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The density flatness phenomenon

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ABSTRACT

This paper investigates a curious phenomenon of some positive random variables having a constant density in some initial subinterval of their support. We discuss two methods of obtaining such a random variable with partly flat density. The main method is to sum powers of independent uniform variables with the number of terms matching the power. We show that the limit of this sum is an interesting infinitely divisible distribution and we study its basic properties, find a differential equation for its density, and obtain a recursive relation satisfied by its moments which allows for the calculation of its moment generating function and cumulants.

1. Introduction

In a recent paper, Weissman (2017) derives a precise formula for the probability density function of the sum of squares, S_n, of n independent and identically distributed standard uniform random variables. Indeed, he proposes to study the random variable $S_n(k) = \sum_1^n U_i^k$, remarking that $S_n(1)$ has the well-known Irwin–Hall density. Weissman's paper thus discusses $S_n(2)$. A special phenomenon noted in that paper is the fact that the density of $S_2(2)$ is flat in the interval $[0, 1]$. This means that $S_2(2)$ is conditionally uniformly distributed, given the event $\{S_2(2) \le 1\}$. Indeed, the author states that he did not find any reference to this phenomenon in the written literature, see Weissman (2017). As he computes the distribution function of $S_n(2)$, he establishes that the mass of the interval $[0, 1]$ decreases to zero as n grows. Using concepts from geometric probability, Forrester (2018) provides more formulas concerning the distribution of $S_n(2)$. Neither of the two named authors discusses $S_n(k)$ for $k > 2$.

The primary motivation of this paper is the above mentioned flatness in the density of $S_2(2)$, which we refer to as the flatness phenomenon. We will give two methods of constructing random variables that exhibit this behavior. An example of the first method is the ratio of uniforms, say $R_1 = U/V$ where U and V are independent uniform random variables in $[0, 1]$. The density of this simple ratio is given by

$$f(z) = \frac{1}{2} I_{(0,1)}(z) + \frac{1}{2z^2} I_{(1,\infty)}(z),$$

where $I_A(\cdot)$ is the indicator function of a set A. Another, more general, example is $R_n = U/Z_n$, where $Z_n = V_1 \cdots V_n$ and $U, V_1 \cdots V_n$ are i.i.d. random variables distributed uniformly in the interval $[0, 1]$. Division by a product of uniforms turns out to be only sufficient. In Section 2.1 we prove a result which gives a source of random variables exhibiting the flatness phenomenon, thus showing the actual reason why the repeated division by uniform works.

Based on the simple observation that both $S_1(1)$ and $S_2(2)$ are flat in $[0, 1]$, one of the authors of the current paper guessed that $S_n(n)$ is also flat in this interval for all n. This guess was verified by drawing histograms of 10^6 simulated sums of the form $S_n(k)$ for various values of k and n. As shown in Fig. 1, it is only when $k = n$ that the flatness is realized.

