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Product Matrix Processes as Limits of Random Plane Partitions

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We consider a random process with discrete time formed by squared singular values of products of truncations of Haar-distributed unitary matrices. We show that this process can be understood as a scaling limit of the Schur process, which gives determinantal formulas for (dynamical) correlation functions and a contour integral representation for the correlation kernel. The relation with the Schur processes implies that the continuous limit of marginals for q-distributed plane partitions coincides with the joint law of squared singular values for products of truncations of Haar-distributed random unitary matrices. We provide structural reasons for this coincidence that may also extend to other classes of random matrices.

1 Introduction

It was observed by many researchers that probability distributions from random matrix theory appear as limit laws in a variety of problems in statistical mechanics and combinatorics. Probably, the most known examples of this phenomenon are Ulam's

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problem for increasing subsequences of random permutations and domino tilings of the Aztec diamond. We refer the reader to the book by Baik *et al.* [7] and references therein for a detailed analysis of these two examples.

In both these problems, distributions of certain key quantities converge to *limiting* distributions of random matrix theory (in an appropriate scaling limit as the size of random matrices tends to infinity), in particular to the Tracy–Widom distribution [44]. However, there are situations where not only limiting random matrix distributions play a role. It can happen that joint laws of eigenvalues (or singular values) corresponding to *finite size* random matrix ensembles arise as scaling limits in a combinatorial or statistical mechanics problem that has no a priori relation with random matrices.

One example is the Gaussian Unitary Ensemble (GUE)-corners process investigated by Johansson and Nordenstam [29] and Okounkov and Reshetikhin [41]. It is easy to define the corners process: start with an infinite random matrix picked from the GUE and consider the eigenvalues of its principal corner submatrices. As was discovered in [29] and [41], the GUE-corners process can be obtained as a scaling limit in tiling models. Okounkov and Reshetikhin further suggested a (heuristic) argument toward the universal appearance of this process. Following this prediction, GUE-corners process was found in more general tilings models by Gorin and Panova [24] and the six-vertex model by Gorin [23] and Dimitrov [19]. It can be also linked to the last passage percolation, see Baryshnikov [8], Gravner *et al.* [27], O'Connell and Yor [37], Bougerol and Jeuli [15], Adler *et al.* [1], and references therein.

The GUE-corners process is a *determinantal* (see, e.g., Borodin [10]) point process with discrete time obtained using random matrices. If, instead of cutting out corners of a single matrix, one starts *adding* independent GUE matrices, then the eigenvalues of the sums also form a determinantal process, and the number of matrices in the sum plays the role of discrete time, see Eynard and Mehta [20]. Another class of (dynamical) determinantal processes with discrete time can be constructed from *products* of random matrices, see Strahov [43] and Akemann and Strahov [6]. (Determinantal processes in products of random matrices were first discovered in Akemann and Burda [2]). Such processes are called *product matrix processes*, and they are formed by the squared singular values of random matrix products. We can use independent complex Gaussian matrices to obtain a simple example of such processes. Namely, let $G_1, \ldots, G_m$ be independent matrices with standard i.i.d. complex Gaussian entries. Assume that G_l is of size $\left(n+\nu_l\right) \times \left(n+\nu_{l-1}\right)$, $\nu_0 = 0$, $\nu_1 \geq 0, \ldots, \nu_{m-1} \geq 0$, and

for each $l = 1, \dots, m$, denote by y_j^l, $j = 1, \dots, n$, the squared singular values of the partial product $Y_l = G_l \cdots G_1$. The configuration

$$\left\{ \left(l, y_j^l \right) | l = 1, \dots, m; j = 1, \dots, n \right\}$$

of all these squared singular values generates a random point process on $\{1, \dots, m\} \times \mathbb{R}_{>0}$. It was shown in Strahov [43] that this process is determinantal (and it can be viewed as a determinantal process with discrete time). Paper [43] gives a contour integral representation for the correlation kernel, together with its hard edge scaling limit, and generalizes results obtained in Akemann *et al.* [5], Akemann *et al.* [4], and Kuijlaars and Zhang [33] to the multi-level situation. A more general class of product matrix processes related to certain multi-matrix models was introduced and studied in Akemann and Strahov [6]. In this class the matrices in the products are no longer independent, but in spite of that the product matrix processes are still determinantal.

From a different viewpoint, various matrix corners processes studied by Johansson and Nordenstam [29], Okounkov and Reshetikhin [41], Adler *et al.* [1], Forrester and Rains [22], Borodin [11], and Gorin [13] were shown to be continuous limits of special *Schur processes* of Okounkov and Reshetikhin [38, 39]. The discrete time determinantal process formed by the eigenvalues of sums of independent GUE matrices can be understood as a limit of a special Schur process as well. It is natural to ask whether product matrix processes also have this property. Motivated by this question, we construct in this paper a product matrix process using corners of independent Haar-distributed unitary matrices (or truncated unitary matrices). We demonstrate that this process is a scaling limit of a certain Schur process, which implies determinantal formulas for (dynamical) correlation functions. Moreover, starting from the general Okounkov–Reshetikhin formula [39] for the correlation kernels of Schur processes, we derive a double contour integral representation for the correlation kernel of the product matrix process formed by truncated unitary matrices. The formula for the correlation kernel we derive in this paper can be understood as a time-dependent generalization of the result obtained in Kieburg *et al.* [31] for the squared singular values of matrix products with truncated unitary matrices. We note that the determinantal point process formed by eigenvalues of matrix products with truncated unitary matrices was studied in Akemann *et al.* [3].

The fact that the product matrix process formed by truncated unitary matrices is a continuous limit of the Schur process enables us to prove Theorem 2.9 below that says that the continuous limit of marginals for q-distributed (skew) plane partitions

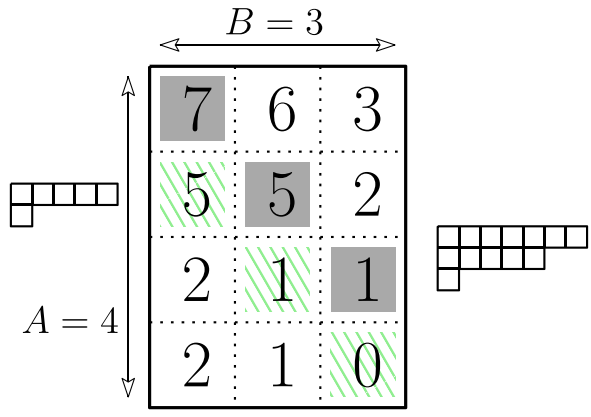

Fig. 1. Plane partition with 4×3 support corresponds to a sequence of (interlacing) Young diagrams. Two of them, $5 \geq 1 \geq 0$ and $7 \geq 5 \geq 1$, are highlighted. The asymptotic behavior of these two diagrams is related to squared singular values of T_1 and T_2T_1, respectively, where T_1 is 4×3 truncation of 7×7 random unitary matrix and T_2 is 3×4 truncation of 4×4 random unitary matrix. In the notations of Section 2.3, $A = 4$, $B = 3$, $\pi = \emptyset$, $p = 2$, $\beta_2 = 1$, $\alpha_1 = 4$, and $\alpha_2 = 5$.

coincides with the joint law of squared singular values for products of corners of Haar-distributed unitary matrices, see Figure 1 for one particular case of the theorem. We consider Theorem 2.9 as the main result of the present paper. It demonstrates that, similarly to the corners process, the time-dependent determinantal processes constructed from products of truncated unitary matrices appear as scaling limits in a model of statistical mechanics of a combinatorial nature. To the best our knowledge, the present paper is the 1st work-relating products of random matrices with scaling limits of models that have no a priori relation to random matrices.

It is also natural to ask about conceptual reasons for such a coincidence. Why should random matrices be directly related to statistical mechanics models? For the GUE-corners process, Okounkov and Reshetikhin [41] suggested the following heuristic argument: if we start with a discrete model of statistical mechanics satisfying a certain Gibbs (i.e., conditional uniformity) property, then one expects the same property to survive in the continuous limit. Olshanski and Vershik [42] classified all such Gibbs measures on triangular arrays of reals, and out of them only the GUE-corners process agrees with the growth conditions implied by the Law of Large Numbers (limit shape behavior) of the discrete model.

For the products of random matrices we do not dispose of an analog of the Okounkov–Reshetikhin argument. However, let us explain the path that led us to

the understanding that a similar connection with combinatorial statistical mechanics is possible. The following fact [13, 22] is easy to prove by comparing the explicit formulas for the distributions: if X is a corner of Haar-random unitary matrix, then the eigenvalues of XX^* (this is often called Jacobi or MANOVA ensemble) are distributed as a continuous limit of a *Schur measure* with two principal specializations. One could argue that this is an instance of the semiclassical asymptotics common in representation theory. Next, we need to understand what happens with these eigenvalues when X is multiplied by another similar matrix. It is known that multiplication of (real/complex/quaternion at $\beta = 1, 2, 4$) matrices is intimately linked to multiplication of corresponding Jack (=zonal) polynomials, which become Schur polynomials in the case of the complex field ($\beta = 2$) that we discuss here. This is discussed by Macdonald [35, Chapter VII], Forrester [21, Section 13.4.3], and more recently used, for example, by Kieburg and Kosters [30] and by Gorin and Marcus [25]. If we consider a version of the multidimensional Fourier transform for the Schur measures (the appropriate version was introduced by Gorin and Panova [24] and Bufetov and Gorin [17] under the name *Schur generating functions*), then being a Schur measure or its continuous limit is equivalent to the factorization of this transform into a product of one variable function. Since such factorization is preserved under multiplication, the squared singular values of products of random matrices have to be described by the Schur processes.

We detail how this argument works in the simplest case of 2×2 matrices in Section 8. For the proof of our main statements, Proposition 2.6 and Theorem 2.9, we choose in Sections 2–7 another path, which is more direct (and leads to a more general result) but, perhaps, more mysterious. Let us remark that while the arguments of Section 8 admit an immediate generalization to products of real and quaternion matrices, yielding their representation as limits of *Macdonald processes*, for the proofs of Sections 2–7 such a generalization is unclear.

The paper is organized as follows. In Section 2 we introduce notation and present the main results. In particular, Proposition 2.4 gives a formula for the correlation kernel of the product matrix process associated with truncated unitary matrices, Proposition 2.6 shows that this product matrix process can be understood as a continuous limit of a special Schur process, and Theorem 2.9 presents our result on convergence of marginals of q-distributed plane partitions to this product matrix process. Sections 2–7 contain the proofs of our statements. In Section 8 we sketch another way to prove our main results by exploiting symmetric functions and zonal polynomials. Finally, the Appendix gives a 2nd proof of Proposition 2.4 based on the Eynard–Mehta theorem.

2 Notation and Statement of Results

2.1 Product matrix processes with truncated unitary matrices

Let $G_1, \ldots, G_p$ be matrices with random complex entries, and assume that each matrix G_k, $k \in \{1, \ldots, p\}$, is of size $N_k \times N_{k-1}$. Set

$$X(k) = G_k \ldots G_1, \quad k \in \{1, \ldots, p\}.$$

If $n = N_0$, then for each k, $k \in \{1, \ldots, p\}$, $X^*(k)X(k)$ are random matrices of the same size $n \times n$. Denote by x_j^k the jth largest eigenvalue of $X^*(k)X(k)$. The configuration of all these eigenvalues,

$$\left\{ \left(k, x_j^k\right) \middle| k = 1, \ldots, p; j = 1, \ldots, n \right\}, \tag{2.1}$$

forms a point process on $\{1, \ldots, p\} \times \mathbb{R}_{>0}$. This point process is called the *product matrix process* associated with the random matrices $X(1), X(2), \ldots, X(p)$.

Here we consider a product matrix process constructed from a collection of truncated unitary matrices. Namely, let $U_1, \ldots, U_l, U_{l+1}, \ldots, U_{l+p-1}$ be independent Haar-distributed unitary matrices. We assume that the size of each matrix U_j, $1 \le j \le p+l-1$, is equal to m_j. Recall that if U is an $m \times m$ matrix, and the integers k, r are chosen such that $1 \le k, r \le m$, the submatrix T of U defined by

$$T = \begin{pmatrix} U_{1,1} & \ldots & U_{1,r} \\ \vdots & & \\ U_{k,1} & \ldots & U_{k,r} \end{pmatrix}$$

is called a $k \times r$ truncation of U. Now, for $1 \le j \le p + l - 1$ let T_j be the truncation of U_j of size $\left(n + \nu_j\right) \times \left(n + \nu_{j-1}\right)$. We agree that $\nu_0 = 0$, and assume that the positive integers n, $\nu_1, \ldots, \nu_{l+p-1}$ are chosen in such a way that the conditions

$$m_1 \ge 2n + \nu_1, \tag{2.2}$$

and

$$m_j \ge n + \nu_j + 1, \quad 2 \le j \le p + l - 1, \tag{2.3}$$

are satisfied. Denote by $x^1 = \left(x_1^1, \ldots, x_n^1\right)$ the vector of the squared singular values of the product matrix $T_l \ldots T_1$, and for $2 \le j \le p$ denote by $x^j = \left(x_1^j, \ldots, x_n^j\right)$ the vector of the squared singular values of the product matrix $T_{j+l-1} \ldots T_1$. Configurations

$\left\{\left(k,x_j^k\right)\middle| k=1,\ldots,p;j=1,\ldots,n\right\}$ form a point process on $\{1,\ldots,p\}\times\mathbb{R}_{>0}$. We will refer to this point process as to the *product matrix process associated with truncated unitary matrices*. We say that the product matrix $T_l\ldots T_1$ determines the *initial conditions* of the product matrix process associated with truncated unitary matrices. The numbers n, m_j, ν_j will be called the *parameters* of the product matrix process associated with truncated unitary matrices.

Proposition 2.1. Consider the product matrix process associated with truncated unitary matrices, and let $x_1^k \le \ldots \le x_n^k$; $k=1,\ldots,p$ denote the set of the squared singular values of the product matrix $T_{k+l-1}\ldots T_1$. The joint probability distribution of $\left(x_1^k,\ldots,x_n^k\right)$ is given by

$$\begin{aligned}&\frac{1}{Z_{n,p+l}}\triangle\left(x^p\right)\\&\times\prod_{r=1}^{p-1}\det\left[\left(x_j^{r+1}\right)^{\nu_{l+r}}\left(x_k^r-x_j^{r+1}\right)_+^{m_{l+r}-n-\nu_{l+r}-1}\left(x_k^r\right)^{n-m_{l+r}}\right]_{k,j=1}^{n}\\&\times\det\left[w_k^{(l)}\left(x_j^1\right)\right]_{k,j=1}^{n}\mathrm{d}x^1\ldots\mathrm{d}x^n,\end{aligned}\tag{2.4}$$

where $(x-y)_+=\max(0,x-y)$, the Vandermonde determinant $\triangle\left(x^p\right)$ is defined by $\triangle\left(x^p\right)=\prod_{1\le i<j\le n}\left(x_j^p-x_i^p\right)$, for $1\le l\le p$ we write $dx^l=dx_1^l\ldots dx_n^l$, $Z_{n,p+l}$ is a normalization constant, and $w_k^{(l)}(x)$ is a sequence of weight functions. The normalization constant $Z_{n,p+l}$ can be written explicitly as

$$Z_{n,p+l}=\frac{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)\Gamma(j)\prod_{k=2}^{p+l-1}\left(\Gamma\left(m_k-n-\nu_k\right)\right)^n}{\prod_{k=1}^{p+l-1}\prod_{j_k=1}^{m_k-n-\nu_k}\left(j_k+\nu_k\right)_n}.\tag{2.5}$$

Here $(a)_m=a(a+1)\ldots(a+m-1)$ stands for the Pochhammer symbol. The function $w_k^{(l)}(x)$ can be expressed as a Meijer G-function,

$$\begin{aligned}w_k^{(l)}(x)&=c_l G_{l,l}^{l,0}\left(\begin{matrix}m_l-n, & \ldots, & m_2-n, & m_1-2n+k\\ \nu_l, & \ldots, & \nu_2, & \nu_1+k-1\end{matrix}\middle| x\right)\\&=\frac{c_l}{2\pi i}\int_C\frac{\Gamma\left(\nu_1+k-1+s\right)\prod_{j=2}^{l}\Gamma\left(\nu_j+s\right)}{\Gamma\left(m_1-2n+k+s\right)\prod_{j=2}^{l}\Gamma\left(m_j-n+s\right)}x^{-s}\mathrm{d}s,\quad 0<x<1.\end{aligned}\tag{2.6}$$

In this formula C denotes a positively oriented contour in the complex s-plane that starts and ends at $-\infty$ and encircles the negative real axis. The constant c_l in the formula for $w_k^{(l)}(x)$ can be written as

$$c_l = \Gamma\left(m_1 - 2n - \nu_1 + 1\right) \prod_{j=2}^{l} \Gamma\left(m_j - n - \nu_j\right). \tag{2.7}$$

Remark 2.2. The Meijer G-function in equation (2.6) is equal to zero for $x \geq 1$. Correspondingly, we set the weight functions $w_k^{(l)}(x)$ to zero for $x \geq 1$.

Remark 2.3. The right-hand side of equation (2.6) can be written as

$$\frac{c_l'}{2\pi i} \int_C \frac{(s)_{\nu_1+k-1} \prod_{j=2}^{l} (s)_{\nu_j}}{(s)_{m_1-2n+k} \prod_{j=2}^{l} (s)_{m_j-n}} x^{-s} \mathrm{d}s,$$

where

$$c_l' = B\left(m_1 - 2n - \nu_1 + 1, \nu_1 + k - 1\right) \prod_{j=2}^{l} B\left(m_j - n - \nu_j, \nu_j\right),$$

and $B(x, y)$ stands for the Beta function.

Our next result provides explicit formulae for the correlation functions of the product matrix process associated with truncated unitary matrices.

Proposition 2.4. The product matrix process with truncated unitary matrices is a determinantal process on $\{1, \ldots, p\} \times \mathbb{R}_{>0}$. Its correlation kernel, $K_{n,p,l}(r, x; s, y)$ (where $r, s \in \{1, \ldots, p\}$, and $x, y \in \mathbb{R}_{>0}$) can be written as

$$\begin{aligned} K_{n,p,l}(r, x; s, y) = &-\frac{1}{x} G_{s-r,s-r}^{s-r,0}\left(\begin{matrix} m_{r+l} - n, & \ldots, & m_{s+l-1} - n \\ \nu_{r+l}, & \ldots, & \nu_{s+l-1} \end{matrix} \middle| \frac{y}{x} \right) \mathbf{1}_{s>r} \\ &+ \frac{1}{(2\pi i)^2} \oint_{C_t} \mathrm{d}t \oint_{C_\zeta} \mathrm{d}\zeta \frac{\prod_{a=0}^{s+l-1} \Gamma\left(\nu_a + \zeta + 1\right)}{\prod_{a=0}^{r+l-1} \Gamma\left(\nu_a + t + 1\right)} \frac{\prod_{a=0}^{r+l-1} \Gamma\left(m_a - n + t + 1\right)}{\prod_{a=0}^{s+l-1} \Gamma\left(m_a - n + \zeta + 1\right)} \frac{x^t y^{-\zeta-1}}{\zeta - t}, \end{aligned} \tag{2.8}$$

where C_t is a closed contour in the complex t-plane encircling the interval $[0, n-1]$ once in the positive direction, C_ζ is a positively oriented closed contour in the complex ζ-plane encircling once an interval containing all the points $-\left(1 + \nu_1\right), \ldots, -\left(m_1 - n\right); \ldots;$

$-(1+\nu_{s+l-1}), \ldots, -(m_{s+l-1}-n)$, which does not intersect C_t. In the formula above it is understood that $m_0 = \nu_0 = 0$.

We note that if $r = s = p$ and $l = 1$, then formula (2.8) reduces to the formula for the correlation kernel derived in Kieburg *et al.* [31, Proposition 2.7].

