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UNCERTAINTY ANALYSIS FOR TIME- AND SPACE-DEPENDENT RESPONSES WITH RANDOM VARIABLES

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ABSTRACT

The performance of a product varies with respect to time and space if the associated limit-state function involves time and space. This study develops an uncertainty analysis method that quantifies the effect of random input variables on the performance (response) over time and space. The combination of the first order reliability method (FORM) and the second order reliability method (SORM) is used to approximate the extreme value of the response with respect to space at discretized instants of time. Then the response becomes a Gaussian stochastic process that is fully defined by the mean, variance, and autocorrelation functions obtained from FORM and SORM, and a sequential single loop procedure is performed for spatial and random variables. The method is successfully applied to the reliability analysis of a crank-slider mechanism, which operates in a specified period of time and space.

1. INTRODUCTION

Uncertainty, which is a gap between the present state of knowledge and the complete knowledge [1], exists in all stages of product development and operation [2]. Examples of uncertainty include random material properties, random loading, random operation conditions; they also include random manufacturing imprecision, as well as the lack of knowledge, such as ignorance, assumptions, and simplifications [1]. Numerous applications and studies have shown that not considering uncertainty properly during the design stage can lead to serious problems, such as low reliability, low robustness, low customer satisfaction, high risk, and high lifecycle cost [1, 3-5].

Reliability methods provide useful tools for uncertainty quantification and management. This is because reliability is not only an important quality characteristic of a product, but also related to other characteristics such as robustness, risk, safety, maintainability, and cost. Reliability is usually quantified by the probability that a product performs its intended function over a specified period of time and under specified service conditions [6]. Reliability problems can be roughly grouped into four categories: (a) time- and space-independent (TSI) problems, (b) space-dependent (SD) problems, (c) time-dependent (TD) problems, and (d) time- and space-dependent (TSD) problems. It is obvious that TSD problems belong to the most general category since other three types are just special cases of the TSD category.

TSI problems are the most traditional problems. They involve only time- and space-independent random variables, such as the geometry or material properties of a structure and applied loads. The responses are also random variables. Reliability methods for TSI problems include, but are not limited to, analytical methods, surrogate model methods, moment methods, and simulation methods. Typical analytical methods include the first-order reliability method (FORM) and the second-order reliability method (SORM) [7-12]. FORM and SORM simplify a limit-state function, which specifies a functional relationship between a response and random input variables, using the first and second order Taylor series expansions, respectively, at the so-called most probable point (MPP) [13]. Surrogate model methods [14-16] use simplified models, which are generally obtained from the design of experiments or variable screening by means of sensitivity analysis, to improve the computation efficiency. Moment methods [13, 17] calculate the moments of the limit-state function and then approximate its distribution with the moments; and then the distribution is used to obtain the reliability. Simulation methods include the direct Monte Carlo simulation (MCS) [18], quasi-Monte Carlo simulation [19], importance sampling [20], and subset simulation [21]. Usually, simulation methods are accurate but computationally expensive.

SD problems have responses that are space dependent. This happens when either input variables are spatially distributed with random fields [22] or the response is a function of spatial variables. Structural reliability analysis for this kind of problems usually requires stochastic finite element methods [22, 23].

Another dimension on which the uncertainty may depend is time. This happens when the response is a function of time or input variables, such as material properties and loads, which are time-variant stochastic processes. For these TD problems, many methodologies are available, including upcrossing rate methods [24-26], surrogate model methods [27-30], simulation methods [31, 32], probability density evolution method [33], envelope function method [34], failure process decomposition based method [35], and extreme value moment method [36]. Generally speaking, upcrossing rate methods are the most dominant methods, surrogate methods can obtain accurate results if the surrogate models are well trained, and simulation methods are also accurate but computationally expensive.

The combination of an SD problem and a TD problem leads to a TSD problem where the response is dependent on both space and time. For TSD problems, only a few methods are available in the literature. Hu and Mahadevan [37, 38] developed a method based on adaptive surrogate modeling. Shi et al. [39] proposed two strategies. One strategy is combing sparse grid technique with the fourth-moment method. And the other is combining the dimension reduction and maximum entropy method. Shi et al. [40]

developed a transferred limit-state function technique to transform the TSD problem into a TSI counterpart. These methods still have limitations for wider applications. Efficiently and accurately dealing with TSD problems remains a challenging issue. There is a need to develop efficient, accurate, and robust methods for TSD problems.

In this work, we aim at developing an efficient and accurate method for a special TSD problem where the response is a function of temporal and spatial variables, as well as random variables. As a result, the response is a time-dependent random field. The main idea is to approximate the extreme value of the response with respect to space at discretized instants of time using the combination of FORM and SORM, thus transforming the TSD response into an equivalent Gaussian stochastic process. The transformation is performed by a sequential single loop procedure [7, 41-43] so that high efficiency is maintained. The Kriging method [44] is also employed. Then MCS is employed to estimate the reliability by sampling the Gaussian process.

