



Relative degree of interconnected SISO nonlinear control systems

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ABSTRACT

The concept of relative degree plays an important role in nonlinear control theory. It provides, for example, a necessary and sufficient condition for the existence of a feedback linearizing control law for a single-input, single-output (SISO) input-affine nonlinear state space system. It also gives a sufficient condition under which a left inverse exists. In applications it is common for systems to be composed of smaller interconnected subsystems. It is known that various feedback structures preserve relative degree, but it is largely unknown how to compute the relative degree of interconnected systems using only their properties. So the goal of this paper is to determine the relative degree of two nonlinear control systems interconnected in a variety of different ways. A collection of illustrative examples is given.

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1. Introduction

In linear system theory, the notion of relative degree plays a fairly modest role. It is usually defined in terms of a system's transfer function written as either a rational function or in terms of a power series about the origin [1]. In the context of nonlinear control systems, the concept is of central importance. For example, it provides a necessary and sufficient condition for the existence of a feedback linearizing control law for a single-input, single-output (SISO) input-affine nonlinear state space system [2]. It also gives a sufficient condition under which a left inverse exists [3], which is useful for solving output tracking control problems. In process control relative degree is used to describe the structural properties of networked systems [4–8]. The relative degree of a nonlinear system is usually defined via a state space model [2], but, as in the linear systems case, relative degree can also be described in a purely input–output setting using Chen–Fliess functional series (also called Fliess operators) [9,10]. This definition is consistent with the state space notion of relative degree, but not every Fliess operator is realizable. So the series definition is in fact more general.

In applications it is common for systems to be composed of smaller interconnected subsystems. Take the linear time-invariant case as an example. Let g and h be two strictly proper rational transfer functions with corresponding relative degrees r_g and r_h . Then the relative degrees of the parallel, series, and feedback interconnections, respectively, $g + h$, gh , and $g(1 - hg)^{-1}$, are easily computed in terms of r_g and r_h as given in Table 1. Feedback in this case can be viewed in terms of the group $\{1 + g\}$ acting on

the plant, where the group product is $(1 + g)(1 + h) = 1 + g + h + gh := 1 + g \odot h$ so that the group inverse is $(1 + g)^{-1} = 1 + \sum_{k \geq 0} (-g)^k := 1 + g^{\circ - 1}$. The relative degree of the product $g \odot h$ and inverse $g^{\circ - 1}$ are also easily determined. In the nonlinear setting, much less is known about this topic. Specifically, it has been shown that static state feedback preserves relative degree [2], and an analogous claim holds for a certain feedback transformation group acting on Fliess operators [11,12]. But beyond that, the topic is largely unexplored. In addition, once the restriction of linearity is removed, other types of system connections are admissible. For example, one could consider the *parallel product* connection where the outputs of two systems having the same input are multiplied pointwise in time. Finally, there is an issue that never occurs for linear systems, namely that a nonlinear system can fail to have a well defined relative degree. This invites the possibility that the interconnection of two systems with relative degree can fail to have relative degree or vice versa. So the general goal of this paper is to provide a nonlinear version of Table 1. It will be expanded to include operations that are admissible in the nonlinear setting, and conditions will be given to ensure that new interconnected systems have a well defined relative degree. Of course, these results will apply to the special case of Fliess operators that are realizable, and thus, the calculation of relative degree can be done directly using a realization of the interconnected system. But the analysis presented here indicates that such a calculation is not necessary. The relative degree of each subsystem can be computed separately by any method and then aggregated. This approach is especially useful for designing an interconnected system with some a priori specified relative degree, such as in the case of rendering a passive system, where a necessary condition is that the composite system must have relative degree one [13]. A variety of simple examples and counterexamples will be presented. A more comprehensive example will also be given with the aid of Mathematica software.

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Table 1
Relative degrees for interconnections of SISO linear time-invariant systems.

r_{g+h}	$\min(r_g, r_h)$ if $r_g \neq r_h$
r_{gh}	$r_g + r_h$
$r_{g(1+h)}$	r_g
$r_{g \otimes h}$	$\min(r_g, r_h)$ if $r_g \neq r_h$
$r_{g \circ -1}$	r_g
$r_{g(1-hg)^{-1}}$	r_g

For brevity, the focus will be on the SISO case. Finally, a preliminary version of this paper appeared in [14].

The paper is organized as follows. The mathematical preliminaries are summarized in the next section. The main results of the paper along with some simple examples are given in Section 4. A larger example is presented in Section 5. The conclusions of the paper and suggestions for future work are summarized in the final section.

2. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \dots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X , $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X . Its *length* is $|\eta| = k$. In particular, $|\eta|_{x_i}$ is the number of times the letter $x_i \in X$ appears in η . The set of all words including the empty word, $\emptyset$, is denoted by X^* , and $X^+ := X^* \setminus \{\emptyset\}$. The set X^* forms a monoid under catenation. The set of all words with prefix η is written as ηX^* . Any mapping $c : X^* \rightarrow \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is denoted by (c, η) and called the *coefficient* of η in c . A series c is *proper* when $(c, \emptyset) = 0$. The *support* of c , $\text{supp}(c)$, is the set of all words having nonzero coefficients. Normally, c is written as a formal sum $c = \sum_{\eta \in X^*} (c, \eta) \eta$. (This does not require or imply any notion of convergence.) The collection of all formal power series over X with coefficients from $\mathbb{R}^\ell$ is denoted by $\mathbb{R}^\ell \langle \langle X \rangle \rangle$. It constitutes an associative $\mathbb{R}$ -algebra under the catenation product and an associative and commutative $\mathbb{R}$ -algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words

$$(x_i \eta) \sqcup (x_j \xi) = x_i(\eta \sqcup (x_j \xi)) + x_j((x_i \eta) \sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup \emptyset = \emptyset \sqcup \eta = \eta$ [15]. It is worth pointing out that any binary product $\square$ on $\mathbb{R}^\ell \langle \langle X \rangle \rangle$ is well defined provided that $c \square d$ is *summable* [16], that is, it can be written in terms of a family of series in $\mathbb{R}^\ell \langle \langle X \rangle \rangle$ so that $(c \square d, \eta)$ is finite for every $\eta \in X^*$. Both the catenation and shuffle products are known to be summable, as are all the binary products to be presented in the next section [17].

