TATE CYCLES ON SOME QUATERNIONIC SHIMURA VARIETIES MOD p

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Abstract

Let F be a totally real field in which a prime number $p > 2$ is inert. We continue the study of the (generalized) Goren–Oort strata on quaternionic Shimura varieties over finite extensions of $\mathbb{F}_p$. We prove that, when the dimension of the quaternionic Shimura variety is even, the Tate conjecture for the special fiber of the quaternionic Shimura variety holds for the cuspidal π-isotypical component, as long as the two unramified Satake parameters at p are not differed by a root of unity.

Contents

1. Introduction

One of the most important conjectures in algebraic geometry is the Tate conjecture on algebraic cycles (see [31]). The general case of this conjecture is far from being proved. In this article, we will consider the Tate conjecture for Hilbert modular varieties modulo an inert prime.

Let F be a totally real field of degree $g = [F : \mathbb{Q}]$, and let $p > 2$ be a prime number inert in F. Let $\mathbb{A}_F$ be the adele ring, and let $\mathbb{A}_F^\infty$ (resp., $\mathbb{A}_F^{\infty,p}$) be the subring of finite adeles (resp., prime-to-p finite adeles) of F. Fix a neat open compact subgroup $K = K^p K_p \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$, where $K_p = \mathrm{GL}_2(\mathcal{O}_{F_p})$ and $K^p \subset \mathrm{GL}_2(\mathbb{A}_F^{\infty,p})$. Let X be the Hilbert modular scheme of level K. This is a quasiprojective smooth scheme over

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$\mathbb{Z}_{(p)}$ of relative dimension g. For a fixed prime $\ell \neq p$, the ℓ-adic étale cohomology group $H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$ is equipped with a commuting action of $\mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the Hecke algebra $\mathscr{H}_K := \overline{\mathbb{Q}}_\ell[K \backslash \mathrm{GL}_2(\mathbb{A}_F^\infty)/K]$. Let $\pi = \pi^\infty \otimes \pi_\infty$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that the Archimedean component π_∞ is a discrete series of parallel weight 2, and such that the K-fixed vectors $\pi^{\infty,K} \neq 0$. We put

$$H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\pi] := \mathrm{Hom}_{\mathscr{H}_K}\big(\pi^{\infty,K}, H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)\big).$$

Let $\rho_\pi :\ \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ denote the Galois representation attached to π (see, e.g., [32]). Then the main result of [2] essentially says that the semisimplification of the $\mathrm{Gal}_{\mathbb{Q}}$-module $H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\pi]$ is isomorphic to the Asai representation $\mathrm{As}(\rho_\pi) := \bigotimes \mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}}(\rho_\pi)$, which is the tensor induction of ρ_π from Gal_F to $\mathrm{Gal}_{\mathbb{Q}}$.[1] By our assumption on p, both ρ_π and $H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$ are unramified at p. It makes sense to view $\mathrm{As}(\rho_\pi)$ and $H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\pi]$ as $\mathrm{Gal}_{\mathbb{F}_p}$-modules.

Assume that g is even so that X is even-dimensional. For q a power of p, we write $\mathrm{Frob}_q \in \mathrm{Gal}_{\mathbb{F}_p}$ for the geometric q-Frobenius. We put

$$\mathcal{T}(\pi, \overline{\mathbb{F}}_p) := \bigcup_{j \geq 1} \mathrm{As}(\rho_\pi)(g/2)^{\mathrm{Frob}_{p^j} = 1}$$

for the space of Tate classes of $\mathrm{As}(\pi)(g/2)$ defined over a finite extension of $\mathbb{F}_p$. If the two (generalized) eigenvalues of $\rho_\pi(\mathrm{Frob}_{p^g})$ are denoted by α_π and β_π, then $(\mathrm{As}(\rho_\pi)(g/2))(\mathrm{Frob}_{p^g})$ has generalized eigenvalues $\alpha_\pi^i \beta_\pi^{g-i}/p^{g^2/2}$ with multiplicity $\binom{g}{i}$ for $i = 0, \dots, g$. We know that $\alpha_\pi \beta_\pi / p^g$ is a root of unity. From this, it is easy to see that $\dim_{\overline{\mathbb{Q}}_\ell} \mathcal{T}(\pi, \overline{\mathbb{F}}_p) \geq \binom{g}{g/2}$, and the equality holds if α_π/β_π is not a root of unity. Therefore, the Tate conjecture predicts that there are $\binom{g}{g/2}$ algebraic cycles on $X_{\overline{\mathbb{F}}_p}$ that contribute to $\mathcal{T}(\pi, \overline{\mathbb{F}}_p)$.

In this article, we take a purely characteristic p approach to construct the desired algebraic cycles on $X_{\overline{\mathbb{F}}_p}$, and we show that these cycles contribute to all the geometric Tate classes in $H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell(g/2))[\pi]$ if the eigenvalues of $\rho_\pi(\mathrm{Frob}_{p^g})$ are sufficiently general.

Definition 1.1

We say that a morphism $X \to Y$ of schemes is an *r-step iterated $\mathbb{P}^1$-bundle* if this morphism admits a factorization $X := Y_r \to Y_{r-1} \to \cdots \to Y_0 := Y$, where each $Y_i \to Y_{i-1}$ is a $\mathbb{P}^1$-bundle. When Y is the spectrum of a field, we say that X is an *r-step iterated $\mathbb{P}^1$-tower.*

[1]Conjecturally, $H^g_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\pi]$ is semisimple so that it is isomorphic to $\mathrm{As}(\rho_\pi)$. This conjecture is true if $\mathrm{As}(\rho_\pi)$ is irreducible. For more discussion, see [26].

The main results of our work include the following.

THEOREM 1.2
Assume that $g = [F : \mathbb{Q}]$ is even and that K is neat. Let B_∞ denote the quaternion algebra over F ramified exactly at all Archimedean places, and let $\mathrm{Sh}_K(B_\infty) := B_\infty^\times \backslash (B \otimes_F \mathbb{A}_F^\infty)^\times / K$ be the associated discrete Shimura variety over $\overline{\mathbb{F}}_p$. Here, we fix an isomorphism $(B_\infty \otimes_F \mathbb{A}_F^\infty)^\times \cong \mathrm{GL}_2(\mathbb{A}_F^\infty)$ so that the Hecke algebra $\mathscr{H}_K$ acts on $H^0(\mathrm{Sh}_K(B_\infty), \overline{\mathbb{Q}}_\ell)$.

(1) *There exist algebraic correspondences*

$$\mathrm{Sh}_K(B_\infty) \xleftarrow{p_i} X_i \xrightarrow{q_i} X_{\overline{\mathbb{F}}_p}, \quad i = 1, \dots, \tbinom{g}{g/2},$$

such that p_i is a $g/2$-step iterated $\mathbb{P}^1$-bundle (so each connected component of X_i is isomorphic to a $g/2$-step iterated $\mathbb{P}^1$-tower), q_i is a closed immersion, and both p_i and q_i are equivariant for prime-to-p Hecke correspondences.

(2) *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to a holomorphic Hilbert cuspidal eigenform of parallel weight 2, and let π_B be the Jacquet–Langlands transfer of π to an automorphic representation of $(B_\infty \otimes \mathbb{A}_F)^\times$. Denote by α_π and β_π the two eigenvalues of $\rho_\pi(\mathrm{Frob}_{p^g})$. We put similarly*

$$H^0\big(\mathrm{Sh}_K(B_\infty), \overline{\mathbb{Q}}_\ell\big)[\pi_B] := \mathrm{Hom}_{\mathscr{H}_K}\big(\pi_B^{\infty,K}, H^0\big(\mathrm{Sh}_K(B_\infty), \overline{\mathbb{Q}}_\ell\big)\big).$$

Then the natural map

$$\bigoplus_{1 \le i \le \binom{g}{g/2}} H^0\big(\mathrm{Sh}_K(B_\infty), \overline{\mathbb{Q}}_\ell\big)[\pi_B] \xrightarrow{\oplus p_i^*} \bigoplus_{1 \le i \le \binom{g}{g/2}} H^0_{\mathrm{et}}(X_{i,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)[\pi_B]$$

$$\xrightarrow{\text{Gysin}} \mathcal{T}(\pi, \overline{\mathbb{F}}_p) \subseteq H^g_{\mathrm{et}}\big(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(g/2)\big)[\pi]$$

is injective if $\alpha_\pi \neq \beta_\pi$, and is isomorphic to $\mathcal{T}(\pi, \overline{\mathbb{F}}_p)$ if α_π/β_π is not a root of unity. In particular, if α_π/β_π is not a root of unity, then the Tate conjecture is true for the π-isotypic component of $H^g_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(g/2))$ over all finite extensions of $\mathbb{F}_p$.

In fact, we prove a result stronger than the one stated here. A full statement will be given later in Theorem 1.7.

Remark 1.3

(1) These cycles $X_1, \dots, X_{\binom{g}{g/2}}$ realize the Jacquet–Langlands correspondence geometrically and, at the same time, they give the Tate classes for the π-isotypical component when π_p is sufficiently general. We expect the union

of them to be the supersingular locus (this will be proved in a future article [22] by Yifeng Liu and the first author). The geometric realization of Jacquet–Langlands correspondence was first studied by Ribet [29], [30] and Helm [13], [14]. They gave some examples of the cycles in the case of modular or Shimura curves and unitary Shimura varieties that realizes the Jacquet–Langlands correspondence geometrically. The geometric aspect of this technique is further developed by the authors in [33]. From this point of view, the theorem above may be understood as: geometric Jacquet–Langlands correspondence can give "generic" Tate classes on the special fiber of the Hilbert modular varieties.

(2) Our construction does not give sufficient algebraic cycles on $X_{\overline{\mathbb{F}}_p}$ when $\alpha_\pi = \beta_\pi$. For instance, for $g = 2$, it follows from the Hodge index theorem and our computation of the intersection matrix of X_1 and X_2 on $X_{\overline{\mathbb{F}}_p}$ (see Example 1.9) that the contribution of X_1 and X_2 to $\mathcal{T}(\pi, \overline{\mathbb{F}}_p)$ is 1-dimensional if $\alpha_\pi = \beta_\pi$. It is an interesting question to find extra "exotic" algebraic cycles that are not cohomologically equivalent to our cycles.

(3) If one instead considers the Tate conjecture of Hilbert modular varieties over the generic fiber (namely over $\mathbb{Q}$), then this topic has a long history dating back to 1980s. But the situation is very different for the discussion in the present article. For a general π that is not CM or the base change from a smaller field, the space of Tate classes $\mathrm{As}(\rho_\pi)(g/2)^{\mathrm{Gal}_\mathbb{Q}}$ is zero. In contrast, the Tate classes in $\mathrm{As}(\rho_\pi)(g/2)$ on the special fiber at an inert prime always have dimension at least $\binom{g}{g/2}$. *So the Tate conjecture of X over $\mathbb{Q}$ is a very different question from the Tate conjecture of $X_{\mathbb{F}_p}$ over $\mathbb{F}_p$.* We list below some known results for the Tate conjecture of Hilbert modular varieties over $\mathbb{Q}$.

- If π is non-CM, then this conjecture was proved by Harder, Langlands, and Rapoport in [12] when $g = 2$. In fact, they show that $\mathrm{As}(\rho_\pi)(1)^{\mathrm{Gal}_\mathbb{Q}}$ is nonzero only if π is the base change of a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, in which case Hirzebruch–Zagier cycles account for the Tate classes. Similar but partial results were obtained by Ramakrishnan [27] and Getz–Hahn [8] in the higher-dimensional cases.
- When $g = 2$ and π is CM, more algebraic cycles are expected to contribute to $\mathrm{As}(\rho_\pi)(1)^{\mathrm{Gal}_\mathbb{Q}}$; this case was solved independently by Murty and Ramakrishnan [25] and by Klingenberg [18] by reducing to the Lefschetz (1,1)-theorem for Hodge classes.
- When $g = 2$ and π is the base change of a cuspidal automorphic representation of $\mathrm{GL}_{2,\mathbb{Q}}$, Langer [19] constructed a variant of the Hirzebruch–Zagier cycle in characteristic 0 and showed that its reduc-

tion modulo p contributes to a 1-dimensional subspace of $\mathcal{T}(\pi, \overline{\mathbb{F}}_p)$. His cycles are strictly contained in the union of our cycles $X_1 \cup X_2$. But it is expected that a lot more cycles will be on the special fiber $X_{\mathbb{F}_p}$ than on the generic fiber X, and so Langer's construction seems to be hard to generalize to general π.

(4) Despite the difference between the Tate conjecture over $\mathbb{Q}$ and that over finite fields, it is an interesting question to study the interrelation between the reduction of cycles in characteristic 0 and cycles in characteristic p that we construct. Such study has interesting corollaries in arithmetic and geometric applications (e.g., bounding Selmer groups; see a series of works of Yifeng Liu and the first author [20]–[22]).

(5) After the first draft of this article, analogues of Theorem 1.2 for special fibers of other Shimura varieties have appeared in recent works (see, e.g., [15], [35]).

(6) A very recent preprint by Ichino and Prasanna [16] constructed certain ℓ-adic Hodge classes over the generic fiber of the product of two quaternionic Shimura varieties to realize the Jacquet–Langlands correspondence. It would be interesting to understand the relations between their Hodge classes and our cycle classes on the special fiber.

1.4. Generalized Goren–Oort cycles

We now explain the construction of the cycles. We allow g to be of arbitrary parity, and we let r be an integer with $1 \le r \le \lfloor g/2 \rfloor$. In the present article we will construct explicitly $\binom{g}{r}$ *generalized Goren–Oort cycles* $X_1, \dots, X_{\binom{g}{r}}$ of codimension r in $X_{\mathbb{F}_{p^g}}$ such that each X_i is isomorphic to an r-step iterated $\mathbb{P}^1$-bundle over (the characteristic p fiber of) some $(g-2r)$-dimensional quaternionic Shimura variety. Moreover, the construction is compatible with prime-to-p Hecke correspondences when the tame level K^p changes. We point out an important feature of these cycles: *the codimension of each X_i is the same as the iterated $\mathbb{P}^1$-bundle dimension.* As pointed out by Xinwen Zhu, the union of these X_i's should be the Zariski closure of the Newton stratum of $X_{\mathbb{F}_{p^g}}$ with slope $(\frac{r}{g}, \dots, \frac{r}{g}, \frac{g-r}{g}, \dots, \frac{g-r}{g})$, where both $\frac{r}{g}$ and $\frac{g-r}{g}$ appear with g times. In particular, if g is even and $r = \frac{g}{2}$, then the union of X_i's should be exactly the supersingular locus of $X_{\mathbb{F}_{p^g}}$.

Fix an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$. Composing with ι_p defines a bijection between the set of p-adic embeddings of F with that of its Archimedean places. Since p is inert in F, the image of every p-adic embedding of F lies in the maximal unramified extension of $\mathbb{Q}_p$; hence the p-Frobenius σ acts naturally on the set of p-adic embeddings. We label the p-adic embeddings of F by $\{\tau_i \mid i \in \mathbb{Z}/g\mathbb{Z}\}$ such that $\tau_{i+1} = \sigma \circ \tau_i$, and we let $\infty_i = \iota_p \circ \tau_i$ denote the corresponding Archimedean place of F. For an even subset $\mathtt{S}$ of Archimedean places of F, we denote by $B_{\mathtt{S}}$ the quaternion algebra over F which ramifies exactly at $\mathtt{S}$. When $\mathtt{S}$ is the set of all Archimedean places, we also write $\mathtt{S}=\infty$. Fix an isomorphism $(B_{\mathtt{S}}\otimes_F \mathbb{A}_F^{\infty})^{\times}\cong \mathrm{GL}_2(\mathbb{A}_F^{\infty})$ so that K can be viewed as an open compact subgroup of $(B_{\mathtt{S}}\otimes_F \mathbb{A}_F^{\infty})^{\times}$. For a certain subset $\mathtt{T}\subseteq \mathtt{S}$, we will define in Section 2.9 a quaternionic Shimura variety $\mathrm{Sh}_K(B_{\mathtt{S},\mathtt{T}})$ over $\mathbb{F}_{p^g}$ attached to the reductive group $\mathrm{Res}_{F/\mathbb{Q}}(B_{\mathtt{S}}^{\times})$ of level K. This is a $(g-\#\mathtt{S})$-dimensional smooth variety, which is proper if $\mathtt{S}$ is nonempty. Here, the subset $\mathtt{T}$ means some modification on the usual Deligne homomorphism in the definition of $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$ (see Section 2.1). The Shimura varieties $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$ with the same $\mathtt{S}$ but different choices of $\mathtt{T}$ will have the same geometry, but the Galois actions on the geometric-connected components of $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$ will be different.

The basic idea under the constructions of the Goren–Oort cycles is as follows. Recall that there are exactly g divisors, say $Y_1,\dots,Y_g$, in the Goren–Oort stratification (or Ekedahl–Oort stratification) of $X_{\mathbb{F}_{p^g}}$. The main result of [33] shows that, when $g\geq 1$, *each* Y_i *is isomorphic to a* $\mathbb{P}^1$*-fibration over* $\mathrm{Sh}_K(B_{\mathtt{S}_i,\mathtt{T}_i})$ *with* $\mathtt{S}_i=\{\infty_i,\infty_{i-1}\}$ *and* $\mathtt{T}_i=\{\infty_i\}$. Actually, the results of [33] apply to more general quaternionic Shimura varieties. For $r=1$, the generalized Goren–Oort cycles of codimension 1 are defined to be these Goren–Oort divisors Y_i's. When $r\geq 2$, we consider the $g-2$ Goren–Oort divisors Z_j of $\mathrm{Sh}_K(B_{\mathtt{S}_i,\mathtt{T}_i})$ for $j\in\{i-2,\dots,i-g+1\}$ (see Proposition 2.31). Taking the inverse image of Z_j in Y_i, we get a codimension 2 cycle $Y_{i,j}$ in $X_{\overline{\mathbb{F}}_p}$, which admits a two-step iterated $\mathbb{P}^1$-bundle morphism $Y_{i,j}\to Z_j\to \mathrm{Sh}_K(B_{i,j})$, where $\mathrm{Sh}_K(B_{i,j})$ is some quaternionic Shimura variety of dimension $g-4$. This gives the construction for $r=2$. In the general case, the codimension r generalized Goren–Oort cycles on $X_{\overline{\mathbb{F}}_p}$ are obtained by repeating this process r times.

Example 1.5

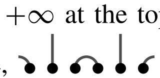
(1) When $g=2$, there are two Goren–Oort divisors X_1,X_2 on $X_{\mathbb{F}_{p^2}}$, and each X_i is isomorphic to a $\mathbb{P}^1$-bundle over the discrete Shimura set $\mathrm{Sh}_K(B^{\times}_{\infty,\mathtt{T}_i})_{\overline{\mathbb{F}}_p}$ with $\mathtt{T}_i=\{\infty_i\}$. We remark that the cycle constructed by Langer in [19] is completely contained in (but not equal to) the union $X_1\cup X_2$.

(2) When $g=3$, there are three Goren–Oort divisors on $X_{\mathbb{F}_{p^3}}$, say Y_1,Y_2,Y_3. For $i\in\mathbb{Z}/3\mathbb{Z}$, each Y_i is a $\mathbb{P}^1$-fibration over $\mathrm{Sh}_K(B_{\mathtt{S}_i,\mathtt{T}_i})$ as discussed above.

(3) When $g=4$, there are six Goren–Oort cycles of codimension 2 on $X_{\mathbb{F}_{p^4}}$. We start with the four Goren–Oort divisors $Y_1,\dots,Y_4$ of $X_{\mathbb{F}_{p^4}}$. Then for each $i\in\mathbb{Z}/4\mathbb{Z}$, we have a $\mathbb{P}^1$-fibration $\pi_i\colon Y_i\to \mathrm{Sh}_K(B_{\mathtt{S}_i,\mathtt{T}_i})$. On each quaternionic Shimura surface $\mathrm{Sh}_K(B_{\mathtt{S}_i,\mathtt{T}_i})$, there are two Goren–Oort divisors, say Z_{i-2} and Z_{i-3}, corresponding to ∞_{i-2} and ∞_{i-3}, respectively. Then each of Z_j with $j\in\{i-2,i-3\}$ is again isomorphic to a $\mathbb{P}^1$-fibration over the 0-dimensional Shimura variety $\mathrm{Sh}_K(B_{\infty,\mathtt{T}_i})$ with $\mathtt{T}_i=\{\infty_i,\infty_{i-2}\}$. Put

$X_{i,j} := \pi_i^{-1}(Z_j) \subseteq Y_i$. This is a codimension 2 cycle on $X_{\mathbb{F}_{p^4}}$. In Theorem 2.32, we will see that $X_{1,3} = X_{3,1} = Y_1 \cap Y_3$ and that $X_{2,4} = X_{4,2} = Y_2 \cap Y_4$, so the six Goren–Oort cycles of codimension 2 are exactly $X_{1,3}, X_{2,4}, X_{1,2}, X_{2,3}, X_{3,4}, X_{4,1}$. Note that the geometry of these six cycles are not the same: each irreducible component of $X_{1,3}$ and $X_{2,4}$ are isomorphic to $(\mathbb{P}^1)^2$, while that of the other four Goren–Oort cycles is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(-p))$ over $\mathbb{P}^1$ (see Example 3.10).

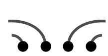
After finishing the present article, we were informed that when $g = 4$, the geometry of these cycles was already known to Yu [36], using a different method.