The rest of the paper is organized as follows: Section 2 discusses the problem addressed in this study, and Section 3 provides an overview of the proposed method followed by the extreme value analysis and the general process in Sections 4 and 5, respectively. Two examples are given in Section 6, and conclusions are made in Section 7.

2. PROBLEM STATEMENT

In this work, we focus on a response that is a function of temporal variables, spatial variables, and random variables. The limit-state function is defined by

$$Y = g\left(\mathbf{X}, \mathbf{S}, t\right) \tag{1}$$

where Y is the response, $\mathbf{X} = \left[X_1, X_2, ..., X_m\right]^T$ is an m-dimensional input random vector, $\mathbf{S} = \left[S_1, S_2, ..., S_n\right]^T$ is an n-dimensional spatial variable vector bounded on $\left[\underline{\mathbf{S}}, \overline{\mathbf{S}}\right]$, and t is the time bounded on $\left[\underline{t}, \overline{t}\right]$.

When $Y < 0$, a failure occurs. The reliability in the space $\left[\underline{\mathbf{S}}, \overline{\mathbf{S}}\right]$ and time span $\left[\underline{t}, \overline{t}\right]$ is then defined by

$$R = \Pr\left\{g(\mathbf{X}, \mathbf{S}, t) > 0,\ \forall \mathbf{S} \in \left[\underline{\mathbf{S}}, \overline{\mathbf{S}}\right], \forall t \in \left[\underline{t}, \overline{t}\right]\right\} \tag{2}$$

where $\forall$ means "for all".

Since the response is a function of random variables and time, Y is a stochastic process, and it is also a random field because it is a function of random variables and space. As a result, Y is a general time-dependent random field. This kind of TSD problem is commonly encountered in engineering applications. For example, the performance of a mechanism, such as the motion error, is a stochastic process due to random mechanism dimensions and joint clearances. The mechanism may also operate in different locations, and the mechanism performance is also space dependent.

This kind of reliability problem is usually more complicated than TSI, SD, and TD problems since it involves both spatial and temporal variables. In this work, we develop a method to effectively perform uncertainty analysis for TSD problems.

3. OVERVIEW

As mentioned in Section 1, the main idea of the proposed method is to approximate the extreme value of the response with respect to space at discretized instants of time using FORM and SORM, thus transforming the TSD response into an equivalent Gaussian stochastic process. Eq. (2) is converted into

$$R = \Pr\left\{Y_{\min}(\mathbf{X}, t) = \min_{\mathbf{S} \in \left[\underline{\mathbf{S}}, \overline{\mathbf{S}}\right]} g(\mathbf{X}, \mathbf{S}, t) > 0,\ \forall t \in \left[\underline{t}, \overline{t}\right]\right\} \tag{3}$$

where $Y_{\min}(\mathbf{X}, t)$ is the minimum value of $g(\mathbf{X}, \mathbf{S}, t)$ with respect to $\mathbf{S}$. $Y_{\min}(\mathbf{X}, t)$ is a general stochastic process, and Eq. (3) can be therefore regarded as the reliability of a TD problem. Since it is nearly impossible to simulate the stochastic process $Y_{\min}(\mathbf{X},t)$ directly, we need to convert it into an equivalent Gaussian process $H(t)$ such that [45]

$$\begin{aligned} R &= \Pr\left\{ Y_{\min}(\mathbf{X},t) = \min_{\mathbf{S}\in[\underline{\mathbf{S}},\overline{\mathbf{S}}]} g(\mathbf{X},\mathbf{S},t) > 0,\ \forall t \in [\underline{t},\overline{t}] \right\} \\ &\approx \Pr\left\{ H(t) > 0, \forall t \in [\underline{t},\overline{t}] \right\} \end{aligned} \tag{4}$$

A possible way to convert $Y_{\min}(\mathbf{X},t)$ into $H(t)$ is to employ FORM at every instant of time on $[\underline{t},\overline{t}]$ as FORM is capable of transforming a non-Gaussian random variable into a Gaussian random variable [45]. However, FORM may result in poor accuracy when $Y_{\min}(\mathbf{X},t)$ is highly nonlinear. A better idea is to employ SORM to improve the accuracy, but SORM does not transform a non-Gaussian random variable into a Gaussian one, as what FORM does. To address this problem, we inversely convert the instantaneous reliability obtained by SORM to its equivalent reliability index with which an equivalent Gaussian variable, which is needed for $H(t)$, can be constructed. However, SORM is less efficient than FORM, especially when the dimension of $\mathbf{X}$ is large. To balance the accuracy and efficiency, we use SORM only at time instants where the corresponding instantaneous reliability is relatively small because the accuracy of the instantaneous reliability at those instants is more important.

Calculating $Y_{\min}(\mathbf{X},t)$ and performing FORM and SORM at every instant of time is impractical. We therefore create surrogate models to reduce the number of extreme value analyses and executions of FORM and SORM. Details will be given in Section 5.

