

AUTOMORPHISMS OF PERFECT POWER SERIES RINGS

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ABSTRACT. Let R be a perfect ring of characteristic p . We show that the group of continuous R -linear automorphisms of the perfect power series ring over R is generated by the automorphisms of the ordinary power series ring together with Frobenius; this answers a question of Jared Weinstein.

1. INTRODUCTION

Let R be a ring. Let $S = R[[t]]$ be the ring of formal power series with coefficients in R , equipped with the t -adic topology, and let \mathfrak{m} be the ideal of S consisting of series with zero constant term. For each $y \in \mathfrak{m}$, the formula

$$(1) \quad \sum_i x_i t^i \mapsto \sum_i x_i y^i$$

defines an element $\text{Sub}(y)$ of the algebra $\text{End}_R^{\text{cts}}(S)$ of continuous R -linear endomorphisms of S preserving \mathfrak{m} . The resulting map $\text{Sub} : \mathfrak{m} \rightarrow \text{End}_R^{\text{cts}}(S)$ is inverse to the map $\text{End}_R^{\text{cts}}(S) \rightarrow \mathfrak{m}$ sending $f : S \rightarrow S$ to $f(t)$. It is well-known (and easy to check) that $\text{Sub}(y)$ is invertible if and only if the coefficient of t in y is a unit in R , i.e., $y \in R^\times t + \mathfrak{m}^2$; that is, Sub identifies $R^\times t + \mathfrak{m}^2$ with the group $\text{Aut}_R^{\text{cts}}(S)$ of continuous R -linear automorphisms of S preserving \mathfrak{m} (this last condition is automatic if R is reduced).

From now on, assume that R is of characteristic $p > 0$ and is perfect, i.e., the Frobenius endomorphism $x \mapsto x^p$ on R is bijective. The analogue of the power series construction in the category of perfect rings is the t -adic completion of $R[t^{1/p}, t^{1/p^2}, \dots]$, which we call S' . The elements of S' may be viewed as formal sums $\sum_{i \in \mathbb{Z}[p^{-1}]_{\geq 0}} x_i t^i$ with $x_i \in R$ whose support (i.e., the set of i for which $x_i \neq 0$) is either finite or an unbounded increasing sequence.

Let \mathfrak{m}' be the ideal of S' consisting of series with zero constant coefficients. Then the formula (1) again defines a substitution homomorphism $\text{Sub}' : \mathfrak{m}' \rightarrow \text{End}_R^{\text{cts}}(S')$ which is inverse to the map $\text{End}_R^{\text{cts}}(S') \rightarrow \mathfrak{m}'$ given by evaluation at t . In particular, we may construct a commutative diagram

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\text{Sub}} & \text{End}_R^{\text{cts}}(S) \\ \downarrow & & \downarrow \\ \mathfrak{m}' & \xrightarrow{\text{Sub}'} & \text{End}_R^{\text{cts}}(S') \end{array}$$

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in which the left vertical arrow is the obvious inclusion. In particular, we get an injective homomorphism $\text{Aut}_R^{\text{cts}}(S) \rightarrow \text{Aut}_R^{\text{cts}}(S')$ of groups of continuous R -linear automorphisms.

We show that while $\text{End}_R^{\text{cts}}(S')$ is much bigger than $\text{End}_R^{\text{cts}}(S)$, the map of automorphism groups is quite close to being an isomorphism.

Theorem 1. *The map $\text{Aut}_R^{\text{cts}}(S) \rtimes \mathbb{Z} \rightarrow \text{Aut}_R^{\text{cts}}(S')$ taking $n \in \mathbb{Z}$ to the map $t \mapsto t^{p^n}$ is an isomorphism of groups.*

That is, an endomorphism of S' over R is an isomorphism if and only if it carries the uncompleted perfect closure of S into itself. For a related result with perfect closures replaced by algebraic closures, see [2].

Theorem 1 answers a question posed to us by Jared Weinstein, motivated by the following considerations. Let S'' be the (t_1, t_2) -adic completion of $R[t_1^{1/p^\infty}, t_2^{1/p^\infty}]$. By a *(one-dimensional commutative) perfect formal group law* over R , we will mean an element $f \in S''$ satisfying the following conditions.