The best way (so far) to parameterize the generalized Goren–Oort cycles is to use some combinatorial data, called *periodic semimeanders* (mostly for the benefit of later computation of the Gysin-restriction matrix). A *periodic semimeander of g nodes* is a graph where g nodes are aligned equidistantly on a section of a vertical cylinder, and are either connected pairwise by nonintersecting curves (called *arcs*) drawn above the section, or connected by a straight line (called *semilines*) to $+\infty$ at the top of the cylinder. We use r to denote the number of arcs. For example,  and  are both semimeanders of six points with $r = 2$ and 3, respectively. An elementary computation shows that there are $\binom{g}{r}$ semimeanders of g nodes with r arcs for $r \leq \frac{g}{2}$. (For a detailed discussion, see Section 3.1.)

To each periodic semimeander $\mathfrak{a}$ with g nodes and r arcs, one can associate a generalized Goren–Oort cycle $X_{\mathfrak{a}}$ of codimension r in $X_{\mathbb{F}_{p^{2g}}}$ (we refer the reader to Section 3.8 for the precise definition). The g nodes of a periodic semimeander $\mathfrak{a}$ correspond to the g Archimedean places $\infty_1, \ldots, \infty_g$ from the left to the right. By construction, $X_{\mathfrak{a}}$ is an r-step iterated $\mathbb{P}^1$-bundle over the quaternionic Shimura variety $\mathrm{Sh}_K(B_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$, where $\mathtt{S}_{\mathfrak{a}}$ consists of all Archimedean places of F corresponding to the end nodes of all r arcs, and $\mathtt{T}_{\mathfrak{a}}$ consists of those corresponding to the *right* ends of the r arcs. We will denote by

$$\pi_{\mathfrak{a}} \colon X_{\mathfrak{a}} \to \mathrm{Sh}_K(B_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$$

the projection map. For instance, when $g = 4$ and $r = 2$, the cycles $X_{1,3}$ and $X_{2,4}$ in Example 1.5 correspond to the semimeanders  and , and the other 4 cycles $X_{1,2}, X_{2,3}, X_{3,4}, X_{4,1}$ correspond respectively to the semimeanders

, , , .

1.6. *Main theorem revisited*

We now describe our main results. We consider a regular multiweight $(\underline{k}, w) \in \mathbb{Z}^{g+1}$ with $\underline{k} = (k_1, \ldots, k_g)$, that is, a collection of integers such that $k_i \geq 2$ and $k_i \equiv w$

mod 2. There is an automorphic étale local system $\mathcal{L}^{(\underline{k},w)}$ on X, which is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf of rank $\prod_{i=1}^{g}(k_i-1)$ pure of Deligne weight $g(w-1)$ (see Section 2.10). We fix a cuspidal automorphic representation $\pi=\pi^\infty\otimes\pi_\infty$ of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to a holomorphic Hilbert modular forms of weight $(\underline{k},w)$ such that $\pi^{\infty,K}\neq 0$. Let $\rho_\pi\colon \mathrm{Gal}_F\to\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be the Galois representation attached to π, and let $\mathrm{As}(\rho_\pi)=\bigotimes\mathrm{Ind}_{\mathrm{Gal}_{\mathbb{Q}}}^{\mathrm{Gal}_F}(\rho_\pi)$ be the Asai representation of ρ_π. Let $\mathscr{H}_{K^p}:=\overline{\mathbb{Q}}_\ell[K^p\backslash\mathrm{GL}_2(\mathbb{A}_F^{\infty,p})/K^p]$ denote the prime-to-p Hecke algebra, and let $\pi^{\infty,p}$ be the prime-to-p part of π. We put

$$H^g_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)})[\pi]:=\mathrm{Hom}_{\mathscr{H}_{K^p}}\big((\pi^{\infty,p})^{K^p},H^g_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)})\big).$$[2]

According to [2], the $\mathrm{Gal}_{\mathbb{F}_p}$-module $H^g_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)})[\pi]$ has the same semisimplification as

$$\mathrm{As}(\rho_\pi)|_{\mathrm{Gal}_{\mathbb{F}_p}}=\bigotimes\mathrm{Ind}_{\mathrm{Gal}_{\mathbb{F}_{p^g}}}^{\mathrm{Gal}_{\mathbb{F}_p}}(\rho_\pi|_{\mathrm{Gal}_{\mathbb{F}_{p^g}}}).$$

Fix an integer r with $1\le r\le g/2$. We denote by $\mathfrak{B}^r_\emptyset$ the set of periodic semimeanders of g nodes and r arcs. As explained above, for each $\mathfrak{a}\in\mathfrak{B}^r_\emptyset$, we have a generalized Goren–Oort cycle $X_{\mathfrak{a}}$ in $X_{\mathbb{F}_{p^{2g}}}$ of codimension r, which admits an r-step iterated $\mathbb{P}^1$-bundle morphism $\pi_{\mathfrak{a}}\colon X_{\mathfrak{a}}\to\mathrm{Sh}_K(B_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$. We can also define an automorphic étale local system $\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}$ on $\mathrm{Sh}_K(B_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$ (see Section 2.10), which is compatible with the local system $\mathcal{L}^{(\underline{k},w)}$ on X in the sense that there is a canonical isomorphism $\pi_{\mathfrak{a}}^*\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}\cong\mathcal{L}^{(\underline{k},w)}|_{X_{\mathfrak{a}}}$. When $(\underline{k},w)=(2,\dots,2)$, both $\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}$ and $\mathcal{L}^{(\underline{k},w)}$ are the constant sheaf $\overline{\mathbb{Q}}_\ell$. We consider the composite map

$$\begin{aligned}\mathrm{Gys}_{\mathfrak{a}}\colon\ & H^{g-2r}_{\mathrm{et}}\big(\mathrm{Sh}_K(B_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{(\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}})}\big)\\ &\xrightarrow{\pi_{\mathfrak{a}}^*} H^{g-2r}_{\mathrm{et}}(X_{\mathfrak{a},\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}|_{X_{\mathfrak{a}}})\xrightarrow{\mathrm{Gysin}} H^g_{\mathrm{et}}\big(X_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}(r)\big),\end{aligned}$$

where the second arrow is the Gysin map. Since the construction of $X_{\mathfrak{a}}$ is compatible with prime-to-p Hecke correspondence, $\mathrm{Gys}_{\mathfrak{a}}$ is equivariant under the action by $\mathscr{H}_{K^p}$. The main result of this article is the following.

THEOREM 1.7

Let α,β denote the two eigenvalues of $\rho_\pi(\mathrm{Frob}_{p^g})$. Consider the map induced by the direct sum of Gysin maps

$$\mathrm{Gys}\colon \bigoplus_{\mathfrak{a}\in\mathfrak{B}^r_\emptyset} H^{g-2r}_{\mathrm{et}}\big(\mathrm{Sh}_K(B_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}\big)[\pi]$$

[2]It should be noted that, by the strong multiplicity 1 theorem, we then have an isomorphism $\mathrm{Hom}_{\mathscr{H}_{K^p}}\big(\pi^{\infty,p,K^p},H^g_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)})\big)\cong\mathrm{Hom}_{\mathscr{H}_K}\big(\pi^{\infty,K},H^g_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)})\big)$.

$$\xrightarrow{\sum_{\mathfrak{a}} \mathrm{Gys}_{\mathfrak{a}}} H^g_{\mathrm{et}}\big(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}(r)\big)[\pi] \quad (1.7.1)$$

on the π-isotypic components. Then the following statements hold:

(1) *If $\alpha \neq \beta$, then the morphism* Gys *is injective.*

(2) *If α/β is not a $2n$th root of unity for any $n \leq g$,*[3] *then* Gys *induces an isomorphism when restricted to the generalized eigenspaces of* $\mathrm{Frob}_{p^{2g}}$ *on both source and target with eigenvalues $\alpha^{2r}\beta^{2(g-r)}/p^{2rg}$.*

This theorem will be proved as a special case of Theorem 4.5. This theorem can be viewed as a version of geometric Jacquet–Langlands transfer from the quaternionic Shimura varieties $\mathrm{Sh}_K(B_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})$'s to X. As for the applications to the Tate conjecture, we assume that g is even. Then for all periodic semimeanders $\mathfrak{a}$ with g nodes and $\frac{g}{2}$ arcs, we have $\mathtt{S}_\mathfrak{a} = \infty$, and the Goren–Oort cycle $X_{\mathfrak{a},\overline{\mathbb{F}}_p}$ is a collection of $(g/2)$-step iterated $\mathbb{P}^1$-bundles parameterized by the common discrete Shimura set[4]

$$\mathrm{Sh}_K(B_\infty)_{\overline{\mathbb{F}}_p} = B_\infty^\times \backslash (B_\infty \otimes_F \mathbb{A}_F^\infty)^\times / K. \quad (1.7.2)$$

Applying Theorem 1.7 to the case $(\underline{k}, w) = (2, \dots, 2)$ gives Theorem 1.2.

1.8. Overview of the proof of Theorem 1.7

We consider the restriction map

$$\begin{aligned}\mathrm{Res}_\mathfrak{a}\colon H^g_{\mathrm{et}}\big(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}(r)\big)[\pi] \quad &\to \quad H^g_{\mathrm{et}}\big(X_{\mathfrak{a},\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}|_{X_\mathfrak{a}}(r)\big)[\pi] \\ &\xrightarrow{\mathrm{Tr}_{\pi_\mathfrak{a}},\cong} H^{g-2r}_{\mathrm{et}}\big(\mathrm{Sh}_K(B_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}\big)[\pi],\end{aligned}$$

where the second map is the trace isomorphism. We get thus a composite map

$$\begin{array}{ccc}\displaystyle\bigoplus_{\mathfrak{b}\in\mathfrak{B}^r_\emptyset} H^{g-2r}_{\mathrm{et}}\big(\mathrm{Sh}_K(B_{\mathtt{S}_\mathfrak{b},\mathtt{T}_\mathfrak{b}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_\mathfrak{b},\mathtt{T}_\mathfrak{b}}\big)[\pi] & \xrightarrow{\mathrm{Gys}} & H^g_{\mathrm{et}}\big(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}(r)\big)[\pi] \\ & & \Big\downarrow {\scriptstyle \mathrm{Res}:=\bigoplus_\mathfrak{a} \mathrm{Res}_\mathfrak{a}} \\ & & \displaystyle\bigoplus_{\mathfrak{a}\in\mathfrak{B}^r_\emptyset} H^{g-2r}_{\mathrm{et}}\big(\mathrm{Sh}_K(B_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}\big)[\pi]\end{array}$$

(1.8.1)

[3]The reason why we have $2n$th (as opposed to nth) root of unity here is purely technical (see Remark 4.6(3)).

[4]Our previous notation for this Shimura set should be $\mathrm{Sh}_K(B_{\infty,\mathtt{T}_\mathfrak{a}})_{\overline{\mathbb{F}}_p}$. Since they are all canonically isomorphic for all $\mathfrak{a}$, we omit $\mathtt{T}_\mathfrak{a}$ from the notation.

When $(\underline{k}, w)$ is of parallel weight 2, this is essentially the intersection matrix of the cycles $X_{\mathfrak{a}}$'s in $X_{\overline{\mathbb{F}}_p}$. The upshot is that each "matrix entry" $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ can be read off from the periodic semimeanders $\mathfrak{a}$ and $\mathfrak{b}$ (see Theorem 4.4), and the determinant of the intersection matrix is closely related to the determinant of the Gram matrix of the link representation of periodic Temperley–Lieb algebras, which has been computed in [24]. Using this result, one can compute explicitly the determinant of $\mathrm{Res} \circ \mathrm{Gys}$, which does not vanish as long as $\alpha \neq \beta$. Theorem 1.7(1) follows immediately, and statement (2) is obtained from statement (1) along with a direct computation of the dimensions of the generalized eigenspaces of $\mathrm{Frob}_{p^{2g}}$ with the given eigenvalue.

Example 1.9

(1) If $g = 2$ and $r = 1$, then the intersection matrix $(\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}})_{\mathfrak{a},\mathfrak{b} \in \mathfrak{B}^1_{\emptyset}}$ (under certain basis) is

$$\begin{pmatrix} -2p & \alpha+\beta \\ p^2\frac{\alpha+\beta}{\alpha\beta} & -2p \end{pmatrix},$$

whose determinant is $p^2(\alpha-\beta)^2/(\alpha\beta)$.

(2) Assume that $g = 3$ and that $r = 1$. Even though the Shimura varieties $\mathrm{Sh}_K(G_{\mathrm{S}_{\mathfrak{a}},\mathrm{T}_{\mathfrak{a}}})$ for $\mathfrak{a} \in \mathfrak{B}^1_{\emptyset}$ are not exactly the same, we nevertheless have an isomorphism (see Proposition A.3)

$$H^1_{\mathrm{et}}\big(\mathrm{Sh}_K(B_{\mathrm{S}_{\mathfrak{a}},\mathrm{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell\big)[\pi] \cong \big[\rho_\pi \otimes \det(\rho_\pi)(1)\big]|_{\mathrm{Gal}_{\mathbb{F}_p}}$$

for each $\mathfrak{a} \in \mathfrak{B}^1_{\emptyset}$ as a $\mathrm{Gal}_{\mathbb{F}_{p^3}}$-module. The intersection matrix (under a suitable basis) is

$$\begin{pmatrix} -2p & \eta^{-1} & \eta \\ \eta & -2p & \eta^{-1} \\ \eta^{-1} & \eta & -2p \end{pmatrix},$$

where η is some operator which acts as scalar multiplication by $(\alpha/\beta)^{1/3}$ (resp., by $(\beta/\alpha)^{1/3}$) on the eigenspace of $\mathrm{Frob}_{p^3} = \alpha\beta^2/p^3$ (resp., $\mathrm{Frob}_{p^3} = \alpha^2\beta/p^3$) in $[\rho_\pi \otimes \det(\rho_\pi)(1)]|_{\mathrm{Gal}_{\mathbb{F}_p}}$. The determinant of the above matrix is $-p^3(\alpha-\beta)^2/(\alpha\beta)$.

Structure of this article

In Section 2, we recall necessary facts about Goren–Oort stratification from [33]. Some of the proofs are mostly bookkeeping, but also technical (the readers may skip these). In Section 3, we first recall the combinatorics about periodic semimeanders and then give the definition of the Goren–Oort cycles associated to periodic semimeanders. In Section 4, we state our main Theorem 4.5 and prove it assuming Theorem 4.4, which says that the Gysin-restriction matrix for Goren–Oort cycles is roughly the same as the Gram matrix of the corresponding periodic semimeanders. This key theorem, Theorem 4.4, is proved in Section 5. The Appendix includes a proof of the description of the cohomology of quaternionic Shimura varieties. This is well known to the experts, but we include it there for completeness.

1.10. Notation

For a field L, we use Gal_L to denote its absolute Galois group. For a number field L, we write $\mathbb{A}_L$ (resp., $\mathbb{A}_L^\infty$, $\mathbb{A}_L^{\infty,p}$) for its ring of adeles (resp., finite adeles, finite adeles away from a rational prime p). When $L = \mathbb{Q}$, we suppress the subscript L (e.g., by writing $\mathbb{A}^\infty$). Let $\underline{p}_L$ denote the idele of $\mathbb{A}_L^\infty$ which is p at all p-adic places and trivial elsewhere. We also normalize the Artin reciprocity map $\mathrm{Art}\colon \mathbb{A}_L^\times/L^\times \to \mathrm{Gal}_L^{\mathrm{ab}}$ so that a local uniformizer at a finite place v corresponds to a *geometric* Frobenius element at v.

Throughout this article, we fix F a totally real field of degree $g > 1$ over $\mathbb{Q}$. Let Σ denote the set of places of F, and let Σ_∞ be the subset of all real places. We fix a prime number $p > 2$ *inert* in the extension $F/\mathbb{Q}$.[5] We also set $\mathfrak{p} = p\mathcal{O}_F$, $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$, $\mathcal{O}_\mathfrak{p}$ the valuation ring, and $k_\mathfrak{p}$ the residue field.

We fix an isomorphism $\iota_p\colon \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_p$. Let $\mathbb{Q}_{p^g}$ denote the unramified extension of $\mathbb{Q}_p$ of degree g in $\overline{\mathbb{Q}}_p$, and let $\mathbb{Z}_{p^g}$ be its valuation ring. Postcomposition with ι_p induces a bijection between the set of Archimedean places and $\Sigma_\infty = \mathrm{Hom}(F, \mathbb{R})$ and the set of p-adic embeddings $\mathrm{Hom}(F, \mathbb{Q}_{p^g}) \cong \mathrm{Hom}(\mathcal{O}_F, \mathbb{F}_{p^g})$. In particular, the absolute Frobenius σ acts on Σ_∞ by sending $\tau \in \Sigma_\infty$ to $\sigma\tau := \sigma \circ \tau$; this makes Σ_∞ into one cycle. Let $\mathbb{Q}_p^{\mathrm{ur}}$ denote the maximal unramified extension of $\mathbb{Q}_p$, and let $\mathbb{Z}_p^{\mathrm{ur}}$ denote its valuation ring. For a finite field $\mathbb{F}_q$, we denote by $\mathrm{Frob}_q \in \mathrm{Gal}_{\mathbb{F}_q}$ the *geometric* Frobenius element.

2. Goren–Oort stratification

We first recall the Goren–Oort stratification of the special fiber of quaternionic Shimura varieties and their descriptions, following [33]. We tailor our discussion to a later application, and hence we will focus on certain special cases discussed in [33].

2.1. Quaternionic Shimura varieties

Let $\mathtt{S}$ be a set of places of F of even cardinality such that $\mathfrak{p} \notin \mathtt{S}$. Put $\mathtt{S}_\infty = \mathtt{S} \cap \Sigma_\infty$

[5] Although most of our argument works equally well when p is only assumed to be unramified, we insist on assuming that p is inert, which largely simplifies the notation so that the proof of the main result is more accessible (but see Remark 4.6(1)).

and $\mathtt{S}_\infty^c = \Sigma_\infty - \mathtt{S}_\infty$,[6] and $d = \#\mathtt{S}_\infty^c$. We also fix a subset $\mathtt{T}$ of $\mathtt{S}_\infty$. We denote by $B_\mathtt{S}$ the quaternion algebra over F ramified exactly at $\mathtt{S}$. Let $G_{\mathtt{S},\mathtt{T}} = \mathrm{Res}_{F/\mathbb{Q}}(B_\mathtt{S}^\times)$ be the associated $\mathbb{Q}$-algebraic group. Here we inserted the subscript $\mathtt{T}$ because we use the following *Deligne homomorphism*:

$$h_{\mathtt{S},\mathtt{T}}\colon \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G_{\mathtt{S},\mathtt{T}}(\mathbb{R}) \cong (\mathbb{H}^\times)^{\mathtt{S}_\infty - \mathtt{T}} \times (\mathbb{H}^\times)^{\mathtt{T}} \times \mathrm{GL}_2(\mathbb{R})^{\mathtt{S}_\infty^c}$$

$$x + y\mathbf{i} \longmapsto \Big((1,\dots,1), (x^2+y^2,\dots,x^2+y^2), \Big(\big(\begin{smallmatrix} x & y \\ -y & x\end{smallmatrix}\big),\dots,\big(\begin{smallmatrix} x & y \\ -y & x\end{smallmatrix}\big)\Big)\Big).$$

When $\mathtt{T} = \emptyset$, the Deligne homomorphism $h_{\mathtt{S},\emptyset}$ is the same as $h_\mathtt{S}$ considered in [33, Section 3.1]. The $G_{\mathtt{S},\mathtt{T}}(\mathbb{R})$-conjugacy class of $h_{\mathtt{S},\mathtt{T}}$ is independent of $\mathtt{T}$ and is isomorphic to $\mathfrak{H}_\mathtt{S} := (\mathfrak{h}^\pm)^{\mathtt{S}_\infty^c}$, where $\mathfrak{h}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. Consider the Hodge cocharacter

$$\mu_{\mathtt{S},\mathtt{T}}\colon \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z\mapsto(z,1)} \mathbb{S}_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_{\mathtt{S},\mathtt{T}}} G_{\mathtt{S},\mathtt{T},\mathbb{C}}.$$

Here, the composite of the natural inclusion $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ with the first (resp., second) projection $\mathbb{S}(\mathbb{C}) \to \mathbb{C}^\times$ is the identity map (resp., the complex conjugation).

The reflex field $F_{\mathtt{S},\mathtt{T}}$—that is, the field of definition of the conjugacy class of $\mu_{\mathtt{S},\mathtt{T}}$—is a finite extension of $\mathbb{Q}$ sitting inside $\mathbb{C}$ and hence inside $\overline{\mathbb{Q}}_p$ via ι_p. It is clear that the p-adic closure of $F_{\mathtt{S},\mathtt{T}}$ in $\overline{\mathbb{Q}}_p$ is contained in $\mathbb{Q}_{p^g}$, the unramified extension of $\mathbb{Q}_p$ of degree g in $\overline{\mathbb{Q}}_p$. Instead of working with an occasional smaller reflex field, we are content with working with Shimura varieties over $\mathbb{Q}_{p^g}$.