In what follows we will give two proofs of Proposition 2.4. The 1st proof will use the fact that the process defined by equation (2.4) can be understood as a continuous limit of a special Schur process, see Proposition 2.6. Since the Schur processes are determinantal, this will imply determinantal formulae for the correlation functions. As for the explicit formula for the correlation kernel (see equation (2.8)), it will be obtained from the general Okounkov–Reshetikhin formula [39] for correlation kernels of the Schur processes by a certain limiting procedure in Section 6. The 2nd proof will be based on the observation that the density of the product matrix process with truncated unitary matrices can be written as a product of determinants. This will enable us to apply the result by Eynard and Mehta [20], and to give a formula for the correlation functions, see the Appendix. This 2nd argument is similar to the proof of Borodin and Rains [14] of the determinantal structure of Schur processes.

2.2 Convergence of the Schur process

In this section we use the notation of Macdonald [35], and follow references [11, 12, 14, 39].

Let Λ be the algebra of symmetric functions in countably many variables $z_1, z_2, \ldots$. We use two sets of generators of Λ: power sums p_k and complete homogeneous symmetric functions h_k, $k = 1, 2, \ldots$, defined through

$$p_k = \sum_{i=1}^{\infty} (z_i)^k, \quad h_k = \sum_{i_1 \le i_2 \le \cdots \le i_k} z_{i_1} z_{i_2} \cdots z_{i_k}.$$

We recall that the Schur functions s_λ form a basis of Λ when λ varies over all Young diagrams (or partitions). We also use skew Schur functions $s_{\lambda/\mu}$ labeled by two Young diagrams λ and μ.

A specialization ϱ of Λ is an algebra homomorphism of Λ to $\mathbb{C}$. A specialization ϱ of Λ is called nonnegative if it takes nonnegative values on the Schur functions, see, for example, Borodin [11, Section 1] for a detailed discussion of nonnegative specializations of the algebra of symmetric functions. The application of a specialization ϱ to $f \in \Lambda$ will be denoted as $f(\varrho)$. The trivial specialization $\emptyset$ of Λ takes value 1 at the constant

function $1 \in \Lambda$, and takes value 0 at any homogeneous $f \in \Lambda$ of degree ≥ 1. In particular $s_\lambda(\emptyset) = 0$ unless $\lambda = \emptyset$, and $s_{\lambda/\mu}(\emptyset) = 0$ unless $\lambda = \mu$.

In this paper we will only use the simplest Schur positive specializations parameterized by arbitrary $m = 1, 2, \dots$ and m–tuple of positive reals $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m_{>0}$. We denote it $\varrho = (\alpha_1, \dots, \alpha_m)$ and set

$$p_k(\varrho) = p_k(\alpha_1, \dots, \alpha_m) = (\alpha_1)^k + (\alpha_2)^k + \dots (\alpha_m)^k.$$

Equivalently, this specialization can be encoded by its generating function

$$H(\varrho; u) := 1 + \sum_{k=1}^{\infty} h_k(\varrho) u^k = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\varrho) u^k}{k} \right) = \prod_{i=1}^{m} \frac{1}{1 - \alpha_i u}, \tag{2.9}$$

where the 2nd identity is the algebraic relation between generators p_k and h_k.

The specializations that we use are often given by geometric series $\varrho = (q^t, q^{t+1}, \dots, q^s)$. When $t > s$, the geometric series is empty and ϱ becomes the trivial specialization.

Definition 2.5. Let p be a natural number, and let $\varrho_0^+, \dots, \varrho_{p-1}^+, \varrho_1^-, \dots, \varrho_p^-$ be nonnegative specializations of Λ. The Schur process of rank p is a probability measure on sequences of Young diagrams

$$\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \lambda^{(p-2)}, \mu^{(p-2)}, \lambda^{(p-1)}, \mu^{(p-1)}, \lambda^{(p)}$$

parameterized by $2p$ Schur positive specializations of the algebra of symmetric functions given by

$$\begin{aligned} &\operatorname{Prob}\left(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \lambda^{(p-2)}, \mu^{(p-2)}, \lambda^{(p-1)}, \mu^{(p-1)}, \lambda^{(p)}\right) \\ &= \frac{1}{Z_{\mathrm{Schur}}} s_{\lambda^{(1)}}\left(\varrho_0^+\right) s_{\lambda^{(1)}/\mu^{(1)}}\left(\varrho_1^-\right) s_{\lambda^{(2)}/\mu^{(1)}}\left(\varrho_1^+\right) s_{\lambda^{(2)}/\mu^{(2)}}\left(\varrho_2^-\right) s_{\lambda^{(3)}/\mu^{(2)}}\left(\varrho_2^+\right) \\ &\times \dots \times s_{\lambda^{(p-1)}/\mu^{(p-1)}}\left(\varrho_{p-1}^-\right) s_{\lambda^{(p)}/\mu^{(p-1)}}\left(\varrho_{p-1}^+\right) s_{\lambda^{(p)}}\left(\varrho_p^-\right). \end{aligned} \tag{2.10}$$

Here Z_{Schur} is a normalization constant.

Since $s_{\lambda/\mu} \equiv 0$ unless $\mu \subset \lambda$, the Schur process lives on the following configurations of Young diagrams:

$$\emptyset \subset \lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \mu^{(2)} \subset \lambda^{(3)} \supset \dots \subset \lambda^{(p-1)} \supset \mu^{(p-1)} \subset \lambda^{(p)} \supset \emptyset.$$

For skew Schur functions we have the following summation formulae:

$$\sum_{\mu\in\mathbb{Y}} s_{\mu/\lambda}(\varrho)s_{\mu/\nu}(\varrho') = H(\varrho;\varrho')\sum_{\kappa\in\mathbb{Y}} s_{\lambda/\kappa}(\varrho')s_{\nu/\kappa}(\varrho), \tag{2.11}$$

and

$$\sum_{\nu\in\mathbb{Y}} s_{\lambda/\nu}(\varrho)s_{\nu/\mu}(\varrho') = s_{\lambda/\mu}(\varrho,\varrho'), \tag{2.12}$$

see Macdonald [35, Section I.5, equation (5.10), and Example I.5.26(1)]. Here

$$H\left(\varrho;\varrho'\right) = \exp\left(\sum_{k=1}^{\infty}\frac{p_k(\varrho)p_k(\varrho')}{k}\right),$$

and the values of the symmetric functions under the union specialization (ϱ,ϱ') are determined by the power sum values given by

$$p_k\left(\varrho,\varrho'\right) = p_k\left(\varrho\right) + p_k\left(\varrho'\right).$$

Hence, for specializations $\varrho_1,\ldots,\varrho_k,\varrho'_1,\ldots,\varrho'_m$ we have

$$H\left(\varrho_1,\ldots,\varrho_k;\varrho'_1,\ldots,\varrho'_m\right) = \prod_{i=1}^{k}\prod_{j=1}^{m} H\left(\varrho_i;\varrho'_j\right). \tag{2.13}$$

Proposition 2.6. Consider the Schur process defined by the probability measure (10). Assume that the specializations $\varrho_0^+,\ldots,\varrho_{p-1}^+$ of the Schur process are defined by

$$\varrho_0^+ = \left(e^{-(1+\nu_1)\epsilon}, e^{-(2+\nu_1)\epsilon},\ldots,e^{-(m_1-n)\epsilon};\ldots;e^{-(1+\nu_l)\epsilon},e^{-(2+\nu_l)\epsilon},\ldots,e^{-(m_l-n)\epsilon}\right), \tag{2.14}$$

$$\varrho_1^+ = \left(e^{-(1+\nu_{l+1})\epsilon}, e^{-(2+\nu_{l+1})\epsilon},\ldots,e^{-(m_{l+1}-n)\epsilon}\right), \tag{2.15}$$

$$\vdots$$

$$\varrho_{p-1}^+ = \left(e^{-(1+\nu_{l+p-1})\epsilon}, e^{-(2+\nu_{l+p-1})\epsilon},\ldots,e^{-(m_{l+p-1}-n)\epsilon}\right). \tag{2.16}$$

The specialization ϱ_p^- is defined by

$$\varrho_p^- = \left(1, e^{-\epsilon},\ldots,e^{-(n-1)\epsilon}\right), \tag{2.17}$$

and all the other specializations $\varrho_1^-, \dots, \varrho_{p-1}^-$ are trivial. With these specializations the Schur process lives on sequences of partitions $\left(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(p)}\right)$, where each $\lambda^{(k)}$, $1 \leq k \leq p$, has at most n nonzero parts almost surely. Set

$$x_j^k = e^{-\epsilon \lambda_j^{(k)}}, \; k = 1, \dots, p; \; j = 1, \dots, n. \tag{2.18}$$

Then the Schur process induces a point process on $\{1, \dots, p\} \times \mathbb{R}_{>0}$, and this process is formed by the configurations

$$\left\{ \left(k, x_j^k\right) \middle| k = 1, \dots, p; \; j = 1, \dots, n \right\}. \tag{2.19}$$

As $\epsilon \to 0$, the point process formed by configurations (2.19) converges to the product matrix process associated with truncated unitary matrices, as defined in Section 2.1.

Remark 2.7. We prove Proposition 2.6 only under the assumption $m_1 \geq 2n + \nu_1$ of (2.2). Although it is very plausible that the statement is true without this condition, we do not address the more general case in this text. A technical difficulty is that without (2.2) we cannot use the result of Proposition 2.1 directly; in particular, the constant c_l of (2.7) is infinite.

2.3 Random skew plane partitions and products of truncated unitary matrices

Let A and B be two natural numbers, and denote by B^A the $A \times B$ rectangle. Let π be a Young diagram such that $\pi \subset B^A$. A skew plane partition Π with support B^A/π is a filling of all boxes of B^A/π such that $\Pi_{i,j} \geq \Pi_{i+1,j}$ and $\Pi_{i,j} \geq \Pi_{i,j+1}$ for all meaningfull values of i and j. Here we assume that $\Pi_{i,j}$ is located in the ith row and jth column of B^A. The volume of a skew plane partition Π is defined by

$$\text{Volume}\,(\Pi) = \sum_{i,j} \Pi_{i,j}. \tag{2.20}$$

Given a parameter $0 < q < 1$, define a probability measure on the set of all skew plane partitions Π with support B^A/π by setting

$$\text{Prob}\,\{\Pi\} \sim q^{\text{Volume}(\Pi)}. \tag{2.21}$$

For a skew plane partition Π we define Young diagrams $\lambda^{(k)}\,(\Pi)$, $1 \leq k \leq A + B + 1$, through

$$\lambda^{(k)}\,(\Pi) = \left\{ \Pi_{i,i+k-A-1} \middle| (i, i+k-A-1) \in B^A/\pi \right\}.$$

Note that $\lambda^{(1)} = \lambda^{(A+B+1)} = \emptyset$. Also, define

$$\mathcal{L}(\pi) = \left\{ A + \pi_i - i + 1 \middle| i = 1, \dots, A \right\}.$$

This is a subset of $\{1, \dots, A+B+1\}$ containing A points, and all such subsets are in bijection with the Young diagrams (or partitions) π contained in the box B^A.

It is not hard to see that the set of all skew plane partitions with support B^A/π consists of sequences $\left(\lambda^{(1)}, \dots, \lambda^{(A+B+1)}\right)$ with

$$\lambda^{(1)} = \lambda^{(A+B+1)} = \emptyset,$$

$$\lambda^{(j)} \prec \lambda^{(j+1)} \text{ if } j \in \mathcal{L}(\lambda), \quad \lambda^{(j)} \succ \lambda^{(j+1)} \text{ if } j \notin \mathcal{L}(\lambda), \tag{2.22}$$

where notation $\mu \prec \nu$ means that μ and ν interlace, that is

$$\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \nu_3 \geq \dots.$$

We refer to Figure 1 for an illustration of $A = 4$, $B = 3$ and $\pi = \emptyset$ case. Moreover, we have

$$\sum_{j=1}^{A+B+1} \left| \lambda^{(j)} \right| = \text{Volume}\,(\Pi),$$

where $|\mu|$ denotes the number of boxes in the Young diagram μ.

The probability measure on the set of all skew plane partitions Π with support B^A/π and defined by equation (2.21) induces a probability measure on sequences $\left(\lambda^{(1)}, \dots, \lambda^{(A+B+1)}\right)$. It is known (see Okounkov and Reshetikhin [39, 40]) that this probability measure can be understood as the Schur process defined in Section 2.2 by equation (2.10) with the rank $p = A + B + 1$, and nonnegative specializations $\{\varrho_i^+\}_{i=0}^{p-1}$, $\{\varrho_i^-\}_{i=1}^{p}$ defined by

$$H\left(\varrho_0^+; u\right) = H\left(\varrho_{A+B+1}^-; u\right) = 1,$$

$$H\left(\varrho_j^+; u\right) = \begin{cases} \dfrac{1}{1 - q^{-j}u}, & j \in \mathcal{L}(\pi), \\ 1, & j \notin \mathcal{L}(\pi); \end{cases} \qquad H\left(\varrho_j^-; u\right) = \begin{cases} 1, & j \in \mathcal{L}(\pi), \\ \dfrac{1}{1 - q^{j}u}, & j \notin \mathcal{L}(\pi). \end{cases}$$

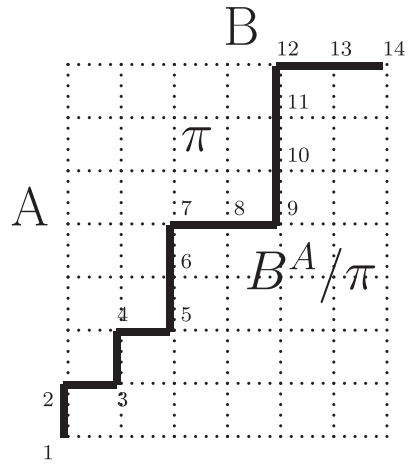

Fig. 2. The set $\{1,\dots,A+B+1\}$ enumerates the boundary of the skew diagram B^A/π. In this example $A=7$, $B=6$, and $l=4$.

Note that for any two neighboring specializations ϱ_k^-, ϱ_k^+ defined above, at least one is trivial, and each $\mu^{(j)}$ coincides either with $\lambda^{(j)}$ or $\lambda^{(j+1)}$. The only nontrivial specializations are one variable specializations $\rho=(\alpha)$ with $\alpha=q^{\pm j}$. A basic property of skew Schur functions is that $s_{\lambda/\mu}(\alpha)=0$ unless $\mu\prec\lambda$; this implies interlacing conditions (2.22).

Let $\pi\in B^A$. The set $\{1,\dots,A+B+1\}$ enumerates the intersections of the boundary of the skew diagram B^A/π with the square grid, as shown on Figure 2. Denote by l the number of vertical segments of of the boundary of the skew diagram B^A/π.

Let $\{\beta_1,\dots,\beta_{2l-1}\}$ be a subset of $\{1,\dots,A+B+1\}$, where the numbers β_1, …, β_{2l-1} parameterize the vertical segments of the boundary of the skew diagram B^A/π, see Figure 3. For example, for the Young diagram π on Figure 2 we have $\beta_1=12$, $\beta_2=9$, $\beta_3=7$, $\beta_4=5$, $\beta_5=4$, $\beta_6=3$, $\beta_7=2$, and $\beta_8=1$. Now, assume that $a_1\geq p$, and pick p numbers $\alpha_1,\dots,\alpha_p$ such that $\beta_2\leq\alpha_1<\ldots<\alpha_p<\beta_1$, see Figure 3.

Consider the sequence $\left(\lambda^{(\alpha_1)},\dots,\lambda^{(\alpha_p)}\right)$ of random Young diagrams associated with a random skew plane partition Π whose support is B^A/π, and whose weight is proportional to $q^{\mathrm{Volume}(\Pi)}$. By assigning to this sequence the point configuration

$$\left\{\left(1,\lambda_i^{(\alpha_1)}-i\right)\right\}_{i\geq1}\cup\ldots\cup\left\{\left(p,\lambda_i^{(\alpha_p)}-i\right)\right\}_{i\geq1} \tag{2.23}$$

we obtain a random point process on $\{1,\dots,p\}\times\mathbb{Z}$.

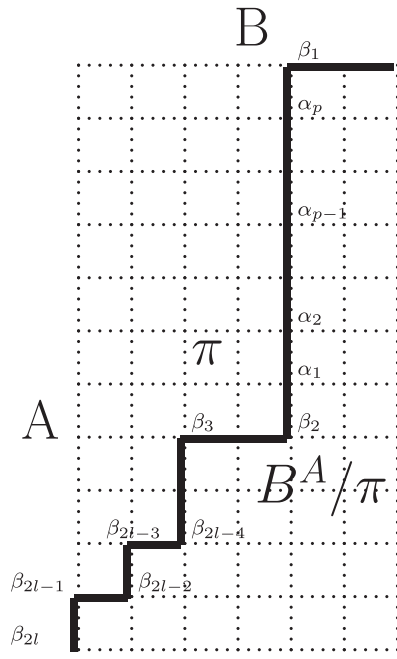

Fig. 3. The parametrization of the the vertical segments of the boundary of B^A/π by $\beta_1, \ldots, \beta_{2l}$, and the choice of $\alpha_1, \ldots, \alpha_p$.

Proposition 2.8. The probability of the point configuration (2.23) is determined by the probability measure

$$\begin{aligned}
&\frac{1}{Z} s_{\lambda^{(\alpha_p)}}\left(1, q, \ldots, q^{B-\pi_1-1}\right) s_{\lambda^{(\alpha_p)}/\lambda^{(\alpha_{p-1})}}\left(q^{A+\pi_1+2-\alpha_p}, q^{A+\pi_1+3-\alpha_p}, \ldots, q^{A+\pi_1+1-\alpha_{p-1}}\right) \\
&\times s_{\lambda^{(\alpha_{p-1})}/\lambda^{(\alpha_{p-2})}}\left(q^{A+\pi_1+2-\alpha_{p-1}}, q^{A+\pi_1+3-\alpha_{p-1}}, \ldots, q^{A+\pi_1+1-\alpha_{p-2}}\right) \\
&\times \ldots \cdot s_{\lambda^{(\alpha_2)}/\lambda^{(\alpha_1)}}\left(q^{A+\pi_1+2-\alpha_2}, q^{A+\pi_1+3-\alpha_2}, \ldots, q^{A+\pi_1+1-\alpha_1}\right) \\
&\times s_{\lambda^{(\alpha_1)}}\Big(q^{A+\pi_1+2-\alpha_1}, q^{A+\pi_1+3-\alpha_1}, \ldots, q^{A+\pi_1+1-\beta_2}; q^{A+\pi_1+2-\beta_3}, q^{A+\pi_1+3-\beta_3}, \ldots, q^{A+\pi_1+1-\beta_4}; \\
&\ldots; q^{A+\pi_1+2-\beta_{2l-3}}, q^{A+\pi_1+3-\beta_{2l-3}}, \ldots, q^{A+\pi_1+1-\beta_{2l-2}}; q^{A+\pi_1+2-\beta_{2l-1}}, q^{A+\pi_1+3-\beta_{2l-1}}, \ldots, q^{A+\pi_1}\Big),
\end{aligned} \tag{2.24}$$

where Z is a normalization constant.

Now we are ready to state the main result of the present work. Let Π be a random skew partition with support B^A/π whose weight is determined by equation (2.21). Let $\left(\lambda^1,\dots,\lambda^{(A+B+1)}\right)$ be a sequence of Young diagrams associated with Π. Consider the subsequence $\left(\lambda^{(\alpha_1)},\dots,\lambda^{(\alpha_p)}\right)$ of $\left(\lambda^1,\dots,\lambda^{(A+B+1)}\right)$, where the indexes α_1, $\dots$, α_p are chosen as it is described above. By assigning to this subsequence the point configuration (2.23) we obtain a random point process on $\{1,\dots,p\}\times\mathbb{Z}$. Set $n=B-\pi_1$, $q=e^{-\epsilon}$, and define

$$x_j^k = e^{-\epsilon\lambda_j^{(\alpha_k)}},\ k=1,\dots,p;\ j=1,\dots,n.$$

Theorem 2.9. As $\epsilon\to 0$, the point process formed by configurations

$$\left\{\left(k,x_j^k\right)|k=1,\dots,p;\ j=1,\dots,n\right\}$$

converges to the product matrix process associated with truncated unitary matrices, described in Section 2.1, and defined by probability distribution (2.4). The parameters of the relevant product matrix process are given by

- $n=B-\pi_1$.
- $m_k=A+B+1-\beta_{2k}$ for $1\le k\le l$ and $m_{l+k}=A+B+1-\alpha_k$ for $1\le k\le p-1$.
- $\nu_1 = A+\pi_1+1-\alpha_1$, $\nu_k = A+\pi_1+1-\beta_{2k-1}$ for $2\le k\le l$, and $\nu_{l+k}=A+\pi_1+1-\alpha_{k+1}$ for $1\le k\le p-1$.

