

Tverberg partitions as weak epsilon-nets

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Abstract

We prove a Tverberg-type theorem using the probabilistic method. Given $\varepsilon > 0$, we find the smallest number of partitions of a set X in $\mathbb{R}^d$ into r parts needed in order to induce at least one Tverberg partition on every subset of X with at least $\varepsilon|X|$ elements. This generalizes known results about Tverberg's theorem with tolerance.

1 Introduction

Tverberg's theorem and the weak ε -net theorem for convex sets are central results describing the combinatorial properties of convex sets. Their statements are the following

Theorem 1.1 (Tverberg 1966, [Tve66]). *Let r, d be positive integers. Given a set X of $(r - 1)(d + 1) + 1$ points in $\mathbb{R}^d$, there is a partition of X into r sets whose convex hulls intersect.*

We call a partition into r sets as above a *Tverberg partition*. For a set $Y \subset \mathbb{R}^d$, we denote by $\text{conv } Y$ its convex hull.

Theorem 1.2 (Weak ε -net; Alon, Bárány, Füredi, Kleitman 1992, [ABFK92]). *Let d be a positive integer and $\varepsilon > 0$. Then, there is an integer $n = n(\varepsilon, d)$ such that the following holds. For any finite set X of points in $\mathbb{R}^d$, there is a set $K \subset \mathbb{R}^d$ of $n(\varepsilon, d)$ points such that for all $Y \subset X$ with $|Y| \geq \varepsilon|X|$, we have that $\text{conv } Y$ intersects K .*

For an overview of both theorems and how they have shaped discrete geometry, consult [Mat02, BS17b]. One key aspect of the weak ε -net theorem is that $n(\varepsilon, d)$ does not depend on $|X|$. The two theorems are closely related to each other. Tverberg's theorem is an important tool in the proof of the “first selection lemma” [Bár82], which in turn is used to prove the weak ε -net

theorem. Finding upper and lower bounds for $n(\varepsilon, d)$ is a difficult problem. As an upper bound, for any fixed d we have $n(\varepsilon, d) = O(\varepsilon^{-d} \text{polylog}(\varepsilon^{-1}))$ [CEG⁺95, MW04]. There are lower bounds superlinear in $1/\varepsilon$, for any fixed d we have $n(\varepsilon, d) = \Omega((1/\varepsilon) \ln^{d-1}(1/\varepsilon))$ [BMN11].

The purpose of this paper is to provide a different link between these two theorems. Just as the weak ε -net gives you a fixed-size set which intersects the convex hull of each not too small subset of X , now we seek a fixed number of partitions of X , such that for every not too small subset $Y \subset X$, at least one of the partitions induces a Tverberg partition on Y . Unlike the weak ε -net problem, we get an exact value for the number of partitions needed.

Given a partition $\mathcal{P}$ of X and $Y \subset X$, we denote by $\mathcal{P}(Y)$ the restriction of $\mathcal{P}$ on Y ,

$$\mathcal{P}(Y) = \{K \cap Y : K \in \mathcal{P}\}.$$

If $\mathcal{P}$ is a partition into r sets, then $\mathcal{P}(Y)$ is also a partition into r sets, though some may be empty. With this notation, we can state the main result of this paper.

Theorem 1.3. *Let $1 \geq \varepsilon > 0$ be a real number and r, d be positive integers. Then, there is an integer $m = m(\varepsilon, r)$ such that the following is true. For every sufficiently large finite set $X \subset \mathbb{R}^d$, there are m partitions $\mathcal{P}_1, \dots, \mathcal{P}_m$ of X into r parts each such that, for every subset $Y \subset X$ with $|Y| \geq \varepsilon|X|$, there is a k such that $\mathcal{P}_k(Y)$ is a Tverberg partition. Moreover, we have*

$$m(\varepsilon, r) = \left\lceil \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln\left(\frac{r}{r-1}\right)} \right\rceil + 1.$$

An equivalent statement is that $\varepsilon > ((r-1)/r)^m$ if and only if $m(\varepsilon, r) \leq m$. One should notice that $1/\ln(r/(r-1)) \sim r$, so $m(\varepsilon, r) \sim r \ln(1/\varepsilon)$. One surprising aspect of this result is that m does not depend on the dimension. The effect of the dimension only appears when we look at how large X must be for the theorem to kick in. The value for $|X|$ where the theorem starts working is, up to polylogarithmic terms, $m d r^3 (\varepsilon - ((r-1)/r)^m)^{-2}$.

