STOCHASTIC TELEGRAPH EQUATION LIMIT
FOR THE STOCHASTIC SIX VERTEX MODEL

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ABSTRACT. In this article we study the stochastic six vertex model under the scaling proposed by Borodin and Gorin, where the weights of corner-shape vertices are tuned to zero, and prove their conjecture that the height fluctuation converges in finite dimensional distributions to the solution of the stochastic telegraph equation.

1. INTRODUCTION

The six vertex model is a model of tiling on subset Ω of $\mathbb{Z}^2$, with each site $(x, y) \in \Omega$ being tiled with the six types as depicted in Figure 1. The tiling obeys the rule that each (solid) line connects to a neighboring line. See Figure 2 for a generic realization for tiling. In this article we focus on the stochastic weights, with $b_1, b_2 \in (0, 1)$, as depicted Figure 1, and take the domain $\Omega := \mathbb{Z}_{\geq 0}^2$ to be the first quadrant. Fix boundary conditions on the axes $\mathbb{Z}_{\geq 0} \times \{0\}$ and $\{0\} \times \mathbb{Z}_{\geq 0}$ that indicate whether a given site along the axes has a line entering into $\mathbb{Z}_{\geq 0}^2$. Starting from the site $(1, 1)$, we tile the given site with one of the six vertices with reference to the incoming (bottom and left) line configurations and with probability given by the weights. This tiling construction then progresses sequentially in the linear order $(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), \ldots$ to the entire quadrant. For a given tiling one associates a height function, $H(x, y)$. This is a $\mathbb{Z}_{\geq 0}$-valued function defined on $(x, y) \in \mathbb{Z}_{\geq 0}^2$, so that, once interpreting a given tiling as non-intersecting lines, the level sets of $H(x, y)$, $x, y \in \mathbb{Z}_{\geq 0}$ are exactly these non-intersecting lines, with the convention $H(0, 0) := 0$. See Figure 2.

I	II	III	IV	V	VI
1	1	b_1	b_2	$1-b_1$	$1-b_2$

FIGURE 1. Six vertices with their weights.

Initiated in [GS92], the Stochastic Six Vertex (S6V) model has caught much attention. Being a special case of the six vertex model, it describes phenomena in equilibrium statistical mechanics. On the other hand, the S6V model also connects to non-equilibrium growth phenomena within the Kardar–Parisi–Zhang (KPZ) universality class. In particular, [BCG16] proved that, starting with step initial condition, the height fluctuation converges at one-point to GUE Tracy–Widom distribution. One-point convergence under different initial conditions (including the stationary case) was obtained in [AB16, Agg16], and [BBCW17] studied a half-space version of the S6V model and demonstrated that its one-point asymptotics match the prediction from other models in the KPZ class. In a related but slightly different direction, there has been study where one tunes the weights simultaneously with spacetime scaling in order to observe the Stochastic Partial Differential Equation (SPDE) limit. In [BO17] it is shown that under a certain tuning of the weights, one-point distribution of the S6V model converges to that of the KPZ equation. For a higher-spin generalization of the S6V model (see [CP16, BP16]), [CT17] obtained a microscopic Hopf–Cole transform and showed convergence to the KPZ equation at the process level. For S6V under the scaling $b_1/b_2 \to 1$, $b_1, b_2 \to b \in (0, 1)$, the convergence to the KPZ equation was obtained in [CGST18] via a Markov duality method.
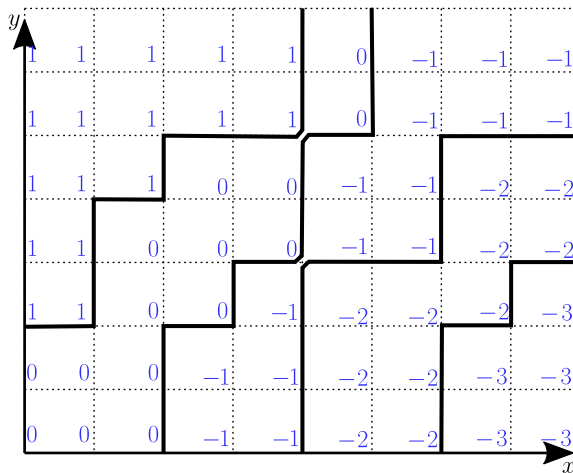

FIGURE 2. The height function.

Recently, Borodin and Gorin [BG18] proposed a new scaling: with $L \to \infty$ being the scaling parameter,

$$(1.1) \qquad b_1 = \exp\left(-\frac{\beta_1}{L}\right), \quad b_2 = \exp\left(-\frac{\beta_2}{L}\right),$$

and scale space being L: $x, y \mapsto L^{-1}x, L^{-1}y$, where $\beta_1, \beta_2 \in (0, \infty)$, $\beta_1 \neq \beta_2$, and fixed. They showed that, under this scaling, the exponential height function converges to the solution of the telegraph equation (TE). To state this result precisely, let us prepare some notation. Set $q := b_1/b_2$, $\mathfrak{q} := e^{\beta_1 - \beta_2}$ and consider

$$(1.2) \qquad \phi(x, y) := q^{H(x,y)} = \mathfrak{q}^{\frac{1}{L}H(x,y)} = e^{\frac{\beta_1 - \beta_2}{L}H(x,y)}.$$

For given Lipschitz functions $\chi, \psi : [0, \infty) \to \mathbb{R}$ with $\chi(0) = \psi(0)$, it is known ([BG18, Proposition 4.1, Theorem 4.4]) that the TE

$$(1.3) \quad \partial_{xy}\Phi + \beta_2\partial_x\Phi + \beta_1\partial_y\Phi = 0, \ x, y > 0, \quad \Phi(x, 0) = \chi(x), \quad \Phi(0, y) = \psi(y)$$

admits a unique solution. More explicitly, consider the Riemann function [BG18, eq. (39)]

$$(1.4) \ \mathcal{R}(x, y) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathcal{C}} \exp\left[(\beta_1 - \beta_2)\left(-x\frac{z}{z + \beta_2} + y\frac{z}{z + \beta_1}\right)\right] \frac{(\beta_2 - \beta_1)\, dz}{(z + \beta_1)(z + \beta_2)},$$

where the integration goes in the positive direction and encircles $-\beta_1$ but not $-\beta_2$. The solution Φ of (1.3) is given by

$$(1.5) \qquad \Phi(x, y) = \psi(0)\mathcal{R}(x, y) + \int_0^y \mathcal{R}(x, y - y')\big(\psi'(y') + \beta_2\psi(y')\big)dy'$$
$$+ \int_0^x \mathcal{R}(x - x', y)\big(\chi'(x') + \beta_1\chi(x')\big)dx'.$$

Definition 1.1. For given $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ and $g \in \mathbb{Z}_{\geq 0}^2 \to \mathbb{R}$, let $f_L(x) := f(Lx)$ and $g_L(x, y) := g(Lx, Ly)$ denote the corresponding scaled functions, and linearly interpolate to be functions on $\mathbb{R}_+$ and $\mathbb{R}_+^2$. Here, the linear interpolation from $\frac{1}{L}\mathbb{Z}_{\geq 0}$ to $\mathbb{R}$ is unique: for $a, b := a + \frac{1}{L} \in \frac{1}{L}\mathbb{Z}_{\geq 0}$ and $x \in (a, b)$,

$$f_L(x) := \frac{x - b}{a - b}f_L(a) + \frac{x - a}{b - a}f_L(b).$$

To linearly interpolate from $(\frac{1}{L}\mathbb{Z}_{\geq 0})^2$ to $\mathbb{R}_+^2$, we fix a diagonal direction, say northeast-southwest, and cut each square $[i, i + 1] \times [j, j + 1]$, $i, j \in \mathbb{Z}_{\geq 0}$, on the integer lattice into two triangles, diagonally along the prescribed direction: This gives a
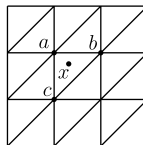
triangulation on $\mathbb{Z}_{\geq 0}^2$ and hence on $(\frac{1}{L}\mathbb{Z}_{\geq 0})^2$, from which we construct a unique linear interpolation: each generic $x \in \mathbb{R}_+^2$ is in a unique triangle, say abc, and $g_L(x)$ is given by linearly interpolating $g_L(a)$, $g_L(b)$, and $g_L(c)$ (three points determines a plane).

Theorem 1.2 ([BG18, Theorem 5.1]). *Fix Lipschitz functions* $\chi, \psi : [0, \infty) \to$ $\mathbb{R}$*, and let* Φ *be the unique solution of the telegraph equation* (1.3) *with boundary conditions* χ, ψ *given by* (1.5). *If, as* $L \to \infty$*, we have*

$$\sup_{x \in [0,a]} |\phi_L(\cdot, x) - \chi(x)| \to 0 \quad and \quad \sup_{x \in [0,a]} |\phi_L(0, \cdot) - \psi(x)| \to 0,$$

for each $a < \infty$*, then, as* $L \to \infty$*,*

$$\sup_{(x,y) \in [0,a]^2} |\phi_L(x,y) - \Phi(x,y)| \longrightarrow_P 0, \quad \sup_{(x,y) \in [0,a]^2} \left| \tfrac{1}{L} H_L(x,y) - \mathbf{h}(x,y) \right| \longrightarrow_P 0,$$

for each $a < \infty$*, where* $\mathbf{h} := \log_{\mathbf{q}} \Phi$*.*

As noted in [BG18, Remark 5.3], rewriting equation (1.3) in terms of $\mathbf{h}$-derivatives, and sending $\mathbf{q} \to 0$, one obtains a nonlinear PDE that was observed in [BCG16, RS16] in the $L \to \infty$ scaling limit but with b_1, b_2 fixed. Such a nonlinear PDE corresponds to the inviscid/hyperbolic scaling limit in the context of hydrodynamic limits. This is in contrast with the aforementioned SPDE-limit results, where the underlying hydrodynamic limits sit in the viscous/hyperbolic regime. Given such an intriguing feature, [BG18] further investigated the random fluctuations of ϕ and H around their respective means. Our work here follows this study of random fluctuations.

Let $u(x,y) := \phi(x,y) - \mathbf{E}[\phi(x,y)]$. Let $U(x,y)$ denote a centered Gaussian field on $\mathbb{R}_+$, with covariance

$$\mathbf{E}[U(x,y)U(x',y')] = \int_0^{x \wedge x'} \int_0^{y \wedge y'} \mathcal{R}(x - \overline{x}, y - \overline{y}) \, \mathcal{R}(x' - \overline{x}, y' - \overline{y}) \, D(\overline{x}, \overline{y}) \, d\overline{x} d\overline{y},$$

where

$$(1.6) \quad D(x,y) := (\beta_1 + \beta_2)\partial_x \Phi \cdot \partial_y \Phi + \beta_2(\beta_2 - \beta_1)\partial_x \Phi \cdot \Phi - \beta_1(\beta_2 - \beta_1)\Phi \cdot \partial_y \Phi.$$

The following is our main result.