We fix an isomorphism $G_{\mathtt{S},\mathtt{T}}(\mathbb{Q}_p) \simeq \mathrm{GL}_2(F_\mathfrak{p})$ and we put $K_p = \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$. We will only consider open compact subgroups $K \subset G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^\infty)$[7] of the form $K = K_pK^p$ with K^p an open compact subgroup of $G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^{\infty,p})$, or occasionally $K = \mathrm{Iw}_pK^p$ with $\mathrm{Iw}_p := \big(\begin{smallmatrix} \mathcal{O}_\mathfrak{p}^\times & \mathcal{O}_\mathfrak{p} \\ p\mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p}^\times\end{smallmatrix}\big)$ when $\mathtt{S}_\infty^c = \emptyset$. For such a K, we have a Shimura variety $\mathcal{S}h_K(G_{\mathtt{S},\mathtt{T}})$ defined over $\mathbb{Q}_{p^g}$, whose $\mathbb{C}$-points (via ι_p) are given by

$$\mathcal{S}h_K(G_{\mathtt{S},\mathtt{T}})(\mathbb{C}) = G_{\mathtt{S},\mathtt{T}}(\mathbb{Q})\backslash\mathfrak{H}_\mathtt{S} \times G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^\infty)/K.$$

We put $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}}) := \varprojlim_{K^p} \mathcal{S}h_{K^pK_p}(G_{\mathtt{S},\mathtt{T}})$. This Shimura variety has dimension $d = \#\mathtt{S}_\infty^c$. There is a natural morphism of geometrically connected components

$$\pi_0\big(\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{Q}}_p}\big) \longrightarrow F_+^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^\times, \tag{2.1.1}$$

where $F_+^\times$ is the subgroup of totally positive elements of $F^\times$, and the superscript cl stands for taking closure in the corresponding topological space. The morphism

[6]Note that the upper script c was used to denote *complex conjugation* in [33]. In the present arrticle, however, we use it to mean *taking the set-theoretic complement*.

[7]In earlier articles of this series, the open compact subgroup K was denoted by $K_\mathtt{S}$. We choose to drop the subscript because, for all $\mathtt{S}$ we encounter later, the group $G_\mathtt{S}(\mathbb{A}^\infty)$ is isomorphic, and hence we can naturally identify the $K_\mathtt{S}$'s for different $\mathtt{S}$'s.

(2.1.1) is an isomorphism if $\mathtt{S}_\infty^c \neq \emptyset$ by [4, Théorème 2.4]. Following the convention in [33, Section 2.11], we call the preimage of an element $\boldsymbol{x} \in F_+^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^\times$ under the map (2.1.1) a *geometrically connected component*, even though it is not geometrically connected when $\mathtt{S}_\infty^c = \emptyset$. The preimage of $\mathbf{1}$ is called the *neutral geometric connected component*, which we denote by $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})^\circ_{\overline{\mathbb{Q}}_p}$.

Note that, for different choices of $\mathtt{T}$, the Shimura varieties $\mathcal{S}h_K(G_{\mathtt{S},\mathtt{T}})$ are isomorphic over $\overline{\mathbb{Q}}_p$ (in fact over $\overline{\mathbb{Q}}$ if we have not p-adically completed the reflex field), but the actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^g})$ depend on $\mathtt{T}$. By Shimura's reciprocity law (see [4] or [33, Section 2.7]), the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^g})$ on $\pi_0(\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{Q}}_p})$ factors through $\mathrm{Gal}_{\mathbb{F}_{p^g}} \cong \mathrm{Gal}(\mathbb{Z}_p^{\mathrm{ur}}/\mathbb{Z}_{p^g})$, so that the connected components of $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{Q}}_p}$ are actually defined over $\mathbb{Q}_p^{\mathrm{ur}}$, the maximal unramified extension of $\mathbb{Q}_p$. More precisely, the action of the geometric Frobenius of $\mathbb{F}_{p^g}$ on $F_+^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^\times$, induced through the homomorphism (2.1.1), is given by multiplication by the finite idele

$$(\underline{p}_F)^{(2\#\mathtt{T}+\#\mathtt{S}_\infty^c)} \in F_+^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^\times.\text{[8]} \tag{2.1.2}$$

This determines a reciprocity map:

$$\mathrm{Rec}_p\colon \mathrm{Gal}_{\mathbb{F}_{p^g}} \longrightarrow F_+^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^\times.$$

Write $\nu\colon G_{\mathtt{S},\mathtt{T}} \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ for the reduced norm homomorphism. Following Deligne's recipe [5] of connected Shimura varieties, we put

$$\mathcal{G}_{\mathtt{S},\mathtt{T},p} := \big(G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^{\infty,p})/\mathcal{O}_{F,(p)}^{\times,\mathrm{cl}}\big) \times \mathrm{Gal}_{\mathbb{F}_{p^g}}\text{ [9]} \tag{2.1.3}$$

and define $\mathcal{E}_{G_{\mathtt{S},\mathtt{T}}}$ to be the subgroup of $\mathcal{G}_{\mathtt{S},\mathtt{T},p}$ consisting of pairs (x,σ) such that $\nu(x)$ is equal to $\mathrm{Rec}_p(\sigma)^{-1}$. Here, $\mathcal{O}_{F,(p)}^{\times,\mathrm{cl}}$ denotes the closure of $\mathcal{O}_{F,(p)}^\times$ in $G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^{\infty,p})$.

The limit $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathbb{Q}_p^{\mathrm{ur}}}$ carries an action by $\mathcal{G}_{\mathtt{S},\mathtt{T},p}$, and $\mathcal{E}_{G_{\mathtt{S},\mathtt{T}}}$ is the stabilizer of each geometrically connected component. Conversely, if $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})^\bullet_{\mathbb{Q}_p^{\mathrm{ur}}}$ is a geometrically connected component, then one can recover $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})$ from $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})^\bullet_{\mathbb{Q}_p^{\mathrm{ur}}}$ by first forming the product

$$\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})^\bullet_{\mathbb{Q}_p^{\mathrm{ur}}} \times_{\mathcal{E}_{G_{\mathtt{S},\mathtt{T}}}} \mathcal{G}_{\mathtt{S},\mathtt{T},p}$$

and then taking the Galois descent to $\mathbb{Q}_{p^g}$.

[8]When $\mathtt{S}_\infty^c = \emptyset$ or, equivalently, when $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})$ is a $\mathbf{0}$-dimensional Shimura variety, the action of Frob_{p^g} is given by multiplication by the finite idele $(\underline{p}_F)^{\#\mathtt{T}}$ in the center $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ of $G_{\mathtt{S},\mathtt{T}}$. This gives the canonical model for the discrete Shimura variety in the sense of [33, Section 2.8].

[9]Comparing with [33, (2.11.3)], we dropped the star extension because the center of $G_{\mathtt{S},\mathtt{T}}$ is $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, which has trivial first cohomology. We also include the Galois part into the definition of $\mathcal{G}$ to simplify notation here.

Notation 2.2
Note that $G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^\infty)$ depends only on the finite places contained in $\mathtt{S}$. In later applications, we will consider only pairs of subsets $(\mathtt{S}',\mathtt{T}')$ such that $\mathtt{S}'$ contains the *same* finite places as $\mathtt{S}$. In that case, we will fix an isomorphism $G_{\mathtt{S}',\mathtt{T}'}(\mathbb{A}^\infty)\cong G_{\mathtt{S},\mathtt{T}}(\mathbb{A}^\infty)$, and denote them uniformly by $G(\mathbb{A}^\infty)$ when no confusions arise. Similarly, we have its subgroup $G(\mathbb{A}^{\infty,p})\subset G(\mathbb{A}^\infty)$ consisting of elements whose p-component is trivial. Thus, we may view K (resp., K^p) as an open compact subgroup of $G(\mathbb{A}^\infty)$ (resp., $G(\mathbb{A}^{\infty,p})$).

Under this identification, the group $\mathcal{G}_{\mathtt{S},\mathtt{T},p}$ is independent of $\mathtt{S},\mathtt{T}$, and we henceforth write $\mathcal{G}_p$ for it. Its subgroup $\mathcal{E}_{G_{\mathtt{S},\mathtt{T}}}$ in general depends on the choice of $\mathtt{S}$ and $\mathtt{T}$. However, the key point is that, if $\mathtt{S}'$ and $\mathtt{T}'$ are another pair of subsets satisfying similar conditions and $\#\mathtt{S}_\infty-2\#\mathtt{T}=\#\mathtt{S}'_\infty-2\#\mathtt{T}'$ (which will be the case we consider later in this article), then the subgroup $\mathcal{E}_{G_{\mathtt{S},\mathtt{T}}}$ is the same as $\mathcal{E}_{G_{\mathtt{S}',\mathtt{T}'}}$.

Remark 2.3
Using Proposition 2.7 and Construction 2.12 later, we have access to most of the statements in [33] which were initially proved for unitary groups and interpreted using connected Shimura varieties. The key point mentioned in Notation 2.2 has the additional benefit that the description of the Goren–Oort strata actually descends to quaternionic Shimura varieties because now the subgroups $\mathcal{E}_{G_{\mathtt{S},\mathtt{T}}}$ are compatible for different $\mathtt{S}$'s and $\mathtt{T}$'s.

2.4. An auxiliary CM field
To use the results in [33] (which rely on Carayol's construction), we fix a CM extension E/F such that

- every place in $\mathtt{S}$ is inert in E/F, and
- the place $\mathfrak{p}$ splits as $\mathfrak{q}\bar{\mathfrak{q}}$ in E/F if $\#\mathtt{S}^c_\infty$ is even, and it is inert in E/F if $\#\mathtt{S}^c_\infty$ is odd.

These conditions imply that $B_\mathtt{S}$ splits over E. In later applications, we will need to consider several subsets $\mathtt{S}$ at the same time. We note that, for all subsets $\mathtt{S}$ involved later, the finite places contained in $\mathtt{S}$ are the same, and thus $\#\mathtt{S}^c_\infty$ will have the same parity. In particular, this means that we can fix for the rest of this article one CM field E that satisfies the above conditions (for the initial $B_\mathtt{S}$).

We will frequently use the following two finite idele elements:

(1) $\underline{p}_F$ denotes the finite idele in $\mathbb{A}_F^\infty$ which is p at $\mathfrak{p}$ and is 1 elsewhere (which we have already introduced in Section 1.10);

(2) when $\mathfrak{p}$ splits into $\mathfrak{q}\bar{\mathfrak{q}}$ in E, $\underline{\mathfrak{q}}$ denotes the finite idele in $\mathbb{A}_E^\infty$ which is p at $\mathfrak{q}$, p^{-1} at $\bar{\mathfrak{q}}$, and 1 elsewhere.

Let $\Sigma_{E,\infty}$ denote the set of complex embeddings of E. We fix a choice of subset $\tilde{\mathtt{S}}_\infty \subseteq \Sigma_{E,\infty}$ such that the natural restriction map $\Sigma_{E,\infty} \to \Sigma_\infty$ induces an isomorphism $\tilde{\mathtt{S}}_\infty \xrightarrow{\cong} \mathtt{S}_\infty$. When $\mathfrak{p}$ splits into $\mathfrak{q}\bar{\mathfrak{q}}$, we use $\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}$ (resp., $\tilde{\mathtt{S}}_{\infty/\bar{\mathfrak{q}}}$) to denote the subset of places in $\tilde{\mathtt{S}}_\infty$ inducing $\mathfrak{q}$ (resp., $\bar{\mathfrak{q}}$) through the isomorphism ι_p. We put

$$\Delta_{\tilde{\mathtt{S}}_\infty} := \#\tilde{\mathtt{S}}_{\infty/\bar{\mathfrak{q}}} - \#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}. \tag{2.4.1}$$

We note that all the subsets $\tilde{\mathtt{S}}_\infty$ that we encounter later in this article will all have the same $\Delta_{\tilde{\mathtt{S}}_\infty}$.

We write $E_\mathfrak{p}$ for $F_\mathfrak{p} \otimes_F E$. It is the quadratic unramified extension of $F_\mathfrak{p}$ if $\mathfrak{p}$ is inert and it is $E_\mathfrak{q} \times E_{\bar{\mathfrak{q}}}$ if $\mathfrak{p}$ splits. We set $\mathcal{O}_{E_\mathfrak{p}} := \mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_F} \mathcal{O}_E$.

We put $\tilde{\mathtt{S}} = (\mathtt{S}, \tilde{\mathtt{S}}_\infty)$. Put $T_{E,\tilde{\mathtt{S}},\mathtt{T}} = T_E = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$, where the subscript $(\tilde{\mathtt{S}}, \mathtt{T})$ means that we take the following Deligne homomorphism:

$$\begin{aligned} h_{E,\tilde{\mathtt{S}},\mathtt{T}} \colon \mathbb{S}(\mathbb{R}) \longrightarrow T_{E,\tilde{\mathtt{S}},\mathtt{T}}(\mathbb{R}) = \bigoplus_{\tau\in\Sigma_\infty} (E \otimes_{F,\tau} \mathbb{R})^\times \cong (\mathbb{C}^\times)^{\mathtt{S}_\infty - \mathtt{T}} \times (\mathbb{C}^\times)^{\mathtt{T}} \times (\mathbb{C}^\times)^{\mathtt{S}_\infty^c} \\ z = x + y\mathbf{i} \longmapsto \big((\bar{z}, \dots, \bar{z}), (z^{-1}, \dots, z^{-1}), (1, \dots, 1)\big). \end{aligned}$$

Here, the isomorphism $E \otimes_{F,\tau} \mathbb{R} \simeq \mathbb{C}$ for $\tau \in \mathtt{S}_\infty$ is given by the chosen embedding $\tilde{\tau} \in \tilde{\mathtt{S}}_\infty$ lifting τ. One has the system of 0-dimensional Shimura varieties $\mathcal{S}h_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ with $\mathbb{C}$-points given by

$$\mathcal{S}h_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})(\mathbb{C}) = E^{\times,\mathrm{cl}} \backslash T_{E,\tilde{\mathtt{S}},\mathtt{T}}(\mathbb{A}^\infty)/K_E,$$

for any open compact subgroup $K_E \subset T_{E,\tilde{\mathtt{S}},\mathtt{T}}(\mathbb{A}^\infty) \cong \mathbb{A}_E^{\infty,\times}$. We put $K_{E,p} = \mathcal{O}_{E,\mathfrak{p}}^\times \subset T_{E,\tilde{\mathtt{S}},\mathtt{T}}(\mathbb{Q}_p)$ and write $\mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) = \varprojlim_{K_E^p} \mathcal{S}h_{K_E^p K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ as the inverse limit over all open compact subgroups $K_E^p \subset T_{E,\tilde{\mathtt{S}},\mathtt{T}}(\mathbb{A}^{\infty,p})$. (As in Notation 2.2, we identify $T_{E,\tilde{\mathtt{S}},\mathtt{T}}(\mathbb{A}^\infty)$ for all $\tilde{\mathtt{S}}$ and $\mathtt{T}$, and we write $T_E(\mathbb{A}^\infty)$ for it, so K_E is naturally its subgroup.)

Under the isomorphism $\iota_p \colon \mathbb{C} \cong \overline{\mathbb{Q}}_p$, the image of the reflex field of $\mathcal{S}h_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ is contained in $\mathbb{Q}_{p^{2g}}$. It makes sense to talk about $\mathcal{S}h_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})_{\mathbb{Q}_{p^{2g}}}$. As $K_{E,p}$ is hyperspecial, the action of $\mathrm{Gal}_{\mathbb{Q}_{p^{2g}}}$ on $\mathcal{S}h_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})(\overline{\mathbb{Q}}_p)$ is unramified. So $\mathcal{S}h_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})_{\mathbb{Q}_{p^{2g}}}$ is the disjoint union of the spectra of some finite unramified extension of $\mathbb{Q}_{p^{2g}}$, and it has an integral canonical model over $\mathbb{Z}_{p^{2g}}$ by taking the spectra of the corresponding rings of integers. Denote by $\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ its special fiber. By Shimura's reciprocity law, the action of the geometric Frobenius $\mathrm{Frob}_{p^{2g}}$ of $\mathbb{F}_{p^{2g}}$ on $\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})(\overline{\mathbb{F}}_p)$ is given by

(i) when $\mathfrak{p}$ is inert in E/F, multiplication by $(\underline{p}_F)^{\#(\mathtt{S}_\infty - \#\mathtt{T}) - \#\mathtt{T}} = (\underline{p}_F)^{\#\mathtt{S}_\infty - 2\#\mathtt{T}}$ and

(ii) when $\mathfrak{p}$ splits into $\mathfrak{q}\bar{\mathfrak{q}}$, multiplication by

$$\underline{\varpi}_{\mathfrak{q}}^{2(\#\tilde{\mathtt{S}}_{\infty/\bar{\mathfrak{q}}}-\#\mathtt{T})}\underline{\varpi}_{\bar{\mathfrak{q}}}^{2(\#\tilde{\mathtt{S}}_{\infty/\mathfrak{q}}-\#\mathtt{T})}=(\underline{p}_F)^{\#\mathtt{S}_\infty-2\#\mathtt{T}}(\underline{\mathfrak{q}})^{\Delta_{\tilde{\mathtt{S}}_\infty}},$$

where $\underline{\varpi}_{\mathfrak{q}}$ (resp., $\underline{\varpi}_{\bar{\mathfrak{q}}}$) is the finite idele in $\mathbb{A}_E^\infty$ which is p at the place $\mathfrak{q}$ (resp., $\bar{\mathfrak{q}}$) and is 1 elsewhere, $\underline{\mathfrak{q}}$ is the idele defined in Section 2.4(2) above, and $\Delta_{\tilde{\mathtt{S}}_\infty}$ is defined in (2.4.1).

In particular, if $(\tilde{\mathtt{S}}',\mathtt{T}')$ is another pair as above such that $\#\mathtt{S}'_\infty-2\#\mathtt{T}'=\#\mathtt{S}_\infty-2\#\mathtt{T}$ and such that $\Delta_{\tilde{\mathtt{S}}_\infty}=\Delta_{\tilde{\mathtt{S}}'_\infty}$ if $\mathfrak{p}$ splits, then there exists an isomorphism of Shimura varieties over $\mathbb{F}_{p^{2g}}$,

$$\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})\xrightarrow{\cong}\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}}',\mathtt{T}'}),\tag{2.4.2}$$

compatible with the Hecke action of $T_E(\mathbb{A}^{\infty,p})$ on both sides as K_E^p varies.

2.5. A unitary Shimura variety

Let $Z=\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ be the center of $G_{\mathtt{S},\mathtt{T}}$. Put $G''_{\tilde{\mathtt{S}}}=G_{\mathtt{S},\mathtt{T}}\times_Z T_{E,\tilde{\mathtt{S}},\mathtt{T}}$, which is the quotient of $G_{\mathtt{S},\mathtt{T}}\times T_{E,\tilde{\mathtt{S}},\mathtt{T}}$ by Z embedded antidiagonally as $z\mapsto(z,z^{-1})$. Consider the product Deligne homomorphism

$$h_{\mathtt{S},\mathtt{T}}\times h_{E,\tilde{\mathtt{S}},\mathtt{T}}\colon\mathbb{S}(\mathbb{R})=\mathbb{C}^\times\to(G_{\mathtt{S},\mathtt{T}}\times T_{E,\tilde{\mathtt{S}},\mathtt{T}})(\mathbb{R}),$$

which can be further composed with the quotient map to $G''_{\tilde{\mathtt{S}}}$ to get

$$h''_{\tilde{\mathtt{S}}}\colon\mathbb{S}(\mathbb{R})=\mathbb{C}^\times\to(G_{\mathtt{S},\mathtt{T}}\times_Z T_{E,\tilde{\mathtt{S}},\mathtt{T}})(\mathbb{R})\cong G''_{\tilde{\mathtt{S}}}(\mathbb{R}).$$

Note that $h''_{\tilde{\mathtt{S}}}$ does not depend on the choice of $\mathtt{T}\subseteq\mathtt{S}_\infty$ (hence the notation), and its conjugacy class is identified with $\mathfrak{H}_{\mathtt{S}}=(\mathfrak{h}^\pm)^{\mathtt{S}^c_\infty}$. Let K''_p denote the (maximal) open compact subgroup $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})\times_{\mathcal{O}_\mathfrak{p}^\times}\mathcal{O}_{E,\mathfrak{p}}^\times$ of $G''_{\tilde{\mathtt{S}}}(\mathbb{Q}_p)$. We will consider open compact subgroups of the form $K''=K''_pK''^p\subset G''_{\tilde{\mathtt{S}}}(\mathbb{A}^\infty)$ with $K''^p\subset G''_{\tilde{\mathtt{S}}}(\mathbb{A}^{\infty,p})$. These data give rise to a Shimura variety $\mathcal{S}h_{K''}(G''_{\tilde{\mathtt{S}}})$ (defined over $\mathbb{Q}_{p^{2g}}$), whose $\mathbb{C}$-points (via ι_p) are given by

$$\mathcal{S}h_{K''}(G''_{\tilde{\mathtt{S}}})(\mathbb{C})=G''_{\tilde{\mathtt{S}}}(\mathbb{Q})\backslash\big(\mathfrak{H}_{\mathtt{S}}\times G''_{\tilde{\mathtt{S}}}(\mathbb{A}^\infty)\big)/K''.$$

We put $\mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}}):=\varprojlim_{K''^p}\mathcal{S}h_{K''}(G''_{\tilde{\mathtt{S}}})$. Its geometrically connected components admit a natural map

$$\pi_0\big(\mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}})_{\overline{\mathbb{Q}}_p}\big)\longrightarrow(F_+^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^\times)\times(E^1\backslash\mathbb{A}_E^1/\mathcal{O}_{E_\mathfrak{p}}^{N_{E/F}=1}),\tag{2.5.1}$$

where $N_{E/F}$ is the norm from E to F, and E^1 (resp., $\mathbb{A}_E^1$) is the subgroup of $E^\times$ (resp., $\mathbb{A}_E^\times$), with norm 1 in $F^\times$ (resp., $\mathbb{A}_F^\times$). As in the quaternionic case, this is an isomorphism if $\mathtt{S}^c_\infty\neq\emptyset$. We write $\mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}})^\circ_{\overline{\mathbb{Q}}_p}$ for the preimage of $\mathbf{1}\times\mathbf{1}$ and call it the *neutral geometrically connected component* of the unitary Shimura variety.