After $H(t)$ is numerically obtained, MCS will be implemented to estimate R or the corresponding probability of failure

$$p_f = 1 - R \tag{5}$$

It is worth mentioning that Eq. (2) can also be rewritten as

$$R=\Pr\left\{\min_{\mathbf{S}\in[\underline{\mathbf{S}},\overline{\mathbf{S}}],t\in[\underline{t},\overline{t}]} g(\mathbf{X},\mathbf{S},t)>0\right\} \tag{6}$$

which means that the TSD problem can also be transformed into a TSI one, with the minimum value of $g(\mathbf{X},\mathbf{S},t)$ with respect to both spatial and temporal variables. But we do not do so for two reasons. First, in many engineering problems, the response $g(\mathbf{X},\mathbf{S},t)$ fluctuates significantly with respect to t and may not be a convex function of t. Thus, calculating the minimum value of $g(\mathbf{X},\mathbf{S},t)$ with respect to t will involve global optimization, which is in general less computationally efficient. Second, even if $\min_{\mathbf{S}\in[\underline{\mathbf{S}},\overline{\mathbf{S}}],t\in[\underline{t},\overline{t}]} g(\mathbf{X},\mathbf{S},t)$ can be obtained, the reliability function with respect to t may not be generated, and only the reliability at the end of the period of time under consideration can be obtained. The proposed method can easily produce the reliability function for the entire period of time. Details will be given in Section 6.

4. EXTREME VALUE ANALYSIS AT AN INSTANT OF TIME

In this section, we provide details about how to obtain $H(\tau),\tau\in[\underline{t},\overline{t}]$. As mentioned in Section 3, to obtain $H(\tau)$, we need to calculate $Y_{\min}(\mathbf{X},\tau)$ and perform FORM and SORM. In Subsection 4.1, the extreme value analysis using FORM will be given and then in Subsection 4.2 details on how to adaptively update the analysis result using SORM will be described.

4.1 Extreme value analysis using FORM

The extreme value analysis at time instant τ using FORM can be modelled as the following optimization problem [7, 42, 43, 46]:

$$\begin{cases}\min\|\mathbf{U}\| \\ s.t.\ \min_{\mathbf{S}\in[\underline{\mathbf{S}},\overline{\mathbf{S}}]} g(T(\mathbf{U}),\mathbf{S},\tau)=0\end{cases} \tag{7}$$

where $\mathbf{U}$ is the vector of standard Gaussian variables transformed from $\mathbf{X}$, and $T(\bullet)$ stands for the transformation. Eq. (7) indicates a two-layer optimization problem whose solution usually requires a double-loop optimization process. The outer loop is the FORM analysis, and the inner loop is the extreme value analysis. Usually, the double-loop optimization can lead to low efficiency. To improve the efficiency, Du et al.[7, 42, 43] developed a sequential single-loop (SSL) approach to decouple the two loops to a sequential single-loop process. The flow chart of employing SSL to solve the optimization problem in Eq. (7) is shown in Fig. 1.

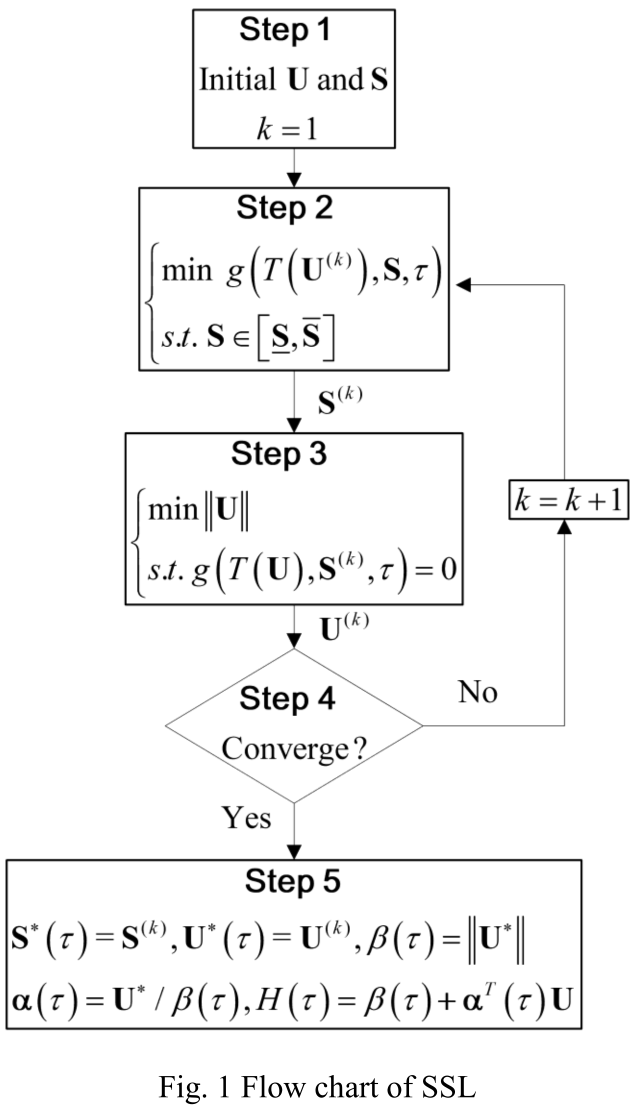

Fig. 1 Flow chart of SSL

Step 5 involves major equations for the MPP search. β and $\boldsymbol{\alpha}$ are the reliability index and sensitivity vector, respectively, and both are dependent on the specific instant of time τ . Obviously, once both $\beta(t)$ and $\boldsymbol{\alpha}(t), t \in [\underline{t}, \overline{t}]$, are obtained, $H(t)$ is available and can then be used for the MCS process to estimate the reliability or the probability of failure.