The proof of Theorem 1.3 follows from a repeated application of the probabilistic method, contained in section 3. We build up on the techniques of [Sob18] to prove Tverberg-type results by making random partitions. The key new observation is that, given m partitions of X , the number of containment-maximal subsets Y such that $\mathcal{P}_k(Y)$ is not a Tverberg partition for any k is polynomial in $|X|$.

This result is also closely related to Tverberg's theorem with tolerance.

Theorem 1.4 (Tverberg with tolerance; García-Colín, Raggi, Roldán-Pensado 2017, [GCRRP17]). *Let r, t, d be positive integers, where r, d are fixed. There is an integer $N(r, t, d) = rt + o(t)$ such that the following holds. For any set X of N points in $\mathbb{R}^d$, there is a partition of X into r sets $X_1, \dots, X_r$ such that, for all $C \subset X$ of cardinality t , we have*

$$\bigcap_{j=1}^r \operatorname{conv}(X_j \setminus C) \neq \emptyset.$$

This is a result that is motivated by earlier work of Larman [Lar72], who studied the case $t = 1, r = 2$. Theorem 1.4 determines the correct leading term as t becomes large. This result has been improved to $N = rt + \tilde{O}(\sqrt{t})$, where the $\tilde{O}$ term hides polylogarithmic factors, and is polynomial in r, d [Sob18]. In the notation of Theorem 1.3, Theorem 1.4 says that if $\varepsilon > 1 - 1/r$, then $m(\varepsilon, r) = 1$. Improved bounds for small values of t can be found in [SS12, MS14].

As the driving engine in the proof of Theorem 1.3 is Sarkaria’s tensoring technique, described in section 2, it can be easily modified to get similar versions of a multitude of variations of Tverberg’s theorem. This includes Tverberg “plus minus” [BS17a], colorful Tverberg with equal coefficients [Sob15] and asymptotic variations of Reay’s conjecture [Sob18]. We do not include those variations explicitly. We do include an ε -version for the colorful Tverberg theorem in section 4, as it is closely related to a conjecture in [Sob18].

A natural question that follows the results of this paper is to determine whether a topological version of Theorem 1.3 also holds.

2 Preliminaries

2.1 Sarkaria’s technique.

We start discussing the preliminaries for the the proof of Theorem 1.3. At the core of the proof is Sarkaria’s technique to prove Tverberg’s theorem via tensor products [Sar92, BO97].

The goal is to reduce Tverberg’s theorem to the colorful Carathéodory theorem.

Theorem 2.1 (Colorful Carathéodory; Bárány 1982 [Bár82]). *Let $F_1, \dots, F_{n+1}$ be sets of points in $\mathbb{R}^n$. If $0 \in \operatorname{conv}(F_i)$ for all $i = 1, \dots, n + 1$, then we can choose points $x_1 \in F_1, \dots, x_{n+1} \in F_{n+1}$ so that $0 \in \operatorname{conv}\{x_1, \dots, x_{n+1}\}$.*

The set $\{x_1, \dots, x_{n+1}\}$ is called a *transversal* of $\mathcal{F} = \{F_1, \dots, F_{n+1}\}$. Each set F_i is called a *color class*. For the sake of brevity we do not reproduce Sarkaria's proof, but point out the main ingredients. We distinguish between Tverberg-type results and colorful Carathéodory-type results by denoting the dimension of their ambient spaces by d and n , respectively.

Let $X = \{x_1, \dots, x_N\}$ be a set of points in $\mathbb{R}^d$ and r a positive integer. We define $n = (d+1)(r-1)$. Let $v_1, \dots, v_r$ be the vertices of a regular simplex in $\mathbb{R}^{r-1}$ centered at the origin. We construct the points

$$\bar{x}_{i,j} = (x_i, 1) \otimes v_j \in \mathbb{R}^{(d+1)(r-1)} = \mathbb{R}^n,$$

where $\otimes$ denotes the standard tensor product. Given two vectors $v_1 \in \mathbb{R}^{d_1}, v_2 \in \mathbb{R}^{d_2}$, their tensor product $v_1 \otimes v_2$ is simply the $d_1 \times d_2$ matrix $v_1 v_2^T$ interpreted as a $d_1 d_2$ -dimensional vector. These tensor products carry all the information about Tverberg partitions into r parts.