Theorem 1.3. *Under the same assumptions as in Theorem* 1.2*, as* $L \to \infty$*,*

$$\sqrt{L} u_L \to U \text{ in finite dimensional distributions.}$$

Corollary 1.4. *Under the same assumptions as in Theorem* 1.2*,*

$$\frac{1}{\sqrt{L}} \big(H(Lx, Ly) - \mathbf{E}[H(Lx, Ly)] \big) \to \widetilde{U}(x,y) := \frac{U(x,y)}{(\beta_1 - \beta_2)\Phi(x,y)}$$

in finite dimensional distributions.

Remark 1.5. It is readily checked that

$$U(x,y) \stackrel{\text{law}}{=} \int_0^x \int_0^y \mathcal{R}(x - x', y - y')\xi(x', y')\sqrt{D(x', y')} dx' dy',$$

where $\xi(x,y)$ denotes the Gaussian white noise on $\mathbb{R}_+^2$. Given such stochastic integral representation, we can also view U as the solution of the stochastic telegraph equation (STE) with zero boundary condition, i.e.,

$$(1.7) \qquad \partial_{xy} U + \beta_2 \partial_x U + \beta_1 \partial_y U = \sqrt{D}\xi, \quad x, y \geq 0, \quad U(x,0) = U(0,y) = 0.$$

Alternatively, substituting $U = (\beta_1 - \beta_2)\Phi\widetilde{U}$ in (1.7) and using (1.3), we have the equation for $\widetilde{U}$:

$$(1.8) \qquad \partial_{xy}\widetilde{U} + \beta_1\partial_y\widetilde{U} + \beta_2\partial_x\widetilde{U} + (\beta_1 - \beta_2)(\partial_y\widetilde{U}\partial_x\mathbf{h} + \partial_x\widetilde{U}\partial_y\mathbf{h})$$
$$= \xi \cdot \sqrt{(\beta_1 + \beta_2)\partial_x\mathbf{h}\partial_y\mathbf{h} - \beta_2\,\partial_x\mathbf{h} + \beta_1\,\partial_y\mathbf{h}}.$$

Corollary 1.4 was conjectured in [BG18, Conjecture 6.1], based on observations through a four-point relation and (separately) through a variational principle and contour integrals. For the low density regime (see [BG18, Section 7] for the precise meaning), the analog of Corollary 1.4 was established in [BG18, Theorem 7.1]. The main step toward proving such Gaussian limits is to show convergence of the variance. Referring to (1.6), we see that the variance involves Φ and its gradients: one term is quadratic in gradients, and the other terms are linear in gradients. In the low density regime, the quadratic-gradient term vanishes in the limit $L \to \infty$, and, through integration by parts, [BG18] reduced convergence of the linear-gradient terms to convergence of ϕ (i.e., the law of large numbers result in Theorem 1.2), thereby showing the convergence of u.

For the general case (i.e., non-low-density) considered here, one needs to address the convergence of the quadratic-gradient term. The main tool we use here is the discrete, integrated form [BG18, eq. (85)] of the STE. From this equation we develop expressions of discrete gradients of ϕ. These expressions permit calculations of moments of the terms in question, and from this we obtain decorrelation through contracting the discrete analog of ξ.

Remark 1.6. We remark that after we posted our paper, the authors of [BG18] also found a proof of this conjecture which is different from our proof. The paper [BG18] will be updated to include their proof. We are thankful for the communications with the authors of [BG18].

2. PRELIMINARIES

In this section we prepare a few tools for subsequent analysis. Recall from [BG18, Theorem 3.1] that ϕ defined in (1.2) satisfies a 'four-point relation':

$$(2.1) \qquad \phi(x + 1, y + 1) - b_1\phi(x, y + 1) - b_2\phi(x + 1, y) + (b_1 + b_2 - 1)\phi(x, y)$$
$$= \xi^{\mathrm{d}}(x + 1, y + 1).$$

Here, $\xi^{\mathrm{d}}(x, y)$ is a process on $\mathbb{Z}^2_{>0}$ that plays the role of ξ (spacetime white noise) in the discrete setting. In particular, with $\nabla_x f(x) := f(x + 1) - f(x)$ denoting the forward discrete gradient acting on a designated variable x, one has

$$(2.2) \qquad \mathbf{E}\big[\xi^{\mathrm{d}}(x + 1, y + 1) \mid H(u, v), u \le x \text{ or } v \le y\big] = 0,$$

$$\mathbf{E}\big[\xi^{\mathrm{d}}(x + 1, y + 1)^2 \mid H(u, v), u \le x \text{ or } v \le y\big]$$
$$(2.3) \qquad = \big(b_2(1 - b_1) + b_1(1 - b_2)\big)\nabla_x\phi(x, y)\nabla_y\phi(x, y)$$
$$+ b_1(1 - b_2)(1 - q)\phi(x, y)\nabla_x\phi(x, y)$$
$$- b_1(1 - b_1)(1 - q)\phi(x, y)\nabla_y\phi(x, y).$$

Recall from [BG18, eq. (45)] the discrete Riemann function

$$(2.4) \quad \mathcal{R}^{\mathrm{d}}(x,y) = \frac{1}{2\pi\mathbf{i}} \oint_{-\frac{1}{b_2(1-b_1)}} \left(\frac{1+b_1(1-b_1)z}{1+b_2(1-b_1)z}\right)^x \left(\frac{1+b_2(1-b_2)z}{1+b_1(1-b_2)z}\right)^y$$
$$\times \frac{(b_2-b_1)\,dz}{(1+b_2(1-b_1)z)(1+b_1(1-b_2)z)},$$

where the integration goes in the positive direction and encircles $-\frac{1}{b_2(1-b_1)}$ but not $-\frac{1}{b_1(1-b_2)}$. We will also be using the notation $\widetilde{\mathcal{R}}^{\mathrm{d}}(x,y) := \mathcal{R}^{\mathrm{d}}(x,y)\mathbf{1}_{\{x\geq 0\}}\mathbf{1}_{\{y\geq 0\}}$. Recall from [BG18, eq. (85)] the following summation representation of ϕ:

$$\phi(x,y) = \phi(0,0)\mathcal{R}^{\mathrm{d}}(x,y) + \sum_{y'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x,y-y')\big(\phi(0,y') - b_2\phi(0,y'-1)\big)$$

$$(2.5) \qquad\qquad + \sum_{x'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x',y)\big(\phi(x',0) - b_1\phi(x'-1,0)\big)$$

$$+ \sum_{x',y'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x',y-y')\xi^{\mathrm{d}}(x',y').$$

Set $\overline{\phi}(x,y) := \mathbf{E}[\phi(x,y)]$. Since on the r.h.s. of (2.5) only the last term is random, we have

$$(2.6) \quad u(x,y) := \phi(x,y) - \mathbf{E}[\phi(x,y)] = \sum_{x',y'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x',y-y')\xi^{\mathrm{d}}(x',y'),$$

$$\overline{\phi}(x,y) = \phi(0,0)\mathcal{R}^{\mathrm{d}}(x,y) + \sum_{y'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x,y-y')\big(\phi(0,y') - b_2\phi(0,y'-1)\big)$$

$$(2.7) \qquad\qquad + \sum_{x'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x',y)\big(\phi(x',0) - b_1\phi(x'-1,0)\big).$$

Hereafter, we use $c(a,b,\ldots) < \infty$ to denote a generic finite constant that may change from line to line but depends only on the designated variables $a, b, \ldots$. The parameters $\beta_1 \neq \beta_2$ are considered fixed, so their dependence will be omitted.

Lemma 2.1. *For any $k = (k_1,k_2) \in \mathbb{Z}_{\geq 0}^2$ we write $|k| = k_1 + k_2$ and $\partial^k = \partial_x^{k_1}\partial_y^{k_2}$ and $\nabla^k = \nabla_x^{k_1}\nabla_y^{k_2}$. Given any $m \in \mathbb{Z}_{>0}$ and $a < \infty$,*

$$(2.8) \qquad \sum_{0\leq|k|\leq m} |\partial^k\mathcal{R}(x,y)| \leq c(a,m), \quad \forall(x,y) \in [0,a]^2,$$

$$(2.9) \qquad \sum_{0\leq|k|\leq m} |L^k\nabla^k\mathcal{R}^d(x,y)| \leq c(a,m), \quad \forall(x,y) \in ([0,aL]\cap\mathbb{Z})^2,$$

$$(2.10) \qquad \lim_{L\to\infty} \sup_{(x,y)\in([0,aL]\cap\mathbb{Z})^2} \sum_{0\leq|k|\leq m} |\partial^k\mathcal{R}(\tfrac{x}{L},\tfrac{y}{L}) - L^k\nabla^k\mathcal{R}^d(x,y)| = 0.$$

Proof. Consider the formula (1.4) for $\mathcal{R}$, and *fix* a contour $\mathcal{C}$ as described therein. This is a closed curve of finite length, and along the contour $z \in \mathcal{C}$ are bounded in absolute value, i.e., $|z| \leq c$. Each of the factors in the integrand is bounded over $[0,a]^2$ uniformly in $z \in \mathcal{C}$. Moreover, each ∂_x brings down a factor $-\frac{(\beta_1-\beta_2)z}{z+\beta_2}$, and each ∂_y brings down a factor $\frac{(\beta_1-\beta_2)z}{z+\beta_1}$; these are all bounded uniformly in $z \in \mathcal{C}$. From these discussions we conclude (2.8).

Noting that (2.9) follows from (2.8) and (2.10), we now move on to proving (2.10). Apply changes of variables to (2.4): $z = L\widetilde{z}/(\beta_1\beta_2)$, $x = L\widetilde{x}$, $y = L\widetilde{y}$. Then

(2.11)

$$\mathcal{R}^{\mathrm{d}}(\widetilde{x}, \widetilde{y}) = \frac{1}{2\pi\mathbf{i}} \oint_{-\frac{\beta_1\beta_2}{Lb_2(1-b_1)}} \left(\frac{\beta_1\beta_2 + Lb_1(1-b_1)\widetilde{z}}{\beta_1\beta_2 + Lb_2(1-b_1)\widetilde{z}} \right)^{L\widetilde{x}} \left(\frac{\beta_1\beta_2 + Lb_2(1-b_2)\widetilde{z}}{\beta_1\beta_2 + Lb_1(1-b_2)\widetilde{z}} \right)^{L\widetilde{y}}$$
$$\cdot \frac{L\beta_1\beta_2(b_2 - b_1)\, d\widetilde{z}}{(\beta_1\beta_2 + Lb_2(1-b_1)\widetilde{z})(\beta_1\beta_2 + Lb_1(1-b_2)\widetilde{z})},$$

where the integration goes in the positive direction and encircles $-\frac{\beta_1\beta_2}{Lb_2(1-b_1)}$ but not $-\frac{\beta_1\beta_2}{Lb_1(1-b_2)}$. Indeed, as $L \to \infty$, $-\frac{\beta_1\beta_2}{Lb_2(1-b_1)} \to -\beta_2$ and $-\frac{\beta_1\beta_2}{Lb_1(1-b_2)} \to -\beta_1$. This being the case, we fix a contour $\mathcal{C}'$ (*independently* of L) that goes in the positive direction encircling $-\beta_2$ but not $-\beta_1$. It is readily checked that *uniformly* over $\widetilde{z} \in \mathcal{C}'$ and $\widetilde{x}, \widetilde{y} \in [0, a]$, as $L \to \infty$,

$$\left(\frac{\beta_1\beta_2 + Lb_1(1-b_1)\widetilde{z}}{\beta_1\beta_2 + Lb_2(1-b_1)\widetilde{z}} \right)^{L\widetilde{x}} \longrightarrow \exp\left((\beta_1 - \beta_2)\left(-\widetilde{x}\frac{\widetilde{z}}{\widetilde{z} + \beta_2} \right) \right),$$

$$\left(\frac{\beta_1\beta_2 + Lb_2(1-b_2)\widetilde{z}}{\beta_1\beta_2 + Lb_1(1-b_2)\widetilde{z}} \right)^{L\widetilde{y}} \longrightarrow \exp\left((\beta_1 - \beta_2)\left(-\widetilde{y}\frac{\widetilde{z}}{\widetilde{z} + \beta_1} \right) \right),$$

$$\frac{L\beta_1\beta_2(b_2 - b_1)}{(\beta_1\beta_2 + Lb_2(1-b_1))(\beta_1\beta_2 + Lb_1(1-b_2)\widetilde{z})} \longrightarrow -\frac{(\beta_2 - \beta_1)}{(\widetilde{z} + \beta_1)(\widetilde{z} + \beta_2)}.$$