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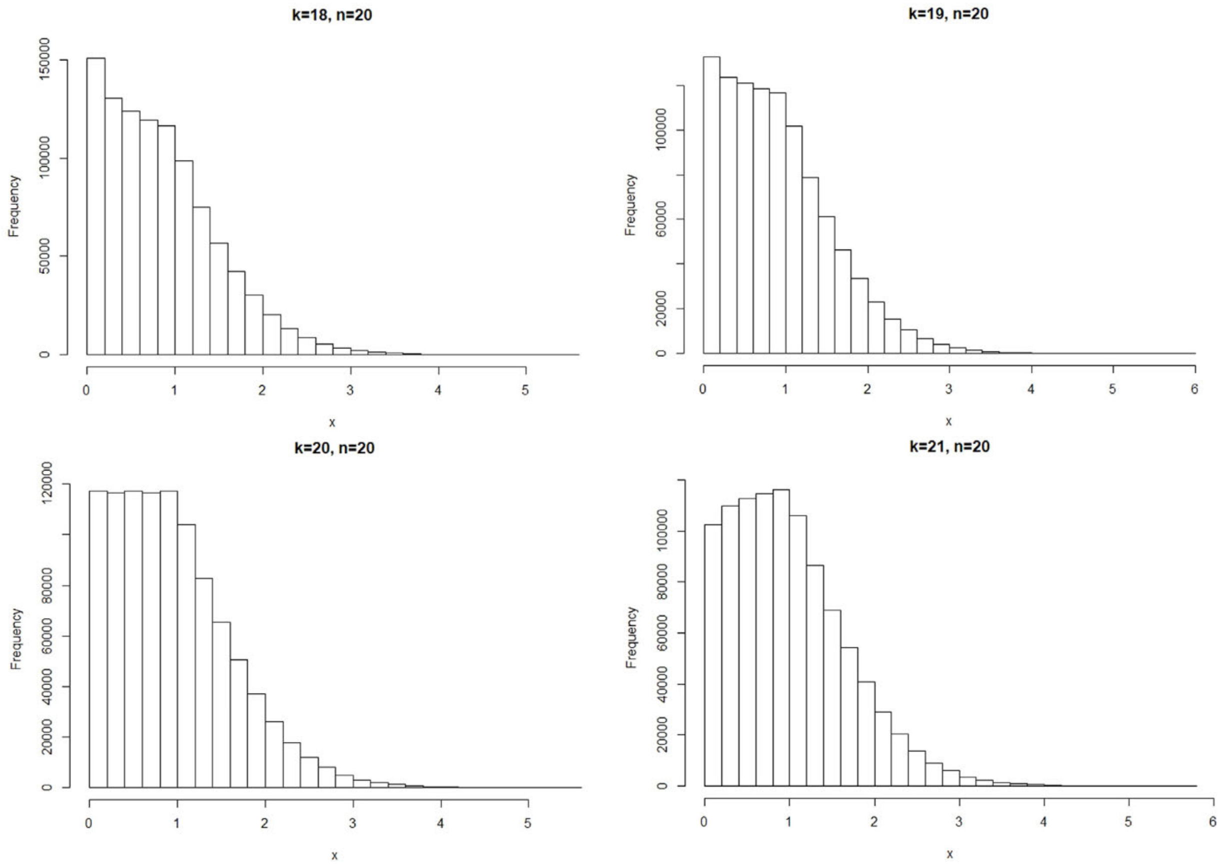

Fig. 1. The graph of $S_n(k)$ for $n = 20$ and several values of k.

In Section 2.2 we provide a direct geometric proof for the flatness of $S_n(n)$. We then derive a basic convolution lemma that provides a more general condition for the sum of not necessarily equal powers of uniform to also exhibit this behavior. Section 3 investigates the limiting distribution of $S_n(n)$ which evidently is infinitely divisible and has the flatness phenomenon. We obtain a representation of this distribution as the limit law of a sum of products of i.i.d. uniform random variables. We then derive a differential equation satisfied by the limiting distribution which can be solved recursively in intervals $(n, n+1)$. Finally, we write a recursive relation for the moments of the limit law and obtain an explicit expression for the moment generating function, which in turn allows us to calculate the cumulants.

2. Two sources of flatness

2.1. Flatness of the type U/V

It is easy to establish directly that the density of R_n, defined above, is given by the constant $1/2^n$ at any point in the interval $(0, 1)$. This constant density drops fast to zero as we divide by more uniform terms in the denominator. This should be compared with $S_n(n)$ below whose constant density stabilizes at a constant value away from zero as n increases. As with $S_n(n)$, R_n was actually guessed and verified by simulation. Proposition 2.1 gives a fairly general construction that includes the case R_n as a special case.

Proposition 2.1. *Let U be a uniform random variable in the interval $(0, a)$ and let Y be any random variable supported in the interval $[b, \infty)$, for some positive constants a and b. Then the density of UY is flat in the interval $(0, ab)$.*