Lemma 2.2. *Let $X = \{x_1, \dots, x_N\}$ be a finite set of points in $\mathbb{R}^d$, r be a positive integer. Then, a partition $X_1, \dots, X_r$ of X is a Tverberg partition if and only if*

$$0 \in \text{conv}\{\bar{x}_{i,j} : i, j \text{ are such that } x_i \in X_j\}$$

A lucid explanation of the lemma above can be found in [Bár15]. Lemma 2.2 implies that, given X , if we consider the sets

$$F_i = \{\bar{x}_{i,j} : j = 1, \dots, r\} \quad i = 1, \dots, N,$$

then finding a Tverberg partition of X into r parts corresponds to finding a transversal of $\mathcal{F} = \{F_1, \dots, F_N\}$ whose convex hull contains the origin in $\mathbb{R}^n$. Since $0 \in \text{conv } F_i$ for each i , Theorem 2.1 or a variation can be applied. Then, by Lemma 2.2, we obtain a Tverberg partition.

For transversals, there is also a natural notion of restriction. Given a family $\mathcal{F}$ of sets in $\mathbb{R}^n$, $\mathcal{G} \subset \mathcal{F}$, and P a transversal of $\mathcal{F}$, we define

$$P(\mathcal{G}) = \{x \in P : x \text{ came from a set in } \mathcal{G}\}.$$

Alternatively, $P(\mathcal{G}) = P \cap (\cup \mathcal{G})$. In order to prove Theorem 1.3, it is sufficient to prove the following.

Theorem 2.3. *Let r, n be positive integers and $1 \geq \varepsilon > 0$ a real number. Then, there is an integer $m = m(\varepsilon, r)$ such that the following is true. For every sufficiently large N , if we are given a family $\mathcal{F}$ of N sets in $\mathbb{R}^n$, such*

that $0 \in \text{conv } F$ and $|F| = r$ for all $F \in \mathcal{F}$, then there are m transversals $P_1, \dots, P_m$ of $\mathcal{F}$ with the following property. For every $\mathcal{G} \subset \mathcal{F}$ with $|\mathcal{G}| \geq \varepsilon|\mathcal{F}|$ there is a k with $0 \in \text{conv } P_k(\mathcal{G})$.

Moreover, we have

$$m(\varepsilon, r) = \left\lceil \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln\left(\frac{r}{r-1}\right)} \right\rceil + 1.$$

Indeed, let us sketch how Theorem 2.3 implies Theorem 1.3.

Proof. Assume r, d, ε, m are given, satisfying the last equality of Theorem 2.3. Let $n = (d+1)(r-1) + 1$. Assume that we are given a set X of N points in $\mathbb{R}^d$, $X = \{x_1, \dots, x_N\}$, where N is a large positive integer. For $v_1, \dots, v_r \in \mathbb{R}^{r-1}$ as before, we construct the sets

$$F_i = \{(x_i, 1) \otimes v_j : j = 1, \dots, r\} \subset \mathbb{R}^n.$$

Then, we apply Theorem 2.3 to the family $\mathcal{F} = \{F_1, \dots, F_N\}$ and find m transversals $P_1, \dots, P_m$. Given a set of indices $I \subset [N]$ such that $|I| \geq \varepsilon N$, consider $\mathcal{G}_I = \{F_i : i \in I\}$. Then, there must be a transversal P_{i_0} such that $0 \in \text{conv } P_{i_0}(\mathcal{G}_I)$. By Lemma 2.2, this means that the partition $\mathcal{P}_{i_0}$ of X induced by P_{i_0} is a Tverberg partition even when restricted to the set $X_I = \{x_i : i \in I\}$. In other words, the partitions induced by $P_1, \dots, P_m$ satisfy the conclusion of Theorem 1.3. $\square$

We also need the following lemma. It bounds the complexity of verifying if $0 \in \text{conv } Y$ if $Y \subset X$ and X is given in advance. For our purposes, we need a slightly weaker version than the one presented in [Sob18] (see also [CEM⁺96]).