2.1. Fliess operators

Let $p \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \rightarrow \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p -norm for a measurable real-valued function, u_i , defined on $[t_0, t_1]$. Let $L_p^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_p$ norm and $B_p^m(R)[t_0, t_1] := \{u \in L_p^m[t_0, t_1] : \|u\|_p \leq R\}$. Given any series $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$, one can uniquely associate a causal m -input, ℓ -output *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0),$$

where $E_\emptyset[u] = 1$ and

$$E_{x_i \eta}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_\eta[u](\tau, t_0) d\tau$$

with $x_i \in X$, $\eta \in X^*$, and $u_0 = 1$ [15]. The letter x_0 is useful for representing nonhomogeneous operators, i.e., those for which $F_c[0] \neq 0$. If there exist constants $K, M > 0$ such that

$$|(c, \eta)| \leq KM^{|\eta|} |\eta|!, \quad \forall \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B_p^m(R)[t_0, t_0 + T]$ into $B_q^\ell(S)[t_0, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $p, q \in [1, \infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ [18]. Here $\mathbb{R}_{LC}^\ell \langle \langle X \rangle \rangle$ will denote the set of all such *locally convergent* generating series.

A Fliess operator F_c defined on $B_p^m(R)[t_0, t_0 + T]$ is said to be *realizable* when there exists a state space model

$$\dot{z}(t) = g_0(z(t)) + \sum_{i=1}^m g_i(z(t)) u_i(t), \quad z(t_0) = z_0 \quad (1a)$$

$$y_j(t) = h_j(z(t)), \quad j = 1, 2, \dots, \ell, \quad (1b)$$

where each g_i is an analytic vector field expressed in local coordinates on some neighborhood $\mathcal{W}$ of z_0 , and each output function h_j is an analytic function on $\mathcal{W}$ such that (1a) has a well defined solution $z(t)$, $t \in [t_0, t_0 + T]$ for any given input $u \in B_p^m(R)[t_0, t_0 + T]$, and $y_j(t) = F_{c_j}[u](t) = h_j(z(t))$, $t \in [t_0, t_0 + T]$, $j = 1, 2, \dots, \ell$. It can be shown that for any word $\eta = x_{i_k} \cdots x_{i_1} \in X^*$

$$(c_j, \eta) = L_{g_{\eta}} h_j(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z_0), \quad (2)$$

where $L_{g_i} h_j$ is the *Lie derivative* of h_j with respect to g_i . In this context, the letter x_0 is identified with the drift vector field g_0 . For any $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$, the $\mathbb{R}$ -linear mapping $\mathcal{H}_c : \mathbb{R} \langle X \rangle \rightarrow \mathbb{R}^\ell \langle \langle X \rangle \rangle$ uniquely specified by $(\mathcal{H}_c(\eta), \xi) = (c, \xi \eta)$, $\xi, \eta \in X^*$ is called the *Hankel mapping* of c . The series c is said to have finite *Lie rank* $\rho_L(c)$ when the range of $\mathcal{H}_c$ restricted to the $\mathbb{R}$ -vector space of Lie polynomials over X , i.e., the free Lie algebra $\mathcal{L}(X) \subset \mathbb{R} \langle X \rangle$, has dimension $\rho_L(c)$. It is well known that F_c is realizable if and only if $c \in \mathbb{R}_{LC}^\ell \langle \langle X \rangle \rangle$ has finite Lie rank [2, 15].

2.2. System interconnections

Given Fliess operators F_c and F_d , where $c, d \in \mathbb{R}_{LC}^\ell \langle \langle X \rangle \rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup d}$, respectively [15]. For example, if $F_c = E_{x_i}$ and $F_d = E_{x_j}$ then $E_{x_i} E_{x_j} = F_{x_i \sqcup x_j} = E_{x_i x_j} + E_{x_j x_i}$. This is exactly the integration by parts formula from integral calculus. When Fliess operators F_c and F_d with $c \in \mathbb{R}_{LC}^\ell \langle \langle X \rangle \rangle$ and $d \in \mathbb{R}_{LC}^m \langle \langle X \rangle \rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta) \psi_d(\eta) \mathbf{1} \quad (3)$$

[19]. Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R} \langle \langle X \rangle \rangle$ to the set of vector space endomorphisms on $\mathbb{R} \langle \langle X \rangle \rangle$, $\text{End}(\mathbb{R} \langle \langle X \rangle \rangle)$, uniquely specified by $\psi_d(x_i \eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with $\psi_d(x_i)(e) = x_0(d_i \sqcup e)$, $i = 0, 1, \dots, m$ for any $e \in \mathbb{R} \langle \langle X \rangle \rangle$, and where d_i is the i th component series of d ($d_0 := \mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R} \langle \langle X \rangle \rangle$. For example, if F_c represents a SISO linear time-invariant system, then its impulse response is $h_c(t) = \sum_{i \geq 0} (c, x_0^i x_1) t^i / i!$. The cascade connection of this system with another in this class, say h_d , renders a composite system with an impulse response given by the convolution product $h_c * h_d$. It is not hard to show in this special case that the composition product as defined above satisfies $(h_c * h_d)(t) = \sum_{i \geq 1} (c \circ d, x_0^i x_1) t^i / i!$, where $(c \circ d, x_0^i x_1) = \sum_{j=0}^{i-1} (c, x_0^{i-j-1} x_1) (d, x_0^j x_1)$. That is, the composition product can be viewed as a nonlinear noncommutative generalization of series convolution.

When two Fliess operators F_c and F_d are interconnected to form a feedback system with F_c in the forward path and F_d in the feedback path, the generating series of the closed-loop system is denoted by the *feedback product* $c@d$. It can be computed explicitly using the Hopf algebra of coordinate functions associated with the underlying *output feedback group* [20]. Specifically, in the SISO case where $X = \{x_0, x_1\}$, define the set of *unital* Fliess operators $\mathcal{F}_\delta = \{I + F_c : c \in \mathbb{R}_{LC}(\langle X \rangle)\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathcal{F}_\delta$ will be denoted by $\mathbb{R}_{LC}(\langle X_\delta \rangle)$. The central idea is that $(\mathcal{F}_\delta, \circ, I)$ forms a group of operators under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \odot d_\delta},$$

where $c_\delta \odot d_\delta := \delta + c \odot d$, $c \odot d := d + c \tilde{\odot} d$, and $\tilde{\odot}$ denotes the *mixed composition product*. That is, the product

$$c \tilde{\odot} d_\delta = \sum_{\eta \in X^*} (c, \eta) \phi_d(\eta)(\mathbf{1}),$$

where ϕ_d is analogous to ψ_d in (3) except here $\phi_d(x_i)(e) = x_i e + x_0(d_i \lrcorner e)$ with $d_0 := 0$ [17]. For linear time-invariant systems, the group product $c \odot d$ reduces to the feedback group product defined in the introduction in terms of transfer functions. The mixed composition product gives the corresponding group action. The coordinate maps for the underlying Hopf algebra H have the form

$$a_\eta : \mathbb{R}(\langle X \rangle) \rightarrow \mathbb{R} : c \mapsto (c, \eta),$$

where $\eta \in X^*$. The commutative product is defined as

$$\mu : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi,$$

where the unit $\mathbf{1}$ is defined to map every c to zero. If the *degree* of a_η is defined as $\deg(a_\eta) = 2|\eta|_{x_0} + |\eta|_{x_1} + 1$, then H is graded and connected with $H = \bigoplus_{n \geq 0} H_n$, where H_n is the set of all elements of degree n and $H_0 = \mathbb{R}\mathbf{1}$. The coproduct Δ is defined so that the formal power series product $c \odot d$ for the group $\mathcal{F}_\delta$ satisfies

$$\Delta a_\eta(c, d) = a_\eta(c \odot d) = (c \odot d, \eta).$$

Of primary importance is the following lemma which describes how the group inverse $c_\delta^{\circ-1} := \delta + c^{\circ-1}$ is computed.

