system. We obtain $J = \bar{J}$ from $a(-q_0^2/z) = -e^{2i\theta_+}\bar{a}(z)$. Combining the integral equations of eigenfunctions, we obtain

$$\begin{aligned}&\begin{pmatrix}N_1(x,z)\\ N_2(x,z)\end{pmatrix} = \begin{pmatrix}-iq_+\\ z\end{pmatrix} + \sum_{j=1}^{J}\frac{z\bar{b}(\bar{z}_j)e^{-i(\bar{z}_j+q_0^2/\bar{z}_j)x}}{(z-\bar{z}_j)\bar{z}_j\bar{a}'(\bar{z}_j)} \times\\ &\times\begin{pmatrix}\bar{z}_j + \sum_{l=1}^{J}\frac{\bar{z}_jb(z_l)e^{i(z_l+q_0^2/z_l)x}}{(\bar{z}_j-z_l)z_la'(z_l)}N_1(x,z_l) + \frac{\bar{z}_j}{2\pi i}\int_\Sigma\frac{\rho(\xi)}{\xi(\xi-\bar{z}_j)}e^{i(\xi+q_0^2/\xi)x}N_1(x,\xi)\,d\xi\\ iq_- + \sum_{l=1}^{J}\frac{\bar{z}_jb(z_l)e^{i(z_l+q_0^2/z_l)x}}{(\bar{z}_j-z_l)z_la'(z_l)}N_2(x,z_l) + \frac{\bar{z}_j}{2\pi i}\int_\Sigma\frac{\rho(\xi)}{\xi(\xi-\bar{z}_j)}e^{i(\xi+q_0^2/\xi)x}N_2(x,\xi)\,d\xi\end{pmatrix} -\\ &- \frac{z}{2\pi i}\int_\Sigma\frac{\bar{\rho}(\xi)}{\xi(\xi-z)}e^{-i(\xi+q_0^2/\xi)x}\times\\ &\times\begin{pmatrix}\xi + \sum_{l=1}^{J}\frac{\xi b(z_l)e^{i(z_l-q_0^2/z_l)x}}{(\xi-z_l)z_la'(z_l)}N_1(x,z_l) + \frac{\xi}{2\pi i}\int_\Sigma\frac{\rho(\eta)}{\eta(\eta-\xi)}e^{i(\eta+q_0^2/\eta)x}N_1(x,\eta)\,d\eta\\ iq_- + \sum_{l=1}^{J}\frac{\xi b(z_l)e^{i(z_l+q_0^2/z_l)x}}{(\xi-z_l)z_la'(z_l)}N_2(x,z_l) + \frac{\xi}{2\pi i}\int_\Sigma\frac{\rho(\eta)}{\eta(\eta-\xi)}e^{i(\eta+q_0^2/\eta)x}N_2(x,\eta)\,d\eta\end{pmatrix}d\xi.\end{aligned} \tag{4.25}$$

We can reconstruct the potential from (4.24).

4.10. Trace formula. Similarly to Sec. 3, we obtain the trace formula as

$$
\begin{aligned}
\log a(z) &= \log\biggl(\prod_{j=1}^{J} \frac{z - z_j}{z + q_0^2/z_j}\biggr) + \frac{1}{2\pi i}\int_{\Sigma} \frac{\log(1+b^2(\xi))}{\xi - z}\,d\xi, \quad z \in D^+, \\
\log \bar{a}(z) &= \log\biggl(\prod_{j=1}^{J} \frac{z + q_0^2/z_j}{z - z_j}\biggr) - \frac{1}{2\pi i}\int_{\Sigma} \frac{\log(1+b^2(\xi))}{\xi - z}\,d\xi, \quad z \in D^-.
\end{aligned}
\tag{4.26}
$$

4.11. Discrete scattering data and their symmetries. To find reflectionless potentials and solitons, we must be able to calculate the relevant discrete scattering data. The coefficients $b(z_j)$ and $\bar{b}(\bar{z}_j)$, $j = 1, 2, \ldots, J$, can be calculated in the same way as in Sec. 3, and we have

$$
\bar{b}(z_j) = b(z_j) = \pm i, \qquad b(\bar{z}_j) = \bar{b}(\bar{z}_j) = \pm i. \tag{4.27}
$$

Because $a(z) \sim -e^{2i\theta_+}$ as $z \to 0$, from the trace formula with $b(\xi) = 0$ in Σ, we obtain the constraint for the reflectionless potentials

$$
\prod_{j=1}^{J} z_j = \pm(-1)^{(J+1)/2} q_0^J e^{i\theta_+}. \tag{4.28}
$$

We claim that $J \geq 2$. Otherwise, if $J = 1$, then $z_1 = \pm q_0 e^{i\theta_+}$. This implies that the eigenvalue z_1 is on the circle, which in this case is the continuous spectrum. Such eigenvalues are not proper; they are not considered here.