- (a) We have $f(t_2, t_1) = f(t_1, t_2)$.
- (b) We have $f(t_1) \equiv t_1 \pmod{t_2^{1/p^\infty}}$.
- (c) In the (t_1, t_2, t_3) -adic completion of $R[t_1^{1/p^\infty}, t_2^{1/p^\infty}, t_3^{1/p^\infty}]$, we have $f(f(t_1, t_2), t_3) = f(t_1, f(t_2, t_3))$.

For example, any ordinary (one-dimensional commutative) formal group law over R , as an element of $R[[t_1, t_2]]$, is a perfect formal group law.

Recall that for every formal group law over a ring of characteristic p , the formal multiplication by integers interpolates continuously to a formal action of \mathbb{Z}_p . The same holds for perfect formal group laws, and Theorem 1 implies that for $m \in \mathbb{Z}_p^\times$, the formal multiplication-by- m map, which *a priori* is a perfect power series in one variable, is in fact always an ordinary power series. This suggests a possible affirmative answer to the following question.

Question 2. *Is every perfect formal group law an ordinary formal group law? That is, is any perfect formal group law contained in $R[[t_1, t_2]]$?*

It may be possible to gain additional insight into Question 2 by classifying continuous R -linear automorphisms of S'' ; however, this approach is complicated by the fact that the map

$$\text{Aut}_R^{\text{cts}}(R[[t_1, t_2]]) \rtimes \mathbb{Z}^2 \rightarrow \text{Aut}_R^{\text{cts}}(S''),$$

in which $(n_1, n_2) \in \mathbb{Z}^2$ maps to the substitution $t_1 \mapsto t_1^{p^{n_1}}, t_2 \mapsto t_2^{p^{n_2}}$, is far from being surjective. For example, for any $f \in S'$, the substitution

$$t_1 \mapsto t_1, t_2 \mapsto t_2 + f(t_1)$$

is an automorphism of S'' with inverse

$$t_1 \mapsto t_1, t_2 \mapsto t_2 - f(t_1).$$

2. PROOF OF THEOREM 1

The remainder of this document consists of the proof of Theorem 1. The argument is loosely inspired by an analogous calculation of automorphism groups of certain rings of Hahn-Mal'cev-Neumann generalized power series [1, §3], although the details turn out to be somewhat different.

By evaluating maps at t , we see that the map $\text{Aut}_R^{\text{cts}}(S) \rtimes \mathbb{Z} \rightarrow \text{Aut}_R^{\text{cts}}(S')$ is injective. To check surjectivity, let $v_t : S' \rightarrow \mathbb{Z}[p^{-1}]_{\geq 0}$ denote the t -adic valuation, and note that

$$v_t(\text{Sub}'(y)(z)) = v_t(y)v_t(z) \quad (y, z \in \mathfrak{m}').$$

Consequently, any $y \in \mathfrak{m}'$ for which $\text{Sub}'(y)$ is an automorphism must satisfy $v_t(y) \in p^{\mathbb{Z}}$; there is thus no harm in assuming that $v_t(y) = 1$.

It suffices to derive a contradiction assuming that there exist $y, z \in \mathfrak{m}' \setminus \mathfrak{m}$ such that $v_t(y) = v_t(z) = 1$ and that $\text{Sub}'(z) \circ \text{Sub}'(y) = \text{id}_{S'}$. (Note that this immediately implies that $\text{Sub}'(y) \circ \text{Sub}'(z) = \text{id}_{S'}$ because $\text{Sub}'(z)$ is injective whenever $z \neq 0$.) Write $y = \sum_i y_i t^i, z = \sum_j z_j t^j$. Let v_p denote the p -adic valuation on $\mathbb{Z}[p^{-1}]$. For $n \in \mathbb{Z}$, put

$$\begin{aligned} a_n &= \min\{i - 1 : v_p(i) \leq -n, y_i \neq 0\}, \\ b_n &= \min\{i - 1 : v_p(i) \leq -n, z_i \neq 0\}, \end{aligned}$$

in each case interpreting the minimum over an empty set as $+\infty$. From our hypotheses,

$$(2) \quad a_n, b_n = 0 \ (n \leq 0); \ 0 < a_1, b_1 < +\infty; \ a_n \leq a_{n+1}, b_n \leq b_{n+1}; \ \lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n = +\infty.$$