Proof. Let $F(y)$ be the distribution of Y. For $t \in (0, ab)$, $t/b < a$, i.e. within the support of U, we have $P[UY \le t] = \int_0^a P[UY \le t|U = u] \cdot \frac{1}{a} du = \int_0^{t/b} P[Y \le t/u] \cdot \frac{1}{a} du + \int_{t/b}^a P[Y \le t/u] \cdot \frac{1}{a} du$. In the second integral, $t/u \le b$ so the integrand collapses to zero because Y is not supported below b, unless Y has an atom at b in which case the second integral takes the constant value $P[Y = b]/a$. In the first integral $t/u \ge b$ so it is $\frac{1}{a}\int_0^{t/b}\int_b^{t/u} dF(y)du$. Applying Fubini's Theorem we get $\frac{1}{a}\int_b^{\infty}\int_0^{t/y} dudF(y) = \frac{t}{a}\int_b^{\infty} 1/ydF(y)$ which is linear in t. $\square$

As pointed out by a referee, this result is closely related to Khintchine's Theorem on unimodal densities, see Khintchine (1937) or Feller (1971). One version of this theorem, the monotone version, states that a non-negative random variable X has a monotone non-increasing density $f(x)$ if and only if it has the form UY where U is uniform in $[0, 1]$ and $Y > 0$. We can use this fact to conclude Proposition 2.1 when Y is absolutely continuous, say with density g. One can check that

$$f(x) = \int_x^\infty \frac{g(y)}{y} dy$$

so that $f'(x) = -g(x)/x$ or $g(x) = -xf'(x)$. Therefore, X is flat in $[0, 1]$ if and only if $g(x) = 0$ in $[0, 1]$, i.e., when Y is supported in $[1, \infty)$. For $a, b > 0$, Proposition 2.1 now follows by noting that $abUY$ rescales the flat portion to $(0, ab)$, that (aU) is uniform in $(0, a)$, and that bY is supported in $[b, \infty)$ when Y is supported in $[1, \infty)$.

2.2. Flatness of $S_n(n)$

In this section we prove the flatness phenomenon of the density of $S_n(n)$. Indeed, we will first give a direct geometric proof of this fact, then we give a second, analytical, proof that allows for some generalization.

In effect, for any $p \ge 1$, the norm of a vector $(x_1, \dots, x_n)$ in $L^p(R^n)$ is given by $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. A ball B_r of radius $r > 0$ and centered at the origin is thus defined by $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \le r$. A classic result attributed to Dirichlet, see Edwards (1922), Xianfu (2005), states that the volume of this ball is given by the formula

$$V(B_r) = \frac{(2\Gamma(\frac{1}{p}+1)r)^n}{\Gamma(\frac{n}{p}+1)}. \tag{2.1}$$

Now for $t \in [0, 1]$ the cumulative distribution of $S_n(n)$ is $F_n(t) = P[U_1^n + \cdots + U_n^n \le t]$.

This probability is precisely the volume of the intersection of B_r with the first octant, when $p = n$ and $r = t^{1/n}$. Using (2.1) we get $F_n(t) = (\Gamma(\frac{1}{n}+1))^n t$. Differentiation gives the result.

A generalization as well as more insight is obtained in the following analytical proof. The next lemma will be pivotal.

Lemma 2.2. *Let X_1 and X_2 be two independent, nonnegative and absolutely continuous random variables, whose densities in the interval $[0, 1]$ are given by $f_i(x) = (1-\alpha_i)/x^{\alpha_i}$, $i = 1, 2$ and $\alpha_i < 1$. Then in the interval $[0, 1]$ $X_1 + X_2$ has the density $f_{X_1+X_2}(t) = C/t^{\alpha_1+\alpha_2-1}$, where $C = (1-\alpha_1)(1-\alpha_2)B(1-\alpha_1, 1-\alpha_2)$ and $B(a, b)$ is the beta function.*

Proof. The convolution formula for $t \in [0, 1]$ gives

$$f_{X_1+X_2}(t) = f_1 * f_2(t) = \int_0^t \frac{(1-\alpha_1)}{x^{\alpha_1}} \frac{(1-\alpha_2)}{(t-x)^{\alpha_2}} dx.$$

Using the change of variable $y = tx$ this becomes

$$\frac{(1-\alpha_1)(1-\alpha_2)}{t^{\alpha_1+\alpha_2-1}} \int_0^1 y^{-\alpha_1}(1-y)^{-\alpha_2} dy. \quad \square$$