4.12. Reflectionless scattering data 2-eigenvalues. In this subsection, we consider scattering data associated with 2-eigenvalues, i.e., $J = 2$, with no reflection. We note that $|z_1| \cdot |z_2| = q_0^2$. Now let $\pi < \theta_+ < 3\pi/2$ and $z_1 = q_1 e^{i\theta_1}$, where $q_1 > q_0$ and $0 < \theta_1 < \pi/2$. Then $z_2 = (q_0^2/q_1)e^{i(\theta_+ - \theta_1 + \pi/2)}$, where $\pi < \arg z_2 < 2\pi$. In particular, we can choose $z_1 = iq_1$, $z_2 = -i(q_0^2/q_1)$, and $\theta_+ = \pi/2$. Then $\bar{z}_1 = i(q_0^2/q_1)$, $\bar{z}_2 = -iq_1$, and $z_1 z_2 = q_0^2$. From the trace formula with $b(\xi) = 0$ in Σ, we obtain

$$
a(z) = \frac{z - iq_1}{z - i(q_0^2/q_1)} \cdot \frac{z + i(q_0^2/q_1)}{z + iq_1}, \qquad \bar{a}(z) = \frac{z - i(q_0^2/q_1)}{z - iq_1} \cdot \frac{z + iq_1}{z + i(q_0^2/q_1)}. \tag{4.29}
$$

We have

$$
\begin{aligned}
a'(iq_1) &= \frac{-i(q_1^2 + q_0^2)}{2q_1(q_1^2 - q_0^2)}, & a'\biggl(-i\frac{q_0^2}{q_1}\biggr) &= \frac{iq_1(q_0^2 + q_1^2)}{2q_0^2(q_0^2 - q_1^2)}, \\
\bar{a}'\biggl(i\frac{q_0^2}{q_1}\biggr) &= -\frac{iq_1(q_0^2 + q_1^2)}{2q_0^2(q_0^2 - q_1^2)}, & \bar{a}'(-iq_1) &= \frac{i(q_1^2 + q_0^2)}{2q_1(q_1^2 - q_0^2)}.
\end{aligned}
\tag{4.30}
$$

Moreover, from the symmetry relation $b(-q_0^2/z) = -\bar{b}(z)$, we obtain

$$
b(iq_1) = \delta_1 i, \qquad b\biggl(-i\frac{q_0^2}{q_1}\biggr) = \delta_2 i, \qquad \bar{b}\biggl(i\frac{q_0^2}{q_1}\biggr) = -\delta_1 i, \qquad \bar{b}(-iq_1) = -\delta_2 i. \tag{4.31}
$$

4.13. Time evolution. In the same way as in Sec. 3, we find that both $a(z,t)$ and $\bar{a}(z,t)$ are independent of time and

$$
\begin{aligned}
b(z;t) &= b(z;0)\exp\biggl\{-2i\biggl[-q_0^2 + \frac{1}{2}\biggl(z^2 - \frac{q_0^4}{z^2}\biggr)\biggr]t\biggr\}, \\
\bar{b}(z;t) &= \bar{b}(z;0)\exp\biggl\{2i\biggl[-q_0^2 + \frac{1}{2}\biggl(z^2 - \frac{q_0^4}{z^2}\biggr)\biggr]t\biggr\}.
\end{aligned}
\tag{4.32}
$$

Hence,

$$
\begin{aligned}
b(iq_1;t) &= \delta_1 i \exp\left\{-2i\left[-q_0^2+\frac{1}{2}\left(-q_1^2+\frac{q_0^4}{q_1^2}\right)\right]t\right\},\\
b\left(-i\frac{q_0^2}{q_1};t\right) &= \delta_2 i \exp\left\{-2i\left[-q_0^2+\frac{1}{2}\left(q_1^2-\frac{q_0^4}{q_1^2}\right)\right]t\right\},\\
\bar{b}\left(i\frac{q_0^2}{q_1};t\right) &= -\delta_1 i \exp\left\{2i\left[-q_0^2+\frac{1}{2}\left(q_1^2-\frac{q_0^4}{q_1^2}\right)\right]t\right\},\\
\bar{b}(-iq_1;t) &= -\delta_2 i \exp\left\{2i\left[-q_0^2+\frac{1}{2}\left(-q_1^2+\frac{q_0^4}{q_1^2}\right)\right]t\right\}.
\end{aligned}
\tag{4.33}
$$