We can define the group $\mathcal{E}_{G''_{\tilde{S}}}$ and $\mathcal{G}''_{\tilde{S},p}$ for the Shimura data $(G''_{\tilde{S}}, h''_{\tilde{S}})$ as in Section 2.1 (see, e.g., [33, Section 2.11] for the recipe). First, we spell out the Shimura reciprocity map:

$$\mathrm{Rec}''_p\colon \mathrm{Gal}_{\mathbb{F}_{p^{2g}}} \longrightarrow (F_+^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_{\mathfrak{p}}^\times) \times (E^1\backslash \mathbb{A}_E^{\infty,N_{E/F}=1}/\mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1}). \quad (2.5.2)$$

The Frobenius image $\mathrm{Rec}''_p(\mathrm{Frob}_{p^{2g}})$ is given as follows:
- when $\mathfrak{p}$ is inert in E/F, $\mathrm{Rec}''_p(\mathrm{Frob}_{p^{2g}}) = (\underline{p}_F)^{2g} \times 1$,
- when $\mathfrak{p}$ splits in E/F, $\mathrm{Rec}''_p(\mathrm{Frob}_{p^{2g}}) = (\underline{p}_F)^{2g} \times (\underline{\mathfrak{q}})^{2\Delta_{\tilde{S}_\infty}}$.

We put $\mathcal{G}''_{\tilde{S},p} = (G''_{\tilde{S}}(\mathbb{A}^{\infty,p})/\mathcal{O}_{E,(p)}^{\times,\mathrm{cl}}) \times \mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$,[10] and we define $\mathcal{E}_{G''_{\tilde{S}}}$ to be its subgroup of pairs (x,σ) such that $\nu''(x)$ is equal to $\mathrm{Rec}''_p(\sigma)^{-1}$, where

$$\begin{array}{rcl}\nu''\colon G''_{\tilde{S}} = G_{S,T} \times_Z T_{E,\tilde{S},T} & \longrightarrow & \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \times \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)^{N_{E/F}=1} \\ (g,t) & \longmapsto & \big(\nu(g)N_{E/F}(t), t/\bar{t}\big)\end{array}$$

is the natural morphism from $G''_{\tilde{S}}$ to its maximal Abelian quotient.

Remark 2.6

Similar to Notation 2.2, if S' is another subset of places of F containing the same finite places as S (together with a choice of $\tilde{S}'_\infty$), then $G''_{\tilde{S}'}(\mathbb{A}^\infty)$ is isomorphic to $G''_{\tilde{S}}(\mathbb{A}^\infty)$. We fix such an isomorphism and denote it uniformly as $G''(\mathbb{A}^\infty)$. Hence we naturally identify groups $\mathcal{G}''_{\tilde{S},p}$ for different $\tilde{S}$'s. When $\#S_\infty = \#S'_\infty$ and $\Delta_{\tilde{S}_\infty} = \Delta_{\tilde{S}'_\infty}$ if $\mathfrak{p}$ splits in E/F, the subgroup $\mathcal{E}_{G''_{\tilde{S}'}} \subset \mathcal{G}''_{\tilde{S}',p}$ can be also identified with $\mathcal{E}_{G''_{\tilde{S}}} \subset \mathcal{G}''_{\tilde{S},p}$. Indeed, in this case the reciprocity map Rec''_p for $\tilde{S}$ and $\tilde{S}'$ is the same.

PROPOSITION 2.7

(1) *We have a canonical isomorphism $\mathcal{E}_{G_{S,T}} \cong \mathcal{E}_{G''_{\tilde{S}}}$, and we have that $\mathcal{S}h_{K_p}(G_{S,T})^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$ together with the action of $\mathcal{E}_{G_{S,T}}$ is isomorphic to $\mathcal{S}h_{K''_p}(G''_{\tilde{S}})^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$ together with the action of $\mathcal{E}_{G''_{\tilde{S}}}$.*

(2) *The Shimura varieties $\mathcal{S}h_K(G_{S,T})$ (resp., $\mathcal{S}h_{K''}(G''_{\tilde{S}})$) admit integral canonical models over $\mathbb{Z}_{p^g}$ (resp., over $\mathbb{Z}_{p^{2g}}$), and the connected Shimura variety $\mathcal{S}h_{K_p}(G_{S,T})^\circ_{\mathbb{Q}_p^{\mathrm{ur}}} \cong \mathcal{S}h_{K''_p}(G''_{\tilde{S}})^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$ admits an integral canonical model over $\mathbb{Z}_p^{\mathrm{ur}}$.*

Proof

For (1), the case when $T = \emptyset$ is treated in [33]. In general, note that the sequence of morphisms

[10] As in the footnote to (2.1.3), we omitted the star product in the definition of this group when compared with [33, (2.11.3)] because the center $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ of $G''_{\tilde{S},T}$ has trivial first cohomology.

$$G''_{\tilde{S}} \leftarrow G_{S,T} \times T_{E,\tilde{S},T} \rightarrow G_{S,T}$$

is compatible with the associated Deligne homomorphism, and the conjugacy classes of Deligne homomorphisms into various algebraic groups defined above are canonically identified. Standard facts (e.g., [33, Corollary 2.17]) about Shimura varieties imply that the series of morphisms of Shimura varieties

$$\mathcal{S}h_{K''_p}(G''_{\tilde{S}}) \leftarrow \mathcal{S}h_{K_p}(G_{S,T}) \times_{\mathbb{Z}_{p^g}} \mathcal{S}h_{K_{E,p}}(T_{E,\tilde{S},T}) \rightarrow \mathcal{S}h_{K_p}(G_{S,T})$$

induce isomorphisms on the neutral connected components. Hence, by [33, Theorem 3.14], there exists an integral canonical model for $\mathcal{S}h_{K''_p}(G''_{\tilde{S}})$ over $\mathbb{Z}_{p^{2g}}$, and thus the neutral connected component $\mathcal{S}h_{K''_p}(G''_{\tilde{S}})^\circ \cong \mathcal{S}h_{K_p}(G_{S,T})^\circ$ admits an integral canonical model over $\mathbb{Z}_p^{\mathrm{ur}}$. Applying $\times_{\mathcal{E}_{G_{S,T}}} \mathcal{G}_{S,T,p}$, the latter induces an integral canonical model of $\mathcal{S}h_{K_p}(G_{S,T})$ over $\mathbb{Z}_p^{\mathrm{ur}}$, which descends to $\mathbb{Z}_{p^g}$ (see [33, Corollary 2.17]). $\square$

Remark 2.8
Item (2) of Proposition 2.7 is a consequence of the much more general theory of Kisin [17]. However, in the following, we will need essentially this explicit relationship between the integral models of $\mathcal{S}h_K(G_{S,T})$ and those of $\mathcal{S}h_{K''}(G''_{\tilde{S}})$.

Notation 2.9
We use $\mathcal{S}h_{K_p}(G_{S,T})$, $\mathcal{S}h_{K_{E,p}}(T_{E,\tilde{S},T})$, $\mathcal{S}h_{K''_p}(G''_{\tilde{S}}), \dots$ to denote the integral models over $\mathbb{Z}_{p^g}$ or $\mathbb{Z}_{p^{2g}}$ of the corresponding Shimura varieties, and we use systematically roman letters to denote the special fibers of Shimura varieties:

$$\mathrm{Sh}_?(G_{S,T})^\star_{\overline{\mathbb{F}}_p} := \mathcal{S}h_?(G_{S,T})^\star_{\mathbb{Z}_p^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p^{\mathrm{ur}}} \overline{\mathbb{F}}_p \qquad \text{and}$$
$$\mathrm{Sh}_?(G_{S,T}) := \mathcal{S}h_?(G_{S,T}) \times_{\mathbb{Z}_{p^g}} \mathbb{F}_{p^g}$$

for $? = K$ or K_p, and $\star = \circ$ or $\emptyset$, and

$$\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{S},T})_{\mathbb{F}_{p^{2g}}} := \mathcal{S}h_{K_{E,p}}(T_{E,\tilde{S},T}) \otimes_{\mathbb{Z}_{p^{2g}}} \mathbb{F}_{p^{2g}},$$
$$\mathrm{Sh}_{K''_p}(G''_{\tilde{S}})_{\mathbb{F}_{p^{2g}}} := \mathcal{S}h_{K''_p}(G''_{\tilde{S}}) \otimes_{\mathbb{Z}_{p^{2g}}} \mathbb{F}_{p^{2g}},$$

and similarly with open compact subgroups $K_E = K_{E,p}K_E^p \subset T_E(\mathbb{A}^\infty)$ and also with $K'' = K''_p K''^p \subset G''(\mathbb{A}^\infty)$. We put $\mathrm{Sh}_{K''_p}(G''_{\tilde{S}})^\circ_{\overline{\mathbb{F}}_p} = \mathcal{S}h_{K''_p}(G''_{\tilde{S}})^\circ_{\mathbb{Z}_p^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p^{\mathrm{ur}}} \overline{\mathbb{F}}_p$.

2.10. Automorphic local systems
We now study the automorphic sheaves on these Shimura varieties. Fix a prime $\ell \neq p$ and an isomorphism $\iota_\ell \colon \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Let $(\underline{k}, w)$ be a *regular multiweight*, which means

a tuple $(\underline{k}, w) \in \mathbb{Z}^{\Sigma_\infty} \times \mathbb{Z}$ such that $k_\tau \equiv w \pmod 2$ and $k_\tau \geq 2$ for all $\tau \in \Sigma_\infty$. Consider the algebraic representation

$$\rho^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}} = \boxtimes_{\tau \in \Sigma_\infty} \big(\mathrm{Sym}^{k_\tau - 2}(\mathrm{std}^\vee) \otimes \det{}^{\frac{k_\tau - w}{2}}\big)$$

of $G_{\mathtt{S},\mathtt{T}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\tau \in \Sigma_\infty} \mathrm{GL}_2(\mathbb{C})$, where std is the standard representation of $\mathrm{GL}_2(\mathbb{C})$. As explained in [23], we have an automorphic $\overline{\mathbb{Q}}_\ell$-lisse sheaf $\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}$ on $\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}})$ associated to $\rho^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}$. Note that $\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}$ is pure of weight $(w-2)(g - \#\mathtt{S}_\infty + 2\#\mathtt{T})$.

We fix a section $\tilde{\Sigma} \subset \Sigma_{E,\infty}$ of the natural restriction map $\Sigma_{E,\infty} \to \Sigma_\infty$ (which is independent of the choices $\tilde{\mathtt{S}}_\infty$). Consider the following 1-dimensional representation of $T_{E,\tilde{\mathtt{S}},\mathtt{T}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\tilde{\tau} \in \tilde{\Sigma}} \mathbb{G}_{m,\tilde{\tau}} \times \mathbb{G}_{m,\bar{\tilde{\tau}}}$,

$$\rho^w_{E,\tilde{\Sigma}} = \bigotimes_{\tilde{\tau} \in \tilde{\Sigma}} x^{2-w} \circ \mathrm{pr}_{E,\tilde{\tau}},$$

where $\bar{\tilde{\tau}}$ is the complex conjugate embedding of $\tilde{\tau}$, $\mathrm{pr}_{E,\tilde{\tau}}$ is the projection to the $\tilde{\tau}$-component, and x^{2-w} is the character of $\mathbb{C}^\times$ given by raising to the $(2-w)$th power. This representation gives rise to a lisse $\overline{\mathbb{Q}}_\ell$-étale sheaf $\mathcal{L}^w_{E,\tilde{\mathtt{S}},\mathtt{T},\tilde{\Sigma}}$ pure of weight $(w-2)(\#\mathtt{S}_\infty - 2\#\mathtt{T})$ on $\mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$. If $\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}}',\mathtt{T}'})$ is another Shimura variety with $\#\mathtt{S}'_\infty - 2\#\mathtt{T}' = \#\mathtt{S}_\infty - 2\#\mathtt{T}$ and $\gamma\colon \mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) \cong \mathrm{Sh}_{K_E}(T_{E,\tilde{\mathtt{S}}',\mathtt{T}'})$ is the isomorphism (2.4.2), then we have a natural isomorphism:

$$\gamma^*(\mathcal{L}^w_{E,\tilde{\mathtt{S}}',\mathtt{T}',\tilde{\Sigma}}) \xrightarrow{\cong} \mathcal{L}^w_{E,\tilde{\mathtt{S}},\mathtt{T},\tilde{\Sigma}}. \tag{2.10.1}$$

Let $\alpha_{\mathtt{T}}\colon G_{\mathtt{S},\mathtt{T}} \times T_{E,\tilde{\mathtt{S}},\mathtt{T}} \to G''_{\tilde{\mathtt{S}}}$ denote the natural quotient morphism. We have the following diagram:

$$\begin{array}{ccccc}
\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}}) & \xleftarrow{\mathrm{pr}_1} & \mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times_{\mathbb{Z}_{p^g}} \mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & \xrightarrow{\boldsymbol{\alpha}_{\mathtt{T}}} & \mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}}) \\
& & \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \\
& & \mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & &
\end{array} \tag{2.10.2}$$

By our definition, the tensor product representation $\rho^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}} \otimes \rho^w_{E,\tilde{\Sigma}}$ of $G_{\mathtt{S},\mathtt{T}} \times T_{E,\tilde{\mathtt{S}},\mathtt{T}}$ factors through $G''_{\tilde{\mathtt{S}}}$. This defines a $\overline{\mathbb{Q}}_\ell$-lisse sheaf $\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma}}$ on $\mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}})$ such that we have a canonical isomorphism:

$$\boldsymbol{\alpha}^*_{\mathtt{T}}(\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma}}) \cong \mathrm{pr}_1^*(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}) \otimes \mathrm{pr}_2^*(\mathcal{L}^w_{E,\tilde{\mathtt{S}},\mathtt{T},\tilde{\Sigma}}). \tag{2.10.3}$$

Put $D = B_{\mathtt{S}} \otimes_F E$. Then our choice of E/F in Section 2.4 implies that $D \cong \mathrm{M}_{2\times 2}(E)$, which explains the omission of $\mathtt{S}$ in our notation. We fix such an isomorphism and then take a maximal order $\mathcal{O}_D \cong \mathrm{M}_{2\times 2}(\mathcal{O}_E)$. Recall that there exists a versal family of Abelian varieties of dimension $4g$ $a\colon \mathbf{A}''_{\tilde{\mathtt{S}}} = \mathbf{A}''_{\tilde{\mathtt{S}},K''_p} \to \mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}})$ (see [33, Section 3.20]) equipped with a natural action by $\mathcal{O}_D$. Here, "versal" means that the Kodaira–Spencer map for the family $\mathbf{A}''_{\tilde{\mathtt{S}}}$ is an isomorphism. Using $\mathbf{A}''_{\tilde{\mathtt{S}}}$, $\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma}}$ can be reinterpreted as follows. Put $H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}}) = R^1 a_*(\overline{\mathbb{Q}}_\ell)$, which is an ℓ-adic local system on $\mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}})$ equipped with an induced action by $\mathrm{M}_{2\times 2}(E)$. For each $\tilde{\tau} \in \Sigma_{E,\infty}$, let $H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})_{\tilde{\tau}}$ denote the direct summand of $H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})$ on which E acts via $E \xrightarrow{\tilde{\tau}} \mathbb{C} \xrightarrow{\iota_\ell} \overline{\mathbb{Q}}_\ell$. Let $\mathfrak{e} = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \in \mathrm{M}_{2\times 2}(E)$ denote the idempotent element. We put $H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})^\circ_{\tilde{\tau}} = \mathfrak{e}\cdot H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})_{\tilde{\tau}}$, which is an ℓ-adic local system on $\mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}})$ of rank 2. We have a canonical decomposition:

$$H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}}) = \bigoplus_{\tilde{\tau}\in\tilde{\Sigma}} \big(H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})_{\tilde{\tau}} \oplus H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})_{\bar{\tilde{\tau}}}\big) = \bigoplus_{\tilde{\tau}\in\tilde{\Sigma}} \big(H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})^{\circ,\oplus 2}_{\tilde{\tau}} \oplus H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})^{\circ,\oplus 2}_{\bar{\tilde{\tau}}}\big).$$

Using this, we obtain the following explicit description:

$$\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma}} = \bigotimes_{\tilde{\tau}\in\tilde{\Sigma}} \Big(\mathrm{Sym}^{k_\tau - 2} H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})^\circ_{\tilde{\tau}} \otimes \big(\wedge^2 H_\ell(\mathbf{A}''_{\tilde{\mathtt{S}}})^\circ_{\tilde{\tau}}\big)^{\frac{w-k_\tau}{2}}\Big). \tag{2.10.4}$$

Remark 2.11
We will introduce a general construction below to relate the unitary Shimura varieties and the quaternionic Shimura varieties. We point out beforehand that the entire construction is modeled on the following question: By Hilbert's Theorem 90, we have an exact sequence

$$1 \to F^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_{\mathfrak{p}}^\times \to E^{\times,\mathrm{cl}}\backslash \mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_{\mathfrak{p}}}^\times \xrightarrow{z\mapsto z/\bar{z}} E^1\backslash \mathbb{A}_E^{\infty,1}/\mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1} \to 1.$$

The construction involves picking a preimage of some element in the target of the surjective map above. In general, there is no canonical choice of this preimage, and all choices form a torsor under the group $F^{\times,\mathrm{cl}}\backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_{\mathfrak{p}}^\times$. In the very special case when the element in the target of the surjective map is trivial, one can have a canonical choice of its preimage, namely, the identity element 1.

Construction 2.12
We now discuss a very important process that allows us to transfer certain correspondences on the unitary Shimura varieties $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ to the quaternionic Shimura varieties $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$. Throughout this section we assume that we are given two sets of data— $\tilde{\mathtt{S}}, \mathtt{T}$ and $\tilde{\mathtt{S}}', \mathtt{T}'$ as above—and that they satisfy the following conditions,

$$\#\mathtt{S}_\infty - 2\#\mathtt{T} = \#\mathtt{S}'_\infty - 2\#\mathtt{T}', \qquad \Delta_{\tilde{\mathtt{S}}_\infty} = \Delta_{\tilde{\mathtt{S}}'_\infty} \quad \text{if } \mathfrak{p} \text{ splits in } E/F, \tag{2.12.1}$$

and we assume that the finite places contained in $\mathtt{S}$ and those in $\mathtt{S}'$ are the same. By (2.4.2), this implies that the Shimura varieties $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ and $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}',\mathtt{T}'})$ are isomorphic.

Suppose now that we are given a correspondence between the two unitary Shimura varieties

$$\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \xleftarrow{\pi''} X \xrightarrow{\eta''} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'}), \tag{2.12.2}$$

where the group $\mathscr{G}''_{\tilde{\mathtt{S}},p} \cong \mathscr{G}''_{\tilde{\mathtt{S}}',p}$ acts on all three spaces and the morphisms are equivariant for the actions. We further assume that the fibers of π'' are geometrically connected.

Step I: We will complete the correspondence above into the commutative diagram

$$\begin{array}{ccccc} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times_{\mathrm{Spec}(\mathbb{F}_{p^d})} \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & \xleftarrow{\pi^\times} & Y & \xrightarrow{\eta^\times} & \mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'}) \times_{\mathrm{Spec}(\mathbb{F}_{p^d})} \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}}',\mathtt{T}'}) \\ \downarrow \boldsymbol{\alpha}_{\mathtt{T}} & & \downarrow \boldsymbol{\alpha}''_{\mathtt{T}} & & \downarrow \boldsymbol{\alpha}'_{\mathtt{T}'} \\ \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) & \xleftarrow{\pi''} & X & \xrightarrow{\eta''} & \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'}) \end{array} \tag{2.12.3}$$

so that Y is defined as the Cartesian product of the left square, and the top line is equivariant for the actions of $\mathscr{G}_{\mathtt{S},\mathtt{T},p} \times \mathbb{A}_E^{\infty,\times} \cong \mathscr{G}_{\mathtt{S}',\mathtt{T}',p} \times \mathbb{A}_E^{\infty,\times}$. For this, it suffices to lift the morphism η'' to $\eta^\times$. We point out that both $\boldsymbol{\alpha}_{\mathtt{T}}$ and $\boldsymbol{\alpha}'_{\mathtt{T}'}$ map every geometrically connected component isomorphically to a geometrically connected component of the target.

We now separate the discussion (but not in an essential way) depending on whether $\mathtt{S}^c_\infty$ is empty.