Because $\|\boldsymbol{\alpha}(t)\| = 1$ and $\mathbf{U}$ is a vector of standard Gaussian variables, the mean of $H(t)$ is $\beta(t)$, the standard deviation of $H(t)$ is constantly 1, and the autocorrelation of $H(t)$ is [26, 45]

$$\rho(t_1, t_2) = \boldsymbol{\alpha}^T(t_1)\boldsymbol{\alpha}(t_2) \tag{8}$$

Note that although $\rho(t_1, t_2)$ is an important statistical characteristic of $H(t)$, it is not necessary for sampling of $H(t)$. In fact, what we need are only the samples of $\mathbf{U}$, and the samples of $H(t)$ can be easily obtained via the following equation

$$H(t) = \beta(t) + \boldsymbol{\alpha}^T(t)\mathbf{U} \tag{9}$$

4.2 Extreme value analysis using SORM

To improve the accuracy of Eq. (9), we also use SORM to update $\beta(t)$ if necessary. Since it is impossible to perform extreme value analyses at all time instants on $[\underline{t}, \overline{t}]$, we only do so at N instants of time denoted by $\mathbf{t} = (t_1, t_2, ...t_i..., t_N)$, and hence what we need to update is $\beta(\mathbf{t}) = (\beta(t_1), \beta(t_2), ...\beta(t_i), ..., \beta(t_N))$. However, SORM is more computationally expensive than FORM, especially when the number of dimension of $\mathbf{X}$ is large. Therefore, we propose to update only some key elements of $\beta(\mathbf{t})$ that influence the target reliability R more than other elements. It is reasonable that those key elements have smaller values than others, because a smaller instantaneous reliability index $\beta(t_i)$ contributes more to the failure event than a larger one.

Fig. 2 shows the procedures to select the key elements of $\beta(\mathbf{t})$ and update them using SORM. In Step 1, $\mathbf{S}^*(\mathbf{t}) = (\mathbf{S}^*(t_1), \mathbf{S}^*(t_2), \ldots \mathbf{S}^*(t_i) \ldots, \mathbf{S}^*(t_N))$, $\mathbf{U}^*(\mathbf{t}) = (\mathbf{U}^*(t_1), \mathbf{U}^*(t_2), \ldots \mathbf{U}^*(t_i) \ldots, \mathbf{U}^*(t_N))$, and $\boldsymbol{\alpha}(\mathbf{t}) = (\boldsymbol{\alpha}(t_1), \boldsymbol{\alpha}(t_2), \ldots \boldsymbol{\alpha}(t_i) \ldots, \boldsymbol{\alpha}(t_N))$. In Step 3, β_p represents the p-th percentile of $\beta(\mathbf{t})$. For example, if $p = 30$, at 30% of the time instants SORM will be performed. Generally speaking, the larger is the value of p, the more accurate will R be, but with lower efficiency. In Step 4, since $\mathbf{S}^*(t_i)$, $\mathbf{U}^*(t_i)$, and $\boldsymbol{\alpha}(t_i)$ are already available from FORM in the SSL procedure, it is quite straightforward to calculate the corresponding instantaneous probability of failure $p_f(t_i)$ using SORM without searching for the MPP $\mathbf{U}^*(t_i)$.