Lemma 2.1 ([20]). *The Hopf algebra (H, μ, Δ) has an antipode S satisfying $a_\eta(c^{\circ-1}) = (S a_\eta)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}(\langle X \rangle)$.*

With this concept, the generating series for the feedback connection, $c@d$, can be computed explicitly as described in the next theorem. The formula has the same form as in linear system theory, where a feedback group element acts from the right on a description of the plant.

Theorem 2.1 ([20]). *For any $c, d \in \mathbb{R}(\langle X \rangle)$ it follows that*

$$c@d = c \tilde{\odot} (-d \circ c)_\delta^{\circ-1}.$$

In addition to the elementary system interconnections described above, there is the quotient connection that is useful in the context of system inversion. This is a type of parallel connection where the quotient of the subsystems' outputs is computed. In terms of generating series, the quotient is realized using the shuffle inverse as described next. Division by zero is avoided by requiring the divisor series to be non proper.

Theorem 2.2 ([9]). *The set of non proper series in $\mathbb{R}(\langle X \rangle)$ is a group under the shuffle product. In particular, the shuffle inverse of any such series c is*

$$c^{\lrcorner-1} = ((c, \emptyset)(1 - c'))^{\lrcorner-1} = (c, \emptyset)^{-1}(c')^{\lrcorner*},$$

where $c' := \mathbf{1} - c/(c, \emptyset)$ is proper and $(c')^{\lrcorner*} := \sum_{k \geq 0} (c')^{\lrcorner k}$.

Theorem 2.3 ([9]). *For $c, d \in \mathbb{R}_{LC}(\langle X \rangle)$, the quotient connection F_c/F_d has a Fliess operator representation if and only if d is non proper. In particular, $F_c/F_d = F_{c/d}$, where $c/d := c \lrcorner d^{\lrcorner-1}$. In addition, the quotient c/d preserves local convergence.*

3. Relative degree

Two concepts are needed to define the notion of relative degree for a series $c \in \mathbb{R}(\langle X \rangle)$ when $X = \{x_0, x_1\}$, that of *linear words* and the *class* of a series.

Definition 3.1. A word in X^* is said to be *linear* if it belongs to the language

$$\mathcal{L} = \{\eta \in X^* : \eta = x_0^{n_1} x_1 x_0^{n_0}, n_1, n_0 \geq 0\}.$$

To define the class of a series, a certain partial ordering is used. Consider the collection of pairwise disjoint sets $\{x_0^i x_1 X^*\}_{i \in \mathbb{N}_0}$. Observe that $X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i) = \bigcup_{i \in \mathbb{N}_0} x_0^i x_1 X^*$ and define the map $\mathcal{T} : X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i) \rightarrow \mathbb{N}_0$ so that $\mathcal{T}(\eta) = k$ if $x_0^{-k}(\eta) \in x_1 X^*$, where $\eta \in X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i)$, and x_0^{-k} denotes the left-shift operator. Clearly $\mathcal{T}$ is a surjective map. Define an order $<$ on $X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i)$ as follows:

$$\zeta < \eta \iff \mathcal{T}(\zeta) > \mathcal{T}(\eta), \quad \forall \zeta, \eta \in X^* \setminus \left(\bigcup_{i \in \mathbb{N}_0} x_0^i \right).$$

Lemma 3.1. $(X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i), <)$ is a strict partial order.

Proof. For all $\eta, \zeta, \beta \in X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i)$ it is immediate that

$$\eta \not< \eta \iff \mathcal{T}(\eta) \not> \mathcal{T}(\eta)$$

and

$$((\eta < \zeta) \wedge (\zeta < \beta)) \implies \eta < \beta \iff$$

$$((\mathcal{T}(\eta) > \mathcal{T}(\zeta)) \wedge (\mathcal{T}(\zeta) > \mathcal{T}(\beta))) \implies \mathcal{T}(\eta) > \mathcal{T}(\beta)).$$

Hence, $(X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i), <)$ is a strict partial order. ■

Since the words in $x_1 X^* \subset X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i)$ act as upper bounds, every chain in $(X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i), <)$ is upper bounded. In which case, by Zorn's lemma, every chain in $X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i)$ has a maximal element.

Lemma 3.2. *If $D \subset X^* \setminus (\bigcup_{i \in \mathbb{N}_0} x_0^i)$, and say η, ζ are the distinct maximal elements in D , then $\mathcal{T}(\eta) = \mathcal{T}(\zeta)$.*

Proof. The claim is proved by contradiction. Assume that $\mathcal{T}(\eta) \neq \mathcal{T}(\zeta)$. Then either $(\mathcal{T}(\eta) < \mathcal{T}(\zeta)) \iff (\zeta < \eta)$ or $(\mathcal{T}(\zeta) < \mathcal{T}(\eta)) \iff (\eta < \zeta)$, which is a contradiction. ■

Now the notion of class is defined. It uses the fact that every $c \in \mathbb{R}(\langle X \rangle)$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ and $c_F := c - c_N$.

Definition 3.2. A series $c \in \mathbb{R}(\langle X \rangle)$ is said to be of *r-class*, denoted by $\mathcal{C}(c) = r$, if $\text{supp}(c_F) \subseteq x_0^{r-1} X^+$ and $\text{supp}(c_F) \not\subseteq x_0^r X^+$. By definition, let $\mathcal{C}(c) = \infty$ if $c_F = 0$.