4.14. Pure two-soliton solution. With $J=2$ and $b(\xi,t)=0$ (reflectionless potential), solving the corresponding discrete system (4.25) and also from (4.24), we find a nonsingular two-soliton solution with $\delta_1\delta_2=-1$.

In this case, the normalization constants are

$$
\begin{aligned}
C_1(t) &:= \frac{b(iq_1;t)}{a'(iq_1;t)} = -\frac{2\delta_1 q_1(q_1^2-q_0^2)}{q_1^2+q_0^2}\exp\left\{-2i\left[-q_0^2+\frac{1}{2}\left(-q_1^2+\frac{q_0^4}{q_1^2}\right)\right]t\right\},\\
C_2(t) &:= \frac{b(-i(q_0^2/q_1);t)}{a'(-i(q_0^2/q_1);t)} = \frac{2\delta_2 q_0^2(q_0^2-q_1^2)}{q_1(q_0^2+q_1^2)}\exp\left\{-2i\left[-q_0^2+\frac{1}{2}\left(q_1^2-\frac{q_0^4}{q_1^2}\right)\right]t\right\},\\
\overline{C}_1(t) &:= \frac{\bar{b}(i(q_0^2/q_1);t)}{\bar{a}'(i(q_0^2/q_1);t)} = \frac{2\delta_1 q_0^2(q_0^2-q_1^2)}{q_1(q_0^2+q_1^2)}\exp\left\{2i\left[-q_0^2+\frac{1}{2}\left(q_1^2-\frac{q_0^4}{q_1^2}\right)\right]t\right\},\\
\overline{C}_2(t) &:= \frac{\bar{b}(-iq_1;t)}{\bar{a}'(-iq_1;t)} = -\frac{2\delta_2 q_1(q_1^2-q_0^2)}{q_1^2+q_0^2}\exp\left\{2i\left[-q_0^2+\frac{1}{2}\left(-q_1^2+\frac{q_0^4}{q_1^2}\right)\right]t\right\}.
\end{aligned}
\tag{4.34}
$$

For $\delta_1=1$ and $\delta_2=-1$, we obtain

$$
\begin{aligned}
q(x,t) &= e^{-2iq_0^2t}\times\\
&\times\frac{\left[i(q_0^4+q_1^4)\cos\left(\frac{q_0^4-q_1^4}{q_1^2}t\right)+iq_0q_1(q_0^2+q_1^2)\cosh\left(\frac{q_0^2-q_1^2}{q_1}x\right)+(q_0^4-q_1^4)\sin\left(\frac{q_0^4-q_1^4}{q_1^2}t\right)\right]}{q_1\left[2q_0q_1\cos\left(\frac{(q_0^4-q_1^4)t}{q_1^2}\right)+(q_0^2+q_1^2)\cosh\left(\frac{(q_0^2-q_1^2)x}{q_1}\right)\right]}.
\end{aligned}
\tag{4.35}
$$

For $\delta_1=-1$ and $\delta_2=1$, we obtain

$$
\begin{aligned}
q(x,t) &= e^{-2iq_0^2t}\times\\
&\times\frac{\left[-i(q_0^4+q_1^4)\cos\left(\frac{q_0^4-q_1^4}{q_1^2}t\right)+iq_0q_1(q_0^2+q_1^2)\cosh\left(\frac{q_0^2-q_1^2}{q_1}x\right)+(-q_0^4+q_1^4)\sin\left(\frac{q_0^4-q_1^4}{q_1^2}t\right)\right]}{q_1\left[-2q_0q_1\cos\left(\frac{(q_0^4-q_1^4)t}{q_1^2}\right)+(q_0^2+q_1^2)\cosh\left(\frac{(q_0^2-q_1^2)x}{q_1}\right)\right]}.
\end{aligned}
\tag{4.36}
$$

In Fig. 2, we display a typical two-soliton solution of breather type with $\delta_1=-\delta_1=1$.