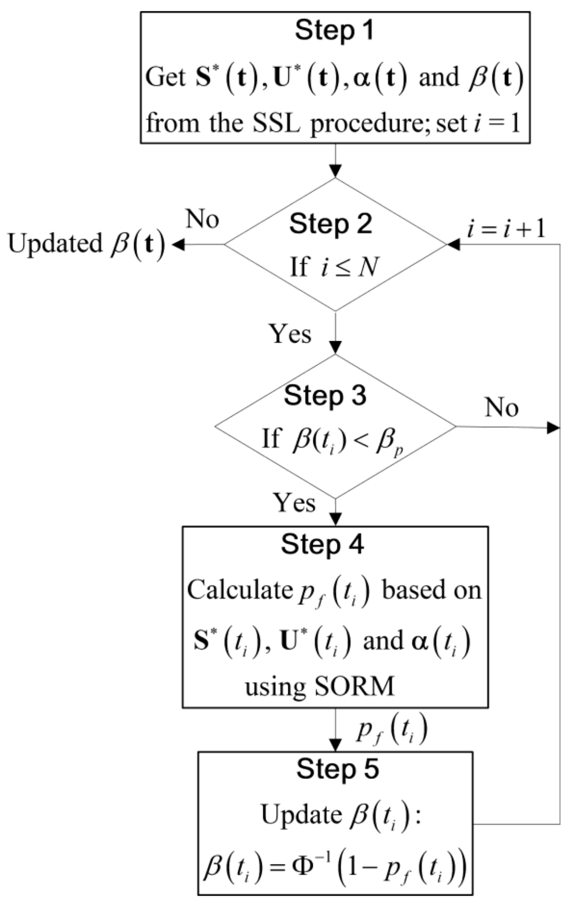

Fig. 2 The procedure of updating $\beta(\mathbf{t})$ using SORM

5. PROCEDURE

In this section, the complete procedure of the proposed method is detailed. Overall, there are three main stages in the procedure. Stage 1 is the SSL procedure discussed in Subsection 4.1. Stage 2 updates $\beta(\mathbf{t})$ using SORM, as detailed in Subsection 4.2. Stage 3 calculates $\beta(t)$ and $\boldsymbol{\alpha}(t), t \in [\underline{t}, \overline{t}]$ with the employment of Kriging models. In the last stage, MCS is implemented to sample $H(t)$ and then estimate the probability of failure.

The flow chart is shown in Fig. 3, and explanations are given in Table 1. In Fig. 3, Steps 1, 2, and 6 are grouped into Stage 1; Steps 3, 8, and 9 are grouped into Stage 2; Stage 3 contains Steps 4 and 5; Stage 4 involves only Step 10. Since Stages 1 and 2 have been discussed in Section 4, and Stage 4 (i.e. the MCS procedure) is straightforward, herein we discuss mainly Stage 3, or the use of Kriging model to approximate $\beta(t)$ and $\boldsymbol{\alpha}(t), t \in [\underline{t}, \overline{t}]$.

The Kriging model can provide not only predictions, but also probabilistic error σ^2 (or the mean square error) of the predictions [44, 45]. Therefore, we can judge if the model is well trained with the error information. For a to-be-approximated function $F(\mathbf{v})$, the Kriging model is expressed as

$$\hat{F}(\mathbf{v}) = f(\mathbf{v}) + \varepsilon(\mathbf{v}) \tag{10}$$

where $f(\mathbf{v})$ includes polynomial terms with unknown coefficients, and $\varepsilon(\mathbf{v})$ is the error term assumed to be a Gaussian stochastic process with mean zero and variance σ^2 [44]. For the problem in this work, $F(\mathbf{v})$ may be $\boldsymbol{\alpha}(t)$ or $\beta(t)$, and $\mathbf{v}$ is t. This means that we build Kriging surrogate models for $\boldsymbol{\alpha}(t)$ and $\beta(t)$ with respect to time. The Kriging models are denoted by $\hat{\beta}(t)$ and $\hat{\boldsymbol{\alpha}}(t)$. We do not provide details about how to create the models, and interested readers can refer to reference [44].