Lemma 3.3. *Every series $c \in \mathbb{R}(\langle X \rangle)$ has a class. In particular, if $c_F \neq 0$, the class can be defined as $\mathcal{C}(c) = \mathcal{T}(\eta) + 1$, where η is a maximal word in $\text{supp}(c_F)$. Thus, class is realized by the map $\mathcal{C} : \mathbb{R}(\langle X \rangle) \rightarrow \mathbb{N} \cup \{\infty\}$.*

Proof. The claim is immediate since $\mathcal{T}$ is surjective on $\mathbb{N}_0$. ■

Example 3.1. Let $c = 1 + x_0x_1 + x_0^2x_1$, so that $c_F = x_0x_1 + x_0^2x_1$. Observe that $\text{supp}(c_F) \subseteq x_0X^+$ but $\text{supp}(c_F) \not\subseteq x_0^2X^+$. Thus, $\mathcal{C}(c) = 2$. $\square$

Finally, the main definition of the section is given.

Definition 3.3. A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has **relative degree** r if $\mathcal{C}(c) = r$ and the linear word $x_0^{r-1}x_1 \in \text{supp}(c_F)$. Otherwise, c does not have relative degree.

This notion of relative degree is equivalent to that given in [9], namely, that c has relative degree r if and only if it has the decomposition

$$c = c_N + Kx_0^{r-1}x_1 + x_0^{r-1}e \quad (4)$$

for some $K \neq 0$ and proper $e \in \mathbb{R}\langle\langle X \rangle\rangle$ with $x_1 \notin \text{supp}(e)$. In addition, this definition of relative degree is consistent with the classical definition whenever $y = F_c[u]$ is realizable [9,10]. In this setting, the requirement that all the words in $\text{supp}(c_F)$ have a prefix x_0^{r-1} is equivalent to the input u not appearing in the first $r - 1$ derivatives of the output y as in the classical definition. The requirement of a linear word in $\text{supp}(c_F)$ provides a sufficient condition for local left invertibility of the input–output map. A system without relative degree may simply not be left invertible by any means, for example, by series inversion or feedback linearization. A stronger notion of relative degree is defined below. In a state space setting it provides a sufficient condition for *global* left invertibility of the input–output map.

Definition 3.4 ([11]). A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has **extended relative degree** r if there exists some $K \neq 0$ and $e \in \mathbb{R}\langle\langle X \rangle\rangle$ such that

$$c = c_N + Kx_0^{r-1}x_1 + x_0^r e.$$

Example 3.2. Consider the generating series for a SISO linear time-invariant system with transfer function $h(s) = \sum_{k \geq r \geq 1} h_k s^{-k}$ and $h_r \neq 0$:

$$c = \sum_{k=r}^{\infty} h_k x_0^{k-1}x_1 = h_r x_0^{r-1}x_1 + x_0^r \sum_{k=0}^{\infty} h_{r+k+1} x_0^k x_1.$$

Such systems always have extended relative degree. $\square$

4. Main results

The main results are organized into three groups: parallel connections, composition connections, and connections involving inversion. The more generic results where the subsystems have relative degree are summarized in Table 2, which is subdivided accordingly. It is assumed throughout that $X = \{x_0, x_1\}$.

4.1. Parallel connections

The following lemma will be useful in this section.

Lemma 4.1. If $r, s \in \mathbb{N}$ with $r \leq s$ then $x_0^{s-1}X^+ \subseteq x_0^{r-1}X^+$.

Proof. Setting $q = s - r \geq 0$, observe that $x_0^{s-1}X^+ = x_0^{r-1}(x_0^q X^+) \subseteq x_0^{r-1}X^+$. $\blacksquare$

Now the relative degree of the parallel sum connection is addressed.

Theorem 4.1. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ have distinct relative degrees r_c and r_d , respectively. Then $c + d$ has relative degree $\min(r_c, r_d)$. On the other hand, if $r_c = r_d =: r$ then $c + d$ has relative degree r if and only if $(c, x_0^{r-1}x_1) + (d, x_0^{r-1}x_1) \neq 0$.

Table 2

Relative degrees for interconnections of SISO nonlinear control systems having relative degree.

r_{c+d}	$\min(r_c, r_d)$ if $r_c \neq r_d$
$r_{c \sqcup d}$	$r = \min(r_c, r_d)$ if $r_c \neq r_d$, and the series with $r \neq r$ is non proper
$r_{c \odot d}$	$r_c + r_d$
$r_{c \circ d}$	r_c
$r_{c \oslash d}$	$\min(r_c, r_d)$ if $r_c \neq r_d$
$r_{c \circ -1}$	r_c
$r_{c \odot d}$	r_c
$r_{c \sqcup -1}$	r_c if c is non proper
$r_{c/d}$	$r = \min(r_c, r_d)$ if $r_c \neq r_d$, and c, d are non proper

Proof. If $r_c \neq r_d$ then from (4)

$$(c + d)_F = K_c x_0^{r_c-1}x_1 + x_0^{r_c-1}e_c + K_d x_0^{r_d-1}x_1 + x_0^{r_d-1}e_d.$$

Assume, for example, that $r_c < r_d$. Using Lemma 4.1 it follows that

$$(c + d)_F = K_c x_0^{r_c-1}x_1 + x_0^{r_c-1}e_{c+d}$$

for some suitable e_{c+d} in $\mathbb{R}\langle\langle X \rangle\rangle$ so that $\text{supp}(c + d)_F \subseteq x_0^{r_c-1}X^+$. Therefore, $c + d$ has relative degree r_c . Now if $r_c = r_d =: r$ then $c + d$ has relative degree r if and only if the linear words do not cancel out. Hence, the necessity and sufficiency of $(c, x_0^{r-1}x_1) + (d, x_0^{r-1}x_1) \neq 0$ is obvious. $\blacksquare$

Example 4.1. Let $c = x_0 + x_0x_1 + x_0x_1^2$ and $d = 1 + x_1$ so that $r_c = 2$ and $r_d = 1$. Then $c + d = 1 + x_0 + x_1 + x_0x_1 + x_0x_1^2$ has relative degree $\min(r_c, r_d) = 1$. $\square$

Example 4.2. Suppose $c = x_0 + x_0x_1$ and $d = 1 - x_0x_1 + x_0x_1^2$ so that $r_c = r_d = 2$. Observe that $c + d = 1 + x_0 + x_0x_1^2$ does not have relative degree since the linear words have canceled. $\square$

Next, the relative degree of the parallel product connection is analyzed. The following notation will be used: if $S_1, S_2 \subseteq X^*$ then $S_1 \sqcup S_2 := \{\text{supp}(\eta_1 \sqcup \eta_2) : \eta_i \in S_i\}$. Four preliminary lemmas are needed first.

Lemma 4.2. If $r, s \in \mathbb{N}$ with $r \leq s$ then $x_0^{r-1}X^+ \sqcup x_0^{s-1}X^+ \subseteq x_0^{r-1}X^+$.