Using this in (2.11) gives *uniformly* over $\widetilde{x}, \widetilde{y} \in [0, a]$ as $L \to \infty$,

(2.12)

$$\mathcal{R}^{\mathrm{d}}(\widetilde{x}, \widetilde{y}) \longrightarrow -\frac{1}{2\pi\mathbf{i}} \oint_{\mathcal{C}'} \exp\left((\beta_1 - \beta_2)\left(-\widetilde{x}\frac{\widetilde{z}}{\widetilde{z} + \beta_2} + \widetilde{y}\frac{\widetilde{z}}{\widetilde{z} + \beta_1} \right) \right) \frac{(\beta_2 - \beta_1)\, d\widetilde{z}}{(\widetilde{z} + \beta_1)(\widetilde{z} + \beta_2)}.$$

Note that, compared to (1.4), the r.h.s. of (2.12) has a different contour $\mathcal{C}'$ and an outstanding negative sign. However, as noted in [BG18] (see comments after equation (39) therein), the integrand in (1.4) and (2.12) has no pole at $|\widetilde{z}| = \infty$, so the contour $\mathcal{C}'$ can be deformed to $-\mathcal{C}$ (the orientation changes after deformation), matching the r.h.s. of (2.12) to (1.4). This proves (2.10) for $|k| = 0$. As for $|k| > 0$, note that each $L\nabla_x$ applied to (2.4) brings a factor $\frac{L(b_1-b_2)(1-b_1)z}{1+b_2(1-b_1)z}$, and each $L\nabla_y$ brings a factor $\frac{L(b_2-b_1)(1-b_2)z}{1+b_1(1-b_2)z}$. These factors converge uniformly over $\widetilde{z} \in \mathcal{C}'$ to $\frac{(\beta_1-\beta_2)\widetilde{z}}{\widetilde{z}+\beta_2}$ and $\frac{(\beta_1-\beta_2)\widetilde{z}}{\widetilde{z}+\beta_1}$, respectively. Hence (2.10) follows. $\qquad\square$

Lemma 2.2. *Given $a < \infty$, we have, for all $(x, y) \in ([0, aL] \cap \mathbb{Z})^2$,*

(2.13)
$$|\phi(x,y)| + |L\nabla_x\phi(x,y)| + |L\nabla_y\phi(x,y)| \le c(a),$$

(2.14)
$$|\overline{\phi}(x,y)| + |L\nabla_x\overline{\phi}(x,y)| + |L\nabla_y\overline{\phi}(x,y)| \le c(a),$$

(2.15)
$$\sup_{(x,y)\in[0,a]^2} |\overline{\phi}_L(x, y) - \Phi(x, y)| \longrightarrow 0, \quad as\ L \to \infty.$$

Proof. Given Lemma 2.1, (2.14)–(2.15) are readily verified from (2.7). As for (2.13), recall that $\mathfrak{q} := e^{\beta_1 - \beta_2}$ is fixed. Indeed, since $H(0,0) := 0$ by definition and $\nabla_x H(x, y) \in \{0, 1\}$ and $\nabla_y H(x, y) \in \{0, 1\}$, we have $\phi(x, y) := \mathfrak{q}^{\frac{1}{L}H(x,y)} \le \mathfrak{q}^{\frac{2La}{L}} = c(a)$ and $|L\nabla_\beta \phi(x, y)| = |L(\mathfrak{q}^{\frac{1}{L}\nabla_a H(x,y)} - 1)\phi(x, y)| \le c(a)$, for $\beta = x, y$. $\qquad\square$

Lemma 2.3. *For any* $k \in \mathbb{Z}_{>0}$,

$$(2.16) \qquad \mathbf{E}\big[\,|\xi^d(x+1,y+1)|^k \,\big|\, H(x',y'), x' \leq x \text{ or } y' \leq y\big] \leq c(k)L^{-k-1},$$

$$(2.17) \qquad\qquad\qquad\qquad |\xi^d(x+1,y+1)| \leq cL^{-1},$$

for all $(x,y) \in \mathbb{Z}_{\geq 0}^2$, $L \geq 1$.

Proof. First, conditioning $H(x',y'), x' \leq x$ or $y' \leq y$ for $\xi^d(x+1,y+1)$ amounts to conditioning on incoming line configuration into the site $(x+1, y+1)$. There are four cases pertaining to such conditions, and in each case $\xi^d(x+1, y+1)$ is computed in [BG18, Proof of Theorem 3.1], using the 'four-point relation' (2.1). We record the results of their computation here and examine the asymptotics in L of the values of ξ^d and their probabilities in each case. In the following, vertices of type I–VI refers to those depicted in Figure 1.

(1) No line enters into the vertex $(x+1, y+1)$ from below or from the left: In this case the vertex is of type I, whereby $\xi^d(x+1, y+1) = 0$.

(2) Two lines enter into the vertex $(x+1, y+1)$, one from below and one from the left: In this case the vertex is of type II, whereby $\xi^d(x+1,y+1) = 0$.

(3) One line enters into the vertex $(x+1, y+1)$ from below, but no line enters from the left: In this case the vertex is of type IV with probability b_2 and

$$|\xi^d(x+1,y+1)| = |q^h(q^{-1} - b_1)(1-q)| \leq cL^{-2}$$

or of type VI with probability $1 - b_2 \leq cL^{-1}$ and

$$|\xi^d(x+1,y+1)| = |q^h b_1(q-1)| \leq cL^{-1}.$$

(4) One line enters into the vertex $(x+1, y+1)$ from the left, but no line enters from below: In this case the vertex is of type III with probability b_1 and

$$|\xi^d(x+1,y+1)| = |q^h(1-b_1)(q-1)| \leq cL^{-2}$$

or of type V with probability $1 - b_1 \leq cL^{-1}$ and

$$|\xi^d(x+1,y+1)| = |q^h b_1(1-q)| \leq cL^{-1}.$$

The conditional moments bound (2.16), and the uniform bound (2.17) readily follows from the preceding discussion. $\qquad\square$

3. Proof of Theorem 1.3 and Corollary 1.4

3.1. **Proof of Theorem 1.3.** Write $\Rightarrow$ for convergence in distribution. Hereafter throughout the article, we fix $(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_n, \mathbf{y}_n) \in \mathbb{R}_+^2$. Our goal is to prove $(\sqrt{L}u(L\mathbf{x}_i, L\mathbf{y}_i))_{i=1}^n \Rightarrow (U(\mathbf{x}_i, \mathbf{y}_i))_{i=1}^n$. To simplify notation, we work under the consent that whenever the arguments of u are not integers, we take the integer parts of them, e.g., $u(L\mathbf{x}_i, L\mathbf{y}_i) := u(\lfloor L\mathbf{x}_i \rfloor, \lfloor L\mathbf{y}_i \rfloor)$. A similar convention is adopted without being explicitly stated for processes over integers.

Given the expression (2.6) of u, we proceed via the martingale Central Limit Theorem (CLT) (as in [BG18] for the low density regime). To this end we linearly order points on $\mathbb{Z}_{>0}^2$ as

$$(3.1) \qquad (x(1), y(1)) := (1, 1), \quad (x(2), y(2)) := (2, 1), \quad (x(3), y(3)) := (1, 2),$$
$$(x(4), y(4)) := (3, 1), \cdots.$$

Consider the discrete time process $M(t) \in \mathbb{R}^d$, $t = 1, 2, \ldots,$

(3.2)

$$M(t) := (M_i(t))_{i=1}^n, \quad M_i(t) := \sum_{s=1}^t \sqrt{L} \widetilde{\mathcal{R}}^{\mathrm{d}}(L\mathbf{x}_i - x(s), L\mathbf{y}_i - y(s)) \, \xi^{\mathrm{d}}(x(s), y(s)).$$

It follows from (2.2) that $M(t)$ is a martingale. Recall that, by definition, $\widetilde{\mathcal{R}}^{\mathrm{d}}(x, y)$ carries indicator functions forcing $x, y \geq 0$. Hence, for some large enough $c_* < \infty$,

(3.3) $\qquad M(c_* L^2) = M(c_* L^2 + 1) = \cdots = M(\infty) = (\sqrt{L} u(L\mathbf{x}_i, L\mathbf{y}_i))_{i=1}^n.$

Let $\mathscr{F}(t) := \sigma(M(1), \ldots, M(t))$ denote the canonical filtration of $M(t)$, and recall that cross variance of M is defined as

(3.4) $\quad \langle M_i, M_j \rangle(t) := \sum_{s=1}^t \mathbf{E}\big[(M_i(s) - M_i(s-1))(M_j(s) - M_j(s-1))\big|\mathscr{F}(s-1)\big].$

Put $\widetilde{\mathcal{R}}(x, y) := \mathcal{R}(x, y)\mathbf{1}_{\{x \geq 0\}}\mathbf{1}_{\{y \geq 0\}}$, and recall the definition of $D(x, y)$ from (1.6). We set

(3.5)

$$Q_{ij} := \int_{\mathbb{R}_+^2} \widetilde{\mathcal{R}}_{ij}(x, y) \, D(x, y) dx dy, \quad \widetilde{\mathcal{R}}_{ij}(x, y) := \widetilde{\mathcal{R}}(\mathbf{x}_i - x, \mathbf{y}_i - y)\widetilde{\mathcal{R}}(\mathbf{x}_j - x, \mathbf{y}_j - y).$$

The martingale CLT from [HH14] applied to $M(t)$ gives

Theorem 3.1 ([HH14, Corollary 3.1]). *If, for any $i, j = 1, \ldots, n$ and $\varepsilon > 0$,*

(Lind) $\qquad \sum_{s=1}^{L^2 c_*} \mathbf{E}\big[(M_i(s) - M_i(s-1))^2 \mathbf{1}_{\{|M_i(s) - M_i(s-1)| > \varepsilon\}}\big] \longrightarrow 0,$

(QV) $\qquad \langle M_i, M_j \rangle(L^2 c_*) \Longrightarrow_P Q_{ij},$

then

$$M(c_* L^2) \Longrightarrow (U(\mathbf{x}_i, \mathbf{y}_i))_{i=1}^n.$$

Remark 3.2. Note that even though [HH14, Corollary 3.1] is stated for an $\mathbb{R}$-valued martingale, generalization to $\mathbb{R}^n$-value is standard, by projection $M(t) \in \mathbb{R}^n$ onto arbitrarily fixed $v \in \mathbb{R}^n$.