The truncated unitary matrices forming the product matrix process in Theorem 2.9 are shown schematically on Figure 4.

Remark 2.10 The condition $m_1\ge 2n+\nu_1$ reads as

$$\alpha_1\ge B-\pi_1+\beta_2. \tag{2.25}$$

The conditions $m_j\ge n+\nu_j+1$ (where $2\le j\le l$) can be rewritten as

$$\beta_{2j-1}-\beta_{2j}\ge 0;\ \alpha_1-\beta_2\ge 0;$$

and

$$\alpha_{k+1}-\alpha_k\ge 0$$

(where $1\le k\le p-1$), and are satisfied automatically.

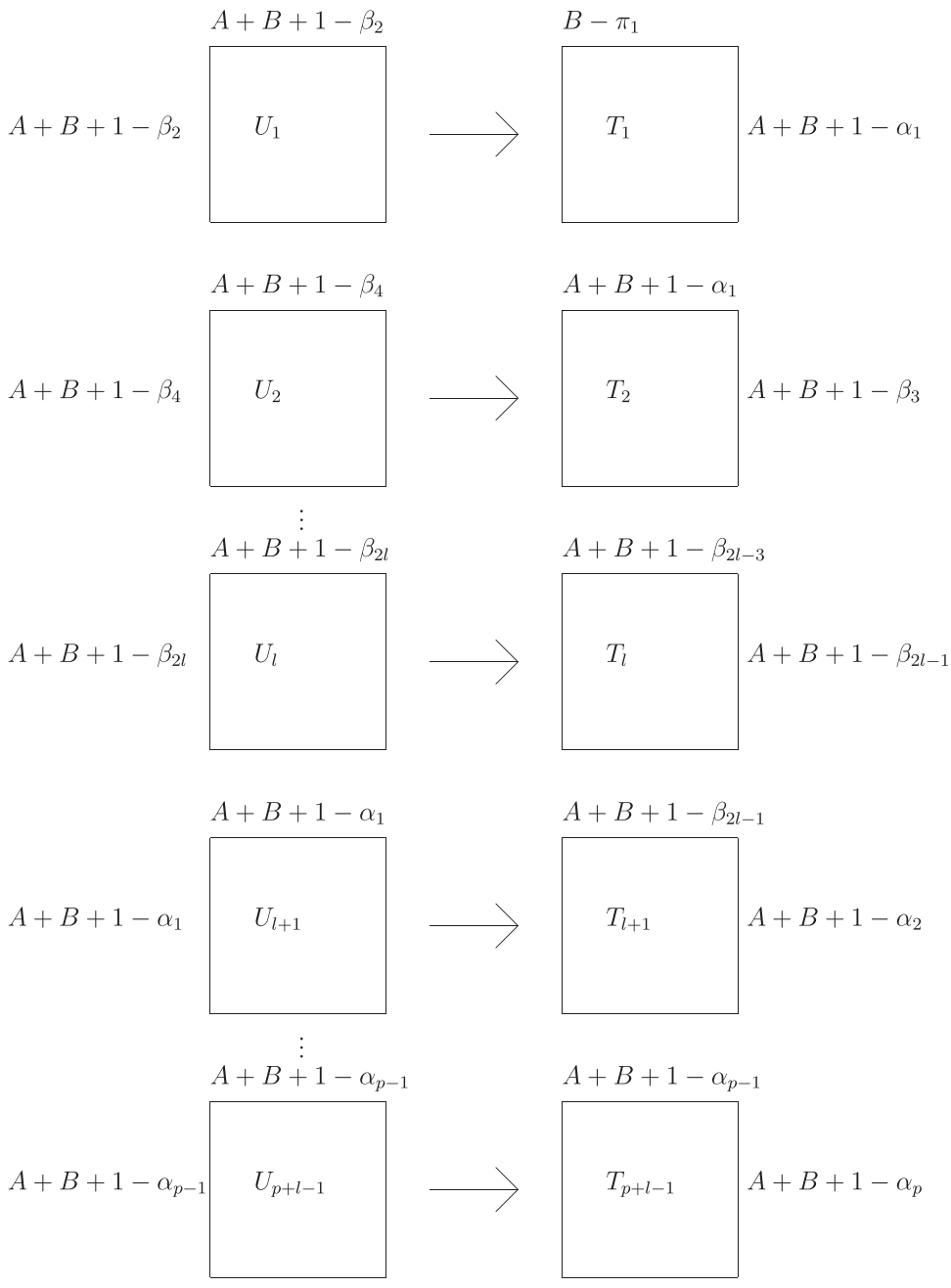

Fig. 4. The truncated unitary matrices forming the product matrix process associated with the random skew plane partitions.

Remark 2.11 The choice of the parameters ν_i, m_i is not unique. Namely, we need to identify the l geometric series in (2.14) with l geometric series in the last two lines of (2.24) and there are $l!$ ways to do so. Formally, our proof goes through only for the choices which agree with the condition $m_1 \geq 2n + \nu_1$ of (2.2), see, however, Remark 2.7.

Example 2.12. Consider the particular case in which $\pi = \emptyset$, as in Figure 1. In this situation $l = 1$, $\beta_1 = A + 1$, $\beta_2 = 1$, and the parameters $\alpha_1, \ldots, \alpha_p$ take values in $\{1, \ldots, A\}$. As a limit, we obtain the product matrix process with truncated unitary matrices whose parameters n; $m_1, \ldots, m_p$; $\nu_1, \ldots, \nu_p$ are given by

- $n = B$;
- $m_1 = A + B$, and $m_k = A + B + 1 - \alpha_{k-1}$ for $2 \leq k \leq p$;
- $\nu_k = A + 1 - \alpha_k$ for $1 \leq k \leq p$.

3 Proof of Proposition 2.1

In order to prove Proposition 2.1 we will use two results obtained in Kieburg, Kuijlaars, and Stivigny [31]. Namely, Corollary 2.6 in Kieburg *et al.* [31] implies that the probability distribution of $x^1 = \left(x_1^1, \ldots, x_n^1\right)$ (which is the vector of the squared singular values of $T_l \ldots T_1$) can be written as

$$\text{const} \triangle\left(x^1\right) \det\left[w_k^{(l)}\left(x_j^1\right)\right]_{k,j=1}^n \mathrm{d}x^1.$$

The 2nd result concerns the density of squared singular values for a product of a nonrandom and a truncated unitary matrix. Namely, assume that U is a Haar-distributed unitary matrix of size $m \times m$, and let T be an $(n + \nu) \times l$ truncation of U. In addition, let X be a nonrandom matrix of size $l \times n$, and impose the following constraints for the parameters n, l, m, and ν:

$$1 \leq n \leq l \leq m, \quad m \geq n + \nu + 1.$$

Denote by $\left(x_1, \ldots, x_n\right)$ the vector of squared singular values of X, and by $\left(y_1, \ldots, y_n\right)$ the vector of squared singular values of TX. If $x_1, \ldots, x_n$ are pairwise distinct and nonzero, then the vector $\left(y_1, \ldots, y_n\right)$ has density

$$\text{const}\left(\prod_{j=1}^n x_j^{-m+n}\right)\left(\prod_{j=1}^n y_j^{\nu}\right) \det\left[\left(x_k - y_j\right)_+^{m-n-\nu-1}\right]_{j,k=1}^n \frac{\triangle(y)}{\triangle(x)}, \tag{3.1}$$

see Kieburg *et al.* [31, Theorem 2.1]. Now assume that $X = T_l \dots T_1$. Applying the results stated above we immediately obtain that the probability distribution of $\left(x_1^k, \dots, x_n^k\right)$ is proportional to the product of determinants as in equation (2.4). In order to compute the normalization constant we use the Andréief integral identity (see, for instance, De Bruijn [16]), and the recurrence relation

$$w_k^{(l+1)}(y) = \int_0^1 \tau^{\nu_{l+1}} (1-\tau)^{m_{l+1}-n-\nu_{l+1}-1} w_k^{(l)}\left(\frac{y}{\tau}\right) \frac{\mathrm{d}\tau}{\tau},$$

see Kieburg *et al.* [31], equation (2.22). The integration over $x_1^1, \dots, x_n^1$ gives

$$\begin{aligned}
&\int_{0\le x_1^1 \le \dots \le x_n^1 < \infty} \dots \int \det\left[\left(x_j^2\right)^{\nu_{l+1}} \left(x_k^1 - x_j^2\right)_+^{m_{l+1}-n-\nu_{l+1}-1} \left(x_k^1\right)^{n-m_{l+1}}\right]_{j,k=1}^n \det\left[w_k^{(l)}\left(x_j^1\right)\right]_{j,k=1}^n \mathrm{d}x_1^1 \dots \mathrm{d}x_n^1 \\
&= \det\left[\int_0^\infty \left(x_j^2\right)^{\nu_{l+1}} \left(t - x_j^2\right)_+^{m_{l+1}-n-\nu_{l+1}-1} t^{n-m_{l+1}} w_k^{(l)}(t)\mathrm{d}t\right]_{j,k=1}^n .
\end{aligned}$$

Changing the integration variable $t = \frac{x_j^2}{\tau}$, we rewrite the integral inside the determinant above as

$$\begin{aligned}
&\left(x_j^2\right)^{\nu_{l+1}} \int_0^1 \left(\frac{x_j^2}{\tau} - x_j^2\right)^{m_{l+1}-n-\nu_{l+1}-1} \frac{\left(x_j^2\right)^{n-m_{l+1}}}{\tau^{n-m_{l+1}}} w_k^{(l)}\left(\frac{x_j^2}{\tau}\right) \frac{x_j^2}{\tau^2}\mathrm{d}\tau \\
&= \int_0^1 \tau^{\nu_{l+1}} (1-\tau)^{m_{l+1}-n-\nu_{l+1}-1} w_k^{(l)}\left(\frac{x_j^2}{\tau}\right)\frac{\mathrm{d}\tau}{\tau} = w_k^{(l+1)}\left(x_j^2\right).
\end{aligned} \tag{3.2}$$

As a result of integration over the variables $x_1^1, \dots, x_n^1; \dots; x_1^{p-1}, \dots, x_n^{p-1}$ we find

$$Z_{n,p+l} = \int_{0\le x_1 \le \dots \le x_n \le \infty} \det\left(x_k^{j-1}\right)_{j,k=1}^n \det\left(w_k^{(l+p-1)}\left(x_j\right)\right)_{j,k=1}^n \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Applying the Andréief integral identity again, we obtain

$$Z_{n,p+l} = \det\left[\int_0^\infty x^{j-1} w_k^{(l+p-1)}(x)\,\mathrm{d}x\right]_{j,k=1}^n$$
$$= \left(c_{l+p-1}\right)^n \det\left[\int_0^\infty x^{j-1} G^{l+p-1,0}_{l+p-1,l+p-1}\left(\begin{matrix} m_{l+p-1}-n, & \dots, & m_2-n, & m_1-2n+k \\ \nu_{l+p-1}, & \dots, & \nu_2, & \nu_1+k-1 \end{matrix}\,\middle|\, x\right)\mathrm{d}x\right]_{j,k=1}^n.$$

The integral inside the determinant can be computed explicitly in terms of Gamma functions. Namely, formula (5.6.1.1) in Luke [34] gives

$$\begin{aligned}&\int_0^\infty x^j G^{l+p-1,0}_{l+p-1,l+p-1}\left(\begin{matrix} m_{l+p-1} & \dots & m_2-n & m_1-2n+k \\ \nu_{l+p-1} & \dots & \nu_2 & \nu_1+k-1 \end{matrix}\,\middle|\, x\right)\mathrm{d}x \\ &= \frac{\prod_{k=2}^{l+p-1}\Gamma\left(\nu_k+j\right)\Gamma\left(\nu_1+k-1+j\right)}{\prod_{k=2}^{l+p-1}\Gamma\left(m_k-n+j\right)\Gamma\left(m_1-2n+k+j\right)},\end{aligned} \tag{3.3}$$

and we find

$$Z_{n,p+l} = \left[\Gamma\left(m_1-2n-\nu_1+1\right)\right]^n \prod_{j=2}^{l+p-1}\left[\Gamma\left(m_j-n-\nu_j\right)\right]^n$$
$$\times \frac{\prod_{k=2}^{l+p-1}\prod_{j=1}^n \Gamma\left(\nu_k+j\right)}{\prod_{k=2}^{l+p-1}\prod_{j=1}^n \Gamma\left(m_k-n+j\right)} \det\left[\frac{\Gamma\left(\nu_1+k+j-1\right)}{\Gamma\left(m_1-2n+k+j\right)}\right]_{j,k=1}^n.$$

The following formula for the determinant with Gamma functions entries is known:

$$\det\left[\frac{\Gamma(c+i+j)}{\Gamma(d+i+j)}\right]_{i,j=0}^{n-1} = \prod_{j=0}^{n-1} j!\,\frac{\Gamma(d-c+j)}{\Gamma(d-c)}\frac{\Gamma(c+j)}{\Gamma(d+n-1+j)}, \tag{3.4}$$

see equation (4.11) in Normand [36]. Using this formula we obtain

$$\det\left[\frac{\Gamma\left(\nu_1+k+j-1\right)}{\Gamma\left(m_1-2n+k+j\right)}\right]_{j,k=1}^n = \prod_{j=1}^n \frac{\Gamma\left(\nu_1+j\right)}{\Gamma\left(m_1-n+j\right)}\frac{\prod_{j=1}^n \Gamma(j)\Gamma\left(m_1-2n-\nu_1+j\right)}{\left(\Gamma\left(m_1-2n-\nu_1+1\right)\right)^n}.$$

This gives

$$Z_{n,p+l} = \prod_{j=2}^{l+p-1} \left[\Gamma\left(m_j - n - \nu_j\right)\right]^n$$
$$\times \frac{\prod_{k=1}^{l+p-1}\prod_{j=1}^{n}\Gamma\left(\nu_k + j\right)}{\prod_{k=1}^{l+p-1}\prod_{j=1}^{n}\Gamma\left(m_k - n + j\right)} \prod_{j=1}^{n}\Gamma(j)\Gamma\left(m_1 - 2n - \nu_1 + j\right).$$

Since

$$\prod_{j=1}^{n}\frac{\Gamma\left(\nu + j\right)}{\Gamma(m - n + j)} = \prod_{j=1}^{m-n-\nu}\frac{\Gamma(\nu + j)}{\Gamma(\nu + j + n)} = \prod_{j=1}^{m-n-\nu}\frac{1}{(j+\nu)_n},$$

we can rewrite the normalization constant $Z_{n,p+l}$ in the same form as in the statement of the Proposition 2.1.

4 Limits of Symmetric Functions

The aim of this section is to obtain certain asymptotic formulae for the Schur functions, and for the skew Schur functions, see Propositions 4.1 and 4.3. We will need these formulae in the proofs of our main results (Proposition 2.6 and Theorem 2.9).

Proposition 4.1. Let $\lambda(\epsilon)$ be a family of Young diagrams with N rows, where $N \leq M$. Assume that $\lambda(\epsilon)$ depends on a positive parameter ϵ in such a way that $\epsilon\lambda_j(\epsilon) \to -\log r_j$, as $\epsilon \to 0+$, for some values $0 < r_1 \leq \ldots \leq r_N < 1$. Then

$$\lim_{\epsilon\to 0+}\left\{\epsilon^{MN-\frac{N(N+1)}{2}} s_\lambda\left(e^{-(1+\nu)\epsilon}, \ldots, e^{-(M+\nu)\epsilon}\right)\right\}$$
$$= \frac{1}{\prod_{j=1}^{N}\Gamma(M - N + j)} \prod_{i=1}^{N} r_i^{1+\nu}\left(1 - r_i\right)^{M-N} \prod_{1\leq i<j\leq N}\left(r_j - r_i\right), \tag{4.1}$$

where $\nu \geq 0$. The convergence is uniform in r_j's.

Proof. Homogeneity of the Schur polynomials implies

$$s_\lambda\left(e^{-(1+\nu)\epsilon}, \ldots, e^{-(M+\nu)\epsilon}\right) = e^{-(1+\nu)\epsilon\sum_{i=1}^{N}\lambda_i} s_\lambda\left(1, e^{-\epsilon}, \ldots, e^{-(M-1)\epsilon}\right). \tag{4.2}$$

Moreover, the principal specialization of the Schur polynomials (see Macdonald [35, § 3, Example 1]) gives

$$s_\lambda\left(1, e^{-\epsilon}, \dots, e^{-(M-1)\epsilon}\right) = e^{-\epsilon \sum_{i=1}^{N}(i-1)\lambda_i} \prod_{1\le i<j\le M} \frac{1-e^{-\epsilon(\lambda_i-\lambda_j-i+j)}}{1-e^{\epsilon(-i+j)}}. \tag{4.3}$$

The product in the right-hand side of the expression above can be written as

$$\begin{aligned}\prod_{1\le i<j\le M} \frac{1-e^{-\epsilon(\lambda_i-\lambda_j-i+j)}}{1-e^{\epsilon(-i+j)}} &= \prod_{1\le i<j\le N}\left(1-e^{-\epsilon(\lambda_i-\lambda_j-i+j)}\right)\\ &\times \prod_{i=1}^{N}\prod_{j=N+1}^{M}\left(1-e^{-\epsilon(\lambda_i-i+j)}\right) \prod_{\substack{i<j\\ 1\le i\le N\\ 1\le j\le M}} \frac{1}{1-e^{-\epsilon(-i+j)}}.\end{aligned} \tag{4.4}$$

Taking the limit $\epsilon \to 0+$, and using the fact that

$$\prod_{\substack{i<j\\ 1\le i\le N\\ 1\le j\le M}} \frac{1}{\epsilon(j-i)} = \frac{1}{\epsilon^{MN-\frac{N(N+1)}{2}}} \frac{1}{\prod_{j=1}^{N}\Gamma(M-N+j)}, \tag{4.5}$$

we obtain the formula in the statement of the Proposition. ∎

Lemma 4.2. Let $k(\epsilon)$, $m(\epsilon)$ be two families of positive integers depending on a positive parameter ϵ in such a way that $\epsilon k(\epsilon) \to -\log r$, $\epsilon m(\epsilon) \to -\log s$, as $\epsilon \to 0+$, for some values $0 < r, s < 1$. Then

$$\lim_{\epsilon\to 0+}\left\{\epsilon^{M-1} h_{k-m}\left(e^{-(1+\nu)\epsilon}, e^{-(2+\nu)\epsilon}, \dots, e^{-(M+\nu)\epsilon}\right)\right\} = \frac{1}{\Gamma(M)} \frac{r^{1+\nu}}{s^{M+\nu}} (s-r)_+^{M-1}, \tag{4.6}$$

where $h_n = s_{(n)}$ is the nth complete homogeneous symmetric function, $M \ge 1$, $\nu \ge 0$, and $(s-r)_+ = \max\{0, s-r\}$. The convergence is uniform in s, r.

Proof. We start by noting that the right-hand side of (4.6) has no singularity at $s = 0$. Indeed, this follows from the observation that for positive r we have $r^{1+\nu}(s-r)_+^{M-1} \le s^{1+\nu} s^{M-1} = s^{M+\nu}$.