Proof. The lemma can be proved by induction on $q = s - r$ and applying Lemma 4.1 $\blacksquare$

Lemma 4.3. If $r \in \mathbb{N}_0$ and $s \in \mathbb{N}$ then $x_0^r \sqcup x_0^{s-1}X^+ \subseteq x_0^{s-1}X^+$.

Proof. The claim follows immediately since every word in $x_0^r \sqcup x_0^{s-1}X^+$ must have a prefix x_0^q with $q \geq s - 1$. $\blacksquare$

Lemma 4.4. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ are such that $\text{supp}(c_F) \subseteq x_0^{r_c-1}X^+$ and $\text{supp}(d_F) \subseteq x_0^{r_d-1}X^+$. If $\min(r_c, r_d) = s$, then $\text{supp}((c \sqcup d)_F) \subseteq x_0^{s-1}X^+$.

Proof. From the stated assumptions,

$$c = c_N + x_0^{r_c-1}e_c, \quad d = d_N + x_0^{r_d-1}e_d$$

for some proper $e_c, e_d \in \mathbb{R}\langle\langle X \rangle\rangle$. Therefore,

$$\begin{aligned} c \sqcup d &= (c_N + x_0^{r_c-1}e_c) \sqcup (d_N + x_0^{r_d-1}e_d) \\ &= (c \sqcup d)_N + (c \sqcup d)_F, \end{aligned}$$

where $(c \sqcup d)_N = c_N \sqcup d_N$ and $(c \sqcup d)_F = (c_N \sqcup d_F) + (x_0^{r_c-1}e_c \sqcup d)$. Using Lemmas 4.1 and 4.3, it follows that $\text{supp}(c_N \sqcup d_F) \subseteq x_0^{r_c-1}X^+ \subseteq x_0^{s-1}X^+$. Similar analysis of the other term in $(c \sqcup d)_F$ gives $\text{supp}((c \sqcup d)_F) \subseteq x_0^{s-1}X^+$ as desired. $\blacksquare$

Lemma 4.5. The class map $\mathcal{C} : \mathbb{R}\langle\langle X \rangle\rangle \rightarrow \mathbb{N} \cup \{\infty\}$ is a monoid epimorphism, where the underlying monoid structures are $(\mathbb{R}\langle\langle X \rangle\rangle, \sqcup, \mathbf{1})$ and $(\mathbb{N} \cup \{\infty\}, \min, \infty)$.

Proof. It follows from Lemma 4.4 that $\mathcal{C}((c \sqcup d)) = \min(\mathcal{C}(c), \mathcal{C}(d))$. By definition, $\mathcal{C}(\mathbf{1}) = \infty$. Therefore, since the map $\mathcal{C}$ is surjective, it is a monoid epimorphism. ■

Now the main result for the parallel product connection is presented. Note, in particular, that only one series is required to have relative degree.

Theorem 4.2. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ such that c has relative degree r and d is of class s , where $s > r$. Then $c \sqcup d$ has relative degree r if and only if d is non proper. If $s = r$, then $c \sqcup d$ has relative degree r if and only if $(c, \emptyset)(d, x_0^{r-1}x_1) + (c, x_0^{r-1}x_1)(d, \emptyset) \neq 0$.

Proof. From Lemma 4.5 it is clear that $\mathcal{C}(c \sqcup d) = \min(r, s) = r$. Now suppose $s > r$. To have relative degree it is necessary that $x_0^{r-1}x_1 \in \text{supp}(c \sqcup d)$. The sufficiency of having d be non proper is obvious. But since $x_0^{r-1}x_1 \notin (x_0^{r-1}X^+ \sqcup \eta)$ for all $\eta \in X^+$, the non properness of d also becomes a necessity. In the case where $s = r$, the requirement that the coefficient of $x_0^{r-1}x_1$

$$(c \sqcup d, x_0^{r-1}x_1) = (c, \emptyset)(d, x_0^{r-1}x_1) + (c, x_0^{r-1}x_1)(d, \emptyset)$$

be nonzero gives the new condition which subsumes that of non properness. ■

Example 4.3. Let $c = x_1$, which has relative degree one, and $d = \mathbf{1} + x_0x_1x_0$, where $\mathcal{C}(d) = 2$ but has no relative degree. Since d is non proper, $c \sqcup d = x_1 + x_1x_0x_1x_0 + x_0x_1^2x_0 + x_0x_1x_0x_1$ has relative degree one. □

Corollary 4.1. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ have distinct relative degrees r_c and r_d , respectively. Then $c \sqcup d$ has relative degree $\min(r_c, r_d)$ if and only if the series with the largest relative degree is non proper. If $r_c = r_d =: r$ then $c \sqcup d$ has relative degree r if and only if $(c, \emptyset)(d, x_0^{r-1}x_1) + (c, x_0^{r-1}x_1)(d, \emptyset) \neq 0$.

Example 4.4. Suppose $c = x_1$ and $d = \mathbf{1} + x_0x_1$ so that $r_c = 1$ and $r_d = 2$. Then $c \sqcup d = x_1 + x_1x_0x_1 + 2x_0x_1^2$ has relative degree $r_{c \sqcup d} = \min(r_c, r_d) = 1$ since d is non proper. □

Example 4.5. Let $c = d = x_1$. Both series are proper and have relative degree one, but $c \sqcup d = 2x_1^2$ does not have relative degree. □

4.2. Composition connections

The main results for the composition connections are given next. Somewhat counterintuitively, it is the basic composition product $c \circ d$ that is the most difficult to handle as addressed next.

Theorem 4.3. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ have relative degrees r_c and r_d , respectively. Then $r_{c \circ d}$ has relative degree $r_c + r_d$.

Proof. Writing c in the form (4) and using the left linearity of the composition product, it follows directly that

$$\begin{aligned} c \circ d &= (c_N + K_c x_0^{r_c-1}x_1 + x_0^{r_c-1}e_c) \circ d \\ &= c_N + K_c x_0^{r_c}d + (x_0^{r_c-1}e_c) \circ d. \end{aligned}$$

From the identity $(x_0^k c) \circ d = x_0^k (c \circ d)$ for any $k \geq 0$, this expression becomes

$$c \circ d = c_N + K_c x_0^{r_c}d + x_0^{r_c-1}(e_c \circ d).$$

In the second term above, write d in the form (4) to get

$$\begin{aligned} c \circ d &= c_N + K_c x_0^{r_c}d_N + K_c K_d x_0^{r_c+r_d-1}x_1 + \\ &\quad K_c x_0^{r_c+r_d-1}e_d + x_0^{r_c-1}(e_c \circ d). \end{aligned}$$

Since $x_1 \notin \text{supp}(e_d)$ by assumption, is it clear that $c \circ d$ has the linear word necessary for $c \circ d$ to have relative degree $r_c + r_d$. It only remains to be shown that $\text{supp}(e_c \circ d) \subseteq x_0^{r_d}X^*$. To see this, observe first that

$$\begin{aligned} \psi_d(x_1)(1) &= x_0(d \sqcup 1) \\ &= x_0d_N + K_d x_0^{r_d}x_1 + x_0^{r_d}e_d. \end{aligned}$$

Therefore, $\text{supp}((\psi_d(\eta x_1))_F) \subseteq x_0^{r_d}X^*$ for any $\eta \in X^*$. Thus, in light of (3), the conclusion follows. ■

The following theorem was proved in [11]. Note that here there is no requirement that d have relative degree.