- When $\mathtt{S}^c_\infty \neq \emptyset$, let Y° denote the preimage $(\pi^\times)^{-1}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})^\circ_{\overline{\mathbb{F}}_p} \times \{\mathbf{1}\})$, where $\mathbf{1}$ denotes the neutral point, namely, the image of $1 \in \mathbb{A}_E^{\infty,\times}$ in $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})_{\overline{\mathbb{F}}_p}$. By our assumption on π'', Y° is a geometrically connected component of Y. Its image under $\eta'' \circ \boldsymbol{\alpha}''_{\mathtt{T}}$ lies in a geometric connected component of $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'})$, say $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'})^\bullet_{\overline{\mathbb{F}}_p}$, corresponding to some $(\boldsymbol{x}, \boldsymbol{s}) \in (F_+^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times}/\mathscr{O}_\mathfrak{p}^\times) \times (E^1 \backslash \mathbb{A}_E^{\infty,1}/\mathscr{O}_{E_\mathfrak{p}}^{N_{E/F}=1})$ via the map (2.5.1). By *Hilbert's Theorem 90*, there exists $\boldsymbol{t} \in E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times}/\mathscr{O}_{E_\mathfrak{p}}^\times$ with $\boldsymbol{t}/\bar{\boldsymbol{t}} = \boldsymbol{s}$, and the choice of $\boldsymbol{t}$ is unique up to $F^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times}/\mathscr{O}_\mathfrak{p}^\times$. We claim that giving a $(\mathscr{G}_{\mathtt{S},\mathtt{T},p} \times \mathbb{A}_E^{\infty,\times})$-equivariant map $\eta^\times$ as above is *equivalent* to choosing such a $\boldsymbol{t}$. Indeed, let $\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})^\bullet_{\overline{\mathbb{F}}_p}$ be the connected component of $\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}$ corresponding to $\boldsymbol{y} = \boldsymbol{x} N_{E/F}(\boldsymbol{t})^{-1}$ via the map (2.1.1). Then $\boldsymbol{\alpha}'_{\mathtt{T}'}$ sends

$\mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})^{\bullet}_{\overline{\mathbb{F}}_p} \times \{\boldsymbol{t}\}$ isomorphically to $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}'})^{\bullet}_{\overline{\mathbb{F}}_p}$. Note that Y (resp., $\mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'}) \times_{\mathrm{Spec}(\mathbb{F}_{p^d})} \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}}',\mathrm{T}'})$) can be recovered from Y° (resp., $\mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})^{\bullet}_{\overline{\mathbb{F}}_p} \times \{\boldsymbol{t}\}$ for any $\boldsymbol{t} \in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times}$ [11]) by applying $- \times_{\mathcal{E}_{G_{\mathrm{S},\mathrm{T}}}} (\mathcal{G}_{\mathrm{S},\mathrm{T},p} \times E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times})$. Here, recall that $\mathcal{E}_{G_{\mathrm{S},\mathrm{T}}}$ is isomorphic to $\mathcal{E}_{G''_{\tilde{\mathrm{S}}}}$ by Proposition 2.7(1), and it embeds into the product $\mathcal{G}_{\mathrm{S},\mathrm{T},p} \times E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times}$ as follows: the morphism from $\mathcal{E}_{G_{\mathrm{S},\mathrm{T}}}$ to $\mathcal{G}_{\mathrm{S},\mathrm{T},p}$ is the natural embedding and that to $E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times}$ is given by first projecting to the Galois factor and then applying the Shimura reciprocity map as specified in Section 2.4(i) and (ii). Therefore, once such a $\boldsymbol{t}$ is chosen, we can define $\eta^\times$ as the morphism obtained by applying $- \times_{\mathcal{E}_{G_{\mathrm{S},\mathrm{T}}}} (\mathcal{G}_{\mathrm{S},\mathrm{T},p} \times E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times})$ to the composite map

$$Y^\circ \xrightarrow{\eta''\circ\boldsymbol{\alpha}''_{\mathrm{T}}} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}'})^{\bullet}_{\overline{\mathbb{F}}_p} \xrightarrow{\sim} \mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})^{\bullet}_{\overline{\mathbb{F}}_p} \times \{\boldsymbol{t}\},$$

where the last isomorphism is the inverse of the restriction of $\boldsymbol{\alpha}'_{\mathrm{T}'}$ to $\mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})^{\bullet}_{\overline{\mathbb{F}}_p} \times \{\boldsymbol{t}\}$. Conversely, it is also clear that such a $\boldsymbol{t}$ is determined by $\eta^\times$.

- When $\mathrm{S}^c_\infty = \emptyset$, a slight rewording is needed. Let X° denote the preimage under π'' of the $\overline{\mathbb{F}}_p$-point $\mathbf{1} \in \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})_{\overline{\mathbb{F}}_p}$. So it is mapped under η'' to a point $\boldsymbol{g}'' \in \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})_{\overline{\mathbb{F}}_p}$. Let Y° denote the preimage under $\pi^\times$ of the $\overline{\mathbb{F}}_p$-point $(\mathbf{1},\mathbf{1}) \in \mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})_{\overline{\mathbb{F}}_p} \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}},\mathrm{T}})_{\overline{\mathbb{F}}_p}$. Then the map $\eta^\times$ must take Y° to an $\overline{\mathbb{F}}_p$-point $(\boldsymbol{x},\boldsymbol{t})$ in $\boldsymbol{\alpha}'^{-1}_{\mathrm{T}'}(\boldsymbol{g}'')$, and, conversely, $\eta^\times$ is determined by this choice of such a point by the same argument as above using the fact that $\eta^\times$ is equivariant for the $(\mathcal{G}_{\mathrm{S},\mathrm{T},p} \times \mathbb{A}_E^{\infty,\times})$-action.

In summary, one can always define such a lift $\eta^\times$, depending on a choice of a certain element $\boldsymbol{t} \in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times}$ which is unique up to multiplication by an element of $F^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^{\times}$. In this case, we say that $\eta^\times$ is constructed with *shift* $\boldsymbol{t}$. In general, we do not have a canonical choice for $\boldsymbol{t}$, and hence neither for $\eta^\times$. However, in the special case when $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}'})^{\bullet}_{\overline{\mathbb{F}}_p}$ is the neutral connected component $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}'})^{\circ}_{\overline{\mathbb{F}}_p}$ in the former case and $\boldsymbol{g}'' = \mathbf{1}$ in the latter case, there is a canonical choice of such a lift, namely, the neutral connected component $\mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})^{\circ}_{\overline{\mathbb{F}}_p} \times \{\mathbf{1}\}$ in the former case and $(\mathbf{1},\mathbf{1})$ in the latter case. So under this additional hypothesis, we do have a canonical map $\eta^\times$.

Step II: Suppose that we have constructed the diagram (2.12.3) with shift $\boldsymbol{t}$ (which is canonical up to an element of $F^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_\mathfrak{p}^{\times}$), and we want to obtain a correspondence

[11] We point out that $E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^{\times}$ is canonically isomorphic to $\mathcal{O}_{E,(p)}^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,p,\times}$.

$$\mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})_{\mathbb{F}_{p^{2g}}} \xleftarrow{\pi} Z \xrightarrow{\eta} \mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})_{\mathbb{F}_{p^{2g}}}. \tag{2.12.4}$$

For this, it suffices to construct (2.12.4) over $\overline{\mathbb{F}}_p$ which carries an equivariant action of $\mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$. Starting with the top row of (2.12.3), composing $\eta^\times$ with multiplication by $\boldsymbol{t}^{-1}$ (note that $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}}',\mathrm{T}'})$ is in fact a group scheme), we get a correspondence[12]

$$\begin{aligned}\mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})_{\overline{\mathbb{F}}_p} &\times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}},\mathrm{T}})_{\overline{\mathbb{F}}_p}\\ &\xleftarrow{\pi^\times} Y \xrightarrow{\boldsymbol{t}^{-1}\circ\eta^\times} \mathrm{Sh}_{K_p}(G_{\mathrm{S}',\mathrm{T}'})_{\overline{\mathbb{F}}_p} \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}}',\mathrm{T}'})_{\overline{\mathbb{F}}_p},\end{aligned} \tag{2.12.5}$$

which respects the projection to $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}},\mathrm{T}})_{\overline{\mathbb{F}}_p} \overset{\gamma}{\cong} \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}}',\mathrm{T}'})_{\overline{\mathbb{F}}_p}$. Taking the fiber of (2.12.5) over $\mathbf{1}$ of $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}},\mathrm{T}})_{\overline{\mathbb{F}}_p}$ would give (2.12.4) (base-changed to $\overline{\mathbb{F}}_p$), but to descend we need to modify the Galois action above (so that the Galois action preserves the fiber over $\mathbf{1}$) as follows: we change the action of $\mathrm{Frob}_{p^{2g}}$ on (2.12.5) by further composing with a Hecke action given by $1 \times (\underline{p}_F)^{2\#\mathrm{T}-\#\mathrm{S}_\infty} \in G(\mathbb{A}^\infty) \times \mathbb{A}_E^{\infty,\times}$ if $\mathfrak{p}$ is inert in E/F, and $1 \times (\underline{p}_F)^{2\#\mathrm{T}-\#\mathrm{S}_\infty}(\underline{\mathfrak{q}})^{-\Delta_{\tilde{\mathrm{S}}_\infty}}$ if $\mathfrak{p}$ splits in E/F. This way, we have constructed a new Galois action on the factor $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathrm{S}},\mathrm{T}})_{\overline{\mathbb{F}}_p}$. By usual Galois descent, we get (2.12.4).

Step III: We will obtain a sheaf version of the construction above; namely, if in addition, we are given an isomorphism of sheaves

$$\eta''^\sharp\colon \pi''^*(\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathrm{S}},\tilde{\Sigma}}) \xrightarrow{\cong} \eta''^*(\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathrm{S}}',\tilde{\Sigma}}), \tag{2.12.6}$$

then we will construct an isomorphism of sheaves

$$\eta^\sharp\colon \pi^*(\mathcal{L}^{(\underline{k},w)}_{\mathrm{S},\mathrm{T}}) \xrightarrow{\cong} \eta^*(\mathcal{L}^{(\underline{k},w)}_{\mathrm{S}',\mathrm{T}'}), \tag{2.12.7}$$

which again depends on the choice of $\boldsymbol{t}$ in Step I. First, pulling back (2.12.6) along $\boldsymbol{\alpha}''_{\mathrm{T}}$ in the commutative diagram (2.12.3), we get

$$\boldsymbol{\alpha}''^*_{\mathrm{T}}(\eta''^\sharp)\colon (\pi^\times)^*\big(\boldsymbol{\alpha}^*_{\mathrm{T}}(\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathrm{S}},\tilde{\Sigma}})\big) \xrightarrow{\cong} (\eta^\times)^*\big(\boldsymbol{\alpha}'^*_{\mathrm{T}'}(\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathrm{S}}',\tilde{\Sigma}})\big).$$

Taking into account the isomorphism (2.10.3), we have

$$\begin{aligned}\boldsymbol{\alpha}''^*_{\mathrm{T}}(\eta''^\sharp)\colon (\pi^\times)^*&\big(\mathrm{pr}_1^*(\mathcal{L}^{(\underline{k},w)}_{\mathrm{S},\mathrm{T}}) \otimes \mathrm{pr}_2^*(\mathcal{L}^{w}_{E,\tilde{\mathrm{S}},\mathrm{T},\tilde{\Sigma}})\big)\\ &\xrightarrow{\cong} (\eta^\times)^*\big(\mathrm{pr}'^*_1(\mathcal{L}^{(\underline{k},w)}_{\mathrm{S}',\mathrm{T}'}) \otimes \mathrm{pr}'^*_2(\mathcal{L}^{w}_{E,\tilde{\mathrm{S}}',\mathrm{T}',\tilde{\Sigma}})\big).\end{aligned}$$

[12] Once again, this correspondence depends on the choice of $\boldsymbol{t}$, which is unique up to multiplication by an element of $\mathcal{O}^{\times,\mathrm{cl}}_{F,(p)}\backslash\mathbb{A}_F^{\times,\infty,p}$.

Composing this with the action of $\boldsymbol{t}^{-1}$, we get an isomorphism

$$(\pi^\times)^*\big(\mathrm{pr}_1^*(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}) \otimes \mathrm{pr}_2^*(\mathcal{L}^{w}_{E,\tilde{\mathtt{S}},\mathtt{T},\tilde{\Sigma}})\big) \xrightarrow{\cong} (\boldsymbol{t}^{-1}\circ\eta^\times)^*\big(\mathrm{pr}_1'^*(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}) \otimes \mathrm{pr}_2'^*(\mathcal{L}^{w}_{E,\tilde{\mathtt{S}}',\mathtt{T}',\tilde{\Sigma}})\big).$$

Since we may also identify the sheaves $\mathcal{L}^w_{E,\tilde{\mathtt{S}},\mathtt{T},\tilde{\Sigma}}$ with $\mathcal{L}^w_{E,\tilde{\mathtt{S}}',\mathtt{T}',\tilde{\Sigma}}$ using (2.10.1), we may restrict the morphism above to the fiber over the neutral point $\mathbf{1}$ and get a morphism of sheaves (2.12.7) that we want over $\overline{\mathbb{F}}_p$. (Once again, this morphism is unique up to multiplication with an element of $F^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_{\mathfrak{p}}^\times$.) To descend it back down to $\mathbb{F}_{p^{2g}}$, we modify the action of the Frobenius by composing it with a central Hecke action as in Step II above. This concludes the needed construction.

We point out that the the above contruction depends on the choice of the element $\boldsymbol{t}\in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E,\mathfrak{p}}^\times$ that appeared in Step I, and such $\boldsymbol{t}$ is determined only up to multiplication by an element of $F^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_{\mathfrak{p}}^\times$. We call η the *morphism associated to η'' with shift $\boldsymbol{t}$*. When $\mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}})^\bullet_{\overline{\mathbb{F}}_p} = \mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}})^\circ_{\overline{\mathbb{F}}_p}$ in Step I, we can take $\boldsymbol{t}=\mathbf{1}$ and we get a canonically determined η with shift $\mathbf{1}$.

Finally, let us mention where the choice made in Step I is specified later in this article. In Section 3.8, we invoke this construction to define the Goren–Oort cycles; this is where the choice will be fixed. Moreover, this choice will retroactively determine the choice we make when applying this construction to define the link morphisms in Section 2.19, whenever that section is quoted. The shift will allow us to keep track of the choices we made.

Remark 2.13
Suppose that we are given two correspondences as in Construction 2.12. Namely, we have

- subsets of $\tilde{\mathtt{S}}_i$, $\mathtt{T}_i$ for $i=1,2,3$ such that $\#\mathtt{S}_{i,\infty}-2\#\mathtt{T}_i$, the subset of $\mathtt{S}_i$ of finite places, and $\Delta_{\tilde{\mathtt{S}}_{i,\infty}}$ are independent of i (if $\mathfrak{p}$ splits in E/F),
- two $\mathcal{G}''_{\tilde{\mathtt{S}}_i,p}$-equivariant correspondences between Shimura varieties

$$\mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}_i}) \xleftarrow{\pi_i''} X_i \xrightarrow{\eta_i''} \mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}_{i+1}})$$

  with $i=1,2$ such that π_i'' is a fiber bundle with geometrically connected fibers.

Then we can compose these two correspondences to get a correspondence

$$\mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}_1}) \xleftarrow{\pi_3''} X_3 := X_1 \times_{\eta_1'',\mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}_2}),\pi_2''} X_2 \xrightarrow{\eta_3''} \mathrm{Sh}_{K_p''}(G''_{\tilde{\mathtt{S}}_3}).$$

Thus we may apply Construction 2.12 to get correspondences (π_1,η_1) and (π_2,η_2) on the quaternionic Shimura varieties

$$
\begin{array}{ccccccccc}
 & & & & X_3 & & & & \\
 & \overset{\pi_3}{\swarrow} & & \swarrow & & \searrow & & \overset{\eta_3}{\searrow} & \\
 & & X_1 & & & & X_2 & & \\
 & \overset{\pi_1}{\swarrow} & & \overset{\eta_1}{\searrow} & & \overset{\pi_2}{\swarrow} & & \overset{\eta_2}{\searrow} & \\
\mathrm{Sh}_{K_p}(G_{\mathtt{S}_1,\mathtt{T}_1}) & & & & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_2,\mathtt{T}_2}) & & & & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_3,\mathtt{T}_3}),
\end{array}
\tag{2.13.1}
$$

with shifts $\boldsymbol{t}_1, \boldsymbol{t}_2$. Then their composition $(\pi_3, \eta_3) = (\pi_2, \eta_2) \circ (\pi_1, \eta_1)$ is the correspondence of quaternionic Shimura varieties associated to (π_3'', η_3'') with shift $\boldsymbol{t}_1\boldsymbol{t}_2$. Conversely, if we apply Construction 2.12 to (π_i'', η_i'') to get three correspondences (π_i, η_i) for $i = 1, 2, 3$ such that $(\pi_3, \eta_3) = (\pi_2, \eta_2) \circ (\pi_1, \eta_1)$, then their shifts satisfy the equality $\boldsymbol{t}_3 = \boldsymbol{t}_1\boldsymbol{t}_2$.

2.14. Hecke operators at p

In this section, we consider the case $\mathtt{S}_\infty^c = \emptyset$, namely, when the Shimura varieties are discrete. We want to relate the Hecke operators at p for the unitary and quaternionic Shimura varieties in a manner similar to that above. *In this section, we assume that* $\mathfrak{p}$ *splits in* E/F, which is the case we will encounter later.

Let $\mathrm{Iw}_p \subset \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ denote the subgroup consisting of matrices which are upper triangular when modulo $\mathfrak{p}$. The discussion in this section is designed to cover this case and give an integral canonical model $\mathcal{S}h_{\mathrm{Iw}_p}(G_{\mathtt{S},\mathtt{T}})$ of the Shimura variety with Iwahori level structure. We denote by $T_\mathfrak{p}$ the Hecke correspondence given by the following diagram:

$$
\begin{array}{ccc}
 & \mathcal{S}h_{\mathrm{Iw}_p}(G_{\mathtt{S},\mathtt{T}}) & \\
\overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} \\
\mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}}) & & \mathcal{S}h_{K_p}(G_{\mathtt{S},\mathtt{T}}),
\end{array}
\tag{2.14.1}
$$

where π_1 is the natural projection and π_2 sends the double coset of $x \in G(\mathbb{A}^\infty)$ to that of $x\left(\begin{smallmatrix} \underline{p}_F^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

For the unitary Shimura variety, we have $G''(\mathbb{Q}_p) \cong \mathrm{GL}_2(F_\mathfrak{p}) \times_{F_\mathfrak{p}^\times} (E_\mathfrak{q}^\times \times E_{\bar{\mathfrak{q}}}^\times)$, and we use Iw_p'' to denote the subgroup $\mathrm{Iw}_p \times_{\mathcal{O}_\mathfrak{p}^\times} (\mathcal{O}_{E_\mathfrak{q}}^\times \times \mathcal{O}_{E_{\bar{\mathfrak{q}}}}^\times)$. Similarly, we have an integral model $\mathcal{S}h_{\mathrm{Iw}_p''}(G_{\tilde{\mathtt{S}}}'')$ of the unitary Shimura variety with this Iwahori level structure. The element $\gamma_\mathfrak{q}'' = \left(\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right), (1, p)\right) \in G''(\mathbb{Q}_p)$ gives rise to a Hecke operator $T_\mathfrak{q}$ corresponding to the double coset $K_p''\gamma_\mathfrak{q}''K_p''$. Geometrically, it is given by the diagram

$$\begin{array}{ccc} & \mathcal{S}h_{\mathrm{Iw}''_p}(G''_{\tilde{\mathtt{S}}}) & \\ {\scriptstyle \pi''_1}\swarrow & & \searrow{\scriptstyle \pi''_2} \\ \mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}}) & & \mathcal{S}h_{K''_p}(G''_{\tilde{\mathtt{S}}}) \end{array} \tag{2.14.2}$$

where π''_1 is the natural projection and π''_2 sends the double coset of $x \in G''(\mathbb{A}^\infty)$ to that of $x\gamma''_{\mathfrak{q}}$.

In language similar to the preceding section (except that we cannot quote it directly because the morphism π'' therein would not have geometrically connected fibers), we may phrase the relation between the Hecke correspondences $T_{\mathfrak{p}}$ and $T_{\mathfrak{q}}$ in terms of the following commutative diagram (with $T_{\mathfrak{q}}$ vertical on the left and $T_{\mathfrak{p}}$ vertical on the right):

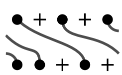
$$\begin{array}{ccccc}
\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) & \xleftarrow{\alpha_{\mathtt{T}}} & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & \xleftarrow{\text{fiber over } \mathbf{1}} & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \\
\uparrow{\scriptstyle \pi''_1} & & \uparrow{\scriptstyle \text{natural}} & & \uparrow{\scriptstyle \pi_1} \\
\mathrm{Sh}_{\mathrm{Iw}''_p}(G''_{\tilde{\mathtt{S}}}) & \xleftarrow{\alpha_{\mathtt{T}}} & \mathrm{Sh}_{\mathrm{Iw}_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & \xleftarrow{\text{fiber over } \mathbf{1}} & \mathrm{Sh}_{\mathrm{Iw}_p}(G_{\mathtt{S},\mathtt{T}}) \\
\downarrow{\scriptstyle \pi''_2} & & \downarrow{\scriptstyle x \mapsto x\left(\left(\begin{smallmatrix} \underline{p}_F^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right), \varpi_{\bar{\mathfrak{q}}}\right)} & & \\
\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) & \xleftarrow{\alpha_{\mathtt{T}}} & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & & \Big\downarrow{\scriptstyle \pi_2} \\
 & & \downarrow{\scriptstyle x \mapsto x(1, \varpi_{\bar{\mathfrak{q}}}^{-1})} & & \\
 & & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}}) & \xleftarrow{\text{fiber over } \mathbf{1}} & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})
\end{array}$$

So we may view $T_{\mathfrak{p}}$ as the correspondence associated to $T_{\mathfrak{q}}$ in a similar fashion to Section 2.12, with shift $\varpi_{\bar{\mathfrak{q}}} \in E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_{\mathfrak{p}}}^{\times}$.