Specializing (4.3) to $\lambda = (m)$ gives

$$h_m\left(1, q, \ldots, q^{M-1}\right) = \prod_{j=1}^{M-1} \frac{1-q^{m+j}}{1-q^j}.$$

Assuming $m \leq k$ we obtain

$$\begin{aligned} &h_{k-m}\left(e^{-(1+\nu)\epsilon}, e^{-(2+\nu)\epsilon}, \ldots, e^{-(M+\nu)\epsilon}\right) \\ &= e^{-(1+\nu)\epsilon(k-m)} h_{k-m}\left(1, e^{-\epsilon}, \ldots, e^{-(M-1)\epsilon}\right) = e^{-(1+\nu)\epsilon(k-m)} \prod_{j=1}^{M-1} \frac{1-e^{-\epsilon(k-m+j)}}{1-e^{-\epsilon j}} \\ &\simeq \left(\frac{r}{s}\right)^{1+\nu} \prod_{j=1}^{M-1} \frac{1-\frac{r}{s}}{\epsilon j} = \frac{1}{\epsilon^{M-1}} \frac{1}{\Gamma(M)} \frac{r^{1+\nu}}{s^{M+\nu}} (s-r)^{M-1}, \end{aligned} \tag{4.7}$$

where $s \geq r$. If $s < r$, then $m > k$, and $h_{k-m} \equiv 0$. ■

Proposition 4.3. Let $\lambda(\epsilon)$, $\mu(\epsilon)$ be two families of Young diagrams depending on a positive parameter ϵ, with $l(\lambda) = l(\mu) = N$. Assume that $\lambda(\epsilon)$, $\mu(\epsilon)$ depend on the parameter ϵ in such a way that $\epsilon\lambda_j(\epsilon) \to -\log r_j$, $\epsilon\mu_j(\epsilon) \to -\log s_j$, as $\epsilon \to 0+$ for some values $0 < r_1 \leq r_2 \leq \ldots \leq r_N < 1$, and $0 < s_1 \leq s_2 \leq \ldots \leq s_N < 1$. Then we have

$$\begin{aligned} &\lim_{\epsilon \to 0+} \left\{ \epsilon^{MN-N} s_{\lambda/\mu}\left(e^{-(1+\nu)\epsilon}, e^{-(2+\nu)\epsilon}, \ldots, e^{-(M+\nu)\epsilon}\right)\right\} \\ &= \frac{1}{(\Gamma(M))^N} \frac{\prod_{i=1}^N (r_i)^{1+\nu}}{\prod_{i=1}^N (s_i)^{M+\nu}} \det\left[\left(s_j - r_i\right)_+^{M-1}\right]_{i,j=1}^N, \end{aligned} \tag{4.8}$$

where $M \geq 1$. The convergence is uniform in r_j's and s_j's.

Proof. Apply the Jacobi–Trudi formula (see [35, Chapter I, (5.4)]),

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - \mu_j - i + j}\right)_{i,j=1}^N,$$

together with Lemma 4.2. ■

Proposition 4.4. Let $\lambda(\epsilon)$ be a family of Young diagrams with n rows. Assume that $\lambda(\epsilon)$ depends on a positive parameter ϵ in such a way that $\epsilon\lambda_j(\epsilon) \to -\log x_j$, as $\epsilon \to 0+$, for some values $0 < x_1 \le \ldots \le x_n < 1$. Then

$$\begin{aligned}
&\lim_{\epsilon\to 0+}\left[\epsilon^{\sum_{k=1}^{p}(m_k-n-\nu_k)n-\frac{n(n+1)}{2}} s_\lambda\left(e^{-(1+\nu_1)\epsilon},\ldots,e^{-(m_1-n)\epsilon};\ldots;e^{-(1+\nu_p)\epsilon},\ldots,e^{-(m_p-n)\epsilon}\right)\right]\\
&=\frac{\left[\Gamma\left(m_1-2n-\nu_1+1\right)\right]^n}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)}\\
&\times\det\left[G^{p,0}_{p,p}\left(\begin{matrix}m_p-n+1 & \ldots & m_2-n+1 & m_1-2n+k+1\\ \nu_p+1 & \ldots & \nu_2+1 & \nu_1+k\end{matrix}\,\middle|\,x_j\right)\right]_{j,k=1}^{n},
\end{aligned}\tag{4.9}$$

where the parameters m_j, ν_j are those specified in Section 2.1. The convergence is uniform in x_j's.

Proof. The proof is by induction over p. Assume that $p = 1$. Then equation (4.9) takes the form

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left[\epsilon^{(m_1-n-\nu_1)n-\frac{n(n+1)}{2}} s_\lambda\left(e^{-(1+\nu_1)\epsilon},\ldots,e^{-(m_1-n)\epsilon}\right)\right]\\
&=\frac{\left[\Gamma\left(m_1-2n-\nu_1+1\right)\right]^n}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)}\det\left[G^{1,0}_{1,1}\left(\begin{matrix}m_1-2n+k+1\\ \nu_1+k\end{matrix}\,\middle|\,x_j\right)\right]_{j,k=1}^{n}.
\end{aligned}\tag{4.10}$$

As follows from equations (2.20–2.25) in Kieburg *et al.* [31],

$$G^{1,0}_{1,1}\left(\begin{matrix}m_1-2n+k+1\\ \nu_1+k\end{matrix}\,\middle|\,x\right)=\frac{1}{\Gamma\left(m_1-2n-\nu_1+1\right)}x^{\nu_1+k}(1-x)^{m_1-2n-\nu_1}.$$

We see that equation (4.10) turns into equation (4.1) (with $N = n$, $M = m_1 - n - \nu$), and conclude that Proposition 4.4 holds true for $p = 1$.

Assume that the statement of Proposition 4.4 holds true for certain natural p. Let us prove this statement for $p+1$. Equation (2.12) implies

$$\begin{aligned} &s_\lambda\left(e^{-(1+\nu_1)\epsilon},\dots,e^{-(m_1-n)\epsilon};\dots;e^{-(1+\nu_{p+1})\epsilon},\dots,e^{-(m_{p+1}-n)\epsilon}\right)\\ &=\sum_\mu s_{\lambda/\mu}\left(e^{-(1+\nu_{p+1})\epsilon},\dots,e^{-(m_{p+1}-n)\epsilon}\right)\times\\ &s_\mu\left(e^{-(1+\nu_1)\epsilon},\dots,e^{-(m_1-n)\epsilon};\dots,e^{-(1+\nu_p)\epsilon},\dots,e^{-(m_p-n)\epsilon}\right). \end{aligned} \tag{4.11}$$

Here we can assume that the sum is over Young diagrams μ with at most n rows. Let $\lambda(\epsilon)$, $\mu(\epsilon)$ be two families of Young diagrams depending on a positive parameter ϵ, with $l(\lambda)=l(\mu)=n$. Assume that $\lambda(\epsilon)$, $\mu(\epsilon)$ depend on the parameter ϵ in such a way that $\epsilon\lambda_j(\epsilon)\to-\log x_j$, $\epsilon\mu_j(\epsilon)\to-\log s_j$, as $\epsilon\to 0+$ for some values $0<x_1\le r_2\le\ldots\le x_n<1$, and $0<y_1\le y_2\le\ldots\le y_n<1$. Then Proposition 4.3 (with $N=n$, $M=m_{p+1}-n-\nu_{p+1}$, and $\nu=\nu_{p+1}$) implies that

$$\epsilon^{-n+(m_{p+1}-n-\nu_{p+1})n}s_{\lambda/\mu}\left(e^{-(1+\nu_{p+1})\epsilon},\dots,e^{-(m_{p+1}-n)\epsilon}\right)$$

converges to

$$\frac{1}{\left[\Gamma\left(m_{p+1}-n-\nu_{p+1}\right)\right]^n}\frac{\prod_{i=1}^n\left(x_i\right)^{1+\nu_{p+1}}}{\prod_{i=1}^n\left(y_i\right)^{m_{p+1}-n}}\det\left[\left(y_j-x_i\right)_+^{m_{p+1}-n-\nu_{p+1}-1}\right]_{i,j=1}^n,$$

and the convergence is uniform in x_j's and y_j's. Moreover, by our assumption

$$\epsilon^{\sum_{k=1}^p(m_k-n-\nu_k)n-\frac{n(n+1)}{2}}s_\mu\left(e^{-(1+\nu_1)\epsilon},\dots,e^{-(m_1-n)\epsilon};\dots,e^{-(1+\nu_p)\epsilon},\dots,e^{-(m_p-n)\epsilon}\right)$$

converges to

$$\frac{\left[\Gamma\left(m_1-2n-\nu_1+1\right)\right]^n}{\prod_{j=1}^n\Gamma\left(m_1-2n-\nu_1+j\right)}\det\left[G_{p,p}^{p,0}\left(\begin{matrix}m_p-n+1 & \dots & m_2-n+1 & m_1-2n+k+1\\ \nu_p+1 & \dots & \nu_2+1 & \nu_1+k\end{matrix}\middle|y_j\right)\right]_{j,k=1}^n,$$

uniformly for $y_1, \dots, y_n \in (0,1)$. Since $\mathrm{d}\mu_j \sim -\frac{\mathrm{d}y_j}{\epsilon y_j}$, we conclude that the right-hand side of equation (4.11) multiplied by $\epsilon^{\sum_{k=1}^{p+1}(m_k-n-\nu_k)n-\frac{n(n+1)}{2}}$ converges to

$$\frac{\left[\Gamma\left(m_1-2n-\nu_1+1\right)\right]^n}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)}\frac{1}{\left[\Gamma\left(m_{p+1}-n-\nu_{p+1}\right)\right]^n}\prod_{i=1}^{n}\left(x_i\right)^{1+\nu_{p+1}}$$
$$\times\int\limits_{0\le y_1\le\dots\le y_n\le 1}\dots\int\prod_{i=1}^{n}\left(y_i\right)^{n-m_{p+1}}\det\left[\left(y_j-x_i\right)_+^{m_{p+1}-n-\nu_{p+1}-1}\right]_{i,j=1}^{n}$$
$$\times\det\left[G^{p,0}_{p,p}\left(\begin{matrix}m_p-n+1 & \dots & m_2-n+1 & m_1-2n+k+1\\ \nu_p+1 & \dots & \nu_{p-1}+1 & \nu_1+k\end{matrix}\,\middle|\,y_j\right)\right]_{j,k=1}^{n}\frac{\mathrm{d}y_1}{y_1}\dots\frac{\mathrm{d}y_n}{y_n},$$

and the convergence is uniform in $x_1, \dots, x_n$. The expression above can be rewritten as

$$\frac{1}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)\prod_{j=2}^{p+1}\left[\Gamma\left(m_j-n-\nu_j\right)\right]^n}$$
$$\times\int\limits_{0\le y_1\le\dots\le y_n\le 1}\dots\int\mathrm{d}y_1\dots\mathrm{d}y_n\prod_{i=1}^{n}\left(y_i\right)^{n-m_{p+1}-1}\det\left[x_k^{1+\nu_p}\left(y_j-x_k\right)_+^{m_{p+1}-n-\nu_{p+1}-1}\right]_{k,j=1}^{n}\times$$
$$\det\left[w_k^{(p)}\left(y_j\right)\right]_{j,k=1}^{n},$$

where we have used equations (2.6) and (2.7) to rewrite the Meijer G-function in terms of the corresponding weight function $w_k^{(p)}(y)$. By the Andréief identity, and by the same calculations as in the proof of Proposition 2.1 it can be shown that (4.12) is equal to

$$\frac{1}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)\prod_{j=2}^{p+1}\left[\Gamma\left(m_j-n-\nu_j\right)\right]^n}\det\left[x_jw_k^{(p+1)}\left(x_j\right)\right]_{j,k=1}^{n}.$$

Rewriting the weight function $w_k^{(p+1)}$ in terms of the corresponding Meijer G-function we obtain that

$$\lim_{\epsilon\to 0+}\left[\epsilon^{\sum_{k=1}^{p+1}(m_k-n-\nu_k)n-\frac{n(n+1)}{2}} s_\lambda\left(e^{-(1+\nu_1)\epsilon},\dots,e^{-(m_1-n)\epsilon};\dots;e^{-(1+\nu_{p+1})\epsilon},\dots,e^{-(m_{p+1}-n)\epsilon}\right)\right]$$
$$=\frac{\left[\Gamma\left(m_1-2n-\nu_1+1\right)\right]^n}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)}$$
$$\times\det\left[G_{p+1,p+1}^{p+1,0}\left(\begin{matrix} m_{p+1}-n+1 & \dots & m_2-n+1 & m_1-2n+k+1 \\ \nu_{p+1}+1 & \dots & \nu_2+1 & \nu_1+k \end{matrix}\middle| x_j\right)\right]_{j,k=1}^{n}, \tag{4.12}$$

uniformly for $x_1,\dots,x_n\in(0,1)$. ■

5 Convergence of the Schur Process to the Product Matrix Process with Truncated Unitary Matrices. Proof of Proposition 2.6

Now we begin to investigate the convergence of the Schur process to the product matrix process with truncated unitary matrices. We start with the case where the initial conditions are defined by a single truncated matrix T_1, see Proposition 5.1. Then (using Proposition 4.4) we generalize Proposition 5.1 to the case where the initial conditions are specified by a product of l truncated matrices $T_l\dots T_1$, and prove Proposition 2.66. We remark that, in principle, the 2nd part can be avoided, as the general l case can be obtained from the $l=1$ case by restriction of a distribution to a subset of matrices. In particular, in this way the consistency of Proposition 2.1 between different values of l can be used to produce an alternative proof of Proposition 4.4.

Consider the Schur process defined in Section 2.2. Define the specializations ϱ_0^+, …, ϱ_{p-1}^+, ϱ_1^-, …, ϱ_p^- of the algebra of symmetric functions as follows:

- The specialization ϱ_p^- is defined by

$$\varrho_p^-=\left(1,e^{-\epsilon},\dots,e^{-(n-1)\epsilon}\right).$$

All other ϱ_1^-, …, ϱ_{p-1}^- are trivial.

- The specializations $\varrho_0^+, \dots, \varrho_{p-1}^+$ are defined by

$$\varrho_j^+ = \left(e^{-(1+\nu_{j+1})\epsilon}, e^{-(2+\nu_{j+1})\epsilon}, \dots, e^{-(m_{j+1}-n)\epsilon}\right),\ 0 \le j \le p-1.$$

Assume that each $\lambda^{(k)}$ is mapped to $\left(\lambda_1^{(k)} - 1, \lambda_2^{(k)} - 2, \dots\right)$, and that $\left(\lambda^{(1)}, \dots, \lambda^{(p)}\right)$ is mapped to the point configurations

$$\left\{\left(1, \lambda_i^{(1)} - i\right)\right\}_{i=1}^n \cup \dots \cup \left\{\left(p, \lambda_i^{(p)} - i\right)\right\}_{i=1}^n. \tag{5.1}$$

With the specializations $\varrho_0^+, \dots, \varrho_{p-1}^+, \varrho_1^-, \dots, \varrho_p^-$ the Schur process can be understood as that living on the point configurations (5.1).

Set

$$x_j^k = e^{-\epsilon\lambda_j^{(k)}},\ k = 1, \dots, p;\ j = 1, \dots, n. \tag{5.2}$$

The above Schur process induces a point process on $\{1, \dots, n\} \times \mathbb{R}_{>0}$, and this process is formed by the configurations

$$\left\{\left(k, x_j^k\right) \middle| k = 1, \dots, p;\ j = 1, \dots, n\right\}. \tag{5.3}$$

Proposition 5.1. As $\epsilon \to 0$, the point process formed by configurations (5.3) converges to the product matrix process associated with truncated unitary matrices, and defined in Section 2.1. The initial conditions of this process are defined by the matrix T_1 (which is the truncation of U_1).

Proof. Since specializations $\varrho_1^-, \dots, \varrho_{p-1}^-$ are trivial, the Schur process turns into the probability measure

$$\begin{aligned}&\text{Prob}\left\{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(p-1)}, \lambda^{(p)}\right\}\\ &= \frac{1}{Z} s_{\lambda^{(1)}}\left(\varrho_0^+\right) s_{\lambda^{(2)}/\lambda^{(1)}}\left(\varrho_1^+\right) s_{\lambda^{(3)}/\lambda^{(2)}}\left(\varrho_2^+\right) \dots s_{\lambda^{(p-1)}/\lambda^{(p-2)}}\left(\varrho_{p-2}^+\right) s_{\lambda^{(p)}/\lambda^{(p-1)}}\left(\varrho_{p-1}^+\right) s_{\lambda^{(p)}}\left(\varrho_p^-\right),\end{aligned} \tag{5.4}$$

where

$$\frac{1}{Z} = \frac{1}{H\left(\varrho_0^+; \varrho_p^-\right) H\left(\varrho_1^+; \varrho_p^-\right) \dots H\left(\varrho_{p-1}^+; \varrho_p^-\right)}.$$

Now we use the asymptotic formulae for the Schur functions obtained in Section 4. In particular, Proposition 4.1 gives

$$\begin{aligned} &s_{\lambda^{(1)}}\left(e^{-(1+\nu_1)\epsilon}, e^{-(2+\nu_1)\epsilon}, \dots, e^{-(m_1-n)\epsilon}\right) \\ &\simeq \frac{\epsilon^{-(m_1-n-\nu_1)n+\frac{n(n+1)}{2}}}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)} \prod_{i=1}^{n}\left(x_i^1\right)^{1+\nu_1}\left(1-x_i^1\right)^{m_1-2n-\nu_1} \prod_{1\le i<j\le n}\left(x_j^1-x_i^1\right), \end{aligned} \tag{5.5}$$

as $\epsilon \to 0+$. In addition, Proposition 4.3 implies that as $\epsilon \to 0+$,

$$\begin{aligned} &s_{\lambda^{(j)}/\lambda^{(j-1)}}\left(e^{-(1+\nu_j)\epsilon}, e^{-(2+\nu_j)\epsilon}, \dots, e^{-(m_j-n)\epsilon}\right) \\ &\simeq \frac{\epsilon^{-(m_j-n-\nu_j)n+n}}{\left[\Gamma\left(m_j-n-\nu_j\right)\right]^n} \prod_{i=1}^{n} \frac{\left(x_i^j\right)^{1+\nu_j}}{\left(x_i^{j-1}\right)^{m_j-n}} \det\left[\left(x_k^{j-1}-x_l^j\right)_+^{m_j-n-\nu_j-1}\right]_{k,l=1}^{n}, \end{aligned} \tag{5.6}$$

where $2 \le j \le p$. Besides, Proposition 4.1 also gives

$$s_{\lambda^{(p)}}\left(1, e^{-\epsilon}, \dots, e^{-(n-1)\epsilon}\right) \simeq \frac{1}{\epsilon^{\frac{n(n-1)}{2}} \prod_{j=1}^{n}\Gamma(j)} \prod_{1\le i<j\le n}\left(x_j^p-x_i^p\right). \tag{5.7}$$

Let us find the asymptotics of the normalization constant (defined by equation (5.4)). This is not strictly necessary as the uniform convergence of the configuration weights implies the convergence of the normalization constants. We perform this limit transition for the sake of completeness. We have

$$\frac{1}{H\left(\varrho_{j-1}^+;\varrho_p^-\right)} = \prod_{i=0}^{n-1}\prod_{l=1}^{m_j-n-\nu_j}\left(1-e^{-(i+l+\nu_j)\epsilon}\right) \simeq \epsilon^{(m_j-n-\nu_j)n} \prod_{i=0}^{n-1}\prod_{l=1}^{m_j-n-\nu_j}\left(i+l+\nu_j\right),$$

where $1 \le j \le p$. Therefore,

$$\begin{aligned} \frac{1}{Z} &\simeq \epsilon^{\sum_{j=1}^{p}(m_j-n-\nu_j)n} \prod_{i=0}^{n-1}\prod_{j=1}^{p}\prod_{l=1}^{m_j-n-\nu_j}\left(i+l+\nu_j\right) \\ &= \epsilon^{\sum_{j=1}^{p}(m_j-n-\nu_j)n} \prod_{j=1}^{p}\prod_{l=1}^{m_j-n-\nu_j}\frac{\Gamma\left(l+\nu_j+n\right)}{\Gamma\left(l+\nu_j\right)}. \end{aligned} \tag{5.8}$$

Taking into account that

$$\mathrm{d}\lambda_i^{(j)} \sim -\frac{\mathrm{d}x_i^j}{\epsilon x_i^j}, \quad 1 \le j \le p, \quad 1 \le i \le n,$$

we find that the probability measure on $\lambda^{(1)}$, $\lambda^{(2)}$,..., $\lambda^{(p)}$ turns into the probability measure

$$\begin{aligned}
&\left(\prod_{j=1}^{p}\prod_{l=1}^{m_j-n-\nu_j}\frac{\Gamma\left(l+\nu_j+n\right)}{\Gamma\left(l+\nu_j\right)}\right)\frac{1}{\prod_{j=1}^{n}\Gamma(j)\Gamma\left(m_1-2n-\nu_1+j\right)}\frac{1}{\prod_{j=2}^{p}\left[\Gamma\left(m_j-n-\nu_j\right)\right]^n}\\
&\times\triangle\left(x^p\right)\prod_{j=2}^{p}\det\left[\left(x_l^j\right)^{\nu_j}\left(x_k^{j-1}-x_l^j\right)^{m_j-n-\nu_j-1}x_k^{n-m_j}\right]_{k,l=1}^{n}\\
&\times\det\left[\left(x_l^1\right)^{\nu_1+k-1}\left(1-x_l^1\right)^{m_1-2n-\nu_1}\right]_{l,k=1}^{n}\mathrm{d}x^1\ldots\mathrm{d}x^n.
\end{aligned} \tag{5.9}$$

This probability measure can be interpreted as the product matrix process associated with truncated unitary matrices, see Proposition 2.1. The initial conditions of this process are defined by the single truncated unitary matrix T_1, which corresponds to $l=1$ in the definition of this product matrix process in Section 2.1. ■

Proposition 9. Consider the Schur measure

$$\frac{1}{Z}s_{\lambda^{(p)}}\left(\varrho_0^+,\varrho_1^+,\ldots,\varrho_{p-1}^+\right)s_{\lambda^{(p)}}\left(\varrho_p^-\right), \tag{5.10}$$

which is the projection (for a discussion of projections of Schur processes see Borodin [11, Section 2]) of the Schur process (equation (5.4)) to the Young diagram $\lambda^{(p)}$. If

$$x_k^p = e^{-\epsilon\lambda_k^{(p)}}, \quad 1 \le k \le n,$$

then as $\epsilon \to 0+$ the probability measure defined by equation (5.10) converges to

$$\begin{aligned}
&\frac{\prod_{k=1}^{p}\prod_{j_k=1}^{m_k-n-\nu_k}\left(j_k+\nu_k\right)_n}{\prod_{j=1}^{n}\Gamma\left(m_1-2n-\nu_1+j\right)\Gamma(j)\prod_{k=2}^{p}\left(\Gamma\left(m_k-n-\nu_k\right)\right)^n}\\
&\times\triangle\left(x^p\right)\det\left[w_k^{(p)}\left(x_j\right)\right]_{k,j=1}^{n}\mathrm{d}x_1^p\ldots\mathrm{d}x_n^p.
\end{aligned} \tag{5.11}$$

The probability measure defined by expression (5.11) can be understood as the ensemble of squared singular values of the product matrix $T_p \dots T_1$ (where T_1, …, T_p are the truncated unitary matrices defined in Section 2.1).