Theorem 4.4 ([11]). Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ and c has relative degree r_c . Then $c \circ d$ has relative degree r_c .

Example 4.6. Let $c = x_0x_1$ and $d = x_1^2$. Here $r_c = 2$ and d has no relative degree. Nevertheless, $c \circ d = x_0x_1 + x_0^2x_1^2$ has relative degree two. □

Theorem 4.5. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ have distinct relative degrees r_c and r_d , respectively. In which case, $c \circ d$ has relative degree $\min(r_c, r_d)$.

Proof. In light of Theorem 4.1, the previous result, and the definition $c \circ d = d + c \circ d$, the conclusion follows immediately. ■

4.3. Connections involving inversion

The main results concerning the composition inverse are presented first. Then the shuffle inverse is addressed.

Theorem 4.6. Relative degree is invariant under the composition inverse.

Proof. In [20] it was shown as a consequence of $c_\delta^{\circ-1}$ being a right inverse of $c_\delta = \delta + c$, i.e., $c_\delta \circ c_\delta^{\circ-1} = \delta$, that $c^{\circ-1} = (-c) \circ c_\delta^{\circ-1}$. Hence, it follows directly from Theorem 4.4 that c and $c^{\circ-1}$ have the same relative degree. ■

Example 4.7. When c is a linear series, the product $c \circ d_\delta$ is both left and right $\mathbb{R}$ -linear. It then follows that

$$(\delta + c)^{\circ-1} = \delta - c + c^{\circ 2} - c^{\circ 3} + \dots,$$

where $c^{\circ k}$ denotes the composition power. If, for example, $c = x_1$, which has relative degree one, then

$$(\delta + x_1)^{\circ-1} = \delta - x_1 + x_0x_1 - x_0^2x_1 + \dots.$$

Clearly $c^{\circ-1}$ has relative degree one as expected. □

In light of Theorem 2.1, it is clear that the following result is just a special case of Theorem 4.4. In which case, there is again no requirement that d have relative degree.

Theorem 4.7. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ and c has relative degree r_c . Then $c \circ d$ has relative degree r_c .

Next the theorems related to the shuffle inverse are given. Two preliminary lemmas are needed.

Lemma 4.6. If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has relative degree r_c , then $\text{supp}((c \sqcup^k)_F) \subseteq x_0^{r_c-1}X^+$ for all $k \in \mathbb{N}$. If, in addition, c is also proper, then $x_0^{r_c-1}x_1 \notin \text{supp}((c \sqcup^k)_F)$ for all $k > 1$.

Proof. The proof is by induction on k . For $k = 1$ clearly $\text{supp}(c_F) \subseteq x_0^{r_c-1}X^+$ from the definition of relative degree. Assume now that the claim is true for $k = n - 1 \geq 1$. Then $c^{\cup n} = c^{\cup} (c^{\cup (n-1)})$, and by Lemma 4.4 it follows that $\text{supp}((c^{\cup n})_F) \subseteq x_0^{r_c-1}X^+$. Hence, one must conclude that $\text{supp}((c^{\cup k})_F) \subseteq x_0^{r_c-1}X^+$ for all $k \in \mathbb{N}$. If c is also proper, then $x_0^{r_c-1}x_1 \notin \text{supp}(x_0^{r_c-1}X^+ \cup \eta)$ where $\eta \in X^+$, and thus, $x_0^{r_c-1}x_1 \notin \text{supp}((c^{\cup k})_F)$ for all $k > 1$. ■

Lemma 4.7. If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is proper and has relative degree r_c , then $c^{\cup *}$ has relative degree r_c .

Proof. The claim follows directly from Lemma 4.6. ■

Theorem 4.8. If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is non proper and has relative degree r_c , then $c^{\cup -1}$ has relative degree r_c .

Proof. From the definition of $(c')^{\cup *}$ in Theorem 2.2, c' is proper. Thus, the result follows from Lemma 4.7. ■

Example 4.8. Suppose $c = 1 + x_1$, which has relative degree one. Then from the identity $x_1^{\cup k} = k!x_1^k$, $k \geq 0$ observe

$$\begin{aligned} (1 + x_1)^{\cup -1} &= \sum_{k=0}^{\infty} (-x_1)^{\cup k} \\ &= 1 - x_1 + 2x_1^2 - 3!x_1^3 + 4!x_1^4 - \dots, \end{aligned}$$

which also has relative degree one. □

Theorem 4.9. Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ such that d is non proper. If c and d have distinct relative degrees r_c and r_d , respectively, then c/d has relative degree $r = \min(r_c, r_d)$. If $r_c = r_d =: r$ then c/d has relative degree r if and only if $(d, \emptyset)(c, x_0^{r-1}x_1) - (c, \emptyset)(d, x_0^{r-1}x_1) \neq 0$.

Proof. If $r_c \neq r_d$, then from Theorem 4.8 it follows that $d^{\cup -1}$ has relative degree r_d . By virtue of Corollary 4.1, $c^{\cup} d^{\cup -1} = c/d$ has relative degree $r = \min(r_c, r_d)$. On the other hand, if $r_c = r_d = r$, then since d is non proper it follows from Corollary 4.1 that $c^{\cup} d^{\cup -1}$ has relative degree r if and only if

$$\begin{aligned} (c^{\cup} d^{\cup -1}, x_0^{r-1}x_1) &= (c, \emptyset)(d^{\cup -1}, x_0^{r-1}x_1) + (d^{\cup -1}, \emptyset)(c, x_0^{r-1}x_1) \\ &\neq 0. \end{aligned}$$

A direct computation gives $(d^{\cup -1}, \emptyset) = (d, \emptyset)^{-1}$, so that from Lemma 4.6

$$(d^{\cup -1}, x_0^{r-1}x_1) = -\frac{(d, x_0^{r-1}x_1)}{(d, \emptyset)^2},$$

and thus,

$$(c^{\cup} d^{\cup -1}, x_0^{r-1}x_1) = \frac{(d, \emptyset)(c, x_0^{r-1}x_1) - (c, \emptyset)(d, x_0^{r-1}x_1)}{(d, \emptyset)^2}.$$