5. Spatially dependent BCs

5.1. RST NLS: Spatial boundary values. We next consider the RST NLS equation

$$
iq_t(x,t)=q_{xx}(x,t)-2\sigma q^2(x,t)q(-x,-t) \tag{5.1}
$$

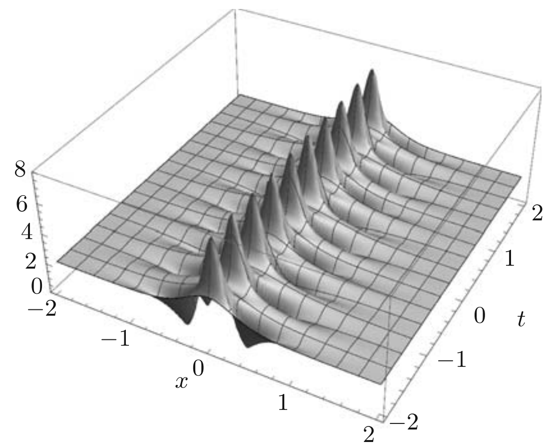

Fig. 2. The amplitude of $q(x,t)$ with $\delta_1 = 1$, $\delta_2 = -1$, $q_1 = 4$, and $q_0 = 2$.

with the BC

$$q(x,t) \to q_0 e^{i(\alpha t + \beta x + \theta_\pm)} \tag{5.2}$$

as $x \to \pm\infty$, where $q_0 > 0$, $0 \le \theta_\pm < 2\pi$, and both α and β are real. We see that $\alpha = \beta^2 + 2\sigma\rho q_0^2$, and the BC then becomes

$$q(x,t) \to q_0 e^{i[(\beta^2 + 2\sigma\rho q_0^2)t + \beta x + \theta_\pm]} \tag{5.3}$$

as $x \to \pm\infty$, where β is real and $\rho = \pm 1$. In particular, $\rho = 1$ if $\theta_+ + \theta_- = 0$, and $\rho = -1$ if $\theta_+ + \theta_- = \pi$. Setting $q(x,t) = \tilde{q}(x,t)e^{i\beta x}$, we obtain

$$i\tilde{q}_t(x,t) = \tilde{q}_{xx}(x,t) + 2i\beta\tilde{q}_x(x,t) - [\beta^2 + 2\sigma\tilde{q}(x,t)\tilde{q}(-x,-t)]\tilde{q}(x,t), \tag{5.4}$$

which is associated with the 2×2 compatible system

$$\begin{aligned} v_x &= Xv = \begin{pmatrix} -ik\tilde{q}(x,t) & \\ \sigma\tilde{q}(-x,-t) & ik \end{pmatrix} v, \\ v_t &= Tv = \\ &= \begin{pmatrix} \begin{matrix} 2ik^2 - 2\beta ik + \\ + i\sigma\tilde{q}(x,t)\tilde{q}(-x,-t) + \dfrac{\beta^2}{2}i \end{matrix} & -2k\tilde{q}(x,t) + 2\beta\tilde{q}(x,t) - i\tilde{q}_x(x,t) \\ \begin{matrix} -2\sigma k\tilde{q}(-x,-t) + \\ + i\sigma\tilde{q}_x(-x,-t) + 2\beta\sigma\tilde{q}(-x,-t) \end{matrix} & \begin{matrix} -2ik^2 + 2\beta ik - \\ - i\sigma\tilde{q}(x,t)\tilde{q}(-x,-t) - \dfrac{\beta^2}{2}i \end{matrix} \end{pmatrix} v. \end{aligned} \tag{5.5}$$

We then find that $a(k,t)$ and $\bar{a}(k,t)$ are independent of time and $b(k,t)$ and $\bar{b}(k,t)$ depend on time as

$$\begin{aligned} b(k,t) &= b(0)\exp\left\{-2i\left[\sigma\rho q_0^2 + \frac{\beta^2}{2} - 2\lambda(\beta - k)\right]t\right\}, \\ \bar{b}(k,t) &= \bar{b}(0)\exp\left\{2i\left[\sigma\rho q_0^2 + \frac{\beta^2}{2} - 2\lambda(\beta - k)\right]t\right\}. \end{aligned} \tag{5.6}$$