Given Theorem 3.1, it suffices to check the conditions (Lind)–(QV). The former follows at once from the fact that $|\xi^{\mathrm{d}}(x, y)| \leq cL^{-1}$ (from Lemma 2.3), which makes the indicator functions in (Lind) zero for all large enough L. We hence devote the rest of the article to proving (QV). From (2.3) we calculate the cross variance (defined in (3.4)) as

(3.6) $\qquad \langle M_i, M_j \rangle(c_* L^2) = L^{-2} \sum_{s=1}^{c_* L^2} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x(s), y(s)) \, D^{\mathrm{d}}(x(s), y(s); \phi),$

where

$$(3.7) \qquad \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x,y) := \widetilde{\mathcal{R}}^{\mathrm{d}}(L\mathbf{x}_i - x, L\mathbf{y}_i - y)\widetilde{\mathcal{R}}^{\mathrm{d}}(L\mathbf{x}_j - x, L\mathbf{y}_j - y),$$

$$(3.8) \qquad D^{\mathrm{d}}(x,y;\phi) := \gamma_{xy} \cdot L\nabla_x \phi(x,y) \cdot L\nabla_y \phi(x,y) + \gamma_x \cdot \phi(x,y) \cdot L\nabla_x \phi(x,y)$$
$$+ \gamma_y \cdot \phi(x,y) \cdot L\nabla_y \phi(x,y),$$

$$(3.9) \qquad \gamma_{xy} := L(b_2(1-b_1) + b_1(1-b_2)), \ \gamma_x := L^2(b_1(1-b_2)(1-q)),$$
$$\gamma_y := -L^2(b_1(1-b_1)(1-q)).$$

Recall that $\overline{\phi}(x,y) := \mathbf{E}[\phi(x,y)]$. Compare (3.5) and (3.6)–(3.8). The main step toward proving (QV) is to show that, in (3.8), we can approximate ϕ, $L\nabla_x\phi$, and $L\nabla_y\phi$ by their continuum counterparts Φ, $\partial_x\Phi$, and $\partial_y\Phi$, in a suitable sense under the limit $L \to \infty$. With this in mind, we decompose $\langle M_i, M_j \rangle (L^2 c_*) - Q_{ij} = S_1 + S_2$, where

$$(3.10) \qquad S_1 := L^{-2} \sum_{s=1}^{c_* L^2} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x(s), y(s)) \, D^{\mathrm{d}}(x(s), y(s); \overline{\phi}) - Q_{ij},$$

$$(3.11) \qquad S_2 := L^{-2} \sum_{s=1}^{c_* L^2} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x(s), y(s)) \left(D^{\mathrm{d}}(x(s), y(s); \phi) - D^{\mathrm{d}}(x(s), y(s); \overline{\phi}) \right).$$

Here, S_2 records the difference of replacing ϕ, $L\nabla_x\phi$, and $L\nabla_y\phi$ with their respective expectations $\overline{\phi}$, $L\nabla_x\overline{\phi}$, and $L\nabla_y\overline{\phi}$, while S_1 accounts for the difference between $\overline{\phi}$, $L\nabla_x\overline{\phi}$, and $L\nabla_y\overline{\phi}$ with their corresponding terms in continuum Φ, $\partial_x\Phi$, and $\partial_y\Phi$. In particular, note that S_1 is deterministic. We will show separately that $S_1 \to 0$ and $S_2 \to_P 0$:

Proposition 3.3. *For fixed $i, j \in \{1, \ldots, n\}$, with S_1 and S_2 defined in (3.10)– (3.11), we have, as $L \to \infty$, (a) $S_1 \to 0$ and (b) $S_2 \to_P 0$.*

Proposition 3.3 verifies the condition (QV) and hence completes the proof of Theorem 1.3. The proof of Proposition 3.3 is carried out in Sections 4–5.

3.2. **Proof of Corollary 1.4.** Fix $a < \infty$. Throughout this proof we assume $x, y \in [0, aL] \cap \mathbb{Z}$ and write $c = c(a)$ to simplify notation. The first step is to express H in terms of ϕ. To this end, write

$$(3.12) \qquad H(x,y) = L \log_{\mathfrak{q}} \left(\phi(x,y) \right) = L \log_{\mathfrak{q}} \left(\overline{\phi}(x,y) + u(x,y) \right).$$

Recall that $\phi = \mathfrak{q}^{\frac{1}{L}H}$ and that $H(0,0) := 0$ and H is 1-Lipschitz (from the definition of height function). Hence

$$(3.13) \qquad \tfrac{1}{c} \leq \phi(x,y) \leq c, \quad \tfrac{1}{c} \leq \overline{\phi}(x,y) \leq c, \quad x, y \in [0, aL] \cap \mathbb{Z}.$$

In (3.12), Taylor expand the function $\log_{\mathfrak{q}}(\overline{\phi} + u)$ in u to the first order. With the aid of preceding bounds, we have

$$(3.14) \qquad H(x,y) = L \log_{\mathfrak{q}}(\overline{\phi}(x,y)) + \frac{L}{\overline{\phi}(x,y) \log \mathfrak{q}} u(x,y) + LR(x,y),$$

for some remainder R such that

$$(3.15) \qquad |R(x,y)| \leq cu^2(x,y).$$

Take expectation in (3.14), and subtract the result from (3.14). With $\mathbf{E}[u(x,y)]$ $= 0$, we have

$$H(x,y) - \mathbf{E}[H(x,y)] = \tfrac{L}{\overline{\phi}(x,y)\log \mathfrak{q}}u(x,y) + L\big(R(x,y) - \mathbf{E}[R(x,y)]\big).$$

Recall the scaling convention from Definition 1.1. Divide both sides by $\sqrt{L}$. With $\log \mathfrak{q} = \beta_1 - \beta_2$, we have

$$(3.16) \qquad \tfrac{1}{\sqrt{L}}\big(H_L - \mathbf{E}[H_L]\big) = \tfrac{1}{\overline{\phi}_L(\beta_1-\beta_2)}\sqrt{L}u_L + \sqrt{L}\big(R_L - \mathbf{E}[R_L]\big).$$

From Theorem 1.3 we already have $\sqrt{L}u_L \to U$ in finite dimensional distributions. This together with (2.15) and (3.13) gives $\frac{1}{\overline{\phi}_L(\beta_1-\beta_2)}\sqrt{L}u_L \to \widetilde{U}$ in finite dimensional distributions. To control the last term in (3.16), we calculate the second moment of $u(x,y)$ from (2.6). By (2.2), the discrete noise $\xi^{\mathrm{d}}(x,y)$, $x,y \in \mathbb{Z}_{>0}$ are uncorrelated, so $\mathbf{E}[u(x,y)^2] = \sum_{x',y'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x', y-y')^2\mathbf{E}[\xi^{\mathrm{d}}(x',y')^2]$. Further using the bounds on $\mathcal{R}^{\mathrm{d}}$ from Lemma 2.1 and the bound on $\mathbf{E}[\xi^{\mathrm{d}}(x',y')^2]$ from Lemma 2.3, we conclude that $\mathbf{E}[u(x,y)^2] \leq cL^{-1}$. Combining this with (3.15) gives $\mathbf{E}|R(x,y)| \leq cL^{-1}$. From this, we see that the last term in (3.16) converges to zero in finite dimensional distributions. This completes the proof.

4. PROOF OF PROPOSITION 3.3(a)

Recall that $(\mathbf{x}_1,\mathbf{y}_1),\ldots,(\mathbf{x}_n,\mathbf{y}_n) \in \mathbb{R}_+^2$ are points fixed previously. Hereafter, we fix further $i,j \in \{1,\ldots,n\}$. Recall from Lemma 2.2 that $\overline{\phi}_L(x,y) = \overline{\phi}(Lx,Ly)$ converges uniformly to Φ. On the other hand, from the summation representation (2.7), it is not hard to check that $L\nabla_x\phi(Lx,Ly) \not\to \partial_x\Phi$ and $L\nabla_y\phi(Lx,Ly) \not\to \partial_y\Phi$ in general.[1] That is, derivatives of $\overline{\phi}$ do *not* converge pointwisely. Given that the quantities D and D^{d} (defined in (1.6) and (3.8)) involve gradients, in order to show $S_1 \to \infty$, one needs to exploit the sum over s in (3.10), as well as the integral over x,y in (3.5). The sum and integral smear out the possibly fluctuating derivatives. In the following two lemmas we expose the aforementioned smearing effect via integration-by-parts and summation-by-parts formulas.

Let $\mathrm{Lip}(\mathbb{R}_+)$ and $\mathrm{Lip}(\mathbb{R}_+^2)$ denote the spaces of functions that are uniformly Lipschitz respectively over compact subsets of $\mathbb{R}_+$ and $\mathbb{R}_+^2$. Following the preceding discussion, instead of Lipschitz norms, we equip $\mathrm{Lip}(\mathbb{R}_+)$ and $\mathrm{Lip}(\mathbb{R}_+^2)$ with the topology of uniform convergence over compact subsets. Recall that $\widetilde{\mathcal{R}}_{ij}(x,y) := \widetilde{\mathcal{R}}(\mathbf{x}_i - x, \mathbf{y}_i - y)\widetilde{\mathcal{R}}(\mathbf{x}_j - x, \mathbf{y}_j - y)$. For $\alpha = x,y$, consider the map

$$(4.1) \qquad V_\alpha : \mathrm{Lip}(\mathbb{R}_+^2) \longrightarrow \mathbb{R}, \quad V_\alpha(\Phi) := \int_{\mathbb{R}_+^2} \widetilde{\mathcal{R}}_{ij}(x,y)\, \partial_\alpha\Phi(x,y) \cdot \Phi(x,y)\, dxdy.$$

For given $\chi,\psi \in \mathrm{Lip}(\mathbb{R}_+)$, let $\Phi = \Phi(\chi,\psi)$ defined through (1.5). Consider the following map $V_{xy} : \mathrm{Lip}(\mathbb{R}_+) \times \mathrm{Lip}(\mathbb{R}_+) \to \mathbb{R}$:

(4.2)

$$V_{xy}(\chi,\psi) := \int_{\mathbb{R}_+^2} \widetilde{\mathcal{R}}_{ij}(x,y)\, \partial_x\Phi(x,y) \cdot \partial_y\Phi(x,y)\, dxdy, \quad \Phi = \Phi(\chi,\psi) \quad \text{via (1.5)}.$$

---

[1] This lack of convergence can also be seen from the fact that the left hand side remains a discrete random variable taking values in $\{0, \pm \ln \mathfrak{q}\}$ in the limit, so there is no way for it to converge to $\partial\Phi$.