Consequently,

$$c = \min \left\{ \frac{p^n}{p^n - 1} a_n, \frac{p^n}{p^n - 1} b_n : n = 1, 2, \dots \right\}$$

exists, is finite and positive, and is achieved by only finitely many indices. Put

$$c_n = \frac{p^n - 1}{p^n} c \quad (n \in \mathbb{Z}).$$

Then

$$a_l \geq c_l, \quad b_m \geq c_m \quad (l, m \in \mathbb{Z});$$

there exist maximal indices l, m for which equalities occur; and these maximal indices are nonnegative and not both zero. Moreover, if $l > 0$, then we must have $v_p(a_l) = -l$, as otherwise we would have the contradiction

$$c_l = a_l = a_{l+1} \geq \frac{p^{l+1} - 1}{p^{l+1}} c > \frac{p^l - 1}{p^l} c = c_l;$$

similarly, if $m > 0$, then $v_p(b_m) = -m$. Since we either have $a_l = c_l$ for some $l > 0$ or $b_m = c_m$ for some $m > 0$, we may deduce that for all $n > 0$, $c_n > 0$ and $v_p(c_n) = -n$.

Since $l+m > 0$, we have $c_{l+m} > 0$, so the coefficient of $t^{1+c_{l+m}}$ in $t = (\text{Sub}'(z) \circ \text{Sub}'(y))(t) = \sum_{i>0} y_i z^i$ must be zero. To obtain the desired contradiction, it will thus suffice to verify that the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ is nonzero for exactly *one* value of i ; we check this by distinguishing options for $d = -v_p(i)$.

- For $d \geq l+m$, we have

$$v_t(y_i z^i) = v_t(y_i t^i) \geq 1 + a_d \geq 1 + c_d \geq 1 + c_{l+m}.$$

For the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ to be nonzero, this chain of inequalities must become a chain of equalities, yielding

$$i = 1 + a_d, \quad d \leq l, \quad d = l + m.$$

Since $m \geq 0$, this is only possible if $d = l$, $m = 0$, $i = 1 + a_l$; in this case, the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ is the nonzero value $y_{1+a_l} z_1^{1+a_l}$.

- For $d < l + m$, we have

$$y_i z^i = y_i (z^{p^{-d}})^{ip^d} = y_i z_1^i \left(1 + \sum_{j>1} (z_j/z_1) t^{p^{-d}(j-1)} \right)^{ip^d} t^i.$$

By the definition of b_n , the sum over j can be rewritten as
(element of $S^{p^{-l-m+1}}$) + (nonzero element of R) $\cdot t^{p^{-d}b_{l+m-d}}$ + (higher order terms).

Since $v_p(ip^d) = 0$ and $t^i \in S^{p^{-l-m+1}}$, the binomial expansion yields

$$y_i z^i = (\text{element of } S^{p^{-l-m+1}}) + (\text{nonzero element of } R) \cdot t^{i+p^{-d}b_{l+m-d}} + (\text{higher order terms}).$$

Since $v_p(c_{l+m}) = -l - m$, the coefficient of $t^{1+c_{l+m}}$ in any element of $S^{p^{-l-m+1}}$ is zero. On the other hand, we have

$$i + p^{-d}b_{l+m-d} \geq 1 + a_d + p^{-d}b_{l+m-d} \geq 1 + c_d + p^{-d}c_{l+m-d} = 1 + c_{l+m}.$$

For the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ to be nonzero, this chain of inequalities must become a chain of equalities, yielding

$$i = 1 + a_d, \quad d \leq l, \quad l + m - d \leq m.$$

This is only possible if $d = l$, $i = 1 + a_l$, $m > 0$; in this case, the coefficient of $t^{1+c_{l+m}}$ in $y_i z^i$ is the nonzero value $y_{1+a_l} z_1^{1+a_l}$.

Since exactly one of the two boundary cases can occur (depending on whether $m = 0$ or $m > 0$), this yields the desired contradiction.

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