

# ENDOMORPHISMS OF POWER SERIES FIELDS AND RESIDUE FIELDS OF FARGUES-FONTAINE CURVES

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ABSTRACT. We show that for  $k$  a perfect field of characteristic  $p$ , there exist endomorphisms of the completed algebraic closure of  $k((t))$  which are not bijective. As a corollary, we resolve a question of Fargues and Fontaine by showing that for  $p$  a prime and  $\mathbb{C}_p$  a completed algebraic closure of  $\mathbb{Q}_p$ , there exist closed points of the Fargues-Fontaine curve associated to  $\mathbb{C}_p$  whose residue fields are not (even abstractly) isomorphic to  $\mathbb{C}_p$  as topological fields.

## 1. INTRODUCTION

In this short note, we address the following question. By an *analytic field*, we will always mean a field complete with respect to a nonarchimedean multiplicative absolute value (assumed to be real-valued and written multiplicatively); by default, we always allow the trivial absolute value.

**Question 1.1.** *Let  $K$  be an analytic field. Let  $k$  be a trivially valued subfield of  $K$ . Is every continuous  $k$ -linear homomorphism from  $K$  to itself which induces automorphisms of residue fields and value groups necessarily surjective (and hence an automorphism)?*

We will view Question

Hereafter, fix a prime number  $p$ . Our main result is a negative answer to Question

The proof depends on a calculation using completed modules of Kähler differentials of analytic fields, as recently studied by the second author [14]. We develop here the bare minimum of this subject needed for the proof of Theorem

Theorem

**Theorem 1.2.** *Let  $\mathbb{C}_p$  be a completed algebraic closure of  $\mathbb{Q}_p$ . Then there exists a perfectoid field  $K$  which is not isomorphic to  $\mathbb{C}_p$  as a topological field, but for which there exists an isomorphism  $K^{\flat} \cong \mathbb{C}_p^{\flat}$ .*

This result admits the following geometric interpretation. For each perfectoid field  $K$ , Fargues and Fontaine define an associated scheme  $X_K$  which is a “complete curve” (i.e., a regular one-dimensional noetherian scheme equipped with a surjection of its Picard group onto  $\mathbb{Z}$ ) in terms of which  $p$ -adic Hodge theory over  $K$  can be simply formulated. Theorem

We conclude this introduction by pointing out that after we prepared our proof of Theorem

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## 2. ANALYTIC FIELDS AND COMPLETED DIFFERENTIALS

As a technical input into the proof of Theorem

**Definition 2.1.** By an *analytic field*, we will mean a field equipped with a multiplicative nonarchimedean absolute value with respect to which the field is complete. By default, we allow the trivial absolute value. When we consider an *extension*  $L/K$  of analytic fields, we require that the absolute value on  $L$  restricts to the absolute value on  $K$ .

**Definition 2.2.** We say that an extension  $L/K$  of analytic fields is *primitive* if there exists  $t \in L^\times$  such that  $K(t)$  is dense in  $L$ ; we will write  $L = \widehat{K(t)}$  if we need to indicate the choice of  $t$ .

With  $t$  given, the extension  $\widehat{K(t)}/K$  corresponds to a point in the projective line over  $K$  in the category of Berkovich nonarchimedean analytic spaces [2]. Without  $t$  given, the points associated to  $L/K$  are all of the same type 1–4 in Berkovich’s classification [2, (1.4.4)]; we thus classify  $L/K$  accordingly. Write

$$E_{L/K} = \dim_{\mathbb{Q}}(|L^\times|/|K^\times|) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad F_{L/K} = \text{trdeg}_{\kappa(K)} \kappa(L),$$

where  $\kappa(*)$  denotes the residue field of  $*$ ; these are determined by the type of  $L/K$  as follows.