Lemma 4.1. *For $\alpha, \beta \in \{x, y\}$, the maps $V_\alpha : Lip(\mathbb{R}_+^2) \to \mathbb{R}$, $V_{xy} : Lip(\mathbb{R}_+)^2 \to \mathbb{R}$ are continuous (under uniform topology, as declared previously).*

Proof. We begin with V_α. Take $\alpha = x$ to simplify notation. The case $\alpha = y$ follows exactly the same. To simplify notation, set $\overline{\mathbf{x}} := \mathbf{x}_i \wedge \mathbf{x}_j$ and $\overline{\mathbf{y}} := \mathbf{y}_i \wedge \mathbf{y}_j$ and $\mathcal{R}_{ij}(x, y) := \mathcal{R}(\mathbf{x}_i - x, \mathbf{y}_i - y)\mathcal{R}(\mathbf{x}_j - x, \mathbf{y}_j - y)$. In (4.1), writing $\partial_x \Phi \cdot \Phi = \frac{1}{2}\partial_x(\Phi^2)$, we have

$$V_x(\Phi) = \frac{1}{2}\int_0^{\overline{\mathbf{x}}}\int_0^{\overline{\mathbf{y}}} \mathcal{R}_{ij}(x, y)\,\partial_x(\Phi^2(x, y))\,dxdy.$$

Integration by parts in x gives

(4.3)
$$V_x(\Phi) = \frac{1}{2}\int_0^{\overline{\mathbf{y}}} \mathcal{R}_{ij}(x, y)\Phi^2(x, y)\Big|_{x=0}^{x=\overline{\mathbf{x}}}dy - \frac{1}{2}\int_0^{\overline{\mathbf{x}}}\int_0^{\overline{\mathbf{y}}} \partial_x\mathcal{R}_{ij}(x, y)\cdot\Phi^2(x, y)dxdy.$$

Given that $\mathcal{R}_{ij}$ is smooth (from Lemma 2.1), from (4.3) it is clear that V_x is continuous in Φ.

Turning to V_{xy}, take the x-derivative in (1.5) to get $\partial_x\Phi(x, y) = \mathcal{R}(0, y)\chi'(x) + G(x, y)$, where

(4.4)
$$\begin{aligned}
G(x, y) := {}& \psi(0)\partial_x\mathcal{R}(x, y) + \beta_1\mathcal{R}(0, y)\chi(x) \\
& + \int_0^y \partial_x\mathcal{R}(x, y - y')\big(\psi'(y') + \beta_2\psi(y')\big)dy' \\
& + \int_0^x \partial_x\mathcal{R}(x - x', y)\big(\chi'(x') + \beta_1\chi(x')\big)dx'.
\end{aligned}$$

Note that G involves the derivatives χ' and ψ'. We integrate by parts to separate the dependence on χ' and ψ' from the dependence on χ and ψ. To state this precisely, consider the set $\mathcal{K}$ that consists of finite linear combinations of the expressions

(4.5)
$$\partial^{k_1}\mathcal{R}(x - x_1, y)\chi(x_1), \quad \partial^{k_2}\mathcal{R}(x, y - y_2)\psi(y_2),$$
$$\int_0^x \partial^{k_3}\mathcal{R}(x - x', y)\chi(x')dx', \quad \int_0^y \partial^{k_4}\mathcal{R}(x, y - y')\psi(y')dy',$$

where $k_i = (k_i, k_i') \in \mathbb{Z}_{\geq 0}^2$ are multi-indices (defined in Lemma 2.1) with $|k_i| \leq 3$, $x_1 \in \{0, x\}$, and $y_2 \in \{0, y\}$. That is,

(4.6)
$$\mathcal{K} := \Big\{ \sum \alpha_{\text{term}} \cdot \big(\text{term in (4.5)}\big) \Big\}.$$

In (4.4), integrating by parts in y' and in x' respectively for the first and second integrals, we have

(4.7)
$$\partial_x\Phi(x, y) = \mathcal{R}(0, y)\chi'(x) + \partial_x\mathcal{R}(x, 0)\psi(y) + \mathcal{K}_y,$$

for some $\mathcal{K}_y$ such that $\mathcal{K}_y, \partial_y\mathcal{K}_y \in \mathcal{K}$. A similar calculation applied to $\partial_y\Phi$ gives

(4.8)
$$\partial_y\Phi(x, y) = \mathcal{R}(x, 0)\psi'(y) + \partial_y\mathcal{R}(0, y)\chi(x) + \mathcal{K}_x,$$

for some $\mathcal{K}_x$ such that $\mathcal{K}_x, \partial_x\mathcal{K}_x \in \mathcal{K}$.

Inserting (4.7)–(4.8) into (4.2) gives

(4.9a)
$$V_{xy}(\chi, \psi) = \mathcal{I}\big(\mathcal{R}(0, y)\chi'(x) \cdot \mathcal{R}(x, 0)\psi'(y)\big)$$

(4.9b)
$$+ \mathcal{I}\big(\mathcal{R}(0, y)\chi'(x) \cdot \partial_y \mathcal{R}(0, y)\chi(x)\big) + \mathcal{I}\big(\partial_x \mathcal{R}(x, 0)\psi(y) \cdot \mathcal{R}(x, 0)\psi'(y)\big)$$

(4.9c)
$$+ \mathcal{I}\big(\mathcal{R}(0, y)\chi'(x) \cdot \mathcal{K}_x\big) + \mathcal{I}\big(\mathcal{K}_y \cdot \mathcal{R}(x, 0)\psi'(y)\big)$$

(4.9d)
$$+ \mathcal{I}\big((\partial_x \mathcal{R}(x, 0)\psi(y) + \mathcal{K}_y) \cdot (\partial_y \mathcal{R}(0, y)\chi(x) + \mathcal{K}_x)\big),$$

where

(4.10)
$$\mathcal{I}(f) := \int_0^{\overline{\mathbf{x}}} \int_0^{\overline{\mathbf{y}}} \mathcal{R}_{ij}(x, y) f(x, y) dx dy.$$

To complete the proof, we next argue that each term in (4.9a)–(4.9d) is a continuous function of (χ, ψ). For (4.9d), we indeed have $\partial_x \mathcal{R}(x, 0)\psi(y), \partial_y \mathcal{R}(0, y)\chi(x) \in \mathcal{K}$. Consequently, the expression

$$(\partial_x \mathcal{R}(x, 0)\psi(y) + \mathcal{K}_y) \cdot (\partial_y \mathcal{R}(0, y)\chi(x) + \mathcal{K}_x)$$

defines a continuous function of $(x, y, \chi, \psi) \in \mathbb{R}_+^2 \times C(\mathbb{R}_+)^2$. Given this property and referring to (4.10), we see that the term in (4.9d) is a continuous function of (χ, ψ). Turning to (4.9c), we note that the terms involve χ' and ψ'. We integrate by parts in x and y, respectively, for the first and second terms in (4.9c). This removes the derivatives on χ and ψ. Further, $\mathcal{K}_x$ and $\mathcal{K}_y$ remain $\mathcal{K}$-valued upon differentiating in x and y, respectively. From this, we see that the terms in (4.9c) are continuous functions of (χ, ψ). Moving on to (4.9b), we write

$$\mathcal{R}(0, y)\chi'(x) \cdot \partial_y \mathcal{R}(0, y)\chi(x) = \mathcal{R}(0, y)\partial_y \mathcal{R}(0, y) \cdot \tfrac{1}{2}\tfrac{d}{dx}\chi(x)^2,$$
$$\partial_x \mathcal{R}(x, 0)\psi(y) \cdot \mathcal{R}(x, 0)\psi'(y) = \mathcal{R}(x, 0)\partial_x \mathcal{R}(x, 0) \cdot \tfrac{1}{2}\tfrac{d}{dy}\psi(x)^2.$$

Given these expressions, integrating by parts in x and y, respectively, for the first and second terms in (4.9b), we conclude that the terms are continuous functions of (χ, ψ). Finally, for (4.9a), straightforward integration by parts in x and y verifies that the term is a continuous function of (χ, ψ). $\square$

Next we turn to the discrete analog of Lemma 4.1. Recall that $\widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x, y) := \widetilde{\mathcal{R}}^{\mathrm{d}}(L\mathbf{x}_i - x, \mathbf{y}_i - y)\widetilde{\mathcal{R}}(L\mathbf{x}_j - x, \mathbf{y}_j - y)$. For $\alpha = x, y$, set

(4.11)
$$\mathsf{v}_\alpha := L^{-2} \sum_{x, y \in \mathbb{Z}_{>0}} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x, y) \, (L\nabla_\alpha \overline{\phi}(x, y)) \, \overline{\phi}(x, y),$$

(4.12)
$$\mathsf{v}_{xy} := L^{-2} \sum_{x, y \in \mathbb{Z}_{>0}} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x, y) \, (L\nabla_x \overline{\phi}(x, y)) \, (L\nabla_y \overline{\phi}(x, y)).$$

Recall the scaling notation and interpolation convention from Definition 1.1.

Lemma 4.2. *Abusing notation, we write* $V_{xy}(\overline{\phi}_L) := V_{xy}(\overline{\phi}_L(\cdot, 0), \overline{\phi}_L(0, \cdot))$. *Then, as* $L \to \infty$,

$$\sum_{\alpha = x, y} \big|\mathsf{v}_\alpha - V_\alpha(\overline{\phi}_L)\big| + \big|\mathsf{v}_{xy} - V_{xy}(\overline{\phi}_L)\big| \longrightarrow 0.$$

Proof. We begin by bounding $|\mathsf{v}_\alpha - V_\alpha(\overline{\phi}_L)|$. Take $\alpha = x$ to simplify notation. The case $\alpha = y$ follows exactly the same. Let $\widetilde{\mathsf{v}}_x$ denote the analog of v_x where the last factor $\overline{\phi}(x,y)$ in (4.11) is replaced by $\overline{\phi}(x+1,y)$. Using

$$(4.13) \qquad \nabla f(x) \cdot f(x+1) + f(x) \cdot \nabla f(x) = \nabla f^2(x)$$

for $f(x) = \overline{\phi}(x,y)$, we have

$$\frac{1}{2}(\mathsf{v}_x + \widetilde{\mathsf{v}}_x) = \frac{1}{2L^2} \sum_{x,y \in \mathbb{Z}_{>0}} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x,y) L \nabla_x\big(\overline{\phi}^2(x,y)\big).$$