We can take $\sigma = 1$ as discussed in Sec. 1 and $J = 1$ and $\delta = 1$ from Sec. 3. Then

$$
\begin{aligned}
b(q_0 e^{i\theta_+}; t) &= i \exp\left\{-2i\left[q_0^2 + \frac{\beta^2}{2} - 2iq_0 \sin\theta_+(\beta - q_0\cos\theta_+)\right]t\right\},\\
\bar{b}(q_0 e^{-i\theta_+}; t) &= -i \exp\left\{2i\left[q_0^2 + \frac{\beta^2}{2} + 2iq_0 \sin\theta_+(\beta - q_0\cos\theta_+)\right]t\right\}.
\end{aligned}
\tag{5.7}
$$

We thus obtain a pure one-soliton solution

$$
\begin{aligned}
q(x,t) &= e^{i\beta x} q_0 e^{2iq_0^2 t} \times \\
&\times \frac{(-e^{-i\theta_+} + e^{i\theta_+ + 4q_0\sin\theta_+(x+2\beta t - 2q_0 t\cos\theta_+)} - 2i\sin\theta_+ e^{-i\beta^2 t + 2q_0\sin\theta_+(x+2\beta t-2q_0 t\cos\theta_+)})}{e^{4q_0\sin\theta_+(x+2\beta t-2q_0t\cos\theta_+)} - 1}.
\end{aligned}
\tag{5.8}
$$

Remark 2. For $\beta \neq 0$, the above solution is singular along the punctured line $x = Ct$, where $C, t \neq 0$.

5.2. Standard NLS equation: Spatial boundary values. The NLS equation has the form

$$
iq_t(x,t) = q_{xx}(x,t) - 2\sigma|q(x,t)|^2 q(x,t), \tag{5.9}
$$

where q^* denotes the complex conjugate of q and $\sigma = \mp 1$. We consider the BC

$$
q(x,t) \to q_0 e^{i(\alpha t + \beta x + \theta_\pm)}, \quad x \to \pm\infty, \tag{5.10}
$$

where $q_0 > 0$, $0 \le \theta_\pm < 2\pi$, and both α and β are real. It is easy to see that $\alpha = \beta^2 + 2\sigma q_0^2$, and the BC then becomes

$$
q(x,t) \to q_0 e^{i[(\beta^2 + 2\sigma q_0^2)t + \beta x + \theta_\pm]} \tag{5.11}
$$

as $x \to \pm\infty$, where β is real. Setting $q(x,t) = \tilde{q}(x,t)e^{i\beta x}$, we obtain

$$
i\tilde{q}_t = \tilde{q}_{xx} + 2i\beta\tilde{q}_x - (\beta^2 + 2\sigma|\tilde{q}|^2)\tilde{q}, \tag{5.12}
$$

which is associated with the 2×2 compatible system

$$
v_x = Xv = \begin{pmatrix} -ik & \tilde{q}(x,t) \\ \sigma\tilde{q}^*(x,t) & ik \end{pmatrix} v, \tag{5.13}
$$

$$
v_t = Tv = \begin{pmatrix} 2ik^2 - 2\beta ik + i\sigma|\tilde{q}|^2 + \dfrac{\beta^2}{2}i & -2\tilde{q}k + 2\beta\tilde{q} - i\tilde{q}_x \\ -2\sigma\tilde{q}^*k + i\sigma\tilde{q}_x^* + 2\beta\sigma\tilde{q}^* & -2ik^2 + 2\beta ik - i\sigma|\tilde{q}|^2 - \dfrac{\beta^2}{2}i \end{pmatrix} v. \tag{5.14}
$$

We then find that both $a(k,t)$ and $\bar{a}(k,t)$ are independent of time,

$$
\begin{aligned}
b(k,t) &= b(0)\exp\left\{-2i\left[\sigma q_0^2 + \frac{\beta^2}{2} - 2\lambda(\beta - k)\right]t\right\},\\
\bar{b}(k,t) &= \bar{b}(0)\exp\left\{2i\left[\sigma q_0^2 + \frac{\beta^2}{2} - 2\lambda(\beta - k)\right]t\right\}.
\end{aligned}
\tag{5.15}
$$

Scattering problem (5.13) is the same as the NLS equation with the NZBC

$$
\tilde{q}(x,t) \to q_0 e^{i(2\sigma q_0^2 t + \theta_\pm)}, \quad x \to \pm\infty. \tag{5.16}
$$