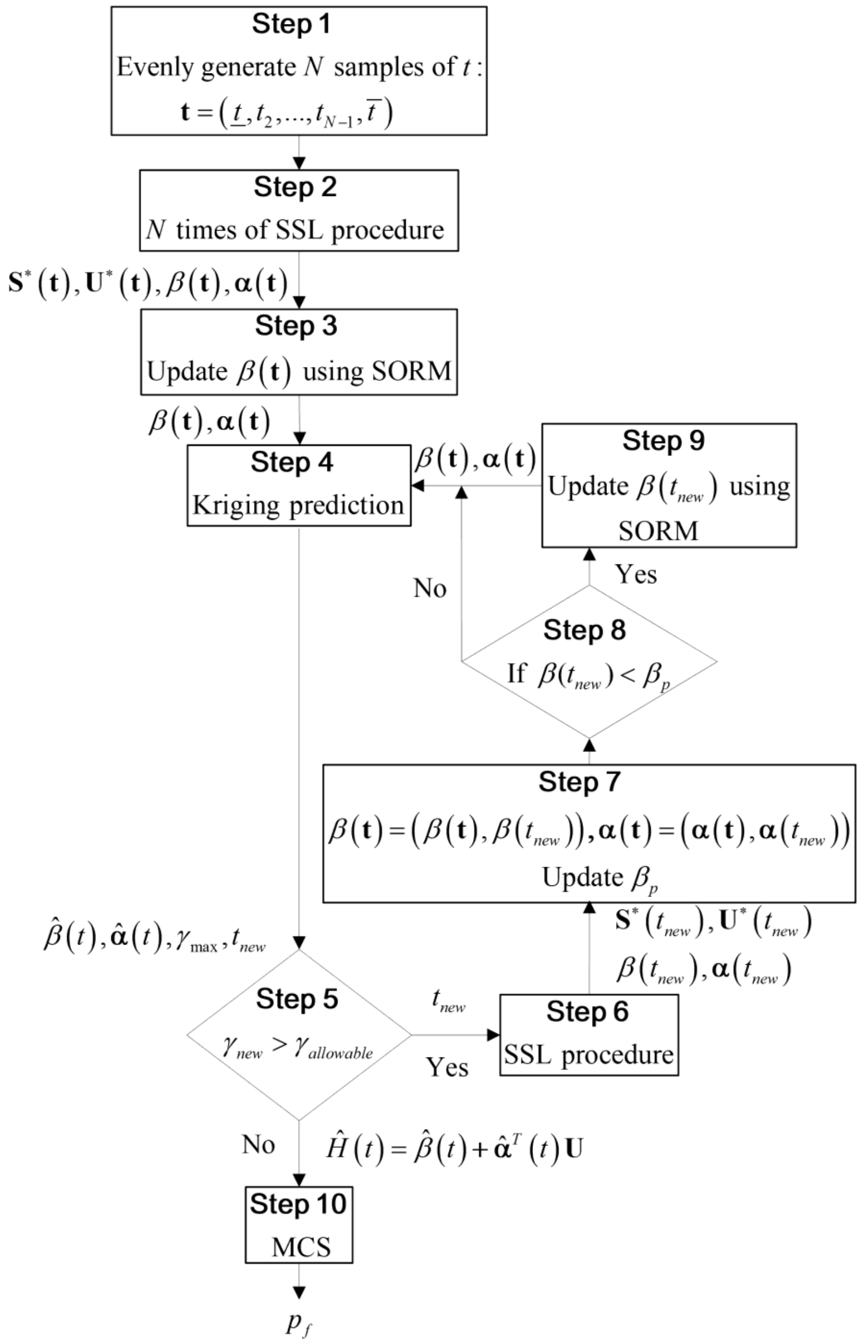

Fig. 3 Flow chart of the complete procedure

Table 1 Explanations for the complete procedure

Steps	Explanations
1	$t_1 = \underline{t}, t_N = \overline{t}$.
2	The detailed procedure of SSL for a given instant of time is shown in Fig.1. Note that after $\mathbf{U}^*(t_i)$ has been obtained, it will be treated as the initial point when searching for $\mathbf{U}^*(t_{i+1})$. The reason is that usually $\mathbf{U}^*(t_{i+1})$ is to some extent close to $\mathbf{U}^*(t_i)$ and that taking $\mathbf{U}^*(t_i)$ as the initial point of $\mathbf{U}^*(t_{i+1})$ may reduce the cost of searching for $\mathbf{U}^*(t_{i+1})$. Similarly, $\mathbf{S}^*(t_i)$ is also treated as the initial point of $\mathbf{S}^*(t_{i+1})$.
3	Details of this step is given in Fig. 2.
4	Kriging models $\hat{\beta}(t)$ and $\hat{\boldsymbol{\alpha}}(t)$ are built. Additionally, the maximum prediction error coefficient $\gamma_{\max}$, and the instant t_{new} of time corresponding to $\gamma_{\max}$ are also obtained.
5	If $\gamma_{\max}$ is larger than the allowable value $\gamma_{allowable}$, the Kriging model is not well trained, and then a new training point at t_{new} is added. There is no rigorous method to determine the value of $\gamma_{allowable}$, but experiments show that 10^{-4} is a good one.
6	Details are given in Fig. 1.
7	The set of training points is updated.
10	N_s samples of $\mathbf{U}$ are generated first, and then N_s samples of $H(t)$ are obtained with $\hat{H}(t) = \hat{\beta}(t) + \hat{\boldsymbol{\alpha}}^T(t)\mathbf{U}$. During the process, $[\underline{t}, \overline{t}]$ is evenly discretized into N_t points $(\underline{t}, t_2, ..., t_{N_t-1}, \overline{t})$.

Some initial samples of $\beta(t)$ and $\boldsymbol{\alpha}(t)$ are generated after the SSL procedure has been performed at instants $\mathbf{t} = (\underline{t}, t_2, ..., t_{N-1}, \overline{t})$. Then the samples of $\beta(t)$ and $\boldsymbol{\alpha}(t)$ are used to train Kriging models, which are then used to approximate or predict $\boldsymbol{\alpha}^T(t)$ and $\beta(t)$ at $\mathbf{t}_p = (\underline{t}, t_2, ..., t_{N_t-1}, \overline{t})$. Since the dimension of $(\boldsymbol{\alpha}^T(t), \beta(t))^T$ is $m+1$, with the Kriging prediction, a prediction matrix $\boldsymbol{\mu}$ and prediction error matrix $\boldsymbol{\sigma}^2$, whose dimensions are both $N_t \times (m+1)$, can be obtained. Then the prediction error coefficients $\boldsymbol{\gamma}$ are calculated by

$$\boldsymbol{\gamma} = \boldsymbol{\sigma}./\boldsymbol{\mu} \tag{11}$$

where "./" denotes an elementwise vector division. To make sure the Kriging models are well trained, the maximum $\gamma_{\max}$ of $\boldsymbol{\gamma}$ should be smaller than the allowable value $\gamma_{\text{allowable}}$. If $\gamma_{\max} > \gamma_{\text{allowable}}$, a new instant t_{new} of time is selected through

$$t_{new} = \arg\ \max \boldsymbol{\gamma}\left(\mathbf{t}_p = \underline{t}, t_2, ..., t_{N_t-1}, \overline{t} \right) \tag{12}$$

and $\beta\left(t_{new}\right)$ and $\boldsymbol{\alpha}\left(t_{new}\right)$ are added to the training point set to refine the Kriging models. Usually, a smaller $\gamma_{\text{allowable}}$ leads to higher accuracy of p_f, but more training points are needed, thus resulting in lower efficiency.