Type of $L/K$	$E_{L/K}$	$F_{L/K}$
1	0	0
2	0	1
3	1	0
4	0	0

In all cases we have  $E_{L/K} + F_{L/K} \leq 1$ , as per Abhyankar’s inequality (e.g., see [13, Lemma 2.1.2]). However, types 1 and 4 cannot be distinguished using  $E_{L/K}$  and  $F_{L/K}$  alone: one must instead observe that  $L/K$  is of type 1 if and only if  $L$  embeds into the completed algebraic closure of  $K$ .

In order to better distinguish between primitive extensions of types 1 and 4, we will use completed modules of differentials.

**Definition 2.3.** Let  $L/K$  be an extension of analytic fields. As described in [14, §4], the module  $\Omega_{L/K}$  admits a maximal seminorm  $\|\bullet\|$  (the *Kähler seminorm*) with respect to which  $d_{L/K} : L \rightarrow \Omega_{L/K}$  is nonexpanding. Let  $\widehat{\Omega}_{L/K}$  denote the completion of  $\Omega_{L/K}$  with respect to  $\|\bullet\|$ ; it receives an induced derivation  $\widehat{d}_{L/K} : L \rightarrow \widehat{\Omega}_{L/K}$ .

**Lemma 2.4.** *Let  $L/K$  be a primitive extension, and choose  $t \in L$  such that  $K(t)$  is dense in  $L$ .*

- (a) *The module  $\widehat{\Omega}_{L/K}$  is generated over  $L$  by the single element  $\widehat{d}_{L/K}(t)$ .*
- (b) *The equality  $\widehat{\Omega}_{L/K} = 0$  holds if and only if the separable closure of  $K$  in  $L$  is dense. (Note that this condition implies that  $L/K$  is of type 1, and conversely whenever  $\text{char}(K) = 0$ .)*

*Proof.* Since  $\Omega_{K(t)/K}$  is generated by  $d_{L/K}(t)$ , (a) is obvious.

Let  $l$  be the separable integral closure of  $K$  in  $L$ . If  $l$  is dense in  $L$  (which forces  $L/K$  to be of type 1), then  $\Omega_{l/K} = 0$  and so  $\widehat{\Omega}_{L/K} = 0$ . This proves the inverse implication in (b).

Suppose that  $L/K$  is not of type 1. Let  $K'$  be a completed algebraic closure of  $K$  and put  $L' = l \widehat{\otimes}_K K'$ ; then the natural map  $\Omega_{L/K} \widehat{\otimes}_L L' \rightarrow \Omega_{L'/K'}$  sends  $\widehat{d}_{L/K}(t) \otimes 1$  to  $\widehat{d}_{L'/K'}(t)$ . The latter is nonzero by [3, Theorem 2.3.2(i)], so  $\widehat{d}_{L/K}(t) \neq 0$ .

It remains to consider the case when  $L/K$  is of type 1 but  $l$  is not dense in  $L$ . (Note that this last step is not needed for the proof of Theorem

Fix an embedding of  $L$  into  $K'$ . Let  $G$  be the group of continuous automorphisms of  $K'$  fixing  $K$ ; this group is naturally identified with the absolute Galois group of  $K$ . The subgroup  $H$  fixing  $L$  is closed in  $G$ , and hence is the absolute Galois group of some separable extension  $L_0$  of  $K$ . If  $\text{char } K = 0$ , then the Ax-Sen theorem [1] applied to both  $L_0$  and  $L$  implies that  $(K')^H = \widehat{L}_0 = L$ , but this contradicts our previous assumption that  $K = l \neq L$ .

We must then have  $\text{char } K = p > 0$ . By Ax-Sen again, we have  $(K')^H = \widehat{L_0^{1/p^\infty}} = \widehat{L^{1/p^\infty}}$ . If  $L_0 \neq K$ , we may choose a separable irreducible polynomial  $P \in K[T]$  of degree  $> 1$  with a root in  $L_0$ ; by Krasner's lemma,  $P$  has a root  $x$  in  $L^{1/p^n}$  for some sufficiently large  $n$ . But then  $x^{p^n} \in L$  generates a nontrivial separable extension of  $K$ , again contradicting our assumption that  $K = l \neq L$ . We conclude that  $L_0 = K$  and so  $t \in \widehat{K^{1/p^\infty}} \setminus K$ .