Lemma 2.4. *Let $X \subset \mathbb{R}^n$ be a finite set. Then, there is a family $\mathcal{H}$ of $|X|^n$ half-spaces in $\mathbb{R}^n$, each containing 0, such that the following holds. For every subset $Y \subset X$, we have $0 \in \text{conv } Y$ if and only if $Y \cap H \neq \emptyset$ for all $H \in \mathcal{H}$.* $\square$

Sketch of proof. 0 belongs to $\text{conv } Y$ if and only if there is no hyperplane separating 0 from Y . There are infinitely many candidate hyperplanes, but they can be grouped into equivalence classes according to which subset of X they separate from 0. We just need one representative from each class. The number of such possible subsets is equal, under duality, to the number of cells into which $|X|$ hyperplanes partition $\mathbb{R}^n$. $\square$

2.2 Hoeffding's inequality

Our main probabilistic tool will be Hoeffding's inequality.

Theorem 2.5 (Hoeffding 1963, [Hoe63]). *Given n independent random variables $x_1, \dots, x_N$ such that $0 \leq x_i \leq 1$, let $y = x_1 + \dots + x_N$. For all $\lambda \geq 0$, we have*

$$\mathbb{P}[y < \mathbb{E}(y) - \lambda] < e^{-2\lambda^2/N}.$$

The expert reader may know that Hoeffding proved a slightly different inequality: $\mathbb{P}[y > \mathbb{E}(y) + \lambda] < e^{-2\lambda^2/N}$. It suffices to apply the inequality to the variables $z_i = 1 - x_i$ to obtain the other bound. This is a special case of Azuma's inequality (with a slightly different constant in the exponent, which would not change the main result significantly) [Azu67]. These inequalities carry at their heart the central limit theorem, which is why such an exponential decay is expected in the tails of the distribution. See [AS16] for references on the subject.

3 Proof of Theorem 2.3

Proof. We first prove that $\varepsilon > ((r-1)/r)^m$ is necessary for Theorem 1.3, which also implies the lower bound for Theorem 2.3. Given N points in $\mathbb{R}^d$ and m partitions $P_1, \dots, P_m$, of them, let us find a subset of size greater than or equal to $N((r-1)/r)^m$ in which no P_k induces a Tverberg partition. First, notice that one of the parts of P_1 must have at most N/r points. If we remove them, then there are at least $N(1 - 1/r)$ points left. We can repeat the same argument, and, among the points we have left, one of the parts induced by P_2 must have at most a $(1/r)$ -fraction of them. Removing those leaves us with at least $N(1 - 1/r)^2$ points. We proceed this way and end up with a set Y of at least $N(1 - 1/r)^m$ points, such that $P_k(Y)$ has at least one empty component for each $k = 1, \dots, m$. Therefore, none of these is a Tverberg partition.

Assume now that $\varepsilon > ((r-1)/r)^m$. We want to prove that there are m transversals as the theorem required. We choose (with foresight) $A = (Nr)^n$, and $\lambda > \sqrt{mN \ln A}$. Define a sequence $N_0, N_1, \dots$ by $N_0 = N$ and $N_k = N_{k-1}(1 - 1/r) + \lambda$ for $k \geq 1$. If we apply Lemma 2.4 to $\cup \mathcal{F}$, we obtain a family $\mathcal{H}$ of A halfspaces, all containing 0, which are enough to check if the convex hulls of the transversals we construct contain 0.

We consider each $F \in \mathcal{F}$ as a color class. For $k = 1, \dots, m$, we will construct P_k and a family $\mathcal{J}_k$ of sets of color classes such that the following properties hold:

- given $\mathcal{G} \subset \mathcal{F}$ such that $0 \notin \text{conv}(P_{k'}(\mathcal{G}))$ for all $k' = 1, \dots, k$, there must be a $\mathcal{V} \in \mathfrak{J}_k$ such that $\mathcal{G} \subset \mathcal{V}$,
- if $\mathcal{V} \in \mathfrak{J}_k$, then $|\mathcal{V}| \leq N_k$, and
- $|\mathfrak{J}_k| \leq A^k$.

We can consider $\mathfrak{J}_0 = \{\mathcal{F}\}$. We construct P_k inductively, assuming $\mathfrak{J}_{k-1}$ and $P_{k'}$ have been constructed for $k' < k$. We start by choosing P_k randomly. For each $F \in \mathcal{F}$, we pick $y_F^k \in F$ uniformly and independently. Then, we denote $P_k = \{y_F^k : F \in \mathcal{F}\}$.