Since $(d, \emptyset) \neq 0$ by assumption, the necessary and sufficient condition for relative degree follows. ■

Example 4.9. Let $c = 1 + x_0x_1$ and $d = 1 + x_1$, which have relative degrees two and one, respectively. Applying the results from Example 4.8 gives

$$\begin{aligned} c/d &= (1 + x_0x_1)^{\cup} (1 + x_1)^{\cup -1} \\ &= (1 + x_0x_1)^{\cup} (1 - x_1 + 2x_1^2 - 3!x_1^3 + \dots) \\ &= 1 - x_1 + x_0x_1 + 2x_1^2 - 2x_0x_1^2 - x_1x_0x_1 - 6x_1^3 + \\ &\quad 6x_0x_1^3 + 4x_1x_0x_1^2 + 2x_1^2x_0x_1 + 24x_1^4 - \dots, \end{aligned}$$

which has relative degree $\min(1, 2) = 1$ as expected. On the other hand, if $c = 1 + x_1$ then trivially $c/d = 1$, which does not have relative degree and is consistent with the condition $(d, \emptyset)(c, x_1) - (c, \emptyset)(d, x_1) = 0$. □

5. State space example

The following two state space systems are considered in [2, p. 151 and p. 167, respectively]:

$$\dot{z} = \begin{bmatrix} 0 \\ z_1 + z_2^2 \\ z_1 - z_2 \end{bmatrix} + \begin{bmatrix} e^{z_2} \\ e^{z_2} \\ 0 \end{bmatrix} u, \quad y = z_3$$

and

$$\dot{z} = \begin{bmatrix} z_3 - z_2^2 \\ -z_2 \\ z_1^2 - z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u, \quad y = z_1.$$

The first system has full relative degree $r_c = 3$ about any point z satisfying $1 + 2z_2 \neq 0$. Its generating series with $z_0 = 0$ is found using (2) to be

$$\begin{aligned} c &= -x_0^2x_1 - 3x_0^2x_1^2 - 4x_0^3x_1^2 - 3x_0^2x_1x_0x_1 - 8x_0^2x_1^3 - \\ &\quad 4x_0^4x_1^2 - 2x_0^3x_1x_0x_1 - 24x_0^3x_1^3 - 19x_0^2x_1x_0x_1^2 - \\ &\quad 8x_0^2x_1^2x_0x_1 - 28x_0^2x_1^4 - 60x_0^4x_1^3 - 42x_0^3x_1x_0x_1^2 - \\ &\quad 22x_0^3x_1^2x_0x_1 - 116x_0^3x_1^4 - \dots \end{aligned}$$

The second system has relative degree $r_d = 2$ about z whenever $1 + 3z_2^2 \neq 0$. Its generating series with $z_0 = 0$ is

$$\begin{aligned} d &= x_0x_1 - x_0^2x_1 + x_0^3x_1 + 6x_0x_1^3 - x_0^4x_1 - 18x_0^2x_1^3 - \\ &\quad 12x_0x_1x_0x_1^2 - 6x_0x_1^2x_0x_1 + x_0^5x_1 + 4x_0^4x_1^2 + \\ &\quad 2x_0^3x_1x_0x_1 + 54x_0^3x_1^3 + 36x_0^2x_1x_0x_1^2 + 18x_0^2x_1^2x_0x_1 + \\ &\quad 24x_0x_1x_0^2x_1^2 + 12x_0x_1x_0x_1x_0x_1 + 6x_0x_1^2x_0^2x_1 - \\ &\quad x_0^6x_1 - 16x_0^5x_1^2 - \dots \end{aligned}$$

The relative degree for any interconnection described in Table 2 can now be determined directly from the theory, i.e., no additional calculations are needed. But, as an independent check, the corresponding series operations are computed with the aid of Mathematica. Their relative degree is then determined directly from Definition 3.3 to validate the results presented in the previous sections. For the parallel connections,

$$\begin{aligned} c + d &= x_0x_1 - 2x_0^2x_1 + x_0^3x_1 - 3x_0^2x_1^2 + 6x_0x_1^3 - x_0^4x_1 - \\ &\quad 4x_0^3x_1^2 - 3x_0^2x_1x_0x_1 - 26x_0^2x_1^3 - 12x_0x_1x_0x_1^2 - \\ &\quad 6x_0x_1^2x_0x_1 + x_0^5x_1 + 30x_0^3x_1^3 + 17x_0^2x_1x_0x_1^2 + \\ &\quad 10x_0^2x_1^2x_0x_1 - 28x_0^2x_1^4 + 24x_0x_1x_0^2x_1^2 + \\ &\quad 12x_0x_1x_0x_1x_0x_1 + 6x_0x_1^2x_0^2x_1 - \dots \end{aligned}$$

$$\begin{aligned} c \cup d &= -6x_0^3x_1^2 - 3x_0^2x_1x_0x_1 - x_0x_1x_0^2x_1 + 12x_0^4x_1^2 + \\ &\quad 6x_0^3x_1x_0x_1 - 27x_0^3x_1^3 + 2x_0^2x_1x_0^2x_1 - \\ &\quad 12x_0^2x_1x_0x_1^2 - 3x_0^2x_1^2x_0x_1 - 3x_0x_1x_0^2x_1^2 - \\ &\quad 20x_0^5x_1^2 - 10x_0^4x_1x_0x_1 + 54x_0^4x_1^3 - 4x_0^3x_1x_0^2x_1 + \\ &\quad 27x_0^3x_1x_0x_1^2 + 9x_0^3x_1^2x_0x_1 - 72x_0^3x_1^4 - \dots \end{aligned}$$