2.15. Links

We recall briefly the notion of links introduced in [33, Section 7]. We put $g = [F : \mathbb{Q}]$ points aligned equidistantly on a section of a vertical cylinder, each point corresponding to an Archimedean place in Σ_∞ (also identified with a p-adic embedding of F via $\iota_p \colon \mathbb{C} \simeq \overline{\mathbb{Q}}_p$) so that the Frobenius action is equivalent to shifting the points in one direction. For a subset $\mathtt{S}$ of places of F as above, we label places in $\mathtt{S}_\infty$ by a *plus sign* and places in $\mathtt{S}_\infty^c$ by a *node*. We call the entire picture a *band* corresponding to $\mathtt{S}$. We often draw the picture in the 2-dimensional xy-plane by thinking of the x-coordinate modulo g. We present the points $\tau_0, \dots, \tau_{g-1}$ on the x-axis with coordinates $x = 0, \dots, g-1$, such that the Frobenius shifts the points to the right by 1, and shifts τ_{g-1} back to τ_0 (by first shifting to $x = g$ and thinking of the x-coordinate modulo g). For example, if F has degree 6 over $\mathbb{Q}$ and $\mathtt{S}_\infty = \{\tau_1, \tau_3, \tau_4\}$, then we draw the band as $\bullet + \bullet + + \bullet$.

Suppose that $\mathtt{S}'$ is another set of places of F with even cardinality such that it contains exactly the same finite places of F as $\mathtt{S}$ and satisfies $\#\mathtt{S}_\infty = \#\mathtt{S}'_\infty$. We put the band for $\mathtt{S}$ above the band for $\mathtt{S}'$ on the same cylinder. In the 2-dimensional picture, we draw the band for $\mathtt{S}$ on the line $y = 1$ and the band for $\mathtt{S}'$ on the line $y = -1$. For each of the nodes of $\mathtt{S}$, we draw a curve starting from it and go monotonically downward, linking to a node of $\mathtt{S}'$ (and ignore the plus signs) such that all the curves do not intersect with each other. Such a graph is called a *link* $\eta\colon \mathtt{S} \to \mathtt{S}'$. Two links are considered the same if the curves can be continuously deformed to each other while keeping all curves from intersecting. We say that a curve is *turning to the left (resp., right)* if it can be deformed into a smooth curve which has positive (resp., negative) tangent slopes in the 2-dimensional picture. The *displacement* of a curve in η is the number of points it travels to the right, which is the difference between the x-coordinates of the ending and starting points of the curve (adding an appropriate multiple of g according to the times the curve wraps around the cylinder). The displacement is negative if the curve turns to the left. The *total displacement* $v(\eta)$ is the sum of displacements of all curves. For example, if $g = 5$ and $\mathtt{S}_\infty = \{\tau_1, \tau_3\}$ and $\mathtt{S}'_\infty = \{\tau_2, \tau_4\}$, then the link given by

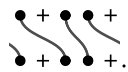
$\eta =$  (2.15.1)

has total displacement $v(\eta) = 3 + 3 + 2 = 8$. For another example, the action of Frobenius σ twists the band and gives rise to a link $\sigma\colon \mathtt{S} \to \sigma(\mathtt{S})$, called the *Frobenius link*, for which every curve is turning to the right with displacement 1. Here $\sigma(\mathtt{S})$ is the set of places containing the same finite places as $\mathtt{S}$, but $\sigma(\mathtt{S})_\infty = \sigma(\mathtt{S}_\infty)$. Its total displacement is $v(\sigma) = d = \#\mathtt{S}^c_\infty$.

For a link $\eta\colon \mathtt{S} \to \mathtt{S}'$, we use $\eta^{-1}\colon \mathtt{S}' \to \mathtt{S}$ to denote the link obtained by reflecting the picture about the equator of the cylinder. For two links $\eta\colon \mathtt{S} \to \mathtt{S}'$ and $\eta'\colon \mathtt{S}' \to \mathtt{S}''$, we have a natural *composition* of links $\eta' \circ \eta\colon \mathtt{S} \to \mathtt{S}''$ given by putting the picture of η on top of the picture of η' and joining the nodes corresponding to $\mathtt{S}'$. It is obvious that $v(\eta^{-1}) = -v(\eta)$ and $v(\eta' \circ \eta) = v(\eta') + v(\eta)$. When discussing the relative positions of two nodes of the band associated to $\mathtt{S}$, it is convenient to use the following.

Notation 2.16

For $\tau \in \mathtt{S}^c_\infty$, let n_τ be the minimal positive integer such that $\sigma^{-n_\tau}\tau \in \mathtt{S}^c_\infty$. We put $\tau^- := \sigma^{-n_\tau}\tau$. We use τ^+ to denote the place in $\mathtt{S}^c_\infty$ such that $(\tau^+)^- = \tau$.

Example 2.17
A link from $\mathtt{S}$ to itself must be an integer power of the *fundamental link* $\eta_{\mathtt{S}}$ (i.e., to link each τ to τ^+ by shifting to the right with displacement n_{τ^+}). An example would be . The total displacement of a fundamental link is exactly $v(\eta_{\mathtt{S}}) = g = [F:\mathbb{Q}]$.

Remark 2.18
As pointed out to us by one of the anonymous referees, one can give an abstract definition of links as follows. Let $\widehat{\mathtt{S}}^c_\infty$ denote the preimage of $\mathtt{S}^c_\infty$ under the projection map $\mathbb{Z} \to \mathbb{Z}/g\mathbb{Z}$, and (for counting purpose) we view $\mathtt{S}^c_\infty$ as a subset of $\widehat{\mathtt{S}}^c_\infty$ by identifying it with its lift in $\{0,\dots,g-1\}$. Then a link from $\mathtt{S}$ to $\mathtt{S}'$ is a bijective and strictly increasing function $\eta\colon \widehat{\mathtt{S}}^c_\infty \to \widehat{\mathtt{S}}'^c_\infty$ (and it would follow automatically that $\eta(x+g) = \eta(x)+g$). The composition of links is the same as the composition of such functions. A link is turning to the left (resp., right) if and only if $\eta(x) \geq x$ (resp., $\eta(x) \leq x$) for every $x \in \widehat{\mathtt{S}}^c_\infty$. The displacement of η is $\sum_{x\in\mathtt{S}^c_\infty} \eta(x) - x$.

2.19. Link morphisms, I
Let $\mathtt{S}$ and $\mathtt{S}'$ be two even subsets of places of F consisting of the same finite places and $\#\mathtt{S}_\infty = \#\mathtt{S}'_\infty$. Let $\eta\colon \mathtt{S} \to \mathtt{S}'$ be a link. We say that η is a *right-turning link* if all its curves (if there are any) are turning to the right. We allow the case $\mathtt{S}_\infty = \Sigma_\infty$ (so that there are no curves in the link η at all), in which case we say that η is the *trivial link*. In this section, we assume that η is right-turning. For each node $\tau \in \mathtt{S}^c_\infty$, we use $m(\tau)$ to denote the displacement of the curve connected to τ in η. Let $\tilde{\mathtt{S}}_\infty$ and $\tilde{\mathtt{S}}'_\infty$ be (any) lifts of $\mathtt{S}_\infty$ and $\mathtt{S}'_\infty$ as in Section 2.4. We have two unitary Shimura varieties, $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ and $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'})$, as defined in Section 2.5.

We now recall the definition of the link morphism on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ associated to the right-turning link η as in [33, Definition 7.5]. Let n be an integer, which is always taken to be 0 if $\mathfrak{p}$ is inert in E. A *link morphism* of *indentation degree n* associated to η on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ is a pair $(\eta''_{(n),\sharp}, \eta''^{\sharp}_{(n)})$, where

(1) $\eta''_{(n),\sharp}\colon \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \to \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'})$ is a morphism of Shimura varieties that induces a bijection on geometric points;

(2) $\eta''^{\sharp}_{(n)}\colon \mathbf{A}''_{\tilde{\mathtt{S}}} \to \eta''^{*}_{(n),\sharp}(\mathbf{A}''_{\tilde{\mathtt{S}}'})$ is a p-quasi-isogeny of Abelian varieties compatible with the $\mathcal{O}_D$-actions, the polarizations, and the tame level structures;

(3) for each geometric point x of $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ with image $x' = \eta''_{(n),\sharp}(x)$, if we write $\tilde{\mathcal{D}}(\mathbf{A}''_{\tilde{\mathtt{S}},x})_{\tilde{\tau}}$ for the $\tilde{\tau}$-component of the *covariant* Dieudonné module of $\mathbf{A}''_{\tilde{\mathtt{S}},x}$ for each $\tilde{\tau} \in \Sigma_{E,\infty}$, then there exists, for each $\tilde{\tau} \in \mathtt{S}^c_{E,\infty}$, some $t_{\tilde{\tau}} \in \mathbb{Z}$ *independent of the point x* such that

$$\eta'^{\sharp}_{(n),*}\big(F^{m(\tau)}_{\mathrm{es},\mathbf{A}''_{\tilde{\mathtt{S}},x}}\big(\tilde{\mathcal{D}}(\mathbf{A}''_{\tilde{\mathtt{S}},x})_{\tilde{\tau}}\big)\big) = p^{t_{\tilde{\tau}}}\,\tilde{\mathcal{D}}(\mathbf{A}''_{\tilde{\mathtt{S}}',x'})_{\sigma^{m(\tau)}\tilde{\tau}},$$

where $F^{m(\tau)}_{\mathrm{es},\mathbf{A}''_{\tilde{\mathtt{S}},x}} : \tilde{\mathcal{D}}(\mathbf{A}''_{\tilde{\mathtt{S}},x})_{\tilde{\tau}} \to \tilde{\mathcal{D}}(\mathbf{A}''_{\tilde{\mathtt{S}},x})_{\sigma^{m(\tau)}\tilde{\tau}}$ is the $m(\tau)$th iteration of the essential Frobenius for $\mathbf{A}''_{\tilde{\mathtt{S}},x}$ defined in [33, Section 4.2]; and

(4) when $\mathfrak{p}$ splits as $\mathfrak{q}\bar{\mathfrak{q}}$ in E, the degree of the quasi-isogeny

$$\eta''^{\sharp}_{(n),\mathfrak{q}} : \mathbf{A}''_{\tilde{\mathtt{S}}}[\mathfrak{q}^\infty] \to \eta''^{*}_{(n),\sharp}\big(\mathbf{A}''_{\tilde{\mathtt{S}}'}[\mathfrak{q}^\infty]\big)$$

of the $\mathfrak{q}$-divisible groups is p^{2n}.

When the indentation degree n is clear by the context, we write simply $(\eta_\sharp, \eta^\sharp)$ for $(\eta_{(n),\sharp}, \eta^\sharp_{(n)})$.

For our purpose, the most important property we need is the following.

LEMMA 2.20 ([33, Proposition 7.8])
Let $\eta\colon \mathtt{S} \to \mathtt{S}'$ be a link as above. Then there exists at most one link morphism $(\eta''_{(n),\sharp}, \eta''^{\sharp}_{(n)})$ with indentation degree n from $\mathrm{Sh}_{K''}(G''_{\tilde{\mathtt{S}}})$ to $\mathrm{Sh}_{K''}(G''_{\tilde{\mathtt{S}}'})$.

Example 2.21
Let $\mathtt{S}$ and $\tilde{\mathtt{S}}$ be as in the preceding section. Let $\sigma^2\colon \mathtt{S} \to \sigma^2(\mathtt{S})$ be the second iteration of the Frobenius link on $\mathtt{S}$. Put $\sigma^2\tilde{\mathtt{S}} = (\sigma^2(\mathtt{S}), \sigma^2(\tilde{\mathtt{S}}_\infty))$. In [33, Section 3.22], we introduced natural morphisms called the *twisted (partial) Frobenius*,

$$\mathfrak{F}''_{\mathfrak{p}^2}\colon\ \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \to \mathrm{Sh}_{K''_p}(G''_{\sigma^2\tilde{\mathtt{S}}}),$$

together with a quasi-isogeny of Abelian varieties,

$$\eta''_{\mathfrak{p}^2} : \mathbf{A}''_{\tilde{\mathtt{S}}} \to (\mathfrak{F}''_{\mathfrak{p}^2})^*\mathbf{A}''_{\sigma^2\tilde{\mathtt{S}}}.$$

Such a morphism is characterized by the fact that the morphism $p\eta''_{\mathfrak{p}^2}$ is given by the p^2-relative Frobenius. Then, in the language of the link morphism introduced above, $(\mathfrak{F}''_{\mathfrak{p}^2}, \eta''_{\mathfrak{p}^2})$ is the link morphism on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ associated to the link $\eta = \sigma^2$ of indentation degree 0 if $\mathfrak{p}$ is inert in E/F and of indentation degree $2\Delta_{\tilde{\mathtt{S}}_\infty}$ if $\mathfrak{p}$ splits in E/F (see [33, Example 7.7(1)]).

Example 2.22
When $\mathfrak{p}$ splits into $\mathfrak{q}\bar{\mathfrak{q}}$ in E/F, consider the Hecke operator $S_\mathfrak{q}$ given by multiplication by $1 \times \underline{\mathfrak{q}}^{-1} \in G''(\mathbb{A}^\infty) = G(\mathbb{A}^\infty) \times_{\mathbb{A}_F^{\infty,\times}} \mathbb{A}_E^{\infty,\times}$ on the unitary Shimura variety. We start with the versal family of Abelian varieties $\mathbf{A}''_{\tilde{\mathtt{S}}}$ on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$, putting $\mathbf{B} := \mathbf{A}''_{\tilde{\mathtt{S}}} \otimes_{\mathcal{O}_E} \bar{\mathfrak{q}} \cdot \mathfrak{q}^{-1}$ equipped with the induced action by $\mathcal{O}_D$. Let $\phi_\mathfrak{q}\colon \mathbf{A}''_{\tilde{\mathtt{S}}} \to \mathbf{B}$ denote the natural p-quasi-isogeny induced by $\mathcal{O}_E \to \bar{\mathfrak{q}}\mathfrak{q}^{-1}$. We equip $\mathbf{B}$ with the natural prime-to-p level structure compatible with $\phi_\mathfrak{q}$. The polarization $\lambda_{\mathbf{A}''_{\tilde{\mathtt{S}}}}$ on $\mathbf{A}''_{\tilde{\mathtt{S}}}$ naturally induces a polarization $\lambda_\mathbf{B}$ on $\mathbf{B}$ such that $\lambda_{\mathbf{A}''_{\tilde{\mathtt{S}}}} = \phi_\mathfrak{q}^\vee \circ \lambda_\mathbf{B} \circ \phi_\mathfrak{q}$. There is a unique morphism,

$$S_{\mathfrak{q}}\colon\ \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \to \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}),$$

which, together with $\phi_{\mathfrak{q}}$, gives a link morphism for the trivial link $\mathrm{id}\colon \mathtt{S} \to \mathtt{S}$ of indentation degree $2g$. If we apply Construction 2.12 to the correspondence

$$\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \xleftarrow{\mathrm{id}} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \xrightarrow{S_{\mathfrak{q}}} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}),$$

then we can lift it to an endomorphism of $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{\tilde{\mathtt{S}},\mathtt{T}})$ given by multiplication by $((\underline{p}_F)^{-1}, \varpi_{\bar{\mathfrak{q}}}^{-2}) \in G(\mathbb{A}^\infty) \times_{\mathbb{A}_F^{\infty,\times}} \mathbb{A}_E^{\infty,\times}$. So the endomorphism $S_{\mathfrak{p}}$ given by multiplication by the central element $(\underline{p}_F)^{-1}$ may be viewed as the morphism on the quaternionic Shimura variety obtained by applying Construction 2.12 to the morphism $\eta'' = S_{\mathfrak{q}}$ with shift $\varpi_{\bar{\mathfrak{q}}}^2$.

2.23. Normalizations of link morphisms

Keep the notation of Section 2.19 and assume moreover that

- the link morphism $(\eta''_{(n),\sharp}, \eta''^{\sharp}_{(n)})$ on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ exists,
- $\Delta_{\tilde{\mathtt{S}}_\infty} = \Delta_{\tilde{\mathtt{S}}'_\infty}$ if $\mathfrak{p}$ splits in E/F, and
- we are given two subsets $\mathtt{T} \subseteq \mathtt{S}_\infty$ and $\mathtt{T}' \subseteq \mathtt{S}'_\infty$ such that $\#\mathtt{T} = \#\mathtt{T}'$.

Let $(\underline{k}, w) \in \mathbb{Z}^{\Sigma_\infty} \times \mathbb{Z}$ be a regular multiweight as in Section 2.10, and let $\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma}}$ and $\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}}',\tilde{\Sigma}}$ be the corresponding ℓ-adic étale local systems on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ and $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'})$, respectively. Then the p-quasi-isogeny $\eta''^{\sharp}_{(n)}$ induces an isomorphism of étale local systems

$$\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}},\tilde{\Sigma}} \xrightarrow{\cong} \eta''^{*}_{(n),\sharp}\mathcal{L}''^{(\underline{k},w)}_{\tilde{\mathtt{S}}',\tilde{\Sigma}}.$$

Applying Construction 2.12 to the link morphism $(\eta''_{(n),\sharp}, \eta''^{\sharp}_{(n)})$ (with $X = \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ and π'' in (2.12.2) equal to the identity), we get a pair of morphisms,

$$\eta_{(n),\sharp}\colon\ \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \to \mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'}) \qquad \text{and} \qquad \eta^{\sharp}_{(n)}\colon \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}} \xrightarrow{\cong} \eta^{*}_{(n),\sharp}(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}),$$

depending on some $\boldsymbol{t} \in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E,\mathfrak{p}}^{\times}$ (See the end of Section 2.12). In the remainder of this article, we call $(\eta_{(n),\sharp}, \eta^{\sharp}_{(n)})$ (or simply $\eta_{(n),\sharp}$ for short) *the link morphism with indentation degree n on the quaternionic Shimura variety* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ *with shift $\boldsymbol{t}$*. Note that by Lemma 2.20 and Construction 2.12, for a fixed lifting $\tilde{\mathtt{S}}$ of $\mathtt{S}$, an indentation degree n, and a shift $\boldsymbol{t}$, there exists at most one link morphism $(\eta_{(n),\sharp}, \eta^{\sharp}_{(n)})$ on $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$.

The link morphism $(\eta_{(n),\sharp}, \eta^{\sharp}_{(n)})$ induces a homomorphism of the cohomology groups,

$$\tilde{\eta}^{\star}_{(n)} \colon H^{\star}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big) \longrightarrow H^{\star}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \eta^{*}_{(n),\sharp}(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'})\big) \xrightarrow{(\eta^{\sharp}_{(n)})^{-1}} H^{\star}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big),$$

which is equivariant under the Hecke action by $G(\mathbb{A}^{\infty})$ and the Galois action by $\mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$.[13] We fix a square root $p^{1/2} \in \overline{\mathbb{Q}}_{\ell}$ of p. We put

$$\eta^{\star}_{(n)} = \frac{1}{p^{v(\eta)/2}} \tilde{\eta}^{\star}_{(n)}, \tag{2.23.1}$$

and we call it the *normalized link morphism* on the cohomology groups of quaternionic Shimura varieties associated to η with indentation degree n and shift $\boldsymbol{t}$. This normalization will be justified in Lemma 2.29(2). When the link morphism $\eta''_{(n),\sharp} : \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) \to \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'})$ preserves the neutral connected components, $\boldsymbol{t} = \mathbf{1}$ is a canonical choice, and in that case, $\eta^{\star}_{(n)}$ is canonically defined.

Let $\eta_1 \colon \mathtt{S}_1 \to \mathtt{S}_2$ and $\eta_2 \colon \mathtt{S}_2 \to \mathtt{S}_3$ be two links with all curves turning to the right, satisfying the conditions above; that is, all $\mathtt{S}_i$ have the same set of finite places, $\#\mathtt{S}_{1,\infty} = \#\mathtt{S}_{2,\infty} = \#\mathtt{S}_{3,\infty}$, $\#\mathtt{T}_1 = \#\mathtt{T}_2 = \#\mathtt{T}_3$, and $\Delta_{\tilde{\mathtt{S}}_{1,\infty}} = \Delta_{\tilde{\mathtt{S}}_{2,\infty}} = \Delta_{\tilde{\mathtt{S}}_{3,\infty}}$ if $\mathfrak{p}$ splits in E/F. Suppose then that there are link morphisms $(\eta''_{i,(n_i),\sharp}, \eta''^{\sharp}_{i,(n_i)})$ for $i = 1,2$ on unitary Shimura varieties with indentation degree n_i. Then the composed morphism

$$\eta''_{12,(n_{12}),\sharp} \colon \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_1}) \xrightarrow{\eta''_{1,(n_1),\sharp}} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_2}) \xrightarrow{\eta''_{2,(n_2),\sharp}} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_3}),$$

together with the composed quasi-isogeny

$$\eta''^{\sharp}_{12,(n_{12})} \colon \mathbf{A}''_{\tilde{\mathtt{S}}_1} \xrightarrow{\eta''^{\sharp}_{1,(n_1)}} \eta''^{*}_{1,(n_1),\sharp}(\mathbf{A}''_{\tilde{\mathtt{S}}_2}) \xrightarrow{\eta''^{*}_{1,(n_1),\sharp}(\eta''^{\sharp}_{2,(n_2)})} \eta''^{*}_{1,(n_1),\sharp}\eta''^{*}_{2,(n_2),\sharp}(\mathbf{A}''_{\tilde{\mathtt{S}}_3}),$$

gives the (unique) link morphism on the unitary Shimura varieties with indentation degree $n_{12} := n_1 + n_2$ associated to the composed link $\eta''_{12,(n_{12})} := \eta''_{2,(n_2)} \circ \eta''_{1,(n_1)}$. From this, we get a link morphism of quaternionic Shimura varieties of indentation degree n_{12},

$$\eta_{12,(n_{12}),\sharp} \colon \mathrm{Sh}_{K_p}(G_{\mathtt{S}_1,\mathtt{T}_1}) \xrightarrow{\eta_{1,(n_1),\sharp}} \mathrm{Sh}_{K_p}(G_{\mathtt{S}_2,\mathtt{T}_2}) \xrightarrow{\eta_{2,(n_2),\sharp}} \mathrm{Sh}_{K_p}(G_{\mathtt{S}_3,\mathtt{T}_3}),$$

such that the shift of $\eta_{12,(n_{12}),\sharp}$ is the product of the shifts of $\eta_{1,(n_1),\sharp}$ and $\eta_{2,(n_2),\sharp}$. Moreover, we have $\eta^{\star}_{12,(n_{12})} = \eta^{\star}_{1,(n_1)} \circ \eta^{\star}_{2,(n_2)}$ on the cohomology groups of quaternionic Shimura varieties.