Proposition 2.3. *Let $U_1, \dots, U_n$ be i.i.d. standard uniform variables. For any n positive numbers $k_1, \dots, k_n$ such that $\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1$, the sum $U_1^{k_1} + \cdots + U_n^{k_n}$ has a p.d.f. that is flat in $[0, 1]$.*

Proof. Let $k_1, \dots, k_n$ be any n positive real numbers and $t \in (0, 1)$. Using the Convolution Lemma above we have

$$f_1 * f_2(t) = \frac{B(1/k_1, 1/k_2)}{k_1 k_2 t^{1-1/k_1-1/k_2}}.$$

Iterating Lemma 2.2 $n-1$ times we see that $f_1 * \cdots * f_n(t)$ is

$$\prod_{i=1}^n \frac{1}{k_i} B(1/k_1, 1/k_2) B(1/k_1 + 1/k_2, 1/k_3) \cdots B(1/k_1 + \cdots + 1/k_{n-1}, 1/k_n) \cdot \frac{1}{t^{1-1/k_1-\cdots-1/k_n}}.$$

As the numbers k_i satisfy the condition stated above, the exponent of t is zero and, with some calculation, the density in $(0, 1)$ has the constant value

$$\prod_{i=1}^n \left(\frac{1}{k_i}\Gamma(\frac{1}{k_i})\right) = \prod_{i=1}^n \left(\Gamma(1+\frac{1}{k_i})\right). \quad \square$$

Our initial observation now follows immediately.

Corollary 2.4. *The density of $U_1^n + \cdots + U_n^n$ is flat in $(0, 1)$ and it approaches the constant $e^{-\gamma}$ as n increases, where $\gamma \approx 0.57721$ is the Euler constant.*

Proof. When $k_i = n$ for all i in Proposition 2.3, the left hand side in the above display becomes $\left(\frac{1}{n}\Gamma(\frac{1}{n})\right)^n$. By the well-known Laurent series expansion of the gamma function about zero, $\Gamma(x) = x^{-1} - \gamma + cx + O(x^2)$, (where $c = \gamma^2/2 + \pi^2/12$), we see that we need the limit of $\left(\frac{1}{n}\left[n - \gamma + \frac{c}{n} + O(1/n^2)\right]\right)^n = \left(1 - \frac{\gamma}{n} + \frac{c}{n^2} + O(1/n^3)\right)^n$, which is $e^{-\gamma}$. □

3. Limiting Distribution of $S_n(n)$

The terms in $S_n(n)$ for $n = 1, 2, 3, \ldots$ form a triangular array, thus the general theory implies that $S_n(n)$ has a limiting distribution that is in fact infinitely divisible.

This limiting distribution is characterized in Theorem 3.2. The following lemma will be needed in the proof.

Lemma 3.1. *Let $\xi_1, \ldots, \xi_k$ be independent and identically distributed exponential random variables with a common mean of 1, and let $S_k = \sum_{i=1}^k \xi_i$. Then P-a.s. $|S_k - k| \le k^{2/3}$.*

Proof. For an exponential random variable $\xi \sim Exp(1)$ it is easy to check the following inequality concerning the moment generating function.

$$E[e^{\pm\lambda(\xi-1)}] = \frac{e^{\mp\lambda}}{1 \mp \lambda} \le e^{3\lambda^2/4}, \tag{3.2}$$

for all $|\lambda| < 1/3$. Indeed, the inequality $\dfrac{e^{\lambda}}{1+\lambda} \le e^{3\lambda^2/4}$ is equivalent to the inequality $g(\lambda) := \frac{3\lambda^2}{4} - \lambda + \ln(1+\lambda) \ge 0$. Via direct calculation, one can check that the derivative $g'(\lambda)$ is negative in $(-1/3, 0)$ and positive above zero. Since $g(0) = 0$ we see that $g(\lambda) \ge 0$ when $\lambda > -1/3$. The case of $\dfrac{e^{-\lambda}}{1-\lambda}$ is similar, but in the interval $(-\infty, 1/3)$. Then $P[|S_k - k| > k^{2/3}]$ is