Proof. We already know that the Schur process defined by equation (5.4) converges to the product matrix process defined by the probability distribution (2.4). The Schur measure defined by expression (5.10) is the projection of this Schur process to the Young diagram $\lambda^{(p)}$, and the probability measure defined by expression (5.11) can be understood as the projection of the product matrix process defined by the probability distribution (2.4) to $x^p = \left(x_1^p, \dots, x_n^p\right)$. The result follows. ■

Proof of Proposition 2.6. Proposition 2.6 is a generalization of Proposition 5.1 to the situation where the initial conditions are defined by a product of l truncated matrices (and not by a single truncated matrix). For the specializations specified in the statement of Proposition 2.6 the Schur process turns into the probability measure

$$\begin{aligned}
&\frac{1}{Z} s_{\lambda^{(p)}}\left(1, e^{-\epsilon}, \dots, e^{-(n-1)\epsilon}\right) s_{\lambda^{(p)}/\lambda^{(p-1)}}\left(e^{-(1+\nu_{l+p-1})\epsilon}, e^{-(2+\nu_{l+p-1})\epsilon}, \dots, e^{-(m_{l+p-1}-n)\epsilon}\right) \\
&s_{\lambda^{(p-1)}/\lambda^{(p-2)}}\left(e^{-(1+\nu_{l+p-2})\epsilon}, e^{-(2+\nu_{l+p-2})\epsilon}, \dots, e^{-(m_{l+p-2}-n)\epsilon}\right) \\
&\vdots \\
&s_{\lambda^{(2)}/\lambda^{(1)}}\left(e^{-(1+\nu_{l+1})\epsilon}, e^{-(2+\nu_{l+1})\epsilon}, \dots, e^{-(m_{l+1}-n)\epsilon}\right) \\
&s_{\lambda^{(1)}}\left(e^{-(1+\nu_1)\epsilon}, e^{-(2+\nu_1)\epsilon}, \dots, e^{-(m_1-n)\epsilon}; \dots; e^{-(1+\nu_l)\epsilon}, e^{-(2+\nu_l)\epsilon}, \dots, e^{-(m_l-n)\epsilon}\right),
\end{aligned} \tag{5.12}$$

where Z is a normalization constant. The rest of the proof of Proposition 2.6 follows the same line as that of Proposition 5.1. The only essential difference is that we use equation (4.9) instead of equation (5.4) for the asymptotics of the relevant Schur function

$$s_\lambda\left(e^{-(1+\nu_1)\epsilon}, \dots, e^{-(m_1-n)\epsilon}; \dots; e^{-(1+\nu_l)\epsilon}, \dots, e^{-(m_l-n)\epsilon}\right)$$

as $\epsilon \to 0+$. ■

6 Convergence of the Correlation Functions and the Proof of Proposition 2.4

Consider the Schur process defined by probability measure (2.10). Assume that the specializations ϱ_0^+, …, ϱ_{p-1}^+ of the Schur process are defined by equations (2.14–2.16), and that the specialization ϱ_p^- is defined by equation (2.17). Let us agree that all other

$\varrho_1^-, \dots, \varrho_{p-1}^-$ are trivial. With these specializations the Schur process can be understood as a point process on $\{1, \dots, p\} \times \mathbb{Z}$. Denote by $\varrho^{\epsilon,\mathrm{Schur}}_{k_1,\dots,k_m}\left(u_1^1, \dots, u_{k_1}^1; \dots; u_1^m, \dots, u_{k_m}^m\right)$ the correlation functions of this Schur process, and by $K^{\epsilon}_{\mathrm{Schur}}(r,u;s,v)$ the correlation kernel of this Schur process.

Proposition 2.6 says that the Schur process under consideration converges to the product matrix process with truncated unitary matrices. This implies convergence of the correlation functions. Namely, if $\varrho_{k_1,\dots,k_m}\left(x_1^1, \dots, x_{k_1}^1; \dots; x_1^m, \dots, x_{k_m}^m\right)$ denotes the correlation function of the product matrix process with truncated unitary matrices defined by probability measure (2.4), then we must have

$$\lim_{\epsilon\to 0+}\left\{\frac{1}{\prod_{i=1}^{k_1}\epsilon x_i^1 \dots \prod_{i=1}^{k_m}\epsilon x_i^m}\varrho^{\epsilon,\mathrm{Schur}}_{k_1,\dots,k_m}\left(-\frac{1}{\epsilon}\log x_1^1, \dots, -\frac{1}{\epsilon}\log x_{k_1}^1; \dots; -\frac{1}{\epsilon}\log x_1^m, \dots, -\frac{1}{\epsilon}\log x_{k_m}^m\right)\right\}$$
$$= \varrho_{k_1,\dots,k_m}\left(x_1^1, \dots, x_{k_1}^1; \dots; x_1^m, \dots, x_{k_m}^m\right),$$

where the denominator came from the coordinate change. This limiting relation between the correlation functions would naturally follow from the limiting relation between the correlation kernels,

$$\lim_{\epsilon\to 0+}\left\{\frac{1}{\epsilon y}\hat{K}^{\epsilon}_{\mathrm{Schur}}\left(r, -\frac{1}{\epsilon}\log x; s, -\frac{1}{\epsilon}\log y\right)\right\} = K_{n,p,l}(r,x;s,y), \tag{6.1}$$

where $K_{n,p,l}(r,x;s,y)$ denotes the correlation kernel of the product matrix process with truncated unitary matrices, and where $\hat{K}^{\epsilon}_{\mathrm{Schur}}(r,u;s,v)$ stands for a kernel equivalent to $K^{\epsilon}_{\mathrm{Schur}}(r,u;s,v)$. Recall that two kernels of a determinantal process are called equivalent if they define the same correlation functions. In what follows we will choose $\hat{K}^{\epsilon}_{\mathrm{Schur}}(r,u;s,v)$ in such a way that the limit in the right-hand side of equation (6.1) will exist.

The Okounkov–Reshetikhin formula for the correlation kernel $K^{\epsilon}_{\mathrm{Schur}}(r,u;s,v)$ is

$$K^{\epsilon}_{\mathrm{Schur}}(r,u;s,v) = \frac{1}{(2\pi i)^2}\oint_{\Sigma_z}\oint_{\Sigma_w}\frac{H\left(\varrho^-_{[r,p]};z\right)H\left(\varrho^+_{[0,s)};w\right)}{(zw-1)H\left(\varrho^+_{[0,r)};z^{-1}\right)H\left(\varrho^-_{[s,p]};w^{-1}\right)}\frac{\mathrm{d}z\mathrm{d}w}{z^{u+1}w^{v+1}}, \tag{6.2}$$

see [14, Theorem 2.2]. The choice of the integration contours Σ_z and Σ_w depends on the time variables r and s, and will be specified below. In the formula for the correlation

kernel $K^{\epsilon}_{\text{Schur}}(r,u;s,v)$ above we have used the notation

$$\varrho^{\pm}_{[i,j]} = \varrho^{\pm}_{i} \cup \varrho^{\pm}_{i+1} \cup \ldots \cup \varrho^{\pm}_{j}.$$

The specializations $\varrho^{+}_{0}, \ldots, \varrho^{+}_{p-1}, \varrho^{-}_{1}, \ldots, \varrho^{-}_{p}$ of the algebra of symmetric functions are specified by equations (2.14–2.17).

For simplicity, let us consider the case corresponding to $l = 1$. The proof of Proposition 2.4 for a general l is essentially the same.

Using equations (2.9) and (2.13) we find

$$\frac{1}{H\left(\varrho^{-}_{[s,p]}; w^{-1}\right)} = \prod_{a_0=1}^{n}\left(1 - e^{-(a_0-1)\epsilon} w^{-1}\right), \; H\left(\varrho^{-}_{[r,p]}; z\right) = \frac{1}{\prod\limits_{a_0=1}^{n}\left(1 - e^{-(a_0-1)\epsilon} z\right)},$$

$$H\left(\varrho^{+}_{[0,s)}; w\right) = \frac{1}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(1 - e^{-b_1\epsilon} w\right) \ldots \prod_{b_s=1+\nu_s}^{m_s-n}\left(1 - e^{-b_s\epsilon} w\right)},$$

and

$$\frac{1}{H\left(\varrho^{+}_{[0,r)}; w\right)} = \prod_{a_1=1+\nu_1}^{m_1-n}\left(1 - e^{-a_1\epsilon} z^{-1}\right) \ldots \prod_{a_r=1+\nu_r}^{m_r-n}\left(1 - e^{-a_r\epsilon} z^{-1}\right).$$

Therefore, we can write

$$\begin{aligned} K^{\epsilon}_{\text{Schur}}(r,u;s,v) &= \frac{1}{(2\pi i)^2} \oint_{\Sigma_z} \oint_{\Sigma_w} \frac{\mathrm{d}z\mathrm{d}w}{(zw-1) z^{u+1} w^{v+1}} \prod_{a_0=1}^{n} \frac{1 - e^{-(a_0-1)\epsilon} w^{-1}}{1 - e^{-(a_0-1)\epsilon} z} \\ &\times \frac{\prod_{a_1=1+\nu_1}^{m_1-n}\left(1 - e^{-a_1\epsilon} z^{-1}\right) \ldots \prod_{a_r=1+\nu_r}^{m_r-n}\left(1 - e^{-a_r\epsilon} z^{-1}\right)}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(1 - e^{-b_1\epsilon} w\right) \ldots \prod_{b_s=1+\nu_s}^{m_s-n}\left(1 - e^{-b_s\epsilon} w\right)}. \end{aligned} \tag{6.3}$$

Assume that $r \geq s$. According to [14, Theorem 2.2], we can choose Σ_w as a counterclockwise circle contour with its center at 0, whose radius is larger than 1. Moreover, Σ_w should be chosen in such a way that all the points $e^{(1+\nu_1)\epsilon}, \ldots, e^{(m_1-n)\epsilon}; \ldots; e^{(1+\nu_s)\epsilon}, \ldots, e^{(m_s-n)\epsilon}$ of the complex w-plane will be situated outside of the circle Σ_w on its right. So we define the contour Σ_w by

$$\Sigma_w = \left\{ w : w = e^{\epsilon\beta + i\varphi}, \varphi \in [-\pi, \pi) \right\},$$

where $0 < \beta < \min\left\{(1+\nu_1), \ldots, (m_1 - n); \ldots; (1+\nu_s), \ldots, (m_s - n)\right\}$.

In addition, we can choose Σ_z as a counterclockwise circle with its center at 0, such that $|z||w| > 1$ for all $z \in \Sigma_z$, for all $w \in \Sigma_w$, and such that the points $1, e^{\epsilon}, \ldots, e^{(n-1)\epsilon}$ of the complex z-plane are situated outside of Σ_z. So we define Σ_z by

$$\Sigma_z = \left\{ z : z = e^{-\alpha\epsilon + i\varphi}, \varphi \in [-\pi, \pi) \right\},$$

where $0 < \alpha < \beta$.

Let us consider the integral over w in (6.3). Since $v \geq -n$, the residue at infinity of the integrand is equal to 0. Moreover, since $\beta > \alpha > 0$, the singular point $w = \frac{1}{z}$ is situated inside the contour Σ_w, for every $z \in \Sigma_z$. This enables us (without changing the integral in the right-hand side of equation (6.3)) to transform Σ_w through the extended w-complex plane into an integration contour Σ'_w that encircles all the points $e^{(1+\nu_1)\epsilon}, \ldots, e^{(m_1-n)\epsilon}; \ldots; e^{(1+\nu_s)\epsilon}, \ldots, e^{(m_s-n)\epsilon}$ once in the clockwise direction, and leaves $w = e^{\epsilon\beta}$ on its left. The contour Σ'_w can be viewed as an image of a contour C_ζ in the complex ζ-plane under the transformation $\zeta \mapsto w = e^{-\epsilon\zeta}$. The contour Σ'_w can be chosen in such a way that C_ζ will be a clockwise oriented closed contour encircling all the points $-(1+\nu_1), \ldots, -(m_1-n); \ldots; -(1+\nu_s), \ldots, -(m_s-n)$ of the negative real axis, and leaving $\zeta = 0$ on its right.

Now let us consider the integral over z in formula (6.3). We note that $u \geq -n$, so the residue of the integrand at infinity is zero. Moreover, since $\beta > \alpha$, and since Σ'_w leaves $e^{\epsilon\beta}$ on its left, the singular point $z = \frac{1}{w}$ is situated inside Σ_z, for every $w \in \Sigma'_w$. Thus, we can deform Σ_z through the extended complex plane into a new contour Σ'_z encircling the points $1, e^{\epsilon}, \ldots, e^{\epsilon(n-1)}$ of the complex z-plane once in the clockwise direction, and this deformation will not affect the integral. The contour Σ'_z can be obtained from a clockwise oriented contour C_t by the transformation $t \mapsto z = e^{\epsilon t}$. Clearly, Σ'_z can be chosen in such a way that C_t will encircle the interval $[0, n-1]$, and will not intersect C_ζ.

We make the change of integration variables,

$$z = e^{t\epsilon}, \quad w = e^{-\zeta\epsilon},$$

and set

$$u = -\frac{1}{\epsilon}\log x, \quad v = -\frac{1}{\epsilon}\log y.$$

This gives

$$K^{\epsilon}_{\mathrm{Schur}}\left(r,-\frac{1}{\epsilon}\log x;s,-\frac{1}{\epsilon}\log y\right)=\frac{\epsilon^2}{(2\pi i)^2}\oint\limits_{C_t}\mathrm{d}t\oint\limits_{C_\zeta}\mathrm{d}\zeta\,\frac{x^t}{y^\zeta}\,\frac{1}{1-e^{-\epsilon(\zeta-t)}}\prod_{a_0=1}^{n}\frac{1-e^{\epsilon(\zeta-a_0+1)}}{1-e^{\epsilon(t-a_0+1)}}$$
$$\times\frac{\prod_{a_1=1+\nu_1}^{m_1-n}\left(1-e^{-\epsilon(t+a_1)}\right)\dots\prod_{a_r=1+\nu_r}^{m_r-n}\left(1-e^{-\epsilon(t+a_r)}\right)}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(1-e^{-\epsilon(\zeta+b_1)}\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(1-e^{-\epsilon(\zeta+b_s)}\right)},\tag{6.4}$$

where the integration contours are chosen as in the statement of Proposition 2.

Set

$$g\left(t,\zeta;\epsilon\right)=\frac{\epsilon}{1-e^{-\epsilon(\zeta-t)}}\prod_{a_0=1}^{n}\frac{\frac{1-e^{\epsilon(\zeta-a_0+1)}}{\epsilon}}{\frac{1-e^{\epsilon(t-a_0+1)}}{\epsilon}}\frac{\prod_{a_1=1+\nu_1}^{m_1-n}\left(\frac{1-e^{-\epsilon(t+a_1)}}{\epsilon}\right)\dots\prod_{a_r=1+\nu_1}^{m_r-n}\left(\frac{1-e^{-\epsilon(t+a_r)}}{\epsilon}\right)}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(\frac{1-e^{-\epsilon(\zeta+b_1)}}{\epsilon}\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(\frac{1-e^{-\epsilon(\zeta+b_s)}}{\epsilon}\right)}.\tag{6.5}$$

As $\epsilon\to 0+$, the function $g\left(t,\zeta;\epsilon\right)$ converges uniformly (with respect to ζ on C_ζ, and with respect to t on C_t) to

$$\frac{1}{\zeta-t}\prod_{a_0=1}^{n}\frac{\zeta-a_0+1}{t-a_0+1}\frac{\prod_{a_1=1+\nu_1}^{m_1-n}\left(t+a_1\right)\dots\prod_{a_r=1+\nu_r}^{m_r-n}\left(t+a_r\right)}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(\zeta+b_1\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(\zeta+b_s\right)},\tag{6.6}$$

as $\epsilon\to 0+$.