[13] Here, $G(\mathbb{A}^{\infty})$ denotes the common finite adelic points of $G_{\mathtt{S},\mathtt{T}}$ and $G_{\mathtt{S}',\mathtt{T}'}$ according to Notation 2.2.

2.24. Automorphic representations

Following [34, Section 5.10], for a regular multiweight $(\underline{k}, w)$ we use $\mathcal{A}_{(\underline{k},w)}$ to denote the set of cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that

- the Archimedean component π_τ for each $\tau \in \Sigma_\infty$ is a holomorphic discrete series of weight $k_\tau - 2$ with central character $x \mapsto x^{w-2}$,
- and the $\mathfrak{p}$-component $\pi_\mathfrak{p}$ is *spherical*.

We write $\pi^{\infty,\mathfrak{p}}$ to denote the prime-to-$\mathfrak{p}$ finite part of π.

For $\pi \in \mathcal{A}_{(\underline{k},w)}$, if v is a finite place of F such that the v-component π_v is spherical (i.e., $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})} \neq 0$), then we write T_v and S_v for the Hecke operators given by the double cosets $\mathrm{GL}_2(\mathcal{O}_{F_v})\left(\begin{smallmatrix}\varpi_v^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)\mathrm{GL}_2(\mathcal{O}_{F_v})$ and $\mathrm{GL}_2(\mathcal{O}_{F_v})\varpi_v^{-1}\mathrm{GL}_2(\mathcal{O}_{F_v})$, respectively. We write $T_v(\pi)$ and $S_v(\pi)$ for the eigenvalues for the actions of T_v and S_v on $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$. We denote by $\rho_\pi\colon \mathrm{Gal}_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ the Galois representation attached to π normalized so that if v is a finite place of F at which π is spherical, then the action of a *geometric* Frobenius at v has trace equal to $T_v(\pi)$. Let $\rho_{\pi,\mathfrak{p}}$ be the restriction of ρ_π to $\mathrm{Gal}_{\mathbb{F}_{p^g}}$ (note that ρ_π is unramified at $\mathfrak{p}$ since $\pi_\mathfrak{p}$ is spherical). The characteristic polynomial of $\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_{p^g})$ is given by

$$X^2 - T_\mathfrak{p}(\pi)X + S_\mathfrak{p}(\pi)p^g. \tag{2.24.1}$$

We say that a cuspidal automorphic representation $\pi \in \mathcal{A}_{(\underline{k},w)}$ *appears* in the cohomology of the Shimura variety $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$ if, for each finite place v of $\mathtt{S}$, the local component π_v of π is square-integrable so that π is the image (under the Jacquet–Langlands correspondence) of a unique automorphic representation $\pi_{B_\mathtt{S}}$ of $G_{\mathtt{S},\mathtt{T}}(\mathbb{A}) = (B_\mathtt{S} \otimes_\mathbb{Q} \mathbb{A})^\times$, and, moreover, $(\pi_{B_\mathtt{S}}^\infty)^K$ is nonzero. When π appears in $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$, the actions of Hecke operators T_v can be expressed as Hecke correspondence on the étale cohomology $H^d_{\mathrm{et}}(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}})$. Moreover, the action of $S_\mathfrak{p}$ is exactly as given in Example 2.22; when $\mathtt{S}_\infty = \Sigma_\infty$, the action of $T_\mathfrak{p}$ is exactly as given in Section 2.14.

Notation 2.25

For $\pi \in \mathcal{A}_{(\underline{k},w)}$ and a $\overline{\mathbb{Q}}_\ell[G(\mathbb{A}^{\infty,p})]$-module M, we write

$$M[\pi] := \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[G(\mathbb{A}^{\infty,p})]}(\pi_{B_\mathtt{S}}^{\infty,\mathfrak{p}}, M)$$

for its π*-isotypical component*. By the strong multiplicity 1 theorem for quaternion algebras, $\pi_{B_\mathtt{S}}$ is determined by $\pi_{B_\mathtt{S}}^{\infty,\mathfrak{p}}$; this justifies the notation for $M[\pi]$. There is also a finite version, as follows. Let $K^p \subset G(\mathbb{A}^{\infty,p})$ be an open compact subgroup so that $(\pi_{B_\mathtt{S}}^{\infty,\mathfrak{p}})^{K^p}$ is a nonzero irreducible module over the prime-to-p Hecke algebra $\mathscr{H}_{K^p} := \overline{\mathbb{Q}}_\ell[K^p\backslash G(\mathbb{A}^{\infty,p})/K^p]$. Then $\pi_{B_\mathtt{S}}$ is determined by the $\mathscr{H}_{K^p}$-module $(\pi_{B_\mathtt{S}}^{\infty,\mathfrak{p}})^{K^p}$. For an $\mathscr{H}_{K^p}$-module M, we have

$$M[\pi] := \mathrm{Hom}_{\mathscr{H}_{K^p}}\big((\pi_{B_\mathtt{S}}^{\infty,\mathfrak{p}})^{K^p}, M\big).$$

PROPOSITION 2.26

Let $\pi \in \mathcal{A}_{(\underline{k},w)}$ be a cuspidal automorphic representation appearing in the cohomology of the Shimura variety $\mathcal{S}h_K(G_{\mathtt{S},\mathtt{T}})$. Then we have a canonical isomorphism

$$H^i_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)[\pi]^{\mathrm{Fr}\text{-}s.s.} = \begin{cases}\rho_{\pi,\mathfrak{p}}^{\otimes d} \otimes [\det(\rho_{\pi,\mathfrak{p}})(1)]^{\otimes \#\mathtt{T}} & \text{if } i = d,\\ 0 & \text{if } i \neq d,\end{cases} \tag{2.26.1}$$

equivariant under the action of the geometric Frobenius Frob_{p^g}. *Here, the superscript* Fr-*s.s. means taking the semisimplification as a* Frob_{p^g}*-module. In particular, if α_π and β_π are the two (generalized) eigenvalues of $\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_{p^g})$, then the (generalized) eigenvalues of the action of* $\mathrm{Frob}_{p^{2g}}$ *on $H^i_{\mathrm{et}}(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}})[\pi]$ are $p^{-2g\#\mathtt{T}}\alpha_\pi^{2(i+\#\mathtt{T})}\beta_\pi^{2(d-i+\#\mathtt{T})}$ with multiplicity $\binom{d}{i}$ for $0 \leq i \leq d$.*

Proof

The first part of the proposition is well known to experts. We defer its proof to the Appendix (see Proposition A.3). The explicit description of the action of $\mathrm{Frob}_{p^{2g}}$ is straightforward. □

PROPOSITION 2.27

Assume that $d = \#\mathtt{S}^c_\infty \neq 0$. Then the following statements hold.

(1) *The $2g$th iteration of the Frobenius link $\sigma^{2g} : \mathtt{S} \to \mathtt{S}$ coincides with the $2d$-fold self-composition of the fundamental link $\eta_{\mathtt{S}}$ introduced in Example 2.17.*

(2) *The link morphism on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ with indentation degree 0 associated to $\sigma^{2g} = \eta^{2d}_{\mathtt{S}}$ exists, and it is given by*

  (a) *g-fold self-composition $(\mathfrak{F}''_{\mathfrak{p}^2})^g$ if $\mathfrak{p}$ is inert in E/F; and*

  (b) *$(\mathfrak{F}''_{\mathfrak{p}^2})^g \cdot S_{\mathfrak{q}}^{-\Delta_{\tilde{\mathtt{S}}\infty}}$ if $\mathfrak{p}$ splits in E/F, where $S_{\mathfrak{q}}$ is defined as in Example 2.22.*

Moreover, this link morphism preserves the neutral geometrically connected component $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})^\circ_{\overline{\mathbb{F}}_p}$ and hence induces a canonical *link morphism $(\eta^{2d}_{\mathtt{S},(0),\sharp}, \eta^{2d,\sharp}_{\mathtt{S},(0)})$ on the quaternionic Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ with shift* **1** *for any fixed subset $\mathtt{T} \subseteq \mathtt{S}_\infty$.*

(3) *Let*

$$(\eta^{2d}_{\mathtt{S}})^\star_{(0)} : H^d_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big) \to H^d_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)$$

be the normalized link morphism (2.23.1) induced by $(\eta^{2d}_{\mathtt{S},(0),\sharp}, \eta^{2d,\sharp}_{\mathtt{S},(0)})$. Then we have an equality of operators on cohomology groups,

$$(\eta^{2d}_{\mathtt{S}})^\star_{(0)} = p^{-dg} \cdot \mathrm{Frob}_{p^{2g}} \circ S_{\mathfrak{p}}^{-d-2\#\mathtt{T}}, \tag{2.27.1}$$

where $S_{\mathfrak{p}}$ is the Hecke operator given by the central element $\underline{p}_F^{-1} \in G(\mathbb{A}^{\infty})$. In particular, for each $\pi \in \mathcal{A}_{(\underline{k},w)}$ and each integer i with $0 \le i \le d$, the (generalized) eigenspace of $\mathrm{Frob}_{p^{2g}}$ *on $H^d_{\mathrm{et}}(\mathrm{Sh}_K(G_{\mathrm{S,T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathrm{S,T}})[\pi]$ with eigenvalue $p^{-2g\#\mathrm{T}}\alpha_\pi^{2(i+\#\mathrm{T})}\beta_\pi^{2(d-i+\#\mathrm{T})}$ is the same as the (generalized) eigenspace of $(\eta_{\mathrm{S}}^{2d})^{\star}_{(0)}$ with eigenvalue $(\alpha_\pi/\beta_\pi)^{2i-d}$.*

Proof
Item (1) is evident. For item (2), we first check that the maps given by (a) and (b) are link morphisms with indentation degree 0 associated to the link η_{S}^{2d}. This follows easily from Examples 2.21 and 2.22. By the uniqueness of link morphisms (Lemma 2.20), they are the link morphisms we sought.

We next show that the link morphism in the unitary case preserves the neutral geometrically connected component $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})^{\circ}_{\overline{\mathbb{F}}_p}$. This is a direct computation using the Shimura reciprocity map (in Section 2.5), which we spell out now. Denote by Φ^{2g} the Frobenius endomorphism of $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})$ relative to $\mathbb{F}_{p^{2g}}$. Then $(\mathfrak{F}''_{\mathfrak{p}^2})^g$ is nothing but the composition of Φ^{2g} with the Hecke operator S_p^{-g}, where S_p is the Hecke correspondence given by the central element $(\underline{p}_F^{-1}, 1) \in G(\mathbb{A}^{\infty}) \times_{\mathbb{A}_F^{\infty,\times}} \mathbb{A}_E^{\infty,\times} \cong G''(\mathbb{A}^{\infty})$. Recall that the set of geometrically connected components of $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})$ is given by

$$\pi_0\big(\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})_{\overline{\mathbb{F}}_p}\big) \cong (F_+^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times}/\mathcal{O}_{\mathfrak{p}}^{\times}) \times (E^1 \backslash \mathbb{A}_E^1/\mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1}).$$

The action of Φ^{2g} on $\pi_0(\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathrm{S}}})_{\overline{\mathbb{F}}_p})$ coincides with the arithmetic Frobenius $\mathrm{Frob}_{p^{2g}}^{-1} \in \mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$, which is computed already by (2.5.2). We now list the actions of these operators on the geometrically connected components.

Operator	When $\mathfrak{p}$ splits	When $\mathfrak{p}$ is inert
Φ^{2g}	$(\underline{p}_F)^{-2g} \times (\underline{\mathfrak{q}})^{-2\Delta_{\tilde{\mathrm{S}}_\infty}}$	$(\underline{p}_F)^{-2g} \times 1$
S_p	$(\underline{p}_F)^{-2} \times 1$	$(\underline{p}_F)^{-2} \times 1$
$S_{\mathfrak{q}}$	$1 \times \underline{\mathfrak{q}}^{-2}$	N/A

It is now clear that the link morphisms given in (1) and (2) preserve the neutral geometrically connected component. This verifies (2).

We now turn to the proof of (3). It suffices to verify (2.27.1) because $S_{\mathfrak{p}}$ acts on the π-component by the scalar $\omega_\pi(\underline{p}^{-1}) = \alpha_\pi\beta_\pi/p^g$ according to (2.24.1), and then item (3) would follow immediately from the following easy computation:

$$p^{-dg} \times p^{-2g\#\mathrm{T}}\alpha_\pi^{2(i+\#\mathrm{T})}\beta_\pi^{2(d-i+\#\mathrm{T})} \times (\alpha_\pi\beta_\pi/p^g)^{-(d+2\#\mathrm{T})} = (\alpha_\pi/\beta_\pi)^{2i-d}.$$

To prove (2.27.1), we first compute the canonical lift of the link morphism $((\eta^{2d}_{\mathtt{S},(0)})''_\sharp, (\eta^{2d}_{\mathtt{S},(0)})''^\sharp)$ to an endomorphism of $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ appearing in Step I of Construction 2.12 (and the shift in Step II is trivial in our case). This lift is clearly a composition of the Frobenius endomorphism relative to $\mathbb{F}_{p^{2g}}$, which we denote by $\Phi^{2g}_\times$, and the action of a Hecke operator given by a central element $\boldsymbol{x}$ in $G(\mathbb{A}^\infty) \times \mathbb{A}_E^{\infty,\times}$. This central element $\boldsymbol{x}$ is characterized by (and uniquely determined by) the following two conditions:

(a) the resulting link morphism on $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ preserves the neutral connected component;

(b) under the natural projection $G(\mathbb{A}^\infty) \times \mathbb{A}_E^{\infty,\times} \to G(\mathbb{A}^\infty) \times_{\mathbb{A}_F^{\infty,\times}} \mathbb{A}_E^{\infty,\times} \cong G''(\mathbb{A}^\infty)$, $\boldsymbol{x}$ is mapped to the central element $((\underline{p}_F)^g, 1)$ if $\mathfrak{p}$ is inert in E/F and to $((\underline{p}_F)^g, (\underline{\mathfrak{q}})^{\Delta_{\tilde{\mathtt{S}}_\infty}})$ if $\mathfrak{p}$ splits in E/F.

We claim that $\boldsymbol{x} = ((\underline{p}_F)^{\#\mathtt{S}^c_\infty + 2\#\mathtt{T}}, (\underline{p}_F)^{\#\mathtt{S}_\infty - 2\#\mathtt{T}})$ if $\mathfrak{p}$ is inert in E/F, and $\boldsymbol{x} = ((\underline{p}_F)^{\#\mathtt{S}^c_\infty + 2\#\mathtt{T}}, (\underline{p}_F)^{\#\mathtt{S}_\infty - 2\#\mathtt{T}}(\underline{\mathfrak{q}})^{\Delta_{\tilde{\mathtt{S}}_\infty}})$ if $\mathfrak{p}$ splits in E/F. Clearly, this element satisfies (b) above. To see (a), we note that the action of $\Phi^{2g}_\times$ on the geometrically connected component is the image of the *arithmetic Frobenius* $\mathrm{Frob}^{-1}_{p^{2g}}$ under the Shimura reciprocity maps in Section 2.1 and Section 2.4; namely,

$$\begin{cases} ((\underline{p}_F)^{-2\#\mathtt{S}^c_\infty - 4\#\mathtt{T}}, (\underline{p}_F)^{2\#\mathtt{T} - \#\mathtt{S}_\infty}) & \text{if } \mathfrak{p} \text{ is inert in } E/F, \\ ((\underline{p}_F)^{-2\#\mathtt{S}^c_\infty - 4\#\mathtt{T}}, (\underline{p}_F)^{2\#\mathtt{T} - \#\mathtt{S}_\infty}(\underline{\mathfrak{q}})^{-\Delta_{\tilde{\mathtt{S}}_\infty}}) & \text{if } \mathfrak{p} \text{ splits in } E/F. \end{cases}$$

But this element is exactly $(\nu \times \mathrm{id})(\boldsymbol{x}^{-1})$.

Now, taking the fiber over $\mathbf{1} \in \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathtt{S}},\mathtt{T}})$ tells us that the (canonical) link morphism $(\eta^d_{\mathtt{S}})_{(0),\sharp}$ is the Frobenius endomorphism $\Phi^{2g}_{G_{\mathtt{S},\mathtt{T}}}$ on $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ relative to $\mathbb{F}_{p^{2g}}$ composed with the Hecke operator given by multiplication by the first coordinate of $\boldsymbol{x}$, namely, $S_{\mathfrak{p}}^{-\#\mathtt{S}^c_\infty - 2\#\mathtt{T}} = S_{\mathfrak{p}}^{-d-2\#\mathtt{T}}$. For the action of $(\eta^d_{\mathtt{S}})^\star_{(0)}$ on $H^d_{\mathrm{et}}(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}})$, we note that the induced action of the Frobenius endomorphism $\Phi^{2g}_{G_{\mathtt{S},\mathtt{T}}}$ on cohomology coincides with $\mathrm{Frob}_{p^{2g}}$ (as opposed to the arithmetic Frobenius). So we have $(\eta^{2d}_{\mathtt{S}})^\star_{(0)} = p^{-dg} \cdot \mathrm{Frob}_{p^{2g}} \circ S_{\mathfrak{p}}^{-d-2\#\mathtt{T}}$, where p^{-dg} is the normalization factor in (2.23.1). This proves (2.27.1) and hence the Proposition. □

2.28. *Link morphisms, II*

Let $\eta\colon \mathtt{S} \to \mathtt{S}'$ be a general link. Then there exists an integer $N \geq 0$ such that the composition of the links $\xi\colon = \eta \circ \sigma^{2gN} = \eta \circ (\eta^{2d}_{\mathtt{S}})^N = (\eta^{2d}_{\mathtt{S}'})^N \circ \eta\colon \mathtt{S} \to \mathtt{S}'$ is right-turning, where $\eta_{\mathtt{S}}$ is the fundamental link for $\mathtt{S}$ (2.17). Suppose that the link morphism on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ associated to ξ with indentation degree n exists. Then we put, for each $\pi \in \mathscr{A}_{(\underline{k},w)}$,

$$\eta^\star_{(n)}\colon H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big)[\pi] \xrightarrow{\left((\eta^{2d}_{\mathtt{S}})^\star_{(0)}\right)^{-N}} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big)[\pi]$$
$$\xrightarrow{\xi^\star_{(n)}} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)[\pi]$$

and refer to it as *the normalized link morphism* on the cohomology group of quaternionic Shimura varieties associated to η with indentation degree n. Here the link morphism $(\eta^{2d}_{\mathtt{S}})^\star_{(0)}$ is taken to be the canonical one, so that it is invertible by Proposition 2.27. The shift of $\eta^\star_{(n)}$ is defined to be the same as that of $\xi^\star_{(n)}$ (as $(\eta^{2d}_{\mathtt{S}})^\star_{(0)}$ has shift $\mathbf{1}$). By Lemma 2.20 on the uniqueness of link morphisms, this definition does not depend on the choice of N (but on the shift of $\xi^\star_{(n)}$) and is compatible with compositions.

LEMMA 2.29

(1) *For any link $\eta\colon \mathtt{S}\to\mathtt{S}'$, there exist an integer $N>0$ and another right-turning link $\xi\colon \mathtt{S}'\to\mathtt{S}$ such that $\xi\circ\eta\colon \mathtt{S}\to\mathtt{S}$ is the same as $\sigma^{2gN}\colon \mathtt{S}\to\mathtt{S}$.*

(2) *If $\eta\colon \mathtt{S}\to\mathtt{S}'$ is a right-turning link and the link morphism $(\eta''_{(n),\sharp}, \eta''^{\sharp}_{(n)})$ on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ with indentation degree n associated to η exists, then there exists $N>0$ such that the link morphism associated to $\eta^{-1}\circ(\eta^{2d}_{\mathtt{S}'})^N\colon \mathtt{S}'\to\mathtt{S}$ of indentation $-n$ exists.*

(3) *Let $\eta\colon \mathtt{S}\to\mathtt{S}'$ be the link as in (2), and let $\eta_{(n),\sharp}\colon \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})\to \mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})$ be the link morphism with some shift $\boldsymbol{t}$ obtained by applying Construction 2.12 to $\eta''_{(n),\sharp}$. If $\eta^{-1}\colon \mathtt{S}'\to\mathtt{S}$ denotes the inverse link, then the morphism*

$$(\eta^{-1})^\star_{(-n)}\colon H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big) \longrightarrow H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big)$$

with shift $\boldsymbol{t}^{-1}$ is the same as the inverse of $\eta^\star_{(n)}$. Moreover, if $\eta''_{(n),\sharp}$ (or, equivalently, $\eta_{(n),\sharp}$) is finite flat of degree $p^{v(\eta)}$, where $v(\eta)$ denotes the total displacement of η, then we have an equality

$$(\eta^{-1})^\star_{(-n)} = p^{-v(\eta)/2}\,\mathrm{Tr}_{\eta_{(n),\sharp}},$$

where $\mathrm{Tr}_{\eta_{(n),\sharp}}$ is the trace map on cohomology induced by the finite flat morphism $\eta_{(n),\sharp}$.