$$\begin{aligned}
&= P[S_k - k > k^{2/3}] + P[S_k - k < -k^{2/3}] \\
&= P[S_k - k > k^{2/3}] + P[-(S_k - k) > k^{2/3}] \\
&= P[e^{\lambda(S_k-k)} > e^{\lambda k^{2/3}}] + P[e^{-\lambda(S_k-k)} > e^{\lambda k^{2/3}}] \\
&\le \frac{E[e^{\lambda(S_k-k)}]}{e^{\lambda k^{2/3}}} + \frac{E[e^{-\lambda(S_k-k)}]}{e^{\lambda k^{2/3}}},
\end{aligned} \tag{3.3}$$

where we used Markov's Inequality. On the other hand, $E[e^{\lambda(S_k-k)}] = E[e^{\sum_{i=1}^k \lambda(\xi_i-1)}] = \Pi_{i=1}^k E[e^{\lambda(\xi_i-1)}]$. In view of Eq. (3.2) the last expression is dominated by $\Pi_{i=1}^k e^{3\lambda^2/4} = e^{3k\lambda^2/4}$, for $|\lambda| < 1/3$.

Likewise, for these values of λ we see that $E[e^{-\lambda(S_k-k)}] \le e^{3k\lambda^2/4}$. It follows from the above and from the inequality in Display (3.3) that

$$P[|S_k - k| > k^{2/3}] \le \min_{\lambda} 2e^{3k\lambda^2/4 - \lambda k^{2/3}} = 2exp\left(-k^{1/3}/3\right).$$

As $\sum_{k=1}^{\infty} exp\left(-k^{1/3}/3\right)$ converges, the Borel–Cantelli Lemma implies that P-a.s. $|S_k - k| \le k^{2/3}$, which establishes the lemma. □

Theorem 3.2. *$S_n(n)$ converges in law to an infinitely divisible distribution F_∞. Furthermore,*

(a) The random variable $W = \sum_{k=1}^{\infty}\prod_{i=1}^{k} X_i$ has the distribution F_∞, where $\{X_i\}$ is a sequence of independent standard uniform random variables.

(b) W obeys the stochastic equation $W \overset{d}{=} U(1 + W)$ where U is standard uniform and such that U and W on the right are independent.

Proof. Consider the triangular array with the nth row formed by $U_1^n, \ldots, U_n^n$. By the general theory, to show that F_∞ is infinitely divisible it is enough to show that in each row of the triangular array the variables are uniformly asymptotically negligible (u.a.n.), that is, for an arbitrary $\epsilon > 0$

$$\lim_{n\to\infty} \max_{1\le k\le n} P[|U_k^n| > \epsilon] = 0.$$

This equation is straightforward to check for U_k^n. To prove (a) consider the order statistics $U_{(1)}, \ldots, U_{(n)}$ of $U_1, \ldots, U_n$. These order statistics subdivide the interval $[0, 1]$ into $n + 1$ subintervals of respective lengths $\Delta_1, \ldots, \Delta_{n+1}$ where $\Delta_1 = U_{(1)}$, $\Delta_{n+1} = 1 - U_{(n)}$ and for $i \neq 1$, $i \neq n + 1$ $\Delta_i = U_{(i)} - U_{(i-1)}$. It is well-known that the vector $(\Delta_1, \ldots, \Delta_{n+1})$ has the same distribution as the vector $\dfrac{1}{S_{n+1}}(\xi_1, \ldots, \xi_{n+1})$, where $\xi_1, \ldots, \xi_{n+1}$ are independent exponential random variables with common mean 1, and $S_{n+1} = \sum_{i=1}^{n+1} \xi_i$. Now

$$\begin{aligned} S_n(n) &= U_{(1)}^n + \cdots + U_{(n)}^n \\ &\overset{d}{=} \left((1 - \frac{\xi_1}{S_{n+1}})^n + (1 - \frac{\xi_1 + \xi_2}{S_{n+1}})^n + \cdots + (1 - \frac{\xi_1 + \cdots + \xi_n}{S_{n+1}})^n \right) \\ &= \sum_{k=1}^{n} (1 - \frac{S_k}{S_{n+1}})^n. \end{aligned}$$