Set $\mathcal{R}_{ij}^{\mathrm{d}}(x,y) := \mathcal{R}^{\mathrm{d}}(L\mathbf{x}_i - x, L\mathbf{y}_i - y)\mathcal{R}(\mathbf{x}_j - x, \mathbf{y}_j - y)$, and recall that $\overline{\mathbf{x}} := \mathbf{x}_i \wedge \mathbf{x}_j$ and $\overline{\mathbf{y}} := \mathbf{y}_i \wedge \mathbf{y}_j$. Further, given the bounds on $\mathcal{R}^{\mathrm{d}}$ from Lemma 2.1 and the bound on $\nabla_x\overline{\phi}$ from Lemma 2.2, we have $|\mathsf{v}_x - \widetilde{\mathsf{v}}_x| \le cL^{-1}$, so

$$(4.14) \qquad \mathsf{v}_x = \frac{1}{2L^2} \sum_{x,y \in \mathbb{Z}_{>0}} \widetilde{\mathcal{R}}_{ij}^{\mathrm{d}}(x,y)\, L\nabla_x(\overline{\phi}(x,y)^2) + r$$

$$= \frac{1}{2L^2} \sum_{x=1}^{L\overline{\mathbf{x}}} \sum_{y=1}^{L\overline{\mathbf{y}}} \mathcal{R}_{ij}^{\mathrm{d}}(x,y)\, L\nabla_x(\overline{\phi}(x,y)^2) + r,$$

for some remainder term r such that $|r| \le cL^{-1}$. In (4.14), applying summation by parts

$$\sum_{i=a}^{b} f(i) \cdot \nabla g(i) = f(i-1)g(i)\big|_{i=a}^{i=b+1} - \sum_{i=a}^{b} \nabla f(i-1) \cdot g(i)$$

in the variable x gives

$$\mathsf{v}_x = \frac{1}{2L} \sum_{y=1}^{L\overline{\mathbf{y}}} \mathcal{R}_{ij}^{\mathrm{d}}(x-1,y)\, \overline{\phi}(x,y)^2 \Big|_{x=1}^{x=L\overline{\mathbf{x}}+1}$$

$$- \frac{1}{2L^2} \sum_{x=1}^{L\overline{\mathbf{x}}} \sum_{y=1}^{L\overline{\mathbf{y}}} L\nabla_x \mathcal{R}_{ij}^{\mathrm{d}}(x-1,y) \cdot \overline{\phi}(x,y)^2 + r.$$

Given Lemmas 2.1–2.2, within the last expression, replacing $\mathcal{R}^{\mathrm{d}}(x,y)$ with $\mathcal{R}(\frac{x}{L}, \frac{y}{L})$ and $L\nabla_x \mathcal{R}^{\mathrm{d}}(x,y)$ with $\partial_x \mathcal{R}(\frac{x}{L}, \frac{y}{L})$ only introduces errors that converge to zero as $L \to \infty$. This gives

$$(4.15) \qquad \mathsf{v}_x = \frac{1}{2L} \sum_{y=1}^{L\overline{\mathbf{y}}} \mathcal{R}_{ij}\big(\tfrac{x-1}{L}, \tfrac{y}{L}\big) \overline{\phi}_L\big(\tfrac{x}{L}, \tfrac{y}{L}\big)^2 \Big|_{x=1}^{x=L\overline{\mathbf{x}}+1}$$

$$- \frac{1}{2L^2} \sum_{x=1}^{L\overline{\mathbf{x}}} \sum_{y=1}^{L\overline{\mathbf{y}}} L\nabla_x \mathcal{R}_{ij}\big(\tfrac{x-1}{L}, \tfrac{y}{L}\big) \cdot \overline{\phi}_L\big(\tfrac{x}{L}, \tfrac{y}{L}\big)^2 + r',$$

for some r' such that $|r'| \to 0$, where $\mathcal{R}_{ij}$ is defined in the proof of Lemma 4.1. Compare (4.3) and (4.15). Since $\{\phi_L\}_L \subset C([0,\overline{\mathbf{x}}] \times [0,\overline{\mathbf{y}}])$ is equicontinuous, and since $\mathcal{R}$ is smooth, in (4.15) replacing sums with integrals and replacing $L\nabla_x$ with ∂_x only introduces errors that converge to zero as $L \to \infty$. From this we conclude that $|\mathsf{v}_x - V_x(\overline{\phi}_L)| \to 0$.

Turning to showing that $|\mathsf{v}_{xy} - V_{xy}(\overline{\phi}_L)| \to 0$, we rewrite (2.7) in a way similar to (1.5) (note that $\overline{\phi} = \phi$ along the axises $\{0\} \times \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\geq 0} \times \{0\}$):

(4.16)
$$\overline{\phi}(x,y) = \overline{\phi}(0,0)\mathcal{R}^{\mathrm{d}}(x,y)$$
$$+ L^{-1} \sum_{y'=1}^{y} \widetilde{\mathcal{R}}^{\mathrm{d}}(x, y-y')\big(L\nabla_{y'}\phi(0, y'-1) + L(1-b_2)\overline{\phi}(0, y'-1)\big)$$
$$+ L^{-1} \sum_{x'=1}^{x} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x', y)\big(L\nabla_{x'}\phi(x'-1, 0) + L(1-b_1)\overline{\phi}(x'-1, 0)\big).$$

Define the discrete analog of $\mathscr{K}$ (as in (4.6)):

$$\mathscr{K}^{\mathrm{d}} := \Big\{ \sum \alpha_{L,\mathrm{term}} \cdot \big(\text{term in (4.17)}\big) : \#\{\alpha_{L,\mathrm{term}} \neq 0\} \leq c, \quad \lim_{L \to 0} \alpha_{L,\mathrm{term}} \in \mathbb{R} \Big\},$$

where, with $k_i = (k_i, k_i') \in \mathbb{Z}_{\geq 0}^2$ being multi-indices with $|k_i| \leq 3$ and with $x_1 \in \{0, x\}$, $y_2 \in \{0, y\}$, and $j_i, j_i', j_i'' \in \{0, \pm 1, \pm 2\}$, the terms read

(4.17)
$$L^{-|k_1|}\nabla^{k_1}\mathcal{R}^{\mathrm{d}}(x - x_1 + j_1, y)\chi(x_1 + j_1'), \quad L^{-|k_2|}\nabla^{k_2}\mathcal{R}^{\mathrm{d}}(x, y - y_2 + j_2)\psi(y_2 + j_2'),$$
$$L^{-1} \sum_{x'=1}^{x} L^{-|k_3|}\nabla^{k_3}\mathcal{R}^{\mathrm{d}}(x - x' + j_3, y + j_3')\chi(x' + j_3''),$$
$$L^{-1} \sum_{y'=1}^{y} L^{-|k_3|}\nabla^{k_3}\mathcal{R}^{\mathrm{d}}(x + j_4, y - y' + j_4')\psi(y' + j_4'').$$

Under the preceding setup, we perform procedures analogous to those leading up to (4.7)–(4.8), with (4.15) in place of (1.5), v_{xy} in place of V_{xy}, $\mathcal{R}^{\mathrm{d}}$ and $L\nabla_x\mathcal{R}^{\mathrm{d}}$ in place of $\mathcal{R}$ and $\partial_x\mathcal{R}$, and $(\overline{\phi}(\cdot, 0), \overline{\phi}(0, \cdot))$ in place of (χ, ψ). This gives

(4.18) $\quad L\nabla_x\overline{\phi}(x,y) = \mathcal{R}^{\mathrm{d}}(0,y) \cdot L\nabla_x\overline{\phi}(x,0) + L\nabla_x\mathcal{R}^{\mathrm{d}}(x,0) \cdot \overline{\phi}(0,y) + \mathcal{K}_y^{\mathrm{d}},$

(4.19) $\quad L\nabla_y\overline{\phi}(x,y) = \mathcal{R}^{\mathrm{d}}(x,0) \cdot L\nabla_y\overline{\phi}(0,y) + L\nabla_y\mathcal{R}^{\mathrm{d}}(0,y) \cdot \overline{\phi}(x,0) + \mathcal{K}_x^{\mathrm{d}},$

for some $\mathcal{K}_\alpha^{\mathrm{d}}$ such that $\mathcal{K}_\alpha^{\mathrm{d}}, L\nabla_\alpha\mathcal{K}_\alpha^{\mathrm{d}} \in \mathscr{K}^{\mathrm{d}}$. For our purpose it is more convenient to change $\overline{\phi}(0,y) \mapsto \frac{1}{2}(\overline{\phi}(0,y) + \overline{\phi}(0, y+1))$ and $\overline{\phi}(x,0) \mapsto \frac{1}{2}(\overline{\phi}(x,0) + \overline{\phi}(x+1, 0))$ in (4.18)–(4.19). To this end, using the bounds on $L\nabla_\alpha\mathcal{R}^{\mathrm{d}}$ from Lemma 2.1 and the bound on $\nabla_\alpha\overline{\phi}$ from Lemma 2.2, we write

(4.18')
$$L\nabla_x\overline{\phi}(x,y) = \mathcal{R}^{\mathrm{d}}(0,y) \cdot L\nabla_x\overline{\phi}(x,0) + L\nabla_x\mathcal{R}^{\mathrm{d}}(x,0) \cdot \tfrac{1}{2}\big(\overline{\phi}(0,y) + \overline{\phi}(0, y+1)\big)$$
$$+ \mathcal{K}_y^{\mathrm{d}} + r_1,$$
(4.19')
$$L\nabla_y\overline{\phi}(x,y) = \mathcal{R}^{\mathrm{d}}(x,0) \cdot L\nabla_y\overline{\phi}(0,y) + L\nabla_y\mathcal{R}^{\mathrm{d}}(0,y) \cdot \tfrac{1}{2}\big(\overline{\phi}(x,0) + \overline{\phi}(x+1, 0)\big)$$
$$+ \mathcal{K}_x^{\mathrm{d}} + r_2,$$

for some r_1, r_2 such that $|r_1|, |r_2| \leq cL^{-1}$.