6. EXAMPLES

In this section, two examples are used to demonstrate the proposed method. MCS is employed to provide accurate solutions for the accuracy comparison.

6.1 A mathematical example

In this mathematical example, the limit-state function is defined by

$$\begin{aligned} g\left(\mathbf{X},\mathbf{S},t\right) = 8 + 10x_1 + 12x_2 + x_1x_2 + 0.1s_1s_2x_1^2 \\ - 0.2x_2^2\cos(t+\pi/2) + \sin(t) \end{aligned} \tag{13}$$

where $\mathbf{X} = (x_1, x_2)^T$ is the vector of two independent random variables $x_i \sim N(0, 0.2^2)$, $i = 1,2$, $\mathbf{S} = (s_1, s_2)^T$, where $s_i \in [1.5, 2.5]$, $i = 1,2$, is the spatial variable vector, and $t \in [0, 2\pi]$ rad is the temporal variable.

The probability of failure is computed over different time intervals with both MCS and the proposed method. In this example, the 50th percentile (i.e. $p = 50$) of $\beta\left(\mathbf{t}\right)$ is used to determine which $\beta\left(t_i\right)$ should be updated using SORM, the allowable maximum prediction error coefficient is $\gamma_{allowable} = 10^{-4}$, the initial value of N is 5 (for Kriging models), the number of simulations for $H\left(t\right)$ is $N_s = 10^6$, and the number of discretized instants of time is $N_t = 126$, which gives a step size of the time 0.05. The number of simulations of MCS N_{MCS} is set to 10^6, which is the same as N_s.

Theoretically, in MCS, for every given realization $\mathbf{s}$ of $\mathbf{S}$, we need to generate N_{MCS} samples of stochastic process $g(\mathbf{X}, \mathbf{s}, t)$, leading to a heavy computational burden. In this example, however, for every given realization $\mathbf{x}$ of $\mathbf{X}$, $\min_{\mathbf{S} \in [1.5, 2.5]^2, t \in [0, 2\pi]} g(\mathbf{x}, \mathbf{S}, t)$ can always be obtained analytically, and so we use $g(\mathbf{X}) = \min_{\mathbf{S} \in [1.5, 2.5]^2, t \in [0, 2\pi]} g(\mathbf{X}, \mathbf{S}, t)$ to replace the limit-state function shown in Eq. (2) and then perform MCS to get accurate results. Results from the proposed method and MCS are listed in Table 2 and plotted in Fig. 4.

As Table 2 and Fig. 4 show, the proposed method has good accuracy. The error is mainly caused by the nonlinearity of the limit-state function. In addition, the number of limit-state function calls by the proposed method is 217, far less than 33×10^6, which is the total number of limit-state function calls by MCS, showing that the proposed method is quite efficient.

Table 2 Probability of failure over different time intervals

$[0, t]$	p_f (proposed) (10^{-3})	p_f (MCS) (10^{-3})	Error (%)
$[0, 3.0]$	4.636	4.663	0.58
$[0, 3.5]$	6.602	6.617	0.23
$[0, 4.0]$	9.579	9.566	0.14
$[0, 4.5]$	11.666	11.581	0.73
$[0, 2\pi]$	11.902	11.808	0.80