Now we define

$$\hat{K}^{\epsilon}_{\mathrm{Schur}}\left(r,u;s,v\right)=\frac{\prod_{k=1}^{s}\epsilon^{m_k-n-\nu_k}}{\prod_{k=1}^{r}\epsilon^{m_k-n-\nu_k}}K^{\epsilon}_{\mathrm{Schur}}\left(r,u;s,v\right).$$

Clearly, the kernels $\hat{K}^{\epsilon}_{\mathrm{Schur}}\left(r,u;s,v\right)$ and $K^{\epsilon}_{\mathrm{Schur}}\left(r,u;s,v\right)$ are equivalent. Let us consider the limit

$$\lim_{\epsilon\to 0+}\left[\frac{1}{\epsilon y}\hat{K}^{\epsilon}_{\mathrm{Schur}}\left(r,-\frac{1}{\epsilon}\log x;s,-\frac{1}{\epsilon}\log y\right)\right].\tag{6.7}$$

The fact that $g(t,\zeta;\epsilon)$ converges uniformly (with respect to ζ on C_ζ, and with respect to t on C_t) to expression (6.6) enables us to interchange the limit and the integrals, and to

compute (6.7) explicitly. By formula (6.1) this limit is equal to $K_{n,p,l=1}(r,x;s,y)$. Thus, we have

$$K_{n,p,l=1}(r,x;s,y) = \frac{1}{(2\pi i)^2}\oint_{C_t}\mathrm{d}t\oint_{C_\zeta}\mathrm{d}\zeta\,\frac{x^t}{y^{\zeta+1}(\zeta-t)}\prod_{a_0=1}^{n}\frac{\zeta-a_0+1}{t-a_0+1} \times \frac{\prod_{a_1=1+\nu_1}^{m_1-n}(a_1+t)\prod_{a_2=1+\nu_2}^{m_2-n}(a_2+t)\dots\prod_{a_r=1+\nu_r}^{m_r-n}(a_r+t)}{\prod_{b_1=1+\nu_1}^{m_1-n}(b_1+\zeta)\prod_{b_2=1+\nu_2}^{m_2-n}(b_2+\zeta)\dots\prod_{b_s=1+\nu_s}^{m_s-n}(b_s+\zeta)}, \tag{6.8}$$

where $r \geq s$. We rewrite the products inside the integrals above in terms of Gamma functions as follows:

$$\prod_{a_0=1}^{n}\frac{\zeta-a_0+1}{t-a_0+1} = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+1-n)}\frac{\Gamma(t+1-n)}{\Gamma(t+1)}, \tag{6.9}$$

$$\prod_{a_1=1+\nu_1}^{m_1-n}(a_1+t) = \frac{\Gamma\left(t+m_1-n+1\right)}{\Gamma\left(t+\nu_1+1\right)},\dots,\prod_{a_r=1+\nu_r}^{m_r-n}(a_r+t) = \frac{\Gamma\left(t+m_r-n+1\right)}{\Gamma\left(t+\nu_r+1\right)}, \tag{6.10}$$

and

$$\prod_{b_1=1+\nu_1}^{m_1-n}(b_1+\zeta) = \frac{\Gamma\left(\zeta+m_1-n+1\right)}{\Gamma\left(\zeta+\nu_1+1\right)},\dots,\prod_{b_s=1+\nu_s}^{m_s-n}(b_s+\zeta) = \frac{\Gamma\left(\zeta+m_s-n+1\right)}{\Gamma\left(\zeta+\nu_s+1\right)}. \tag{6.11}$$

Taking this into account we see that the right-hand side of equation (6.8) can be rewritten as that of equation (2.8). This proves Proposition 2.4 for $r \geq s$ (and $l = 1$).

Assume that $r < s$. In this case we can choose both Σ_z and Σ_w as counter-clockwise circle contours whose centers are at 0, and whose radii are less than 1, see [14, Theorem 2.2]. Let us agree that $|z| < |w| < 1$ for all $z \in \Sigma_z$, and $w \in \Sigma_w$. In addition, we will choose Σ_z in such a way that all the points $e^{-(1+\nu_1)\epsilon}, \dots, e^{-(m_1-n)\epsilon}; \dots; e^{-(1+\nu_s)\epsilon}, \dots, e^{-(m_s-n)\epsilon}$ will be situated inside Σ_z.

We will deform the contour Σ_w through the extended complex plane into a new contour Σ_w' encircling all the points $e^{(1+\nu_1)\epsilon}, \dots, e^{(m_1-n)\epsilon}; \dots; e^{(1+\nu_s)\epsilon}, \dots, e^{(m_s-n)\epsilon}$ in the clockwise direction, and leaving the points $w = \frac{1}{z}$, $z \in \Sigma_z$ outside. As we deform Σ_w

into Σ'_w, we should pick up the contribution at the residue at $w = \frac{1}{z}$. Thus, we rewrite equation (3.3) as

$$K^{\epsilon}_{\text{Schur}}(r,u;s,v) = -\frac{1}{2\pi i}\oint_{\Sigma_z} \frac{\mathrm{d}z}{z^{u-v+1}} \frac{1}{\prod_{b_{r+1}=1+\nu_{r+1}}^{m_{r+1}-n}\left(1-e^{-b_{r+1}\epsilon}z^{-1}\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(1-e^{-b_s\epsilon}z^{-1}\right)}$$
$$+\frac{1}{(2\pi i)^2}\oint_{\Sigma_z}\oint_{\Sigma'_w}\frac{\mathrm{d}z\mathrm{d}w}{(zw-1)z^{u+1}w^{v+1}}\prod_{a_0=1}^{n}\frac{1-e^{-(a_0-1)\epsilon}w^{-1}}{1-e^{-(a_0-1)\epsilon}z}$$
$$\times\frac{\prod_{a_1=1+\nu_1}^{m_1-n}\left(1-e^{-a_1\epsilon}z^{-1}\right)\dots\prod_{a_r=1+\nu_r}^{m_r-n}\left(1-e^{-a_r\epsilon}z^{-1}\right)}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(1-e^{-b_1\epsilon}w\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(1-e^{-b_s\epsilon}w\right)}. \tag{6.12}$$

Denote by $K^{\epsilon,I}_{\text{Schur}}(r,u;s,v)$ the 1st term in the right-hand side of the above equation, and by $K^{\epsilon,II}_{\text{Schur}}(r,u;s,v)$ the 2nd term, so that

$$K^{\epsilon,I}_{\text{Schur}}(r,u;s,v) = -\frac{1}{2\pi i}\oint_{\Sigma_z} \frac{\mathrm{d}z}{z^{u-v+1}} \frac{1}{\prod_{b_{r+1}=1+\nu_{r+1}}^{m_{r+1}-n}\left(1-e^{-b_{r+1}\epsilon}z^{-1}\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(1-e^{-b_s\epsilon}z^{-1}\right)}, \tag{6.13}$$

and

$$K^{\epsilon,II}_{\text{Schur}}(r,u;s,v) = \frac{1}{(2\pi i)^2}\oint_{\Sigma_z}\oint_{\Sigma'_w}\frac{\mathrm{d}z\mathrm{d}w}{(zw-1)z^{u+1}w^{v+1}}\prod_{a_0=1}^{n}\frac{1-e^{-(a_0-1)\epsilon}w^{-1}}{1-e^{-(a_0-1)\epsilon}z}$$
$$\times\frac{\prod_{a_1=1+\nu_1}^{m_1-n}\left(1-e^{-a_1\epsilon}z^{-1}\right)\dots\prod_{a_r=1+\nu_r}^{m_r-n}\left(1-e^{-a_r\epsilon}z^{-1}\right)}{\prod_{b_1=1+\nu_1}^{m_1-n}\left(1-e^{-b_1\epsilon}w\right)\dots\prod_{b_s=1+\nu_s}^{m_s-n}\left(1-e^{-b_s\epsilon}w\right)}. \tag{6.14}$$

In the formula for $K^{\epsilon,II}_{\text{Schur}}(r,u;s,v)$ we deform the contour Σ_z through the extended complex plane into a new contour Σ'_z encircling the points 1, e^{ϵ}, …, $e^{(n-1)\epsilon}$ in the clockwise direction. Since we agree that all the points $e^{-(1+\nu_1)\epsilon}$, …, $e^{-(m_1-n)\epsilon}$ …; $e^{-(1+\nu_s)\epsilon}$, …, $e^{-(m_s-n)\epsilon}$ are situated inside Σ_z, the contour Σ'_w can be chosen such that this deformation will not affect the value of $K^{\epsilon,II}_{\text{Schur}}(r,u;s,v)$. The contours Σ'_w and Σ'_z can be viewed as images of contours C_ζ, C_t under the transformations $\zeta \mapsto w = e^{-\epsilon\zeta}$, and $t \to z = e^{\epsilon t}$, and the contours C_ζ, C_t will be those described in the statement of the Proposition.

Set

$$\hat{K}^{\epsilon,I}_{\mathrm{Schur}}(r,u;s,v)=\frac{\prod_{k=1}^{s}\epsilon^{m_k-n-\nu_k}}{\prod_{k=1}^{r}\epsilon^{m_k-n-\nu_k}}K^{\epsilon,I}_{\mathrm{Schur}}(r,u;s,v)\,,$$

$$\hat{K}^{\epsilon,II}_{\mathrm{Schur}}(r,u;s,v)=\frac{\prod_{k=1}^{s}\epsilon^{m_k-n-\nu_k}}{\prod_{k=1}^{r}\epsilon^{m_k-n-\nu_k}}K^{\epsilon,II}_{\mathrm{Schur}}(r,u;s,v)\,,$$

and

$$\hat{K}^{\epsilon}_{\mathrm{Schur}}(r,u;s,v)=\hat{K}^{\epsilon,I}_{\mathrm{Schur}}(r,u;s,v)+\hat{K}^{\epsilon,II}_{\mathrm{Schur}}(r,u;s,v)\,.$$

After the change of variables we find

$$\widehat{K}^{\epsilon,II}_{\mathrm{Schur}}\left(r,-\frac{1}{\epsilon}\log x;s,-\frac{1}{\epsilon}\log y\right)=\frac{\epsilon}{(2\pi i)^2}\oint_{C_t}\mathrm{d}t\oint_{C_\zeta}\mathrm{d}\zeta\,\frac{x^t}{y^\zeta}g(t,\zeta;\epsilon),\tag{6.15}$$

where $g(t,\zeta;\epsilon)$ is defined by equation (6.5). We thus have

$$\lim_{\epsilon\to 0+}\left[\frac{1}{\epsilon y}\widehat{K}^{\epsilon,II}_{\mathrm{Schur}}\left(r,-\frac{1}{\epsilon}\log x;s,-\frac{1}{\epsilon}\log y\right)\right]=\frac{1}{(2\pi i)^2}\oint_{C_t}\mathrm{d}t\oint_{C_\zeta}\mathrm{d}\zeta\,\frac{x^t}{y^{\zeta+1}(\zeta-t)}\prod_{a_0=1}^{n}\frac{\zeta-a_0+1}{t-a_0+1}$$
$$\times\frac{\prod_{a_1=1+\nu_1}^{m_1-n}(a_1+t)\prod_{a_2=1+\nu_2}^{m_2-n}(a_2+t)\ldots\prod_{a_r=1+\nu_r}^{m_r-n}(a_r+t)}{\prod_{b_1=1+\nu_1}^{m_1-n}(b_1+\zeta)\prod_{b_2=1+\nu_2}^{m_2-n}(b_2+\zeta)\ldots\prod_{b_s=1+\nu_s}^{m_s-n}(b_s+\zeta)},\tag{6.16}$$

where again we have used the uniform convergence of $g(t,\zeta;\epsilon)$ to take the limit inside the integrals. Note that (as in the case $r\geq s$) the right-hand side of equation (6.16) can be written as the 2nd term in the right-hand side of equation (2.8).

Now consider the formula for $\hat{K}^{\epsilon,I}_{\mathrm{Schur}}(r,u;s,v)$ given by (6.13). Assume that $u>v$. In this case the residue at infinity is equal to zero, and all finite poles are situated inside the contour Σ_z. This implies that $\hat{K}^{\epsilon,I}_{\mathrm{Schur}}(r,u;s,v)$ is equal to 0 for $u>v$. If $u\leq v$, then the residue at 0 is equal to 0, and we can deform Σ_z into a contour Σ'_z encircling the poles $e^{-(1+\nu_{r+1})\epsilon},\ldots,e^{-(m_{r+1}-n)\epsilon};\ldots;e^{-(1+\nu_s)\epsilon},\ldots,e^{-(m_s-n)\epsilon}$ once in the counterclockwise direction, and leaving 0 on the left. The contour Σ'_z can be viewed as an image of a

contour C_t under the transformation $t \mapsto z = e^{\epsilon t}$, where C_t will be that described in the statement of the Proposition. After the change of variables we find that

$$\hat{K}^{\epsilon,I}_{\text{Schur}}(r,u;s,v) = -\frac{\epsilon}{2\pi i}\oint_{C_t} \mathrm{d}t \left(\frac{y}{x}\right)^{-t} \frac{1}{\prod_{b_{r+1}=1+\nu_{r+1}}^{m_{r+1}-n} \frac{1-e^{\epsilon(t+b_{r+1})}}{\epsilon} \cdots \prod_{b_s=1+\nu_s}^{m_s-n} \frac{1-e^{\epsilon(t+b_s)}}{\epsilon}}, \quad (6.17)$$

where $y < x$. We have

$$\begin{aligned}
&\lim_{\epsilon\to 0+}\left[\frac{1}{\epsilon y}\widehat{K}^{\epsilon,I}_{\text{Schur}}\left(r,-\frac{1}{\epsilon}\log x;s,-\frac{1}{\epsilon}\log y\right)\right] \\
&= -\frac{1}{2\pi i}\oint_{C_t} dt \frac{x^t}{y^{t+1}} \frac{1}{\prod_{b_{r+1}=1+\nu_{r+1}}^{m_{r+1}-n}(b_{r+1}+t)\cdots\prod_{b_s=1+\nu_s}^{m_s-n}(b_s+t)} \\
&= -\frac{1}{2\pi i}\oint_{C_t} dt \frac{x^t}{y^{t+1}} \frac{\Gamma(t+\nu_{r+1}+1)\ldots\Gamma(t+\nu_s+1)}{\Gamma(t+m_{r+1}-n+1)\ldots\Gamma(t+m_s-n+1)} \\
&= -\frac{1}{x}G^{s-r,0}_{s-r,s-r}\left(\begin{matrix} m_{r+1}-n, & \ldots, & m_s-n \\ \nu_{r+1}, & \ldots, & \nu_s \end{matrix}\middle|\frac{y}{x}\right),
\end{aligned} \quad (6.18)$$

where $y < x$. As follows from equations (2.22–2.24) in Kieburg *et al.* [31], the function

$$G^{s-r,0}_{s-r,s-r}\left(\begin{matrix} m_{r+1}-n, & \ldots, & m_s-n \\ \nu_{r+1}, & \ldots, & \nu_s \end{matrix}\middle|\frac{y}{x}\right)$$

is equal to 0 for $y \geq x$. Therefore, equation

$$\begin{aligned}
&\lim_{\epsilon\to 0+}\left[\frac{1}{\epsilon y}\widehat{K}^{\epsilon,I}_{\text{Schur}}\left(r,-\frac{1}{\epsilon}\log x;s,-\frac{1}{\epsilon}\log y\right)\right] \\
&= -\frac{1}{x}G^{s-r,0}_{s-r,s-r}\left(\begin{matrix} m_{r+1}-n, & \ldots, & m_s-n \\ \nu_{r+1}, & \ldots, & \nu_s \end{matrix}\middle|\frac{y}{x}\right)
\end{aligned} \quad (6.19)$$

holds true for $y \geq x$ as well. Finally, formula (6.1), equations (6.16) and (6.18) give the desired formula for the correlation kernel in the case $r < s$.

7 Proof of Proposition 2.8 and Theorem 2.9

Proof of Proposition 2.8. In order to compute the probability of the point configuration (2.23), we need to compute the projection of the Schur process associated with a skew plane partition Π to the diagrams $\lambda^{(\alpha_1)}, \ldots, \lambda^{(\alpha_p)}$. It is convenient to obtain

first the projection on $\lambda^{(\beta_{2l+1})}, \ldots, \lambda^{(\beta_2)}, \lambda^{(\alpha_1)}, \ldots, \lambda^{(\alpha_p)}$, see Figure 3. Using equation (2.12) we find

$$\begin{aligned}
&\frac{1}{Z'} s_{\lambda^{(\beta_{2l+1})}}\left(q^{-1}, \ldots, q^{-(\beta_{2l+1}-1)}\right) s_{\lambda^{(\beta_{2l+1})}/\lambda^{(\beta_{2l})}}\left(q^{\beta_{2l+1}}, \ldots, q^{\beta_{2l}-1}\right) \\
&\times s_{\lambda^{(\beta_{2l-1})}/\lambda^{(\beta_{2l})}}\left(q^{-\beta_{2l}}, \ldots, q^{-(\beta_{2l-1}-1)}\right) s_{\lambda^{(\beta_{2l-1})}/\lambda^{(\beta_{2l-2})}}\left(q^{\beta_{2l-1}}, \ldots, q^{\beta_{2l-2}-1}\right) \\
&\vdots \\
&\times s_{\lambda^{(\beta_3)}/\lambda^{(\beta_4)}}\left(q^{-\beta_4}, \ldots, q^{-(\beta_3-1)}\right) s_{\lambda^{(\beta_3)}/\lambda^{(\beta_2)}}\left(q^{\beta_3}, \ldots, q^{\beta_2-1}\right) \\
&\times s_{\lambda^{(\alpha_1)}/\lambda^{(\beta_2)}}\left(q^{-\beta_2}, \ldots, q^{-(\alpha_1-1)}\right) s_{\lambda^{(\alpha_2)}/\lambda^{(\alpha_1)}}\left(q^{-\alpha_1}, \ldots, q^{-(\alpha_2-1)}\right) \\
&\vdots \\
&\times s_{\lambda^{(\alpha_p)}/\lambda^{(\alpha_{p-1})}}\left(q^{-\alpha_{p-1}}, \ldots, q^{-(\alpha_p-1)}\right) s_{\lambda^{(\beta_1)}/\lambda^{(\alpha_p)}}\left(q^{-\alpha_p}, \ldots, q^{-(\beta_1-1)}\right) \\
&\times s_{\lambda^{(\beta_1)}}\left(q^{\beta_1}, \ldots, q^{A+B}\right).
\end{aligned} \tag{7.1}$$

Equations (2.11) and (2.12) enable us to sum over the Young diagrams $\lambda^{(\beta_{2l+1})}$, $\lambda^{(\beta_{2l})}$, …, $\lambda^{(\beta_2)}$, and $\lambda^{(\beta_1)}$. The result is

$$\begin{aligned}
&\frac{1}{Z''} s_{\lambda^{(\alpha_1)}}\left(q^{-1}, \ldots, q^{-(\beta_{2l+1}-1)}; q^{-\beta_{2l}}, \ldots, q^{-(\beta_{2l-1}-1)}; \ldots; q^{-\beta_2}, \ldots, q^{-(\alpha_1-1)}\right) \\
&\times s_{\lambda^{(\alpha_2)}/\lambda^{(\alpha_1)}}\left(q^{-\alpha_1}, \ldots, q^{-(\alpha_2-1)}\right) \ldots s_{\lambda^{(\alpha_p)}/\lambda^{(\alpha_{p-1})}}\left(q^{-\alpha_{p-1}}, \ldots, q^{-(\alpha_p-1)}\right) \\
&\times s_{\lambda^{(\alpha_p)}}\left(q^{\beta_1}, \ldots, q^{A+B}\right).
\end{aligned} \tag{7.2}$$

Taking into account the homogeneity of the Schur polynomials, and noting that $\beta_1 = A + \pi_1 + 1$, we see that the expression above can be rewritten as in the statement of the Proposition 2.8. ■

Proof of Theorem 2.8. Proposition 2.8 says that the probability of the point configuration

$$\left\{\left(1, \lambda_i^{(\alpha_1)} - i\right)\right\}_{i\geq 1} \cup \ldots \cup \left\{\left(1, \lambda_i^{(\alpha_p)} - i\right)\right\}_{i\geq 1}$$

can be written as a product of skew Schur functions, see equation (2.24). If $q = e^{-\epsilon}$, the parameters n; $m_1, \ldots, m_{l+p-1}$; $\nu_1, \ldots, \nu_{l+p-1}$ are related with parameters A, B; π_1; $\alpha_1, \ldots, \alpha_p$; $\beta_1, \ldots, \beta_{2l+1}$ as in the statement of Theorem 2.9, and $\lambda^{(\alpha_1)}, \ldots, \lambda^{(\alpha_p)}$ are identified with $\lambda^{(1)}, \ldots, \lambda^{(p)}$, then the probability measure defined by equation (5.12) turns into the

probability measure defined by equation (2.24). Application of Proposition 2.6 gives the result. ■

8 Alternative Proof Through Symmetric Functions and Zonal Polynomials

In this section we sketch an alternative path to derive Proposition 2.6. For the clarity of the exposition we only detail one simplest case. At the end of the section we outline ways for generalizations.

Simplest case: product of two 2×2 matrices. Let U_1 and U_2 be two 4×4 independent Haar-distributed random unitary complex matrices. Let T_1 and T_2 be principal 2×2 corners of U_1 and U_2, respectively. Our aim is to link the distribution of the squared singular values of $T_1 T_2$ to a Schur measure.

First, note that T_1 and T_2 are almost surely nondegenerate. The squared singular values of $T_1 T_2$ are eigenvalues of $T_1 T_2 T_2^* T_1^*$. Since eigenvalues are preserved under conjugations, they are the same as the eigenvalues of $(T_1^* T_1)(T_2 T_2^*)$. Since, T_1 and T_1^* have the same distribution, we can further rewrite the law of interest as the distribution of eigenvalues for the matrix $(T_1 T_1^*)(T_2 T_2^*)$.