Choose  $a_0 = 0, a_1, \dots \in K$  such that the sequence  $r_n = |t - a_n^{1/p^n}|$  converges to zero. Then  $L$  is the completion of its subalgebra  $\bigcup_n k\{r_0^{-1}t, r_n^{-p^n}(t^{p^n} - a_n)\}$ ; in particular,  $k[t]$  is dense in  $L$ . Consider the Banach ring  $\mathcal{A} := L \widehat{\otimes}_K L$  provided with the tensor product norm  $\|\bullet\|$  and note that the ideal  $J = \text{Ker}(\mathcal{A} \rightarrow L)$  is generated by  $T := 1 \otimes t - t \otimes 1$ .

We claim that  $\|T^{p^n}\| \leq r_n^{p^n}$ . Indeed, since  $|t^{p^n} - a_n| = r_n^{p^n}$ , we have that  $\|1 \otimes t^{p^n} - a_n\| \leq r_n^{p^n}$  and  $\|t^{p^n} \otimes 1 - a_n\| \leq r_n^{p^n}$ . (Note, for the sake of completeness, that  $T$  is quasi-nilpotent, i.e. its spectral norm vanishes, and hence  $L$  is the uniform completion of  $\mathcal{A}$ , i.e. the completion with respect to the spectral seminorm. This is a topological extension of the classical fact that  $T$  is nilpotent and  $L$  is the reduction of  $\mathcal{A}$  when  $L/K$  is finite and purely inseparable.)

By [14, Remark 4.3.4(ii)], there is an isomorphism  $J/J^2 \xrightarrow{\sim} \widehat{\Omega}_{L/K}$  that takes  $T$  to  $\widehat{d}_{L/K}(t)$ . Thus, we should only show that  $T \neq aT^2$  in  $\mathcal{A}$ . Assume, to the contrary, that  $T = aT^2$  and set  $s = \|a\|$ . Then,  $\|T\| = \|a^{p^n-1}T^{p^n}\| \leq s^{p^n-1}r_n^{p^n}$  for any  $n$  and hence  $\|T\| = 0$ . Thus  $L \widehat{\otimes}_K L = L$  and since  $L \otimes_K L$  embeds into  $L \widehat{\otimes}_K L$  by [6, 3.2.1(4)], we obtain a contradiction.  $\square$

### 3. PROOFS AND EXAMPLES

We now settle the questions raised in the introduction.

**Proposition 3.1.** *Question*

**Example 3.2.** Let  $k$  be an analytic field whose absolute value is nondiscrete, and choose a sequence  $x_1, x_2, \dots \in k^\times$  such that  $|x_i| < 1$  and  $\lim_n |x_1 \cdots x_n| > 0$ . (For a more concrete example, take  $k$  to be a completed algebraic closure of  $\mathbb{C}((t))$  and take  $x_n = t^{2^{-n}}$ .) Let  $K$  be the completion of  $k(t_1, t_2, \dots)$  for the Gauss valuation (i.e., the valuation of a nonzero polynomial is the maximum valuation of its coefficients); then  $K$  admits a unique valuation-preserving endomorphism  $\tau$  fixing  $k$  and taking  $t_n$  to  $t_n - x_n t_{n+1}$  for each  $n$ . We will show that the image of  $\tau$  does not contain  $t_1$ , and hence  $\tau$  is not an isomorphism.

Suppose to the contrary that there exists  $y \in K$  with  $\tau(y) = t_1$ . By hypothesis, there exists some  $\lambda \in k$  such that  $|\lambda| < |x_1 \cdots x_n|$  for all  $n$ . We may then choose  $y' \in K_0(t_1, \dots, t_n)$  for some positive integer  $n$  in such a way that  $|y - y'| < |\lambda|$ . Put  $y'' = t_1 + x_1 t_2 + \cdots + x_1 \cdots x_n t_{n+1}$ ; then  $\tau(y'') = t_1 - x_1 \cdots x_{n+1} t_{n+2}$ , so  $|y'' - y| = |\tau(y'' - y)| = |x_1 \cdots x_{n+1}| > |\lambda|$ . Hence

$|y'' - y'| = |x_1 \cdots x_{n+1}|$ , but  $y'' - y'$  equals  $x_1 \cdots x_n t_{n+1}$  plus an element of  $k(t_1, \dots, t_n)$  and so cannot have valuation less than  $|x_1 \cdots x_n|$ . This yields the desired contradiction.