As expected, $r_{c+d} = \min(r_c, r_d) = 2$, and $r_{c \cup d}$ is not defined since both c and d are proper. On the other hand, the parallel product connection of $1 + c$ and $1 + d$ has relative degree $r_{c \cup d} = \min(r_c, r_d) = 2$. For the composition type connections,

$$\begin{aligned} c \circ d &= -x_0^4x_1 + x_0^5x_1 - x_0^6x_1 - 6x_0^4x_1^2 + x_0^7x_1 - \\ &\quad 18x_0^6x_1^2 - 9x_0^5x_1x_0x_1 + 18x_0^5x_1^3 - 3x_0^4x_1x_0^2x_1 + \end{aligned}$$

$$12x_0^4x_1x_0x_1^2 + 6x_0^4x_1^2x_0x_1 - x_0^8x_1 + 8x_0^7x_1^2 + \\ 4x_0^6x_1x_0x_1 - 54x_0^6x_1^3 + 2x_0^5x_1x_0^2x_1 - \\ 36x_0^5x_1x_0x_1^2 - 18x_0^5x_1^2x_0x_1 - 24x_0^4x_1x_0^2x_1^2 - \\ 12x_0^4x_1x_0x_1x_0x_1 - 6x_0^4x_1^2x_0^2x_1 + \dots$$

$$c \circ d_\delta = -x_0^2x_1 - 3x_0^2x_1^2 - x_0^4x_1 - 4x_0^3x_1^2 - 3x_0^2x_1x_0x_1 - \\ 8x_0^2x_1^3 + x_0^5x_1 - 10x_0^4x_1^2 - 5x_0^3x_1x_0x_1 - 24x_0^3x_1^3 - \\ 3x_0^2x_1x_0^2x_1 - 19x_0^2x_1x_0x_1^2 - 8x_0^2x_1^2x_0x_1 - \\ 28x_0^2x_1^4 - x_0^6x_1 - 14x_0^5x_1^2 - 7x_0^4x_1x_0x_1 - \\ 90x_0^4x_1^3 - x_0^3x_1x_0^2x_1 - 58x_0^3x_1x_0x_1^2 - \dots$$

$$c \odot d = x_0x_1 - 2x_0^2x_1 + x_0^3x_1 - 3x_0^2x_1^2 + 6x_0x_1^3 - 2x_0^4x_1 - \\ 4x_0^3x_1^2 - 3x_0^2x_1x_0x_1 - 26x_0^2x_1^3 - 12x_0x_1x_0x_1^2 - \\ 6x_0x_1^2x_0x_1 + 2x_0^5x_1 - 6x_0^4x_1^2 - 3x_0^3x_1x_0x_1 + \\ 30x_0^3x_1^3 - 3x_0^2x_1x_0^2x_1 + 17x_0^2x_1x_0x_1^2 + \\ 10x_0^2x_1^2x_0x_1 - 28x_0^2x_1^3 + 24x_0x_1x_0^2x_1^2 + \\ 12x_0x_1x_0x_1x_0x_1 + 6x_0x_1^2x_0^2x_1 - \dots$$

In this case, $r_{c \odot d} = r_c + r_d = 5$, $r_{c \circ d_\delta} = r_c = 3$, and $r_{c \odot d} = \min(r_c, r_d) = 2$, which is consistent with the theory. Finally, for the connections involving inversion,

$$d^{\circ-1} = -x_0x_1 + x_0^2x_1 - 6x_0x_1^3 - x_0^4x_1 + 18x_0^2x_1^3 + \\ 12x_0x_1x_0x_1^2 + 6x_0x_1^2x_0x_1 + x_0^5x_1 - 4x_0^4x_1^2 - \\ 2x_0^3x_1x_0x_1 - 30x_0^3x_1^3 - 24x_0^2x_1x_0x_1^2 - \\ 12x_0^2x_1^2x_0x_1 - 12x_0x_1x_0^2x_1^2 - 6x_0x_1x_0x_1x_0x_1 + \dots$$

$$c@d = -x_0^2x_1 - 3x_0^2x_1^2 - 4x_0^3x_1^2 - 3x_0^2x_1x_0x_1 - 8x_0^2x_1^3 - \\ 4x_0^4x_1^2 - 2x_0^3x_1x_0x_1 - 24x_0^3x_1^3 - 19x_0^2x_1x_0x_1^2 - \\ 8x_0^2x_1^2x_0x_1 - 28x_0^2x_1^4 - 60x_0^4x_1^3 - 42x_0^3x_1x_0x_1^2 - \\ 22x_0^3x_1^2x_0x_1 - 116x_0^3x_1^4 - 22x_0^2x_1x_0^2x_1^2 - \\ 14x_0^2x_1x_0x_1x_0x_1 - 105x_0^2x_1x_0x_1^3 - 64x_0^2x_1^2x_0x_1^2 - \\ 28x_0^2x_1^3x_0x_1 - 124x_0^2x_1^5 + x_0^7x_1 - \dots$$

$$(1+d)^{\omega-1} = 1 - x_0x_1 + x_0^2x_1 - x_0^3x_1 + 4x_0^2x_1^2 + 2x_0x_1x_0x_1 - \\ 6x_0x_1^3 - 12x_0^3x_1^2 - 6x_0^2x_1x_0x_1 - 2x_0x_1x_0^2x_1 + \\ 28x_0^4x_1^2 + 14x_0^3x_1x_0x_1 - 36x_0^3x_1^3 + 6x_0^2x_1x_0^2x_1 - \\ 24x_0^2x_1x_0x_1^2 - 12x_0^2x_1^2x_0x_1 + 96x_0^2x_1^4 + 2x_0x_1x_0^3x_1 - \\ 12x_0x_1x_0^2x_1^2 - 6x_0x_1x_0x_1x_0x_1 + 48x_0x_1x_0x_1^3 + \\ 24x_0x_1^2x_0x_1^2 + 12x_0x_1^3x_0x_1 - \dots$$

$$(1+c)/(1+d) = 1 - x_0x_1 - x_0^3x_1 + x_0^2x_1^2 + 2x_0x_1x_0x_1 - 6x_0x_1^3 - \\ 10x_0^3x_1^2 - 6x_0^2x_1x_0x_1 - 8x_0^2x_1^3 - x_0x_1x_0^2x_1 + \\ 12x_0^4x_1^2 + 6x_0^3x_1x_0x_1 - 33x_0^3x_1^3 + 4x_0^2x_1x_0^2x_1 - \\ 31x_0^2x_1x_0x_1^2 - 17x_0^2x_1^2x_0x_1 + 68x_0^2x_1^4 + 2x_0x_1x_0^3x_1 - \\ 9x_0x_1x_0^2x_1^2 - 6x_0x_1x_0x_1x_0x_1 + 48x_0x_1x_0x_1^3 + \\ 24x_0x_1^2x_0x_1^2 + 12x_0x_1^3x_0x_1 + \dots$$

Again, as predicted by the theory, $r_{d^{\circ-1}} = r_d = 2$, $r_{c@d} = r_c = 3$, $r_{(1+d)^{\omega-1}} = r_{(1+d)} = 2$, and $r_{(1+c)/(1+d)} = \min(r_{(1+c)}, r_{(1+d)}) = 2$.

6. Conclusions and future work

In this paper the relative degree of two interconnected SISO nonlinear control systems was determined under a number of different scenarios. The analysis applies in both the state space setting and for input–output systems represented as Fliess operators. Future work will include the multivariable setting using the series notion of vector relative degree given in [21,22]. One could also consider stronger versions of relative degree in the SISO case, such as extended relative degree, and the interconnection of more than two systems.

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