Set $A = T_1 T_1^*$ and $B = T_2 T_2^*$. The (ordered) eigenvalues of A are real numbers a_1, a_2 distributed with probability density

$$\rho(a_1, a_2) = 12(a_2 - a_1)^2, \quad 0 \le a_1 \le a_2 \le 1. \tag{8.1}$$

This computation is a particular case of the identification of singular values of a corner of a random unitary matrix with Jacobi ensemble, see [18] and references therein. This is also the $k = p = \ell = 1$ case in Proposition 2.1. The eigenvalues of B, $0 \le b_1 \le b_2 \le 1$ have the same distribution.

Next, we fix $0 < q < 1$ and consider a distribution on pairs of integers $\lambda_1 \ge \lambda_2 \ge 0$ with weight

$$P_q(\lambda_1, \lambda_2) = (1-q)(1-q^2)^2(1-q^3)\, s_{(\lambda_1,\lambda_2)}(1, q)\, s_{(\lambda_1,\lambda_2)}(q, q^2). \tag{8.2}$$

On one hand, (8.2) is a particular case of the Schur measure. On the other hand, the explicit evaluations

$$s_{(\lambda_1,\lambda_2)}(1, q) = \frac{\det \begin{pmatrix} 1 & 1 \\ q^{\lambda_1+1} & q^{\lambda_2} \end{pmatrix}}{1-q} = \frac{q^{\lambda_2} - q^{\lambda_1+1}}{1-q}, \qquad s_{(\lambda_1,\lambda_2)}(q, q^2) = q^{\lambda_1+\lambda_2} \frac{q^{\lambda_2} - q^{\lambda_1+1}}{1-q},$$

imply that upon the change of variables $q^{\lambda_i} = a_i$, $i = 1, 2$, in the limit $q \to 1$, (8.2) becomes (8.1). The same computation works for matrices of arbitrary size and for any values of the random matrix parameter β, see [22, Section 3.1] and [13, Theorem 2.8].

The next step is to use the Cauchy–Littlewood identity (see [35, Chapter I])

$$\sum_{(\lambda_1,\lambda_2)} s_{(\lambda_1,\lambda_2)}(u_1,u_2)s_{(\lambda_1,\lambda_2,0^{k-2})}(v_1,v_2,\dots,v_k) = \prod_{i=1}^{2}\prod_{j=1}^{k}\frac{1}{1-u_iv_j}, \quad k \geq 2. \tag{8.3}$$

Equation (8.2) and the identity (8.3) lead to the expectation evaluation

$$\mathbb{E}_{P_q}\left[\frac{s_{(\lambda_1,\lambda_2)}(u_1,u_2)}{s_{(\lambda_1,\lambda_2)}(1,q)}\right] = \prod_{i=1}^{2}\frac{(1-q^i)(1-q^{i+1})}{(1-u_iq)(1-u_iq^2)}. \tag{8.4}$$

Here u_1 and u_2 can be any complex numbers such that the series defining the expectation absolutely converges. We make a particular choice, $(u_1,u_2) = (q^{\mu_1+1}, q^{\mu_2})$ for two integers $\mu_1 \geq \mu_2 \geq 0$. Then, using the label–variable duality (which is an immediate consequence of the definition of the Schur polynomials as ratios of two determinants)

$$\frac{s_{(\lambda_1,\lambda_2)}(q^{\mu_1+1}, q^{\mu_2})}{s_{(\lambda_1,\lambda_2)}(1,q)} = \frac{s_{(\mu_1,\mu_2)}(q^{\lambda_1+1}, q^{\lambda_2})}{s_{(\mu_1,\mu_2)}(1,q)}, \tag{8.5}$$

we get

$$\mathbb{E}_{P_q}\left[\frac{s_{(\mu_1,\mu_2)}(q^{\lambda_1+1}, q^{\lambda_2})}{s_{(\mu_1,\mu_2)}(1,q)}\right] = \prod_{i=1}^{2}\frac{(1-q^i)(1-q^{i+1})}{(1-q^{\mu_i-i+3})(1-q^{\mu_i-i+4})}, \qquad \mu_1 \geq \mu_2 \geq 0. \tag{8.6}$$

Note that as one varies μ_1, μ_2, the left-hand side of (8.6) uniquely determines all the moments of P_q-random vector $(q^{\lambda_1+1}, q^{\lambda_2})$. Indeed, since $\lambda_1 + 1 > \lambda_2$, it is enough to consider only symmetric linear combinations of moments, and those are finite combinations of Schur polynomials. Since $(q^{\lambda_1+1}, q^{\lambda_2})$ is supported inside $[0,1] \times [0,1]$, the moments uniquely determine the distribution P_q. The conclusion is that (8.6) is *equivalent* to the definition (8.2). Further, the same equivalency holds in the limit $q \to 1$, as $(q^{\lambda_1+1}, q^{\lambda_2}) \to (a_1, a_2)$. Equation (8.6) becomes

$$\mathbb{E}_{\rho(a_1,a_2)}\left[\frac{s_{(\mu_1,\mu_2)}(a_1,a_2)}{s_{(\mu_1,\mu_2)}(1,1)}\right] = \prod_{i=1}^{2}\frac{i(i+1)}{(\mu_i-i+3)(\mu_i-i+4)}, \qquad \mu_1 \geq \mu_2 \geq 0. \tag{8.7}$$

We can now describe what is happening with expectations (8.4), (8.6) and (8.7), when we multiply the matrices. The computation relies on the following integral identity:

$$\int_{U(2)} \frac{s_{(\mu_1,\mu_2)}(VUWU^{-1})}{s_{(\mu_1,\mu_2)}(1,1)} \mathrm{d}U = \frac{s_{(\mu_1,\mu_2)}(v_1,v_2)}{s_{(\mu_1,\mu_2)}(1,1)} \cdot \frac{s_{(\mu_1,\mu_2)}(w_1,w_2)}{s_{(\mu_1,\mu_2)}(1,1)}, \quad \mu_1 \geq \mu_2 \geq 0. \tag{8.8}$$

In (8.8), the integration goes over the group $U(2)$ of 2×2 unitary complex matrices, V and W are two fixed complex matrices with eigenvalues (v_1, v_2) and (w_1, w_2), respectively, and by $s_{(\mu_1,\mu_2)}(VUWU^{-1})$ we mean the Schur polynomial $s_{(\mu_1,\mu_2)}$ evaluated on two eigenvalues of the matrix $VUWU^{-1}$. When V and W are unitary, the relation (8.8) is known as *the functional equation* for the characters of $U(2)$. More generally, (8.8) is the identification of *zonal polynomials* of the symmetric space $GL(2;\mathbb{C})/U(2)$ with Schur polynomials, see [35, Chapter VII] and [21, Section 13.4.3]. For real and quaternion matrices, an analog of (8.8) holds with Schur polynomials replaced by the Jack polynomials. Using $N = 2$ in $U(N)$ also plays no special role in (8.1) and the identity extends to all $N > 0$.

Coming back to AB, the matrices $A = T_1 T_1^*$ and $B = T_2 T_2^*$ are independent, and the distribution of each of them is $U(2)$-invariant, with respect to the action by conjugation. In other words, while the eigenvalues have a specific distribution (1), the eigenvectors are chosen uniformly at random (in the set of all possible pairs of orthogonal unit vectors in $\mathbb{C}^2$). Thus, if we plug $V = A$, $W = B$ into (8.8) and take expectation with respect to A and B, we get

$$\mathbb{E}\left[\frac{s_{(\mu_1,\mu_2)}(AB)}{s_{(\mu_1,\mu_2)}(1,1)}\right] = \mathbb{E}\left[\frac{s_{(\mu_1,\mu_2)}(A)}{s_{(\mu_1,\mu_2)}(1,1)}\right] \mathbb{E}\left[\frac{s_{(\mu_1,\mu_2)}(B)}{s_{(\mu_1,\mu_2)}(1,1)}\right], \quad \mu_1 \geq \mu_2 \geq 0.$$

Combining with (8.7), this implies

$$\mathbb{E}\left[\frac{s_{(\mu_1,\mu_2)}(AB)}{s_{(\mu_1,\mu_2)}(1,1)}\right] = \left(\prod_{i=1}^{2} \frac{i(i+1)}{(\mu_i - i + 3)(\mu_i - i + 4)}\right)^2, \qquad \mu_1 \geq \mu_2 \geq 0,$$

which is (again by (8.3) and (8.5)) precisely the $q \to 1$ limit of

$$\mathbb{E}_{\tilde{P}_q}\left[\frac{s_{(\mu_1,\mu_2)}(q^{\nu_1+1}, q^{\nu_2})}{s_{(\mu_1,\mu_2)}(1,q)}\right], \quad \mu_1 \geq \mu_2 \geq 0,$$

for the integral vector $\nu_1 \geq \nu_2 \geq 0$ distributed according to the Schur measure

$$\tilde{P}_q(\nu_1,\nu_2) = (1-q)^2(1-q^2)^4(1-q^3)^2\, s_{(\nu_1,\nu_2)}(1,q)\, s_{(\nu_1,\nu_2)}(q,q^2,q,q^2). \tag{8.9}$$

We conclude that the eigenvalue distribution for AB is the $q \to 1$ limit of (q^{ν_1+1}, q^{ν_2}) distributed according to the Schur measure (8.9).

Generalizations. Let us discuss how to see the structure of the full Schur processes, rather than just its slices given by the Schur measures. Note that the transition between P_q of (8.2) and $\tilde{P}_q$ of (8.9) can be seen as one step of Markov chain on two-row Young diagrams $\lambda = (\lambda_1, \lambda_2)$ with transition probabilities $P(\lambda \to \nu)$ found from the decomposition

$$\frac{s_\lambda(u_1,u_2)}{s_\lambda(1,q)} \cdot \left[\prod_{i=1}^{2} \frac{(1-q^i)(1-q^{i+1})}{(1-u_i q)(1-u_i q^2)}\right] = \sum_\nu P(\lambda \to \nu) \frac{s_\nu(u_1,u_2)}{s_\nu(1,q)}. \tag{8.10}$$

Summing (8.10) using (8.4), we get $\sum_\lambda P_q(\lambda) P(\lambda \to \nu) = \tilde{P}_q(\nu)$. In the $q \to 1$ limit, the same structure of the Markov chain can be seen for the projection of the joint law of A and AB onto their eigenvalues; this is again a corollary of (8.8).

Comparison of (8.10) with the *skew Cauchy identity* (see, e.g., [35, Section I.5, Example 26]) yields that $P(\lambda \to \nu)$ is given by the fomula

$$P(\lambda \to \nu) = (1-q)(1-q^2)^2(1-q^3)\frac{s_\nu(1,q) s_{\nu/\lambda}(q,q^2)}{s_\lambda(1,q)}. \tag{8.11}$$

Combining the definition of P_q with (8.11), we conclude that the two times Markov chain with initial state P_q, transitional probability $P(\lambda \to \nu)$ (and final state $\tilde{P}_q$) is the Schur process. Sending $q \to 1$, we see that that the joint law of squared singular values of A and AB is the continuous limit of this Schur process.

At this moment we can generalize the argument to products of more matrices. Each additional factor gives another time step of the Markov chain. These transition probabilities generalize (8.11) and therefore, the link to Schur processes persists. Let us make a remark about the sizes of the matrices that are being multiplied. The computation leading to (8.1) and its connection to (8.2) can be generalized to rectangular corners of random unitary matrices of arbitrary sizes (and we can also deal with real/squaternion $\beta = 1, 4$ cases). The identity (8.8) has similar extensions. However, when we start iterating (8.8) it is convenient to assume that all the involved matrices have the same square size, as then the present arguments extend word-for-word. This

is less general than the setting of Proposition 2.6. It is plausible that the arguments of this section can be adapted to the changing sizes as well, but we will not pursue this direction. When we pass between a rectangular matrix T and its square counterpart TT^* encapsulating the singular values, the following distinction becomes important: the matrices TT^* and T^*T have the same nontrivial eigenvalues, but additional 0s show up because of different sizes. One needs to translate this into the language of Schur processes and expectations (8.4) and (8.6).

Finally, one can go beyond corners of unitary matrices and consider factors with more complicated unitarily–invariant distributions. We refer to [26] for a progress in this direction.

9 Appendix. Measures given by products of determinants, the Eynard–Mehta theorem, and the 2nd proof of Proposition 2.4

The aim of this Appendix is to give another, more direct proof of Proposition 2.4 based on an application of the Eynard–Mehta theorem [20]. The starting point of this proof is the fact that the density of the product matrix process under considerations is given by a product of determinants, see Proposition 2.1. This enables us to apply the Eynard–Mehta theorem to the product matrix process with truncated unitary matrices. Although, the proof below is similar to the arguments of [14] and [43], we decided to reproduce it in the present setting for pedagogical reasons.

9.1 *The Eynard–Mehta theorem*

Let us first recall the formulation of the Eynard–Mehta theorem. Let $n, p \geq 1$ be two fixed natural numbers, and let $\mathfrak{X}_0$, $\mathfrak{X}_{p+1}$ be two given sets. Let $\mathfrak{X}$ be a complete separable metric space, and consider a probability measure on $(\mathfrak{X}^n)^p$ given by

$$\begin{aligned}\mathrm{Prob}_{n,p}(\underline{x})\mathrm{d}\mu(\underline{x}) = \frac{1}{Z_{n,p}} \det\left(\phi_{0,1}(x_i^0, x_j^1)\right)_{i,j=1}^n \det\left(\phi_{p,p+1}(x_i^p, x_j^{p+1})\right)_{i,j=1}^n \\ \times \prod_{r=1}^{p-1} \det\left(\phi_{r,r+1}(x_i^r, x_j^{r+1})\right)_{i,j=1}^n \mathrm{d}\mu(\underline{x}).\end{aligned} \tag{9.1}$$

In the formula above, $Z_{n,p}$ is the normalization constant, the functions $\phi_{r,r+1} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$, $r = 1, \ldots, p-1$ are given *intermediate one-step transition functions*, $\phi_{0,1} : \mathfrak{X}_0 \times \mathfrak{X} \to \mathbb{C}$ is a given *initial one-step transition function*, and $\phi_{p,p+1} : \mathfrak{X} \times \mathfrak{X}_{p+1} \to \mathbb{C}$ is a given *final one-step transition function*. Also,

$$\underline{x} = \left(x^1, \ldots, x^p\right) \in \left(\mathfrak{X}^n\right)^p; \;\; x^r = \left(x_1^r, \ldots, x_n^r\right), r = 1, \ldots, p,$$

the vectors

$$x^0 = (x_1^0, \dots, x_n^0) \in \mathfrak{X}_0^n, \quad x^{p+1} = (x_1^{p+1}, \dots, x_n^{p+1}) \in \mathfrak{X}_{p+1}^n,$$

are fixed initial and final vectors, and

$$\mathrm{d}\mu(\underline{x}) = \prod_{r=1}^{p} \prod_{j=1}^{n} \mathrm{d}\mu(x_j^r).$$

Here μ is a given Borel measure on $\mathfrak{X}$. Given two transition functions ϕ and ψ set

$$\phi * \psi(x, y) = \int_{\mathfrak{X}} \phi(x, t)\psi(t, y) d\mu(t).$$

The theorem below is the Eynard-Mehta theorem.

Theorem 9.1. The probability measure $\mathrm{Prob}_{n,p}(\underline{x})\mathrm{d}\mu(\underline{x})$ given by equation (9.1) defines a determinantal point process on $\{1, \dots, p\} \times \mathfrak{X}$. The correlation kernel of this determinantal point process, $K_{n,p}(r, x; s, y)$ (where $r, s \in \{1, \dots, p\}$, and $x, y \in \mathfrak{X}$), is given by the formula

$$K_{n,p}(r, x; s, y) = -\phi_{r,s}(x, y) + \sum_{i,j=1}^{n} \phi_{r,p+1}(x, x_i^{p+1}) \left(A^{-1}\right)_{i,j} \phi_{0,s}(x_j^0, y). \tag{9.2}$$

The functions $\phi_{r,s}$, and the matrix $A = (a_{i,j})$ (where $i, j = 1, \dots, n$) are defined in terms of transition functions as follows:

$$\phi_{r,s}(x, y) = \begin{cases} \left(\phi_{r,r+1} * \dots * \phi_{s-1,s}\right)(x, y), & 0 \le r < s \le p+1, \\ 0, & r \ge s, \end{cases} \tag{9.3}$$

and

$$a_{i,j} = \phi_{0,p+1}(x_i^0, x_j^{p+1}). \tag{9.4}$$

Remark 9.2. For the process defined by probability measure (9.1), and described by Theorem 9.1, the correlation functions can be written as determinants of block matrices, namely

$$\varrho_{k_1,\dots,k_p}\left(x_1^1, \dots, x_{k_1}^1; \dots; x_1^p, \dots, x_{k_p}^p\right) = \det \begin{bmatrix} \left(K_{n,p}(1, x_i^1; 1, x_j^1)\right)_{i=1,\dots,k_1}^{j=1,\dots,k_1} & \dots & \left(K_{n,p}(1, x_i^1; p, x_j^p)\right)_{i=1,\dots,k_1}^{j=1,\dots,k_p} \\ \vdots & & \\ \left(K_{n,p}(p, x_i^p; 1, x_j^1)\right)_{i=1,\dots,k_p}^{j=1,\dots,k_1} & \dots & \left(K_{n,p}(p, x_i^p; p, x_j^p)\right)_{i=1,\dots,k_p}^{j=1,\dots,k_p} \end{bmatrix},$$

where $1 \le k_1, \dots, k_p \le n$, and for $1 \le l, r \le p$

$$\left(K_{n,p}(l, x_i^l; r, x_j^r)\right)_{i=1,\dots,k_l}^{j=1,\dots,k_r} = \begin{pmatrix} K_{n,p}(l, x_1^l; r, x_1^r) & \dots & K_{n,p}(l, x_1^l; r, x_{k_r}^r) \\ \vdots & & \\ K_{n,p}(l, x_{k_l}^l; r, x_1^r) & \dots & K_{n,p}(l, x_{k_l}^l; r, x_{k_r}^r) \end{pmatrix}.$$

In what follows the functions

$$\phi_{0,s}(i, y), \quad 2 \le s \le p,$$

will be called *initial transition functions*, and the functions

$$\phi_{r,p+1}(x, j), \quad 1 \le r \le p-1,$$

will be called *final transition functions*. In addition, the functions of the form

$$\phi_{r,s}(x, y), \quad 1 \le r \le p-2, \quad r+2 \le s \le p,$$

will be called *intermediate transition functions.* Finally, the function

$$\phi_{0,p+1}(i, j)$$

will be called *the total transition function*.

In order to prove Proposition 2.4 we rewrite the density of the product matrix process with truncated unitary matrices obtained in Proposition 2.1 as in the statement of the Eynard–Mehta theorem, and obtain explicit expressions for the transition functions.

9.2 *Explicit formulae for the transition functions*

In our situation $\mathfrak{X}_0 = \{1, \dots, n\}$, $\mathfrak{X}_{p+1} = \{1, \dots, n\}$, $\mathfrak{X} = \mathbb{R}_{>0}$. The initial one-step transition function is defined by

$$\phi_{0,1} : \{1, \dots, n\} \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}; \ \phi_{0,1}(i, x) = w_i^{(l)}(x).$$

The final one-step transition function is defined by

$$\phi_{p,p+1} : \mathbb{R}_{>0} \times \{1, \dots, n\} \longrightarrow \mathbb{R}; \ \phi_{p,p+1}(x, k) = x^{k-1}.$$

In addition, the intermediate transition functions can be written as

$$\phi_{r,r+1} : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}, \quad r = 1, \dots, p-1,$$

$$\phi_{r,r+1}(x, y) = y^{\nu_{r+l}} (x-y)_+^{m_{r+l}-n-\nu_{r+l}-1} x^{-m_{r+l}+n},$$

where $(x-y)_+ = \max(0, x-y)$.