Given a half-space $H \in \mathcal{H}$, consider the random variable

$$x_F^k(H) = \begin{cases} 1 & \text{if } y_F^k \in H \\ 0 & \text{otherwise} \end{cases}$$

Since $0 \in \text{conv}(F)$, we know that $\mathbb{E}(x_F^k(H)) \geq 1/r$. By linearity of expectation, for each $\mathcal{V} \in \mathfrak{J}_{k-1}$ we have

$$\mathbb{E} \left[\sum_{F \in \mathcal{V}} x_F^k(H) \right] \geq \frac{1}{r} |\mathcal{V}|.$$

Since all variables $x_F^k(H), x_{F'}^k(H)$ are independent for $F \neq F'$, Hoeffding's inequality gives

$$\mathbb{P} \left[\sum_{F \in \mathcal{V}} x_F^k(H) < \frac{|\mathcal{V}|}{r} - \lambda \right] < e^{-2\lambda^2/|\mathcal{V}|} \leq e^{-2\lambda^2/N}$$

Therefore the union bound gives

$$\begin{aligned} \mathbb{P} \left[\exists H \in \mathcal{H} \exists \mathcal{V} \in \mathfrak{J}_{k-1} \text{ such that } \sum_{F \in \mathcal{V}} x_F^k(H) < \frac{|\mathcal{V}|}{r} - \lambda \right] &\leq A \cdot |\mathfrak{J}_{k-1}| \cdot e^{-2\lambda^2/N} \\ &\leq A^k e^{-2\lambda^2/N} < 1 \end{aligned}$$

by the choice of λ .

Therefore, there is a choice of P_k such that for all $\mathcal{V} \in \mathfrak{J}_{k-1}$ and all half-spaces $H \in \mathcal{H}$, we have

$$\sum_{F \in \mathcal{V}} x_F^k(H) \geq \frac{|\mathcal{V}|}{r} - \lambda.$$

We fix P_k to be this choice. We are ready to construct $\mathfrak{J}_k$. For each $\mathcal{V} \in \mathfrak{J}_{k-1}$ and each half-space $H \in \mathcal{H}$, we construct the set $\mathcal{V}' = \{F \in \mathcal{V} : x_F^k(H) = 0\}$. We call $\mathfrak{J}_k$ to the family of all sets that can be formed this way. Let us prove that $\mathfrak{J}_k$ satisfies all the desired properties.

Claim 3.1. *Given $\mathcal{G} \subset \mathcal{F}$ such that $0 \notin \text{conv}(P_{k'}(\mathcal{G}))$ for all $k' = 1, \dots, k$, there must be a $\mathcal{V}' \in \mathfrak{J}_k$ such that $\mathcal{G} \subset \mathcal{V}'$.*

Proof. If $0 \notin \text{conv}(P_{k'}(\mathcal{G}))$ for all $k' = 1, \dots, k$, we already know that there must be a $\mathcal{V} \in \mathfrak{J}_{k-1}$ such that $\mathcal{G} \subset \mathcal{V}$. Since $0 \notin \text{conv}(P_k(\mathcal{G}))$, there must be a half-space $H \in \mathcal{H}$ containing 0 such that $x_F^k(H) = 0$ for all $F \in \mathcal{G}$. Therefore, there is a $\mathcal{V}' \in \mathfrak{J}_k$ with $\mathcal{G} \subset \mathcal{V}'$. $\square$

Claim 3.2. *If $\mathcal{V}' \in \mathfrak{J}_k$, then $|\mathcal{V}'| \leq N_k$.*

Proof. Let $\mathcal{V} \in \mathfrak{J}_{k-1}$, $H \in \mathcal{H}$ be the family and half-space that defined $\mathcal{V}'$, respectively. Then,

$$|\mathcal{V}'| = \sum_{F \in \mathcal{V}} (1 - x_F^k(H)) \leq |\mathcal{V}| \left(1 - \frac{1}{r}\right) + \lambda \leq N_{k-1} \left(1 - \frac{1}{r}\right) + \lambda = N_k.$$

$\square$

Claim 3.3. *We have $|\mathfrak{J}_k| \leq A^k$.*

Proof. By construction, $|\mathfrak{J}_k| \leq |\mathfrak{J}_{k-1}| \cdot A \leq A^k$. $\square$

This concludes the construction of $P_1, \dots, P_m$.