**Remark 3.3.** Let  $k$  be a field of characteristic 0. For each positive integer  $n$ , the derivation  $\frac{d}{dt}$  on  $k((t))$  extends to the derivation  $\partial_n = n^{-1}t^{1/n-1} \frac{d}{dt^{1/n}}$  on  $k((t^{1/n}))$  satisfying  $|\partial_n f| \leq |t|^{-1}|f|$  for any  $f \in k((t^{1/n}))$ . Let  $K$  be a completed algebraic closure of  $k((t))$ ; by Puiseux's theorem,  $K$  is the completion of  $\bigcup_{n=1}^{\infty} \bar{k}((t^{1/n}))$  for  $\bar{k}$  the algebraic closure of  $k$  in  $K$ , so the derivation  $\frac{d}{dt}$  extends uniquely to a continuous derivation on  $K$ . Consequently,  $\widehat{\Omega}_{K/k}$  is generated by  $\widehat{d}_{K/k}(x)$  for any  $x \in K - \bar{k}$ . For any  $k$ -linear automorphism  $\tau$  of  $K$ , let  $L$  be the completion of  $\tau(K)(t)$  within  $K$ ; taking  $x = \tau(t)$  in the previous discussion shows that  $\widehat{\Omega}_{L/\tau(K)} = 0$ . By Lemma

*Proof of Theorem* Choose a sequence  $\{d_i\}_{i=1}^{\infty}$  of positive integers in such a way that:

- (a)  $d_i$  is not divisible by  $p$ ;
- (b)  $\lim_{i \rightarrow \infty} (d_{i+1} - pd_i) = \infty$ ; and
- (c) the sequence  $\{p^{-i}d_i\}_{i=1}^{\infty}$  is strictly increasing (for large  $i$ , this follows from (b)) and bounded.

For a concrete example, take

$$d_i := 1 + pi + p^2(i-1) + p^3(i-2) + \cdots + p^i.$$

Choose a sequence  $\{c_i\}_{i=1}^{\infty}$  of elements of  $k$  such that each field  $\mathbb{F}_p(c_i)$  is finite, but the field  $\mathbb{F}_p(c_1, c_2, \dots)$  is infinite (in fact any  $c_i \neq 0$  will do, but this assumption shortens the argument). Set

$$\alpha_n := \sum_{i=0}^n c_i t^{p^{-i}d_i} \in K$$

and

$$r_n := |\alpha_{n+1} - \alpha_n| = |t|^{p^{-n-1}d_{n+1}};$$

by construction,  $\{r_n\}_{n=1}^{\infty}$  is a strictly decreasing sequence with nonzero limit. Consider the Berkovich affine line  $\mathbb{A}_{k((t))}^1$  with coordinate  $x$  and let  $E_n$  be the closed disc of radius  $r_n$  centered at  $\alpha_n$ . The intersection of the  $E_n$  does not contain any element of any finite extension of  $k((t))$ , so it consists of a single point  $z$  of type 4. (Otherwise, by [7, Lemma 10.1, Corollary 11.9] the generalized power series  $x = \sum_{i=0}^{\infty} c_i t^{p^{-i}d_i}$  would be algebraic over  $k(t)$ , hence over  $\overline{\mathbb{F}_p}(t)$  because the minimal polynomial must be invariant under coefficientwise automorphisms, hence over  $\mathbb{F}_p(t)$ ; but then [7, Corollary 11.9] would force the  $c_i$  to belong to a finite extension of  $\mathbb{F}_p$ .) The completed residue field  $L = \mathcal{H}(z)$  of this point is a primitive extension of  $k((t))$  topologically generated by  $x$ , and the conditions  $z \in E_n$  mean that  $|x^{p^n} - \alpha_n^{p^n}| = r_n^{p^n}$  for each  $n$ . Since  $\widehat{d}_{L/k}$  is nonexpanding, we have  $\|\widehat{d}_{L/k}(\alpha_n^{p^n})\| \leq r_n^{p^n}$ . Furthermore, in the expression  $\alpha_n^{p^n} = \sum_{i=0}^n t^{p^{n-i}d_i}$  only the term  $t^{d_n}$  is not a  $p$ -th power, so