For the initial transition functions $\phi_{0,s}, s = 2, \dots, p$, we obtain the following recurrence relation:

$$\phi_{0,s}(i,x) = \int_0^1 \tau^{\nu_{s-1+l}} (1-\tau)^{m_{s-1+l}-n-\nu_{s-1+l}-1} \phi_{0,s-1}\left(i, \frac{x}{\tau}\right) \frac{\mathrm{d}\tau}{\tau}. \tag{9.5}$$

This is the recurrence relation for $\phi_{0,s}(i,x) = w_i^{(s-1+l)}(x)$ (where $1 \le s \le p$), see Kieburg *et al.* [31, Equation (2.23)].

The total transition function $\phi_{0,p+1}(i,j)$ can be written as

$$\begin{aligned} \phi_{0,p+1}(i,j) &= \int_0^\infty \phi_{0,p}(i,p)\, \phi_{p,p+1}(t,j)\mathrm{d}t = \int_0^\infty t^{j-1} w_i^{(p-1+l)}(t)\mathrm{d}t \\ &= c_{p-1+l} \int_0^\infty t^{j-1} G^{p+l-1,0}_{p+l-1,p+l-1}\left(\begin{matrix} m_{p+l-1}-n, & \dots, & m_2-n, & m_1-2n+i \\ \nu_{p+l-1}, & \dots, & \nu_2, & \nu_1+i-1 \end{matrix} \middle| t \right) \mathrm{d}t, \end{aligned} \tag{9.6}$$

where c_{p-1+l} is defined by equation (2.7). The Mellin transform of a Meijer G-function is

$$\int_0^\infty \mathrm{d}x x^{s-1} G^{m,n}_{p,q}\left(\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} \middle| xy \right) = \frac{1}{y^s} \frac{\prod_{i=1}^m \Gamma(b_i+s) \prod_{j=1}^n \Gamma\left(1-a_j-s\right)}{\prod_{k=m+1}^q \Gamma(1-b_k-s) \prod_{l=n+1}^p \Gamma(a_l+s)}. \tag{9.7}$$

This gives an expression of the *total transition function* $\phi_{0,p+1}(i,j)$ in terms of Gamma functions

$$\phi_{0,p+1}(i,j) = c_{p-1+l} \frac{\prod_{k=2}^{p+l-1} \Gamma(\nu_k+j)\, \Gamma(\nu_1+i+j-1)}{\prod_{k=2}^{p+l-1} \Gamma(m_k-n+j)\, \Gamma(m_1-2n+i+j)}. \tag{9.8}$$

Similar calculations give us the formula for the *final transition functions*. In particular, we can write

$$\begin{aligned} \phi_{p-1,p+1}(x,j) &= \int_0^\infty \phi_{p-1,p}(x,y) \phi_{p,p+1}(y,j)\mathrm{d}y \\ &= \int_0^\infty y^{\nu_{p-1+l}} (x-y)_+^{m_{p-1+l}-n-\nu_{p-1+l}-1} x^{-m_{p-1+l}+n} y^{j-1}\mathrm{d}y \\ &= x^{-m_{p-1+l}+n} \int_0^x y^{\nu_{p-1+l}} (x-y)^{m_{p-1+l}-n-\nu_{p-1+l}-1} y^{j-1}\mathrm{d}y. \end{aligned} \tag{9.9}$$

The change of the integration variable $y = \tau x$ gives

$$\begin{aligned}\phi_{p-1,p+1}(x,j) &= x^{j-1}\int_0^1 \tau^{\nu_{p-1+l}+j-1}\,(1-\tau)^{m_{p-1+l}-n-\nu_{p-1+l}-1}\,\mathrm{d}\tau \\ &= x^{j-1}\frac{\Gamma\left(\nu_{p-1+l}+j\right)\Gamma\left(m_{p-1+l}-n-\nu_{p-1+l}\right)}{\Gamma\left(m_{p-1+l}-n+j\right)}.\end{aligned} \tag{9.10}$$

Repeating this calculation, we arrive to a general formula for the final transition function

$$\phi_{r,p+1}(x,j) = \prod_{a=r}^{p-1}\frac{\Gamma\left(\nu_{a+l}+j\right)\Gamma\left(m_{a+l}-n-\nu_{a+l}\right)}{\Gamma\left(m_{a+l}-n+j\right)}x^{j-1}, \quad r\in\{1,\dots,p-1\}. \tag{9.11}$$

Finally, let us find the *intermediate transition functions* $\phi_{r,s}(x,y)$. Since

$$G^{1,0}_{1,1}\left(\begin{array}{c} a \\ b \end{array}\middle| x\right) = \frac{(1-x)^{a-b-1}x^b}{\Gamma(a-b)}, \quad 0<x<1,$$

we can rewrite the transition function $\phi_{r,r+1}(x,y)$ as

$$\phi_{r,r+1}(x,y) = \frac{1}{x}\Gamma\left(m_{r+l}-n-\nu_{r+l}\right)G^{1,0}_{1,1}\left(\begin{array}{c} m_{r+l}-n \\ \nu_{r+l} \end{array}\middle|\frac{y}{x}\right). \tag{9.12}$$

In addition, we have the following recurrence relation

$$\begin{aligned}&\int_0^1 x^{\nu_r}(1-x)^{m_r-n-\nu_r-1}G^{r-1,0}_{r-1,r-1}\left(\begin{array}{cccc} m_{r-1}-n, & \dots, & m_2-n, & m_1-n \\ \nu_{r-1}, & \dots, & \nu_2, & \nu_1+n-1 \end{array}\middle|\frac{y}{x}\right)\frac{\mathrm{d}x}{x} \\ &= \Gamma\left(m_r-n-\nu_r\right)G^{r,0}_{r,r}\left(\begin{array}{cccc} m_r-n, & \dots, & m_2-n, & m_1-n \\ \nu_r, & \dots, & \nu_2, & \nu_1+n-1 \end{array}\middle| y\right),\end{aligned} \tag{A13}$$

see Beals and Szmigielski [9, Equation (5)]. Starting from (9.12), and applying (9.13), we obtain

$$\phi_{r,s}(x,y) = \frac{1}{x}\prod_{k=r+1}^{s}\Gamma\left(m_{k+l-1}-n-\nu_{k+l-1}\right)G^{s-r,0}_{s-r,s-r}\left(\begin{array}{ccc} m_{r+l}-n, & \dots, & m_{s+l-1}-n \\ \nu_{r+l}, & \dots, & \nu_{s+l-1} \end{array}\middle|\frac{y}{x}\right), \tag{9.14}$$

where $1\le r\le p-2$, and $r+2\le s\le p$.

A.2 *The inverse of A*

Here we find an explicit formula for $\left(A^{-1}\right)_{i,j}$ in the formula for the correlation kernel in Theorem 2.4. If $A = \left(a_{i,j}\right)_{i,j=1}^{n}$, then $a_{i,j}$ is equal to the total transition function

$\phi_{0,p+1}(i,j)$ given by equation (9.8). We need to find the inverse of the matrix $\check{A} = \left(\check{a}_{i,j}\right)_{i,j=1}^{n}$ defined by

$$\check{a}_{i,j} = \frac{\Gamma(\nu_1 - 1 + i + j)}{\Gamma(m_1 - 2n + i + j)}, \quad i,j \in \{1,\dots,n\}.$$

Proposition 9.3. For $N = 1,2,\dots$ and $-\alpha, -\beta \in \mathbb{C} \setminus \mathbb{N}$, the matrix

$$\left[\frac{(\alpha+1)_{i+j}}{(\alpha+\beta+2)_{i+j}}\right]_{i,j=0}^{N-1}$$

is invertible and its inverse matrix $\left(\gamma_{i,j}\right)_{i,j=0}^{N-1}$ is given by

$$\begin{aligned}\gamma_{i,j} &= \frac{(-1)^{i+j}(\alpha+\beta+1)_i(\alpha+\beta+1)_j}{(\alpha+1)_i(\alpha+1)_j(\alpha+\beta+1)} \\ &\times \sum_{p=0}^{N-1} \frac{(2p+\alpha+\beta+1)(\alpha+1)_p p!}{(\alpha+\beta+1)_p(\beta+1)_p(p-i)!\,i!\,(p-j)!\,j!}(\alpha+\beta+i+1)_p(\alpha+\beta+j+1)_p.\end{aligned} \tag{9.15}$$

Proof. See Theorem 10 in Zhang and Chen [45]. ■

We apply Proposition 9.3, and find

$$A^{-1} = \left(b_{i,j}\right)_{i,j=1}^{n}, \tag{9.16}$$

where

$$\begin{aligned}b_{i,j} &= \left(c_{p-1+l}\right)^{-1} \frac{\prod_{k=2}^{l+p-1}\Gamma\left(m_k - n + i\right)}{\prod_{k=2}^{l+p-1}\Gamma\left(\nu_k + i\right)} \frac{\Gamma\left(m_1 - 2n + 2\right)}{\Gamma\left(\nu_1 + 1\right)} \\ &\times (-1)^{i+j} \frac{\left(m_1 - 2n + 1\right)_{i-1}\left(m_1 - 2n + 1\right)_{j-1}}{\left(\nu_1 + 1\right)_{i-1}\left(\nu_1 + 1\right)_{j-1}\left(m_1 - 2n + 1\right)} \\ &\times \sum_{k=0}^{n-1} \frac{\left(2k + m_1 - 2n + 1\right)\left(\nu_1 + 1\right)_k}{\left(m_1 - 2n + 1\right)_k\left(m_1 - 2n - \nu_1 + 1\right)_k} \frac{k!\left(m_1 - 2n + i\right)_k\left(m_1 - 2n + j\right)_k}{(k-i+1)!\,(i-1)!\,(k-j+1)!\,(j-1)!}.\end{aligned} \tag{9.17}$$

Now we can write the 2nd term in the formula for the correlation kernel (equation (9.2)) as

$$\begin{aligned}\widetilde{K}_{n,p}(r,x;s,y) &= \sum_{i,j=1}^{n} \phi_{r,p+1}(x,i)\left(A^{-1}\right)_{i,j}\phi_{0,s}(j,y)\\ &= \frac{\Gamma\left(m_1-2n+1\right)}{\Gamma\left(m_1-2n-\nu_1+1\right)\Gamma\left(\nu_1+1\right)}\frac{1}{\prod_{k=2}^{l+p-1}\Gamma\left(m_k-n-\nu_k\right)}\\ &\times\sum_{k=0}^{n-1}\frac{k!\left(\nu_1+1\right)_k\left(2k+m_1-2n+1\right)}{\left(m_1-2n+1\right)_k\left(m_1-2n-\nu_1+1\right)_k}P_{r,k}(x)Q_{s,k}(y),\end{aligned} \tag{9.18}$$

where

$$P_{r,k}(x)=\sum_{i=0}^{k}\frac{(-1)^{k-i}\left(m_1-2n+i+1\right)_k\left(m_1-2n+1\right)_i\prod_{a=2}^{p+l-1}\Gamma\left(m_a-n+i+1\right)}{(k-i)!\,i!\left(\nu_1+1\right)_i\prod_{a=2}^{p+l-1}\Gamma\left(\nu_a+i+1\right)}\phi_{r,p+1}(x,i+1), \tag{9.19}$$

and

$$Q_{s,k}(y)=\sum_{j=0}^{k}\frac{(-1)^{k}\left(m_1-2n+j+1\right)_k\left(m_1-2n+1\right)_j}{(k-j)!\,j!\left(\nu_1+1\right)_j}\phi_{0,s}\left(j+1,y\right). \tag{9.20}$$

9.4 *The contour integral representation for* $P_{r,k}(x)$

Using explicit formula for the final transition function $\phi_{r,p+1}(x,i)$ (see equation (9.11)) we rewrite $P_{r,k}(x)$ as

$$\begin{aligned}P_{r,k}(x) &= \frac{\Gamma\left(\nu_1+1\right)\prod_{a=r+l}^{p+l-1}\Gamma\left(m_a-n-\nu_a\right)}{\Gamma\left(m_1-2n+1\right)}\\ &\times\sum_{i=0}^{k}\frac{(-1)^{k-i}}{(k-i)!\,i!}\frac{\Gamma\left(m_1-2n+i+k+1\right)\prod_{a=2}^{r+l-1}\Gamma\left(m_a-n+i+1\right)}{\prod_{a=1}^{r+l-1}\Gamma\left(\nu_a+i+1\right)}x^i.\end{aligned} \tag{9.21}$$

The Residue Theorem gives the following contour integral representation for $P_{r,k}(x)$:

$$\begin{aligned}P_{r,k}(x) &= \frac{\Gamma\left(\nu_1+1\right)\prod_{a=r+l}^{p+l-1}\Gamma\left(m_a-n-\nu_a\right)}{\Gamma\left(m_1-2n+1\right)}\\ &\times\frac{1}{2\pi i}\oint_{\Sigma_k}\frac{\Gamma(t-k)\Gamma\left(m_1-2n+t+k+1\right)\prod_{a=2}^{r+l-1}\Gamma\left(m_a-n+t+1\right)}{\prod_{a=0}^{r+l-1}\Gamma\left(\nu_a+t+1\right)}x^t\mathrm{d}t,\end{aligned} \tag{9.22}$$

where Σ_k is a closed contour encircling the interval $[0,k]$ once in the positive direction, and where $\nu_0=0$.

A.4 *The contour integral representation for* $Q_{s,k}(Y)$

Equation (9.20) together with the formula for the initial transition functions obtained in Section 9.2 give

$$\begin{aligned} Q_{s,k}(Y) &= \frac{\Gamma\left(m_1-2n-\nu_1+1\right)\Gamma\left(\nu_1+1\right)\prod_{a=2}^{s+l-1}\Gamma\left(m_a-n-\nu_a\right)}{\Gamma\left(m_1-2n+1\right)} \\ &\times \sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!\,j!}\frac{\Gamma\left(m_1-2n+j+k+1\right)}{\Gamma\left(\nu_1+1+j\right)} \\ &\times G^{s+l-1,0}_{s+l-1,s+l-1}\left(\begin{matrix} m_{s+l-1}-n, & \dots, & m_2-n, & m_1-2n+j+1 \\ \nu_{s+l-1}, & \dots, & \nu_2, & \nu_1+j \end{matrix}\,\middle|\, Y\right). \end{aligned} \tag{9.23}$$

The contour integral representation for the Meijer G-function inside the sum above is

$$\begin{aligned} &G^{s+l-1,0}_{s+l-1,s+l-1}\left(\begin{matrix} m_{s+l-1}-n, & \dots, & m_2-n, & m_1-2n+j+1 \\ \nu_{s+l-1}, & \dots, & \nu_2, & \nu_1+j \end{matrix}\,\middle|\, Y\right) \\ &\quad=\frac{1}{2\pi i}\int_C \frac{\Gamma\left(\nu_1+j+u\right)\prod_{a=2}^{s+l-1}\Gamma\left(\nu_a+u\right)}{\Gamma\left(m_1-2n+j+u+1\right)\prod_{a=2}^{s+l-1}\Gamma\left(m_a-n+u\right)}Y^{-u}\mathrm{d}u, \end{aligned} \tag{9.24}$$

where the contour C is a positively oriented curve in the complex u-plane that starts and ends at $-\infty$, and encircles the negative real axis. In equation (9.23) the resulting sum is

$$\begin{aligned} &\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!\,j!}\frac{\Gamma\left(m_1-2n+j+k+1\right)}{\Gamma\left(\nu_1+1+j\right)}\frac{\Gamma\left(\nu_1+j+u\right)}{\Gamma\left(m_1-2n+j+u+1\right)} \\ &=\frac{(-1)^k}{k!}\frac{\Gamma\left(u+\nu_1\right)}{\Gamma\left(1+\nu_1\right)}\frac{\Gamma\left(m_1-2n+k+1\right)}{\Gamma\left(m_1-2n+u+1\right)}\sum_{j=0}^{k}\frac{(-k)_j}{j!}\frac{\left(\nu_1+u\right)_j}{\left(\nu_1+1\right)_j}\frac{\left(m_1-2n+k+1\right)_j}{\left(m_1-2n+k+1\right)_j} \\ &=\frac{(-1)^k}{k!}\frac{\Gamma\left(u+\nu_1\right)}{\Gamma\left(1+\nu_1\right)}\frac{\Gamma\left(m_1-2n+k+1\right)}{\Gamma\left(m_1-2n+u+1\right)} \\ &\times {}_3F_2\left(-k,u+\nu_1,m_1-2n+k+1;1+\nu_1,m_1-2n+u+1;1\right). \end{aligned}$$

The Pfaff–Saalschültz Theorem says that

$${}_3F_2\left(-k,a,b;c,d;1\right)=\frac{(c-a)_k(c-b)_k}{(c)_k(c-a-b)_k},$$

if the balanced condition, $c+d=1-k+a+b$, is satisfied, see, for example, Ismail [28, Section 1.4]. In our case

$$a=u+\nu_1,\ b=l-2n+1+k,\ c=1+\nu_1,\ d=m_1-2n+1+u,$$

and the balanced condition is satisfied. Thus, we have

$$ {}_3F_2\left(-k, u+\nu_1, m_1-2n+1+k; 1+\nu_1, m_1-2n+1+u; 1\right) = \frac{(1-u)_k(\nu_1-m_1+2n-k)_k}{(1+\nu_1)_k(2n-u-m_1-k)_k}. $$

Taking into account that

$$ \frac{(1-u)_k}{(2n-m_1-u-k)_k} = \frac{(u-k)_k}{(u-2n+m_1+1)_k}, $$

we obtain the formula

$$ \begin{aligned} Q_{s,k}(Y) &= (-1)^k \frac{\Gamma(m_1-2n-\nu_1+1)}{\Gamma(m_1-2n+1)} \frac{\Gamma(m_1-2n+1+k)(\nu_1-m_1+2n-k)_k}{k!\,(1+\nu_1)_k} \\ &\times \frac{\prod_{a=2}^{s+l-1}\Gamma\left(m_a-n-\nu_a\right)}{2\pi i}\int_C \frac{\prod_{a=1}^{s+l-1}\Gamma\left(\nu_a+u\right)}{\prod_{a=2}^{s+l-1}\Gamma\left(m_a-n+u\right)} \frac{(u-k)_k}{\left(u-2n+m_1+1\right)_k} \\ &\times \frac{Y^{-u}\mathrm{d}u}{\Gamma\left(m_1-2n+1+u\right)}. \end{aligned} \tag{9.25} $$

A.5 *Derivation of the correlation kernel*

Equation (9.14) gives the 1st term in equation (9.2) for the correlation kernel. To write explicitly the 2nd term in equation (9.2) we insert the formulae for $P_{r,k}$ (equation (9.22)), and $Q_{s,k}$ (equation (9.25)) into equation (9.18). After simplifications we see that the 2nd term in equation (9.2) can be written as

$$ \begin{aligned} \widetilde{K}_{n,p}(r,x;s,y) &= \frac{\prod_{a=r+l}^{p+l-1}\Gamma\left(m_a-n-\nu_a\right)}{\prod_{a=s+l}^{p+l-1}\Gamma\left(m_a-n-\nu_a\right)} \\ &\times \frac{1}{(2\pi i)^2}\oint_{\Sigma_k}\mathrm{d}t\int_C \mathrm{d}u \frac{\prod_{a=2}^{r+l-1}\Gamma\left(m_a-n+t+1\right)}{\prod_{a=0}^{r+l-1}\Gamma\left(t+\nu_a+1\right)} \frac{\prod_{a=0}^{s+l-1}\Gamma\left(\nu_a+u\right)}{\prod_{a=2}^{s+l-1}\Gamma\left(m_a-n+u\right)} \\ &\times \sum_{k=0}^{n-1}(m_1-2n+2k+1)\frac{\Gamma(t-k)\Gamma(t+m_1-2n+k+1)}{\Gamma(u-k)\Gamma(u+m_1-2n+k+1)}x^t y^{-u}. \end{aligned} \tag{9.26} $$

The sum inside the integral is the same as that in Kuijlaars and Stivigny [32] (see the proof of Proposition 4.4 in Kuijlaars and Stivigny [32]), and the rest of the proof is the same as that of Proposition 4.4 and Theorem 4.7 in Kuijlaars and Stivigny [32].

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