If $\mathcal{G} \subset \mathcal{F}$ is such that $0 \notin \text{conv } P_k(\mathcal{G})$ for $k = 1, \dots, m$, then there must be a $\mathcal{V} \in \mathfrak{J}_m$ such that $\mathcal{G} \subset \mathcal{V}$.

Recall that m was chosen so that $((r-1)/r)^m < \varepsilon$. Therefore

$$|\mathcal{G}| \leq |\mathcal{V}| \leq N_m \leq N \left(\frac{r-1}{r}\right)^m + r\lambda < \varepsilon N,$$

where the last inequality holds if N is large enough, as $\lambda = O(\sqrt{N \ln N})$. $\square$

4 A Colorful version

Another important variation of Tverberg's theorem is the following conjecture by Bárány and Larman.

Conjecture 4.1 (Colorful Tverberg; Bárány, Larman 1992 [BL92]). *For any given $d+1$ sets $F_1, \dots, F_{d+1}$ of r points each in $\mathbb{R}^d$, there is a Tverberg partition $X_1, \dots, X_r$ of their union such that for all i, j we have $|F_i \cap X_j| = 1$.*

A partition $X_1, \dots, X_r$ with $|F_i \cap X_j| = 1$ for all i, j is called a *colorful partition*. Consult [BFZ14, BMZ11, BMZ15] and the references therein the current solved cases and techniques. We present an ε -version of the conjecture above in the following theorem. Let $p_r \sim 1 - 1/e$ be the probability that a random permutation of a set with r elements has fixed points.

Theorem 4.2. *Let r, d be positive integers and $\varepsilon > 0$ be a real number. There is an $m_{\text{col}} = m_{\text{col}}(\varepsilon, r)$ such that the following holds. For a sufficiently large N , if we are given N sets $F_1, \dots, F_N$ of r points in $\mathbb{R}^d$ each, then there are m_{col} colorful partitions of $\mathcal{F} = \{F_1, \dots, F_N\}$ such that for any $\mathcal{G} \subset \mathcal{F}$ with $|\mathcal{G}| \geq \varepsilon |\mathcal{F}|$, at least one of the partitions induces a colorful Tverberg partition on $|\mathcal{G}|$. Moreover, we have*

$$m_{\text{col}} \leq \left\lfloor \frac{\ln(\varepsilon)}{\ln(1 - p_r)} \right\rfloor + 1$$

We should note that the theorem above gives $m \sim 1 + \ln(1/\varepsilon)$ if r is large enough. This is related to the colorful version from [Sob18], which seeks the smallest ε for which $m_{\text{col}}(\varepsilon, r) = 1$. Using our notation, the main conjecture in that paper states the following.

Conjecture 4.3. *For all $\varepsilon > 0$ and any positive integer r , we have*

$$m_{\text{col}}(\varepsilon, r) = 1.$$

To prove Theorem 4.2, we also use Sarkaria's transformation. In order to translate the conditions on the colors through the tensor products, we need the following definition.

A set B is an r -block if it is an $r \times r$ array of points in $\mathbb{R}^n$ such that the convex hull of each column contains the origin. A *colorful transversal* of an r -block B is a subset of r points of B that has exactly one point of each column and exactly one point of each row. Given a family $\mathcal{B}$ of r -blocks, a *colorful transversal* for $\mathcal{B}$ is the result of putting together a colorful transversal for each block. If we apply Sarkaria's technique, colorful partitions in $\mathbb{R}^d$ become colorful transversals of r -blocks in $\mathbb{R}^n$. Theorem 4.2 is then implied by the following.

Theorem 4.4. *Let n, r be positive integers and $\varepsilon > 0$ be a real number. $m_{\text{col}} = m_{\text{col}}(\varepsilon, r)$ such that the following holds. For a sufficiently large N , if we are given N r -blocks $B_1, \dots, B_N$ in $\mathbb{R}^n$, there are m_{col} colorful transversals $P_1, \dots, P_m$ of $\mathcal{B} = \{B_1, \dots, B_N\}$ such that for any $\mathcal{G} \subset \mathcal{B}$ with $|\mathcal{G}| > \varepsilon |\mathcal{B}|$ for at least one k we have $0 \in \text{conv}(P_k(\mathcal{G}))$. Moreover, we have*

$$m_{\text{col}} \leq \left\lfloor \frac{\ln(\varepsilon)}{\ln(1-p_r)} \right\rfloor + 1.$$

We also need the observation from [Sob18] that, for any r -block and any half-space H that contains the origin, the probability that a random colorful transversal has points in H is greater than or equal to p_r .