$$\widehat{d}_{L/k}(\alpha_n^{p^n}) = \widehat{d}_{L/k}(t^{d_n}) = d_n t^{d_n-1} \widehat{d}_{L/k}(t)$$

and hence (since  $d_n$  is not divisible by  $p$ )

$$\|\widehat{d}_{L/k}(t)\| \leq r_n^{p^n} |t|^{1-d_n} = |t|^{p^{-1}d_{n+1}-d_n+1}.$$

Since this holds for all  $n$ , we conclude that  $\|\widehat{d}_{L/k}(t)\| = 0$ .

By the previous paragraph,  $\widehat{d}_{L/k}(t) = 0$  and hence  $\widehat{d}_{L/k((x))}(t) = 0$ . By Lemma

*Proof of Theorem* We use [8, Theorem 1.5.6] as our blanket reference concerning the perfectoid correspondence. By [8, Example 1.3.5], there is an algebraic extension of  $\mathbb{Q}_p$  whose completion is perfectoid with tilt isomorphic to the completed perfect closure of  $\mathbb{F}_p((t))$ ; hence  $\mathbb{C}_p$  is perfectoid and  $\mathbb{C}_p^b$  is isomorphic to the completed algebraic closure of  $\mathbb{F}_p((t))$ . By Theorem

## REFERENCES

- [1] J. Ax, Zeros of polynomials over local fields—the Galois action, *J. Alg.* **15** (1970), 417–428.
- [2] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.
- [3] A. Cohen, M. Temkin, and D. Trushin, Morphisms of Berkovich curves and the different function, *Adv. Math.* **303** (2016), 800–858.
- [4] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge  $p$ -adique, in preparation; draft (September 2015) available at <http://webusers.imj-prg.fr/~laurent.fargues/>.
- [5] L. Fargues and J.-M. Fontaine, Vector bundles on curves and  $p$ -adic Hodge theory, in *Automorphic Forms and Galois Representations, Volume 1*, London Math. Soc. Lect. Note Ser. 414, Cambridge Univ. Press, 2014.
- [6] L. Gruson, Théorie de Fredholm  $p$ -adique, *Bull. Soc. Math. France* **94** (1966), 67–95.
- [7] K.S. Kedlaya, On the algebraicity of generalized power series, *Beitr. Algebra Geom.* (2016).
- [8] K.S. Kedlaya, New methods for  $(\varphi, \Gamma)$ -modules, *Res. Math. Sci.* **2:20** (2015).
- [9] K.S. Kedlaya, Automorphisms of perfect power series rings, arXiv:1602.09051v1 (2016).
- [10] K.S. Kedlaya and R. Liu, Relative  $p$ -adic Hodge theory, I: Foundations, *Astérisque* **371** (2015), 239 pages.
- [11] M. Mategnon and M. Reversat, Sous-corps fermés d’un corps valué, *J. Algebra* **90** (1984), 491–515.
- [12] P. Scholze, Perfectoid spaces, *Publ. Math. IHÉS* **116** (2012), 245–313.
- [13] M. Temkin, Stable modifications of relative curves, *J. Alg. Geom.* **19** (2010), 603–677.
- [14] M. Temkin, Metrization of differential pluriforms on Berkovich analytic spaces, *Nonarchimedean and Tropical Geometry*, Simons Symposia, Springer, 2016, 195–285.