The only difference between the standard NLS equation and (5.12) in their IST formulations is the time evolution. Based on the scattering problem and one-soliton solution of the defocusing NLS equation [15], with $\sigma = 1$, we obtain

$$
\begin{aligned}
q(x,t) &= \tilde{q}e^{i\beta x} = \\
&= e^{i\beta x}\left[q_0 e^{i[(\beta^2+2q_0^2)t+\theta_+]} + \frac{iC_1^*(0)\alpha_1^* \exp[2i(q_0^2+\beta^2/2)t - 4v_1(\beta-k_1)t - 2v_1x]}{1+(q_0|C_1(0)|/2v_1)e^{-2v_1x-4v_1(\beta-k_1)t}}\right],
\end{aligned} \tag{5.17}
$$

where $\alpha_1 = k_1 + iv_1$, $v_1 = \sqrt{q_0^2 - k_1^2}$, $-q_0 < k_1 < q_0$, $e^{2v_1x_0} = q_0|C_1(0)|/2v_1$, and $C_1^*(0) = -|C_1(0)|e^{i\theta_+}$. We can rewrite it in the form

$$
q(x,t) = q_0 e^{2iq_0^2 t} e^{i(\beta x+\beta^2 t)}\left[e^{i\theta_+} + \frac{iC_1^*(0)(\alpha_1^*/q_0)e^{-2v_1(x+2\beta t-2k_1t)}}{1+(q_0|C_1(0)|/2v_1)e^{-2v_1(x+2\beta t-2k_1t)}}\right]. \tag{5.18}
$$

A property of the NLS equation is its Galilean invariance, i.e., if $q_1(x,t)$ solves the NLS equation and satisfies the BC $q_1(x,t) \to q_0 e^{i(2\sigma q_0^2 t+\theta_\pm)}$ as $x \to \pm\infty$, then $q_2(x,t) := q_1(x+2\beta t, t)e^{i(\beta x+\beta^2 t)}$ also solves the NLS equation and satisfies the BC $q_2(x,t) \to q_0 e^{i[(\beta^2+2\sigma q_0^2)t+\beta x+\theta_\pm]}$ as $x \to \pm\infty$. For $\sigma = 1$, the one-soliton solution is given by

$$
q_1(x,t) = q_0 e^{2iq_0^2 t}\left[e^{i\theta_+} + \frac{iC_1^*(0)(\alpha_1^*/q_0)e^{-2v_1(x-2k_1t)}}{1+(q_0|C_1(0)|/2v_1)e^{-2v_1(x-2k_1t)}}\right]. \tag{5.19}
$$

We have $q(x,t) = q_1(x+2\beta t, t)e^{i(\beta x+\beta^2 t)}$, which means that the IST result agrees with the result based on the Galilean invariance of the NLS equation. By taking $\alpha = k_1/q_0$ and $\gamma = -v_1/q_0$, possibly up to a phase, we can simplify $q_1(x,t)$ and $q(x,t)$ as

$$
\begin{aligned}
q_1(x,t) &= q_0 e^{2iq_0^2 t}[\gamma \tanh(q_0\gamma(x - 2q_0\alpha t - x_0)) - i\alpha], \\
q(x,t) &= q_0 e^{2iq_0^2 t} e^{i(\beta x+\beta^2 t)}[\gamma \tanh(q_0\gamma(x + 2\beta t - 2q_0\alpha t - x_0)) - i\alpha],
\end{aligned} \tag{5.20}
$$

where $\alpha^2 + \gamma^2 = 1$.

If we ignore the nonlinear term, then the linear partial differential equation

$$
iq_t(x,t) - q_{xx}(x,t) = 0 \tag{5.21}
$$

also satisfies the Galilean invariance. Indeed, by the Fourier transform,

$$
u_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b^{(1)}(\xi)e^{i(\xi x+\xi^2 t)}\, d\xi \tag{5.22}
$$

is a solution of (5.21). We set $u_2(x,t) = u_1(x+2\beta t, t)e^{i(\beta x+\beta^2 t)}$, and then

$$
u_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b^{(1)}(\xi)e^{i(\xi+\beta)x}e^{i(\xi+\beta)^2 t}\, d\xi. \tag{5.23}
$$

Redefining the variables ($\xi' = \xi + \beta$) shows that $u_2(x,t)$ is also a solution of (5.21), which implies that the Galilean invariance is also satisfied for the linear problem.

Acknowledgments. The authors are pleased to acknowledge Justin T. Cole for assistance with the figures in this paper and with `Mathematica`.

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