Inserting (4.18')–(4.19') into (4.12) gives

(4.20a)

$$\mathsf{v}_{xy}(\chi,\psi) = \mathcal{I}^{\mathrm{d}}\Big(\mathcal{R}^{\mathrm{d}}(0,y)L\nabla_x\overline{\phi}(x,0)\cdot\mathcal{R}^{\mathrm{d}}(x,0)L\nabla_y\overline{\phi}(0,y)\Big)$$

(4.20b)
$$+ \mathcal{I}^{\mathrm{d}}\Big(\mathcal{R}^{\mathrm{d}}(0,y)\cdot L\nabla_x\overline{\phi}(x,0)\cdot L\nabla_y\mathcal{R}^{\mathrm{d}}(0,y)\cdot\tfrac{1}{2}\big(\overline{\phi}(x,0)+\overline{\phi}(x+1,0)\big)\Big)$$
$$+ \mathcal{I}^{\mathrm{d}}\Big(L\nabla_x\mathcal{R}^{\mathrm{d}}(x,0)\cdot\tfrac{1}{2}\big(\overline{\phi}(0,y)+\overline{\phi}(0,y+1)\big)\cdot\mathcal{R}^{\mathrm{d}}(x,0)\cdot L\nabla_y\overline{\phi}(0,y)\Big)$$

(4.20c)
$$+ \mathcal{I}^{\mathrm{d}}\Big(\mathcal{R}^{\mathrm{d}}(0,y)L\nabla_x\overline{\phi}(x,0)\cdot\mathcal{K}_x^{\mathrm{d}}\Big) + \mathcal{I}^{\mathrm{d}}\Big(\mathcal{K}_y^{\mathrm{d}}\cdot\mathcal{R}^{\mathrm{d}}(x,0)L\nabla_y\overline{\phi}(0,y)\Big)$$

(4.20d)
$$+ \mathcal{I}^{\mathrm{d}}\Big(\big(L\nabla_x\mathcal{R}^{\mathrm{d}}(x,0)\cdot\tfrac{1}{2}\big(\overline{\phi}(0,y)+\overline{\phi}(0,y+1)\big)+\mathcal{K}_y^{\mathrm{d}}\big)$$
$$\cdot\big(L\nabla_y\mathcal{R}^{\mathrm{d}}(0,y)\cdot\tfrac{1}{2}\big(\overline{\phi}(x,0)+\overline{\phi}(x+1,0)+\mathcal{K}_x^{\mathrm{d}}\big)\big)\Big)$$
$$+ r',$$

where $\mathcal{I}^{\mathrm{d}}(f) := L^{-2}\sum_{x=1}^{L\overline{\mathbf{x}}}\sum_{y=1}^{L\overline{\mathbf{y}}}\mathcal{R}_{ij}^{\mathrm{d}}(x,y)f(x,y)$. Here r' collects all the terms that involve r_1 and r_2 from the expansion. Given the bounds from Lemmas 2.1 and 2.2, we indeed have $|r'| \le cL^{-1}$. Also, using (4.13) for $f(x) = \overline{\phi}(x,0)$ and for $f(y) = \overline{\phi}(0,y)$, we rewrite the terms in (4.20b) as

(4.20b')
$$\mathcal{I}^{\mathrm{d}}\Big(\mathcal{R}^{\mathrm{d}}(0,y)\cdot L\nabla_y\mathcal{R}^{\mathrm{d}}(0,y)\cdot L\nabla_x\big(\overline{\phi}(x,0)^2\big)\Big)$$
$$+ \mathcal{I}^{\mathrm{d}}\Big(L\nabla_x\mathcal{R}^{\mathrm{d}}(x,0)\cdot\mathcal{R}^{\mathrm{d}}(x,0)\cdot L\nabla_y\big(\overline{\phi}(0,y)^2\big)\Big).$$

Recall that, in the proof of Lemma 4.1, we processed the terms in (4.9a)–(4.9d) through integration by parts so that the resulting expressions do not involve derivatives of χ or of ψ. Here, similarly processing (4.20a), (4.20b'), and (4.20c)–(4.20d) (via summation by parts) gives expressions that do not involve discrete gradients of $\overline{\phi}(x,0)$ or of $\overline{\phi}(0,y)$. Given that $\{\phi_L(x,y)\}_L \subset C([0,\overline{\mathbf{x}}]\times[0,\overline{\mathbf{y}}])$ is equicontinuous, and given the bounds from Lemma 2.1, within the *processed* expressions of (4.20a), (4.20b'), and (4.20c)–(4.20d), replacing $\mathcal{R}^{\mathrm{d}}$ and $L^{|k|}\nabla^k\mathcal{R}^{\mathrm{d}}$ with $\mathcal{R}$ and $\partial^k\mathcal{R}$ and replacing the sums with integrals only causes errors that converge to zero as $L \to \infty$. From this we conclude that $|\mathsf{v}_{xy} - V_{xy}(\overline{\phi}_L)| \to 0$. $\qquad\square$

Based on Lemmas 4.1–4.2, we finish the proof of Proposition 3.3(a). With S_1 defined in (3.10), referring to (1.6), (3.8), (4.1)–(4.11), and (4.11)–(4.12), we decompose $S_1 = \sum_{\beta=xy,x,y}\sum_{i=1,2}S_{1\beta}^i$, where

$$S_{1xy}^1 := (\gamma_{xy} - \beta_1 + \beta_2)\mathsf{v}_{xy}, \quad S_{1x}^1 := (\gamma_x - \beta_2(\beta_2 - \beta_1))\mathsf{v}_x,$$
$$S_{1y}^1 := (\gamma_y + \beta_1(\beta_2 - \beta_1))\mathsf{v}_y,$$
$$S_{1xy}^2 := (\beta_1 + \beta_2)\big(\mathsf{v}_{xy} - V_{xy}(\Phi)\big), \quad S_{1x}^2 := \beta_2(\beta_2 - \beta_1)\big(\mathsf{v}_x - V_x(\Phi)\big),$$
$$S_{1y}^2 := -\beta_1(\beta_2 - \beta_1)\big(\mathsf{v}_y - V_y(\Phi)\big).$$

For $S_{1\beta}^1$, $\beta = xy, x, y$, it is readily checked from Lemma 2.2 that $|\mathsf{v}_\beta| \le c$. From (3.9), we have that $\gamma_{xy} \to (\beta_1 + \beta_2)$, $\gamma_x \to \beta_2(\beta_2 - \beta_1)$, and $\gamma_y \to -\beta_1(\beta_2 - \beta_1)$. Hence $S_{1\beta}^1 \to 0$. As for $S_{1\beta}^2$, $\beta = xy, x, y$, further decompose $\mathsf{v}_\beta - V_\beta(\Phi) = (\mathsf{v}_\beta - V_\beta(\overline{\phi}_L)) + (V_\beta(\overline{\phi}_L) - V_\beta(\Phi))$. Using Lemmas 4.1–4.2 and (2.15) to bound the respective terms, we conclude that $S_{1\beta}^2 \to 0$.

5. Proof of Proposition 3.3(b)

The proof begins by deriving a summation representation for $L\nabla_x\phi$ and $L\nabla_y\phi$. To this end, rewrite (2.5) as

$$\phi(x,y) = \overline{\phi}(x,y) + \sum_{x'\in(0,x]} \sum_{y'\in(0,y]} \mathcal{R}^{\mathrm{d}}(x-x', y-y')\xi^{\mathrm{d}}(x',y').$$

Take discrete derivatives on both sides to get

$$(5.1) \qquad L\nabla_x\phi(x,y) = L\nabla_x\overline{\phi}(x,y) + B_x(x,y) + F_x(x,y),$$
$$L\nabla_y\phi(x,y) = L\nabla_y\overline{\phi}(x,y) + B_y(x,y) + F_y(x,y),$$

where

$$(5.2) \qquad B_\alpha(x,y) := \sum_{x',y'\in\mathbb{Z}_{>0}} L\nabla_\alpha\widetilde{\mathcal{R}}^{\mathrm{d}}(x-x', y-y')\xi^{\mathrm{d}}(x',y'),$$

$$(5.3) \qquad F_x(x,y) := L\sum_{y'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(0, y-y')\xi^{\mathrm{d}}(x+1, y'),$$

$$F_y(x,y) := L\sum_{x'\in\mathbb{Z}_{>0}} \widetilde{\mathcal{R}}^{\mathrm{d}}(x-x', 0)\xi^{\mathrm{d}}(x', y+1).$$

Lemma 5.1. *For any fixed $a \in [1,\infty)$ and $f : \mathbb{Z}_{\geq 0}^2 \to \mathbb{R}$, $\alpha = x, y$, we have*

$$\sup_{x,y\in[0,aL]\cap\mathbb{Z}} \mathbf{E}[(B_\alpha(x,y))^2] \leq c(a)L^{-1}, \qquad \sup_{x,y\in[0,aL]\cap\mathbb{Z}} \mathbf{E}[(F_\alpha(x,y))^2] \leq c(a).$$

Proof. For simpler notation, throughout the proof we write $c = c(a)$,

Calculate the second moment of $B_\alpha(x,y)$ (5.2). By Lemma 2.1, the variables $\xi^{\mathrm{d}}(x,y)$, $x,y \in \mathbb{Z}_{>0}^2$ are uncorrelated, so

$$(5.4) \qquad \mathbf{E}[(B_\alpha(x,y))^2] = \sum_{x',y'\in\mathbb{Z}_{>0}} \left(L\nabla_\alpha\widetilde{\mathcal{R}}^{\mathrm{d}}(x-x', y-y')\right)^2 \mathbf{E}\left[\xi^{\mathrm{d}}(x',y')^2\right].$$

By Lemma 2.1, the term $L\nabla_\alpha\widetilde{\mathcal{R}}^{\mathrm{d}}(x-x', y-y')$ is bounded by c. With $x', y' \in [0,aL]\cap\mathbb{Z}$, the number of terms within the sum is $\leq cL^2$. By Lemma 2.3, the $\mathbf{E}[\xi^{\mathrm{d}}(x,y)^2] \leq cL^{-3}$. From these discussions, we conclude the desired bound for B_α.

We now turn to bounding F_α. Take $\alpha = x$ to simplify notation. Following the same argument for obtaining (5.5), here we have

$$(5.5) \qquad \mathbf{E}[(F_x(x,y))^2] = L^2 \sum_{y'\in\mathbb{Z}_{>0}} \left(\widetilde{\mathcal{R}}^{\mathrm{d}}(0, y-y')\right)^2 \mathbf{E}\left[\xi^{\mathrm{d}}(x+1, y')^2\right].$$

By Lemma 2.1, the term $\widetilde{\mathcal{R}}^{\mathrm{d}}(0, y-y')$ is bounded by c. With $y' \in [0,aL]\cap\mathbb{Z}$, the number of terms within the sum is $\leq cL$. By Lemma 2.3, the $\mathbf{E}[\xi^{\mathrm{d}}(x,y)^2] \leq cL^{-3}$. From these discussions, we conclude the desired bound for F_α. $\square$

Having established Lemma 5.1, we now proceed to bounding S_2. To simplify notation, set

$$L^{-2} \sum_{s=1}^{c_* L^2} f(x(s), y(s)) := \Sigma_L^*(f).$$

First, recall from (3.11) and (3.8) that S_2 involves the term $\phi(x,y)L\nabla_x\phi(x,y)$ and $\phi(x,y)L\nabla_y\phi(x,y)$ via $D^{\mathrm{d}}(x,y;\phi)$. From Theorem 1.2 and (2.15), we have that

$\|\phi_L - \overline{\phi}_L\|_{C(\mathbb{R}^2_+)} \to 0$, as $L \to \infty$. From this, together with the bound (2.13) on ϕ and the bounds on $\mathcal{R}^d$ from Lemma 2.1, we see that

$$\Sigma_L^* \Big(\widetilde{\mathcal{R}}_{ij}^d \cdot \big(\gamma_x \phi L \nabla_x \phi - \gamma_x \overline{\phi} L \nabla_x \phi + \gamma_y \phi L \nabla_y \phi - \gamma_y \overline{\phi} L \nabla_y \phi \big) \Big) \longrightarrow_P 0.$$