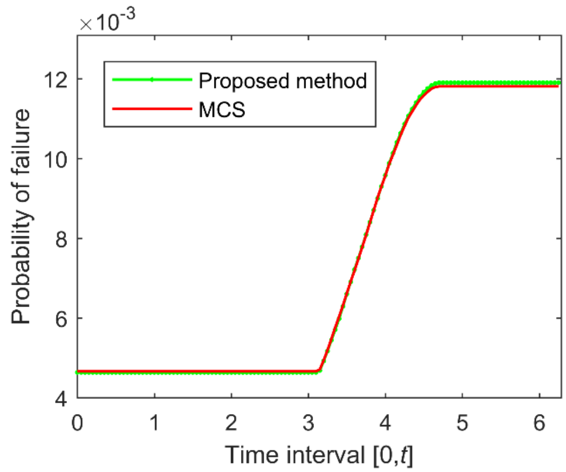

Fig. 4 Probability of failure over different time intervals

6.2 A slider mechanism

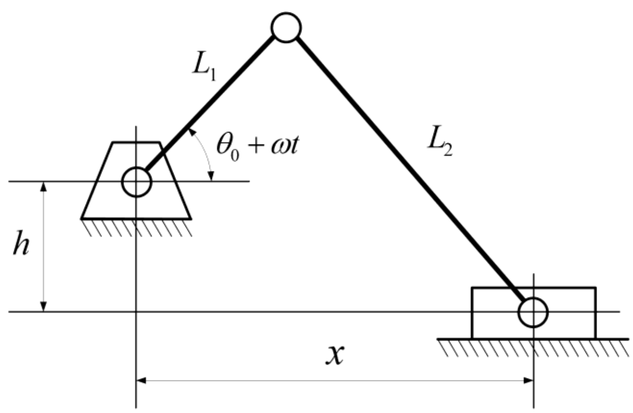

Fig. 5 A slider mechanism

Shown in Fig. 5 is a slider mechanism. It is used for difference applications (locations). The locations or spatial variables are the offset h and the initial angle θ_0 with the following ranges: $h \in [14.9, 15.1]$ m and $\theta_0 \in [0^\circ, 5^\circ]$; the spatial variable vector is then $\mathbf{S} = (h, \theta_0)^T$. The random variable vector is $\mathbf{X} = (L_1, L_2)^T$, which includes two independent random link lengths $L_1 \sim N(15, 0.15^2)$ m and $L_2 \sim N(35, 0.35^2)$ m. The time span is $t \in [0, 0.2\pi]$s. The limit-state function is defined by

$$g = 1.1 - (x_{actual} - x_{required}) \tag{14}$$

in which the actual position x_{actual} and the required position $x_{required}$ of the slider are

$$x_{actual} = L_1 \cos(\theta_0 + \omega t) + \sqrt{L_2^2 - (h + L_1 \sin(\theta_0 + \omega t))^2} \tag{15}$$

$$x_{required} = 15\cos(\omega t) + \sqrt{35^2 - (15 + 15\sin(\omega t))^2} \tag{16}$$

respectively, where $\omega = 1 \text{ rad/s}$ is the angular velocity.

The probability of failure is computed over different time intervals with both MCS and the proposed method. In this example, $p = 50$, $\gamma_{allowable} = 10^{-4}$, $N_s = N_{MCS} = 10^6$, $N_t = 40$ (i.e., the time step of the discretization of $H(t)$ is 0.005π), and the initial value of N is 7.

Results from the proposed method and MCS are listed in Table 3 and are plotted in Fig. 6. The proposed method obtains accurate results. As for the efficiency, the proposed method evaluates the limit-state function 214 times while MCS approximately 40.6×10^6. This indicates that the proposed method is much more efficient.

Table 3 Probability of failure over different time intervals

$[0, t]$ (0.01π s)	p_f (proposed) (10^{-3})	p_f (MCS) (10^{-3})	Error (%)
$[0, 5]$	6.765	6.729	0.53
$[0, 10]$	8.750	8.729	0.24
$[0, 15]$	11.930	11.811	1.01
$[0, 20]$	16.975	17.015	0.24

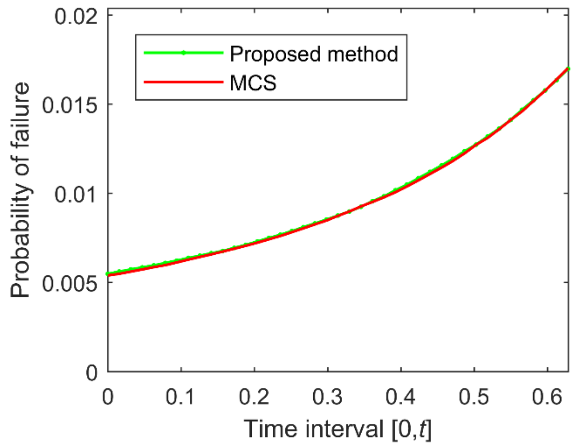

Fig. 6 Probability of failure over different time intervals

7. CONCLUSIONS

In this work, a combination of the first-order and the second-order methods (FORM and SORM) is proposed to perform uncertainty analysis for a time- and space-dependent response with random input variables. With the employment of FORM, SORM and the sequential single-loop method, we firstly transform the time- and space-dependent response into an equivalent Gaussian stochastic process, thus converting the time- and space-dependent reliability problem into an equivalent time-dependent reliability problem. Then the equivalent Gaussian process is simulated to estimate the time- and space-dependent probability of failure. To mitigate the computation burden, Kriging models are created to approximate the characteristics of the equivalent Gaussian stochastic process.

Transforming the time- and space-dependent response into an equivalent Gaussian stochastic process can avoid global optimization process which aims at obtaining the minimum value of the limit-state function with respect to the temporal variable.

Numerical examples show that the proposed method has both good accuracy and efficiency. If the limit-state function, however, is a nonconvex function with respect to spatial variables, the true extreme value of the response may not be easily found, and in this case, the proposed method may result in large errors, or low efficiency, or both. The extreme value of a limit-state function may not be differentiable, and in this case the MPP search for both FORM and SORM may not converge if a gradient-based MPP search algorithm is used.

Future research may focus on two directions. The first direction is to develop efficient global optimization methods for the minimum response with respect to both special and temporal variables, thus transforming the time- and space-dependent problem into a traditional time- and space-independent problem. And the second one is to investigate optimization-free methods to efficiently deal with general problems where the limit-state function is highly nonlinear with respect to input random variables and nonconvex with respect to both spatial and temporal variables.

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