Proof. We proceed in a similar fashion to the proof of Theorem 2.3.

Assume that $\varepsilon > (1-p_r)^m$. We want to prove that there are m transversals as the theorem requires. We choose (with foresight) $A = (Nr^2)^n$, and $\lambda > \sqrt{mN \ln A}$. Define a sequence recursively by $N_0 = N$ and $N_k = N_{k-1}(1-p_r) + \lambda$. If we apply Lemma 2.4 to $\cup \mathcal{B}$, we obtain a family $\mathcal{H}$ of A half-spaces, all containing 0, which are enough to check if the convex hulls of the colorful transversals we construct contain 0.

For $k = 1, \dots, m$, we will construct P_k and a family $\mathfrak{J}_k$ of sets of r -blocks with the following properties.

- Given $\mathcal{G} \subset \mathcal{B}$ such that $0 \notin \text{conv}(P_{k'}(\mathcal{G}))$ for all $k' = 1, \dots, k$, there must be a $\mathcal{V} \in \mathfrak{J}_k$ such that $\mathcal{G} \subset \mathcal{V}$,
- if $\mathcal{V} \in \mathfrak{J}_k$, then $|\mathcal{V}| \leq N_k$, and
- $|\mathfrak{J}_k| \leq A^k$.

We can consider $\mathfrak{J}_0 = \{\mathcal{B}\}$ to start the induction. We construct P_k inductively, assuming $\mathfrak{J}_{k-1}$ and $P_{k'}$ have been constructed for $k' < k$. We first choose P_k randomly. For each $B \in \mathcal{B}$, we pick a colorful transversal y_B^k randomly and independently. Then, we denote $P_k = \{y_B^k : B \in \mathcal{B}\}$.

Given a half-space $H \in \mathcal{H}$, consider the random variable

$$x_B^k(H) = \begin{cases} 1 & \text{if } y_B^k \cap H \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{E}[x_B^k(H)] \geq p_r$ for each $B \in \mathcal{B}, H \in \mathcal{H}$, we have that for any $\mathcal{V} \in \mathfrak{J}_{k-1}$

$$\mathbb{E} \left[\sum_{B \in \mathcal{V}} x_B^k(H) \right] \geq |\mathcal{V}| p_r$$

Since all variables $x_B^k(H)$, $x_{B'}^k(H)$ are independent for $B \neq B'$, Hoeffding's inequality gives

$$\mathbb{P} \left[\sum_{B \in \mathcal{V}} x_B^k(H) < |\mathcal{V}|p_r - \lambda \right] < e^{-2\lambda^2/|\mathcal{V}|} \leq e^{-2\lambda^2/N}.$$

Therefore

$$\begin{aligned} \mathbb{P} \left[\exists H \in \mathcal{H} \exists \mathcal{V} \in \mathfrak{J}_{k-1} \sum_{B \in \mathcal{V}} x_B^k(H) < |\mathcal{V}|p_r - \lambda \right] &< A \cdot |\mathfrak{J}_{k-1}| e^{-2\lambda^2/N} \\ &\leq A^k e^{-2\lambda^2/N} < 1 \end{aligned}$$

by the choice of λ .

Therefore, there must be a choice of P_k such that for all $H \in \mathcal{H}$ and all $\mathcal{V} \in \mathfrak{J}_{k-1}$ we have

$$\sum_{B \in \mathcal{V}} x_B^k(H) \geq |\mathcal{V}|p_r - \lambda.$$

We fix P_k to be this choice. In order to form $\mathfrak{J}_k$, for each $\mathcal{V} \in \mathfrak{J}_{k-1}$ and $H \in \mathcal{H}$, we include the set $\{B \in \mathcal{V} : x_B^k(H) = 0\}$. Proving that $\mathfrak{J}_k$ satisfies the desired properties and that this implies the conclusion of Theorem 4.4 follows from arguments analogous to those at the end of section 3. $\square$

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