By Lemma 3.1 we have, P-a.s., that $S_k \geq k - k^{2/3}$ and $S_n \leq n + n^{2/3}$. Thus $1 - S_k/S_n \leq 1 - \frac{k-k^{2/3}}{n+n^{2/3}}$. Using the obvious inequality $1 - \epsilon \leq e^{-\epsilon}$ – in a neighborhood of zero – we have

$$0 \leq S_n(n) \leq \sum_{k=1}^{n} \left(1 - \frac{k - k^{2/3}}{n + n^{2/3}}\right)^n \leq \sum_{k=1}^{n} \exp\left(-\frac{n(k - k^{2/3})}{n + n^{2/3}}\right).$$

Since $\exp\left(-\frac{n(k-k^{2/3})}{n+n^{2/3}}\right) \leq \exp\left(-k + k^{2/3}\right)$ and $\sum_{k=1}^{n} \exp\left(-k + k^{2/3}\right)$ converges as $n \to \infty$, the dominated convergence theorem for series, see Billingsley (1995), implies that the limit of $S_n(n)$ equals

$$\sum_{k=1}^{\infty} \lim_{n\to\infty} (1 - \frac{S_k}{S_{n+1}})^n = \sum_{k=1}^{\infty} e^{-S_k}.$$

The result now follows by noting that $e^{-S_k} = \prod_{i=1}^{k} e^{-\xi_i}$, and that $e^{-\xi_i}$ follows a standard uniform distribution.

To get (b) we observe that $W = X_1(1 + X_2 + X_2X_3 + X_2X_3X_4 + \cdots)$ and employ self similarity. □

Remark 3.3. Theorem 3.2(b) establishes a relationship between the two flatness types discussed earlier. On the one hand, the density of W is flat in $[0, 1]$ because $S_n(n)$ has a flat density for each n. On the other hand, $1 + W$ is completely supported in $[1, \infty)$ and so W has the form UY as required in Proposition 2.1 with $a = b = 1$.

Remark 3.4. The distributional equation between W and $U(1 + W)$ is a special case of a more general, well-studied, distributional equation between two random variables X and $AX + B$ where (A, B) is a random vector independent of X in the second expression. The recent book by Buraczewski et al. (2016) surveys many known results about this equivalence.

3.1. The density of W

By Theorem 3.2(b), W is a product of two random variables one of which is absolutely continuous. Hence W is absolutely continuous. Let $f_\infty(y)$ be the density of W, which is evidently supported in $[0, \infty)$. Thus, the density of $Y = 1 + W$ is $f_\infty(y - 1)$ and the joint density of X and Y is $g(x, y) = I_{[0,1]}(x) \times f_\infty(y - 1)$. In view of Theorem 3.2(b) again, for $t > 1$, $F_\infty(t) = P[XY \leq t] = \int_0^1 \int_1^{t/x} f_\infty(y - 1)dydx = \int_0^1 F_\infty(t/x - 1)dx$. Using the change of variable $u = t/x - 1$ we get the integral equation

$$\frac{F_\infty(t)}{t} = \int_{t-1}^{\infty} \frac{F_\infty(u)}{(1 + u)^2} du.$$

Differentiating the above equation with respect to t and regrouping terms we obtain

$$f_\infty(t) = \frac{F_\infty(t) - F_\infty(t - 1)}{t}.$$

This equation already shows that the density f_∞ is differentiable at every positive, non-integer, real value. Differentiating the equivalent form $tf_\infty(t) = F_\infty(t) - F_\infty(t - 1)$ again we then obtain the differential equation

$$f'_\infty(t) = -\frac{f_\infty(t - 1)}{t} \tag{3.4}$$

We write $f_\infty(t) = \sum_{k=0}^{\infty} f_k(t)$, where $f_k(t) = f_\infty(t)I_{(k,k+1)}(t)$. Recalling that $f_0(t) = e^{-\gamma}$, we see that $f_\infty(t)$ can be found essentially by recursively solving the initial value problems $f'_{k+1}(t) = -\frac{f_k(t-1)}{t}$ with $f_{k+1}(k+1) = f_k(k+1)$. The following are the first two parts f_1 and f_2, while f_3 already cannot be written in closed form as it involves the integral of the logarithmic integral function $Li(x)$:

$$\begin{aligned} f_1(t) &= e^{-\gamma}(1 - \log t), \\ f_2(t) &= \log(t - 1)\log t + Li_2(1 - t) + C, \end{aligned}$$

where $Li_2(z)$ is the dilogarithm function and $C = e^{-\gamma}(1 - \log 2) + \frac{\pi^2}{12}$.