Granted this, instead of showing $S_2 \to_P 0$, it suffices to show $\widehat{S}_2 \to_P 0$, where

$$(5.6) \quad \widehat{S}_2 := L^{-2} \sum_{s=1}^{c_* L^2} \widetilde{\mathcal{R}}_{ij}^d(x(s), y(s)) \cdot \big(\widehat{D}^d(x(s), y(s); \overline{\phi}, \phi) - D^d(x(s), y(s); \overline{\phi}) \big),$$

$$(5.7) \quad \widehat{D}^d(x, y; \overline{\phi}, \phi) := \gamma_{xy} \cdot L \nabla_x \phi(x, y) \cdot L \nabla_y \phi(x, y) + \gamma_x \cdot \overline{\phi}(x, y) \cdot L \nabla_x \phi(x, y)$$
$$+ \gamma_y \cdot \overline{\phi}(x, y) \cdot L \nabla_y \phi(x, y).$$

Now, insert the expressions (5.1) for $L \nabla_x \phi(x, y)$ and $L \nabla_y \phi(x, y)$ into the r.h.s. of (5.7), and plug the result into (5.6). Expanding the result accordingly, we have that, for some bounded, deterministic $f_{BB}, f_{BF}, g_{BF}, \ldots : \mathbb{Z}_{>0}^2 \to \mathbb{R}$,

$$\widehat{S}_2 = \Sigma_L^* \big(f_{BB}(B_x B_y) \big) + \Sigma_L^* \big(f_B B_x + g_B B_y \big) + \Sigma_L^* \big(f_{BF} B_x F_y + g_{BF} B_y F_x \big)$$
$$+ \Sigma_L^* \big(f_F F_x + g_F F_y \big) + \Sigma_L^* \big(f_{FF} F_x F_y \big)$$
$$:= K_{BB} + K_B + K_{BF} + K_F + K_{FF}.$$

Write $\| \cdot \|_k := (\mathbf{E}[| \cdot |^k])^{1/k}$ for the k-th norm. By triangle inequality and Cauchy–Schwarz inequality, for $p \geq 1$,

$$\|K_{BB}\|_p \leq \Sigma_L^* \big(|f_{BB}| \cdot (\|B_x\|_{2p} \|B_y\|_{2p}) \big),$$
$$\|K_B\|_p \leq \Sigma_L^* \big((|f_B| + |g_B|) \cdot (\|B_x\|_p + \|B_y\|_p) \big),$$
$$\|K_{BF}\|_p \leq \Sigma_L^* \big((|f_{BF}| + |g_{BF}|) \cdot (\|B_x\|_{2p} \|F_y\|_{2p} + \|B_y\|_{2p} \|F_x\|_{2p}) \big).$$

Given that f_{BB} and f_{BF} are bounded, applying Lemma 5.1 gives $\|K_{BB}\|_1, \|K_B\|_2, \|K_{BF}\|_1 \to 0$.

It now remains to show that $K_F, K_{FF} \to_P 0$. From (5.5) and (2.3), it is not hard to check that $\mathbf{E}[F_x(x, y)^2] \not\to 0$ (so our bound in Lemma 5.1 is sharp). Given this situation, unlike in the preceding, here we cannot apply triangle inequality to pass $\| \cdot \|_1$ into the sum Σ_L^*. Instead, we need to exploit the averaging effect of Σ_L^*. This is done in the following lemma, which completes the proof.

Lemma 5.2. *Given deterministic $f : \mathbb{Z}_{>0}^2 \to \mathbb{R}$ and $a < \infty$, we have that*

$$(5.8) \quad \mathbf{E}\Big[\Big(L^{-2} \sum_{x,y \in [0,aL] \cap \mathbb{Z}} f(x,y) F_\alpha(x,y) \Big)^2 \Big] \leq c(a) L^{-1} \|f\|_{L^\infty(\mathbb{Z}_{>0}^2)}^2, \quad \alpha = x, y,$$

$$(5.9)$$
$$\mathbf{E}\Big[\Big(L^{-2} \sum_{x,y \in [0,aL] \cap \mathbb{Z}} f(x,y) F_x(x,y) F_y(x,y) \Big)^2 \Big] \leq c(a) L^{-1} \|f\|_{L^\infty(\mathbb{Z}_{>0}^2)}^2.$$

In particular, $\|K_B\|_2 + \|K_{FF}\|_2 \to 0$.

Proof. Fix $a \in [1, \infty)$ and $f : \mathbb{Z}_{\geq 0}^2 \to \mathbb{R}$. To simplify notation, throughout the proof we write $c = c(a)$ and always assume (without explicitly stating) that variables x, y, x_1, etc., are in $[0, aL] \cap \mathbb{Z}$.

We begin with the bound (5.8). Take $\alpha = x$ to simplify notation. Calculate the l.h.s. of (5.8) from (5.3). By Lemma 2.1, the variables $\xi^d(x, y)$, $x, y \in \mathbb{Z}_{>0}^2$ are uncorrelated, so

l.h.s. of (5.8)

$$
= L^{-2} \sum_{(x_1, y_1), (x_2, y_2)} \sum_{y_1', y_2'} \Big(\prod_{i=1}^{2} f(x_i, y_i) \widetilde{\mathcal{R}}^d(0, y_i - y_i') \Big) \mathbf{E}\big[\xi^d(x_1 + 1, y_1') \xi^d(x_2 + 1, y_2') \big]
$$

$$
= L^{-2} \sum_{(x_1, y_1), (x_2, y_2), x_1 = x_2} \sum_{y} \Big(\prod_{i=1}^{2} f(x_i, y_i) \widetilde{\mathcal{R}}^d(0, y_i - y) \Big) \mathbf{E}\big[\xi^d(x_1 + 1, y)^2 \big].
$$

By Lemma 2.1, the Riemann function $\widetilde{\mathcal{R}}^d$ is bounded, and by Lemma 2.3,

$$
\mathbf{E}[\xi^d(x_1 + 1, y)^2] \leq c L^{-3}.
$$

With $x_i, y_i, y \in [0, aL] \cap \mathbb{Z}$, the number of terms within the sum is $\leq c L^{3+1}$. From this we conclude that

$$
\text{l.h.s. of (5.8)} \leq c L^{-2} L^{3+1} \| f \|_{L^\infty(\mathbb{Z}_{>0}^2)}^2 L^{-3} \leq c L^{-1} \| f \|_{L^\infty(\mathbb{Z}_{>0}^2)}^2.
$$

We now move on to (5.9). Similarly to the preceding, we calculate

(5.10)

$$
\text{l.h.s. of (5.9)} = \sum_{(x_1, y_1), (x_2, y_2)} \sum_{x_1', x_2', y_1', y_2'} \Big(\prod_{i=1}^{2} f(x_i, y_i) \widetilde{\mathcal{R}}^d(0, y_i - y_i') \widetilde{\mathcal{R}}^d(x_i - x_i', 0) \Big)
$$

$$
\cdot \mathbf{E}\Big[\prod_{i=1}^{2} \xi^d(x_i + 1, y_i') \prod_{i=1}^{2} \xi^d(x_i', y_i + 1) \Big].
$$

To bound the r.h.s. of (5.10), we proceed by discussing the relative location of the following four points where ξ^d is evaluated:

$$
(x(s_1), y(s_1)) := (x_1 + 1, y_1'), \qquad (x(s_2), y(s_2)) := (x_2 + 1, y_2'),
$$
$$
(x(s_3), y(s_3)) := (x_3', y_3 + 1), \qquad (x(s_4), y(s_4)) := (x_4', y_4 + 1).
$$

Here, $s_i \in \mathbb{Z}_{>0}$ denotes the order of the point under the linear ordering (3.1). For example, if $(x_2 + 1, y_2') = (2, 2)$, $s_2 = 3$. Let $s_* = \max\{s_1, \ldots, s_4\}$ denote the maximal order among the four points, and let $\mathscr{G}(t) := \sigma(\xi^d(x(1), y(1)), \ldots, \xi^d(x(t), y(t)))$ denote the canonical filtration of $\xi^d(x, y)$ under the linear ordering (3.1).

(1) The point $(x(s_*), y(s_*))$ is separated from the other three points. In this case, first take conditional expectation $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_* - 1)]$. With the aid of (2.2), we have

$$
\mathbf{E}\Big[\prod_{s \in \{s_1, \ldots, s_4\}} \xi^d(x(s), y(s)) \Big] = \mathbf{E}\Big[\prod_{s \neq s_*} \xi^d(x(s), y(s)) \, \mathbf{E}\big[\xi^d(x(s_*), y(s_*)) | \mathscr{G}(s_* - 1) \big] \Big]
$$
$$
= 0.
$$

(2) The point $(x(s_*), y(s_*))$ is identical with another point, and the other two points are separated. Take $s_1 = s_2 > s_3 > s_4$ to simplify notation, and other permutations follow exactly the same. In this case, take conditional

expectations $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_1-1)]$, $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_3-1)]$, and $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_4-1)]$ in order. Using Lemma 2.3 for $k=2,1,1$, respectively, we have

$$\mathbf{E}\Big[\prod_{s\in\{s_1,\ldots,s_4\}}\xi^{\mathrm{d}}(x(s),y(s))\Big]\le cL^{-2-1}L^{-1-1}L^{-1-1}=cL^{-7}.$$

(3) The point $(x(s_*),y(s_*))$ is identical with another point, and the other two points are identical. Take $s_1=s_2>s_3=s_4$ to simplify notation, and other permutations follow exactly the same. In this case, take conditional expectations $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_1-1)]$, $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_3-1)]$ in order. Using Lemma 2.3 for $k=2,2$, respectively, we have

$$\mathbf{E}\Big[\prod_{s\in\{s_1,\ldots,s_4\}}\xi^{\mathrm{d}}(x(s),y(s))\Big]\le cL^{-1-2}L^{-1-2}=cL^{-6}.$$

(4) The point $(x(s_*),y(s_*))$ is identical with two other points, and the fourth point is separated. Take $s_1=s_2=s_3>s_4$ to simplify notation, and other permutations follow exactly the same. In this case, take conditional expectations $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_1-1)]$, $\mathbf{E}[\,\cdot\,|\mathscr{G}(s_4-1)]$ in order. Using Lemma 2.3 for $k=3,1$, respectively, we have

$$\mathbf{E}\Big[\prod_{s\in\{s_1,\ldots,s_4\}}\xi^{\mathrm{d}}(x(s),y(s))\Big]\le cL^{-1-3}L^{-1-1}=cL^{-6}.$$

(5) All four points are together. Using Lemma 2.3 for $k=4$ gives

$$\mathbf{E}\Big[\prod_{s\in\{s_1,\ldots,s_4\}}\xi^{\mathrm{d}}(x(s),y(s))\Big]\le cL^{-1-4}=cL^{-5}.$$

Now, with $x_i,y_i\in[0,aL]\cap\mathbb{Z}$, the number of terms within the sum in (5.10) is of order L^8. Each contraction of points reduces the number of terms by L^{-2}. For example, the number of terms corresponding to case (2) is $\le cL^{8-2}$, because $(x(s_*),y(s_*))$ being joined once amounts to contracting one point. Following this line of reasoning, the number of terms within each case (2)–(5) is bounded by cL^6, cL^4, cL^4, cL^2, respectively. From these discussions, we bound the r.h.s. of (5.10) by

$$\text{l.h.s. of (5.8)} \le cL^{6-7}+cL^{4-6}+cL^{4-6}+cL^{2-5}\le cL^{-1}.$$

This concludes the proof. $\qquad\square$

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