3.2. The moments of W

For positive integer values of n let m_n be the nth moment of W. Then by Theorem 3.2(b) we have

$$m_n = E[U^n] \cdot E[(1+W)^n] = \frac{1}{n+1} \cdot E\left(\sum_{i=0}^{n} \binom{n}{i} W^i\right)$$
$$= \frac{1}{n+1} \cdot \sum_{i=0}^{n} \binom{n}{i} m_i.$$

Re-arranging terms we get the recursive equation

$$nm_n = \sum_{i=0}^{n-1} \binom{n}{i} m_i. \tag{3.5}$$

In particular, the first four moments are 1, 3/2, 17/6 and 19/3 respectively. We note here that the median is already less than 1 and it is $e^{\gamma}/2 \approx 0.891$.

Summing up the two sides in Eq. (3.5) we get

$$\sum_{n=1}^{\infty} \frac{nm_n t^n}{n!} = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n}{i} m_i \frac{t^n}{n!} \text{which implies } t \sum_{n=1}^{\infty} \frac{nm_n t^{n-1}}{n!} = \sum_{i=0}^{\infty} \sum_{n=i+1}^{\infty} \binom{n}{i} m_i \frac{t^n}{n!},$$

by interchanging the double sum on the right. Observing that the sum on the left is the derivative of the moment generating function of W, say $M_\infty(t)$, we thus obtain

$$tM'_\infty(t) = \sum_{i=0}^{\infty} \frac{m_i t^i}{i!} \sum_{n=i+1}^{\infty} \frac{t^{n-i}}{(n-i)!} = \sum_{i=0}^{\infty} \frac{m_i t^i}{i!} \sum_{j=1}^{\infty} \frac{t^j}{j!}$$
$$= \sum_{i=0}^{\infty} \frac{m_i t^i}{i!} (e^t - 1).$$

We arrive at the differential equation $tM'_\infty(t) = \left(e^t - 1\right) M_\infty(t)$, which readily gives $\log M_\infty(t) = \int \frac{e^t - 1}{t} dt + C$. Integrating component-wise the Taylor expansion of the integrand on the right and using the fact that $M_\infty(0) = 1$ we see that $C = 0$ and that the cumulant generating function is

$$\log M_\infty(t) = \int_0^t \frac{e^x - 1}{x} dx = \sum_{i=1}^{\infty} \frac{t^i}{i \cdot i!}.$$

Hence the cumulants are given, for all $i \geq 1$, by $\kappa_i = 1/i$.

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References

Billingsley, Patrick, 1995. Probability and Measure, third ed. In: Wiley Series in Probability and Mathematical Statistics, New York.
Buraczewski, D., Damek, E., Mikosch, T., 2016. Stochastic Models with Power-Law Tails; the Equation $X = AX + B$. Springer.
Edwards, J., 1922. A Treatise on the Integral Calculus, II. Chelsea Publishing Company, New York.
Feller, W., 1971. An Introduction To Probability Theory and Its Applications, Vol. 2, 3rd Ed. John Wiley and Sons.
Forrester, P.J., 2018. Comment on sum of squares of uniform random variables by I. Weissman. Stat. Probab. Lett. 142, 118–122.
Khintchine, A.Y., 1937. On unimodal distributions. Izvestiya Nauchno- Issledovatelskogo Instituta Matematiki i Mekhaniki 2, 1–7.
Weissman, I., 2017. Sum of squares of uniform random variables. Statist. Probab. Lett. 129, 147–154.
Xianfu, Wang, 2005. Volumes of generalized unit balls. Math. Mag. 78 (5), 390–395.