Proof

Item (1) is obvious. For item (2), we may first find N so that $\xi := \eta^{-1}\circ(\eta^{2d}_{\mathtt{S}'})^N$ has all curves turning to the right. Then we consider the two morphisms

$$\begin{array}{ccc} \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}}) & & \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'}) \\ & \searrow^{\eta_{(n),\sharp}} \quad {}^{(\eta_{\mathtt{S}'}^{2d})^N_{(0),\sharp}}\swarrow & \\ & \mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}'}) & \end{array}$$

Since the link morphism $\eta_{(n),\sharp}$ induces a bijection on the closed points, [14, Proposition 4.8] implies that, after possibly enlarging N, the map $(\eta_{\mathtt{S}'}^{2d})^N_{(0),\sharp}$ factors through $\eta_{(n),\sharp}$, as $\eta_{(n),\sharp} \circ \xi_\sharp$. It is easy to see that $\xi_\sharp$ gives the required link morphism.

The first part of (3) follows from the uniqueness of the link morphism (Lemma 2.20). For the second part of (3), note that the composed morphism

$$\begin{aligned} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big) &\xrightarrow{p^{v(\eta)/2}\eta^\star_{(n)}} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big) \\ &\xrightarrow{\mathrm{Tr}_{\eta_{(n),\sharp}}} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big) \end{aligned}$$

is nothing but the multiplication by $p^{v(\eta)}$, according to our normalization of $\eta^\star_{(n),\sharp}$ in (2.23.1). It follows immediately that $(\eta^{-1})^\star_{(-n)} = p^{-v(\eta)/2}\,\mathrm{Tr}_{\eta_{(n),\sharp}}$. □

2.30. Goren–Oort divisors

We recall the definition of the Goren–Oort stratification from [33, Section 4]. We will make essential use of the case of divisors. Let $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ be the special fiber of a quaternionic Shimura variety of the type considered in Section 2.1. We fix throughout this article a choice of lifting $\tilde{\mathtt{S}}_\infty$ of $\mathtt{S}_\infty$, and let $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ be the associated unitary Shimura variety.

In [33, Definition 4.6, Section 4.9], we defined, for each $\tau \in \mathtt{S}^c_\infty$, the *Goren–Oort divisor* $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_\tau$ of $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$ at τ as the vanishing locus of the τ-partial Hasse invariant of the versal family $\mathbf{A}''_{\tilde{\mathtt{S}}}$. Each $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_\tau$ is projective and smooth by [33, Proposition 4.7]. Transferring these structures to the quaternionic Shimura varieties using Proposition 2.7, we get a Goren–Oort divisor $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau$ on $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ for each $\tau \in \mathtt{S}^c_\infty$. When $\mathtt{T} = \emptyset$, this is done in [33, Section 4.9], and the general case is the same.

For a subset $J \subseteq \mathtt{S}^c_\infty$, we put $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_J = \bigcap_{\tau\in J}\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau$ and $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_J = \bigcap_{\tau\in J}\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_\tau$. The closed subvarieties $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_J$ (resp., $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_J$) with J running through the subsets of $\mathtt{S}^c_\infty$ form the Goren–Oort stratification of $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ (resp., $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$).

The main results of [33] give an explicit description of all closed Goren–Oort strata $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_J$ (resp., $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_J$) as a $\mathbb{P}^1$-power bundle over another unitary (resp., quaternionic) Shimura variety. We list results from [33] that we will make use of later. (One more result will be used later in proving Lemma 5.14.)

PROPOSITION 2.31

Let $\tau \in \mathtt{S}^c_\infty$. Assume that $\tau^- = \sigma^{-n_\tau}\tau$ is different from τ (see Notation 2.16 for the notation). We put $\mathtt{S}_\tau = \mathtt{S} \cup \{\tau, \tau^-\}$ and $\mathtt{T}_\tau = \mathtt{T} \cup \{\tau\}$. Let $\tilde{\mathtt{S}}_{\tau,\infty}$ be the lifting of $\mathtt{S}_{\tau,\infty}$ derived from $\tilde{\mathtt{S}}_\infty$ according to the rule of [33, Section 5.3], and put $\tilde{\mathtt{S}}_\tau = (\mathtt{S}_\tau, \tilde{\mathtt{S}}_{\tau,\infty})$. In particular, $\Delta_{\tilde{\mathtt{S}}_{\tau,\infty}} = \Delta_{\tilde{\mathtt{S}}_\infty}$ when $\mathfrak{p}$ splits in E/F.

(1) *There exists a $\mathbb{P}^1$-bundle fibration*

$$\pi''_\tau : \operatorname{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_\tau \to \operatorname{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_\tau})_\tau$$

equivariant for the action of $\mathscr{G}''_{\tilde{\mathtt{S}},p} = \mathscr{G}''_{\tilde{\mathtt{S}}_\tau,p}$, and a p-quasi-isogeny of Abelian schemes on $\operatorname{Sh}_{K''}(G''_{\tilde{\mathtt{S}}})_\tau$,

$$\Phi_{\pi''_\tau} : \mathbf{A}''_{\tilde{\mathtt{S}}} \to \pi''^{*}_\tau(\mathbf{A}''_{\tilde{\mathtt{S}}'}).$$

By Construction 2.12, this gives rise to a $\mathbb{P}^1$-bundle fibration,

$$\pi_\tau : \operatorname{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau \longrightarrow \operatorname{Sh}_{K_p}(G_{\mathtt{S}_\tau,\mathtt{T}_\tau}),$$

with some shift $\boldsymbol{t}_\tau = \boldsymbol{t}_\tau(\mathtt{S},\mathtt{T}) \in E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^\times$ that is compatible with the Hecke action of $G(\mathbb{A}^{\infty,p})$. Moreover, there is an isomorphism of étale sheaves for each given regular multiweight $(\underline{k}, w)$,

$$\pi^\sharp_\tau : \mathscr{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}|_{\operatorname{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau} \xrightarrow{\cong} \pi^*_\tau(\mathscr{L}^{(\underline{k},w)}_{\mathtt{S}_\tau,\mathtt{T}_\tau}).$$

The morphisms π_τ and $\pi^\sharp_\tau$ are uniquely determined once $\boldsymbol{t}_\tau$ is fixed.

(2) *Let $\mathcal{O}(1)$ be the tautological quotient line bundle on $\operatorname{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau$ for the $\mathbb{P}^1$-bundle given by π_τ. If τ^- is different from τ, then the normal bundle of the closed immersion $\operatorname{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau \hookrightarrow \operatorname{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ is, up to tensoring a line bundle which is a torsion class in the Picard group of $\operatorname{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau$, the same as $\mathcal{O}(-2p^{n_\tau}) = \mathcal{O}(1)^{\otimes(-2p^{n_\tau})}$.*

Proof

In item (1), the existence of π''_τ is a special case of [33, Corollary 5.9]. Roughly speaking, this $\mathbb{P}^1$-bundle π''_τ parameterizes the lines (the Hodge filtration) in the reduced $\tilde{\tau}^- := \sigma^{-n_\tau}\tilde{\tau}$-component of the relative de Rham homology of the versal family $\mathbf{A}''_{\tilde{\mathtt{S}}_\tau}$ over $\operatorname{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_\tau})_\tau$. It is straightforward to check that the condition (2.12.1) is satisfied for the pairs $(\tilde{\mathtt{S}}, \mathtt{T})$ and $(\tilde{\mathtt{S}}_\tau, \mathtt{T}_\tau)$. We apply Construction 2.12 to deduce the existence of $(\pi_\tau, \pi^\sharp_\tau)$ from that of $(\pi''_\tau, \Phi_{\pi''_\tau})$.

Item (2) follows from [33, Proposition 6.4], when noting that the quaternionic Shimura varieties and the unitary Shimura varieties have isomorphic geometrically connected components. □

Proposition 2.26(1) implies that we have a morphism,

$$\pi_\tau^*\colon H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S}_\tau,\mathtt{T}_\tau})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_\tau,\mathtt{T}_\tau}\big)\longrightarrow H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\tau,\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big),$$

equivariant under the actions of the prime-to-p Hecke algebra $\mathscr{H}_{K^p}$. It is canonical up to the ambiguity of choosing the shift in Construction 2.12.

THEOREM 2.32

(1) *Let $\tau_1,\tau_2\in\mathtt{S}^c_\infty$ be two places such that $\tau_1,\tau_2,\tau_1^-,\tau_2^-$ are distinct. We have a Cartesian diagram:*

$$\begin{array}{ccc}
\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\{\tau_1,\tau_2\}} & \xrightarrow{\pi_{\tau_1}} & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau_1},\mathtt{T}_{\tau_1}})_{\tau_2}\\
\downarrow{\scriptstyle \pi_{\tau_2}} & & {\scriptstyle \pi_{\tau_2}}\downarrow\\
\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau_2},\mathtt{T}_{\tau_2}})_{\tau_1} & \xrightarrow{\pi_{\tau_1}} & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau_1,\tau_2},\mathtt{T}_{\tau_1,\tau_2}})
\end{array}$$

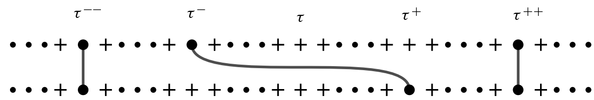
If we use the notation of shifts of these π_{τ_i} as in Proposition 2.31(1), then we have an equality,

$$\boldsymbol{t}_{\tau_1}(\mathtt{S},\mathtt{T})\boldsymbol{t}_{\tau_2}(\mathtt{S}_{\tau_1},\mathtt{T}_{\tau_2})=\boldsymbol{t}_{\tau_2}(\mathtt{S},\mathtt{T})\boldsymbol{t}_{\tau_1}(\mathtt{S}_{\tau_2},\mathtt{T}_{\tau_2}).$$

Moreover, we have a commutative diagram of induced morphisms on the cohomology groups:

$$\begin{array}{ccc}
H^\star_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau_1}\cup\mathtt{S}_{\tau_2},\mathtt{T}_{\tau_1}\cup\mathtt{T}_{\tau_2}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\tau_1}\cup\mathtt{S}_{\tau_2},\mathtt{T}_{\tau_1}\cup\mathtt{T}_{\tau_2}}\big) & \xrightarrow{\pi^*_{\tau_2}} & H^\star_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau_1},\mathtt{T}_{\tau_1}})_{\tau_2,\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\tau_1},\mathtt{T}_{\tau_1}}\big)\\
\downarrow{\scriptstyle \pi^*_{\tau_1}} & & \downarrow{\scriptstyle \pi^*_{\tau_1}}\\
H^\star_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau_2},\mathtt{T}_{\tau_2}})_{\tau_1,\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\tau_2},\mathtt{T}_{\tau_2}}\big) & \xrightarrow{\pi^*_{\tau_2}} & H^\star_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\{\tau_1,\tau_2\},\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)
\end{array}$$

(2) *Let $\tau\in\mathtt{S}^c_\infty$ be a place such that τ,τ^+,τ^- are distinct. Put $n=n_{\tau^+}-n_\tau$ if $\mathfrak{p}$ splits in E/F and $n=0$ if $\mathfrak{p}$ is inert in E/F. Let $\eta\colon\mathtt{S}_{\tau^+}=\mathtt{S}\cup\{\tau^+,\tau\}\to\mathtt{S}_\tau=\mathtt{S}\cup\{\tau,\tau^-\}$ be the link given by straight lines, except sending τ^- to τ^+ over τ:*



Let $\eta_{(n),\sharp}$ be the morphism defined by the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau^+} & \hookleftarrow & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\{\tau^+,\tau\}} & \hookrightarrow & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau} \\ \downarrow{\scriptstyle \pi_{\tau^+}} & \swarrow{\scriptstyle \cong} & & & \downarrow{\scriptstyle \pi_\tau} \\ \mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau^+},\mathtt{T}_{\tau^+}}) & & \xrightarrow{\quad \eta_{(n),\sharp} \quad} & & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau},\mathtt{T}_{\tau}}). \end{array} \tag{2.32.1}$$

Then the following statements hold.

(a) *The map $\eta_{(n),\sharp}$ is the morphism obtained by applying Construction 2.12 to a link morphism on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_{\tau^+}})$ with indentation degree n.*

(b) *If $\boldsymbol{t}_? \in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E,\mathfrak{p}}^\times$ for $? = \tau, \tau^+$ denotes the shift of the correspondence*

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_?,\mathtt{T}_?}) \xleftarrow{\pi_?} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_? \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}),$$

then $\eta_{(n),\sharp}$ has shift $\boldsymbol{t}_{\tau^+}\boldsymbol{t}_\tau^{-1}$.

(c) *The morphism $\eta_{(n),\sharp}$ is finite flat of degree $p^{v(\eta)}$.*

(d) *The p-quasi-isogeny between the versal families of Abelian varieties on $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}_{\tau^+}})$ given by*

$$\pi''^{*}_{\tau^+}(\mathbf{A}''_{\tilde{\mathtt{S}}_{\tau^+}})|_{\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_{\{\tau^+,\tau\}}} \xleftarrow{\Phi_{\pi''_{\tau^+}}} \mathbf{A}''_{\tilde{\mathtt{S}}}|_{\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_{\{\tau^+,\tau\}}} \xrightarrow{\Phi_{\pi''_{\tau}}} \pi''^{*}_{\tau}(\mathbf{A}''_{\tilde{\mathtt{S}}_{\tau}})|_{\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})_{\{\tau^+,\tau\}}}$$

induces a link morphism on the sheaves $\eta^\sharp_{(n)}\colon \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\tau^+},\mathtt{T}_{\tau^+}} \longrightarrow \eta^*_{(n),\sharp}(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_\tau,\mathtt{T}_\tau})$. Then the induced normalized link morphism $\eta^\star_{(n)}$ on the cohomology groups constructed as in Section 2.23 fits into the following commutative diagram:

$$\begin{array}{ccccc} H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau^+,\overline{\mathbb{F}}_p}) & \longrightarrow & H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\{\tau^+,\tau\},\overline{\mathbb{F}}_p}) & \longleftarrow & H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau,\overline{\mathbb{F}}_p}) \\ \uparrow{\scriptstyle \pi^*_{\tau^+}} & \nearrow{\scriptstyle \cong} & & & \uparrow{\scriptstyle \pi^*_{\tau}} \\ H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau^+},\mathtt{T}_{\tau^+}})_{\overline{\mathbb{F}}_p}) & & \xleftarrow{\quad p^{(n_\tau+n_{\tau^+})/2}\eta^\star_{(n)} \quad} & & H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau},\mathtt{T}_{\tau}})_{\overline{\mathbb{F}}_p}) \end{array} \tag{2.32.2}$$

where the upper horizontal arrows are natural restriction maps. Here, for simplification, we have suppressed the sheaves from the notation. For instance, $H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau^+,\overline{\mathbb{F}}_p})$ should be understood as $H^\star_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau^+,\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}|_{\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\tau^+}})$.

(3) *Assume that $\mathtt{S}^c_\infty = \{\tau, \tau^-\}$ (and hence $\mathfrak{p}$ splits in E/F). Then $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^-\}}$ is isomorphic to the special fiber of the 0-dimensional*

Shimura variety $\mathrm{Sh}_{\mathrm{Iw}_p}(G_{\mathtt{S}_\tau,\mathtt{T}_\tau})$ *of Iwahori level at* $\mathfrak{p}$. *Let* $\eta\colon \mathtt{S}_{\tau^-} \to \mathtt{S}_\tau$ *denote the link map (with no curve). Then the link morphism* $\eta_{(n_{\tau^-}),\sharp}$: $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau^-},\mathtt{T}_{\tau^-}}) \xrightarrow{\cong} \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\tau,\mathtt{T}_\tau})$ *of indentation degree* $2n_{\tau^-}$ *associated to* η *exists, and the following diagram*

$$\begin{array}{ccccc} & & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^-\}} & & \\ & \swarrow^{\pi_\tau} & & \searrow^{\pi_{\tau^-}} & \\ \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\tau,\mathtt{T}_\tau}) & & & & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\tau^-},\mathtt{T}_{\tau^-}}) \xrightarrow[\eta_{(n_{\tau^-})}]{\cong} \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\tau,\mathtt{T}_\tau}) \end{array}$$

is (the base change to $\mathbb{F}_{p^g}$ *of) the Hecke correspondence* $T_\mathfrak{p}$ *on* $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\tau,\mathtt{T}_\tau})$. *If* $\boldsymbol{t}_? \in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^\times$ *for* $? = \tau, \tau^+$ *denotes the shift of the correspondence*

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_?,\mathtt{T}_?}) \xleftarrow{\pi_?} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_? \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}),$$

then $\eta_{(n_{\tau^-}),\sharp}$ *has shift* $\varpi_{\bar{\mathfrak{q}}}\boldsymbol{t}_\tau^{-1}\boldsymbol{t}_{\tau^-}$. *Moreover, the map induced by the diagram above on cohomology groups,*

$$H^0_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S}_\tau,\mathtt{T}_\tau})_{\overline{\mathbb{F}}_p}\big) \xrightarrow{(\eta_{(n_{\tau^-})}\circ\pi_{\tau^-})^*} H^0_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^-\},\overline{\mathbb{F}}_p}\big)$$
$$\xrightarrow{\mathrm{Tr}_{\pi_\tau}} H^0_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S}_\tau,\mathtt{T}_\tau})_{\overline{\mathbb{F}}_p}\big),$$

is the usual Hecke action $T_\mathfrak{p}$. *Here, as in (2), we have suppressed the sheaves from the notation.*

Proof

The analogues of (1), (2)(a), and (2)(c) for unitary Shimura varieties were proved in [33, Proposition 7.12, Theorem 7.16]. The statements here follow from Construction 2.12.

Item (2)(b) regarding shifts follows directly from Remark 2.13. Item (2)(d) directly follows from the construction of $\eta^\sharp_{(n)}$ and $\eta^\star_{(n)}$. For item (3), the analogous statement for unitary Shimura varieties $\mathrm{Sh}_{K''}(G''_{\tilde{\mathtt{S}}'})$ (with $T_\mathfrak{p}$ replaced by $T_\mathfrak{q}$) was proved in [33, Theorem 7.16(2)]. One deduces (3) using the construction in Section 2.14, and computes the shifts by Remark 2.13. □

3. Goren–Oort cycles

In this section, we investigate certain generalizations of the Goren–Oort strata studied in [9], which are called the *Goren–Oort cycles*. They are parameterized by certain combinatorial data, which are called the *periodic semimeanders*. We will show later that the intersection matrix of the Goren–Oort cycles turns out to be closely related to the Gram matrix associated to these periodic semimeanders (which explains our choice of the combinatorial model).

3.1. Periodic semimeanders

The combinatorial construction that we will use later is related to the so-called *link representations* of periodic Temperley–Lieb algebras, which appear naturally in the study of mathematical physics (see, e.g., [6], [10], [24]). We will simply state here the main result with minimal input, and refer the reader to [24] for a detailed discussion of the mathematical physics background and the proofs.

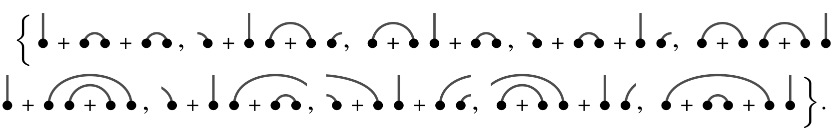
We slightly modify the usual definition of periodic semimeanders to adapt to our situation. Recall that F is a totally real field of degree g and that $\mathtt{S}$, $\mathtt{T}$ are introduced as in Section 2.1, and $d = \#\mathtt{S}_\infty^c$. We consider the band associated to $\mathtt{S}$ defined as in Section 2.15, and recall that the band is placed on a cylinder, but we often draw it over the 2-dimensional xy-plane with the x-coordinate taken modulo g.

A periodic semimeander for $\mathtt{S}$ is a collection of curves (arcs) that link two nodes of the band for $\mathtt{S}$, and straight lines (semilines) that link a node to infinity ($+\infty$ in the y-direction) subject to the following conditions.

- All the arcs and semilines lie on the cylinder above the band (that is to have positive y-coordinate in the 2-dimensional picture).
- Each node of the band for $\mathtt{S}$ is exactly one endpoint of an arc or a semiline.
- There are no intersection points among these arcs and semilines.

The number of arcs is denoted by r (so $r \le d/2$), and the number of semilines $d - 2r$ is called the *defect* of the periodic semimeander. Two periodic semimeanders are considered the same if they can be continuously deformed into each other while keeping the above three properties in the process. We use $\mathfrak{B}_\mathtt{S}^r$ to denote the set of semimeanders for $\mathtt{S}$ with r arcs (up to the deformations). For example, if F has degree 7 over $\mathbb{Q}$, $r = 2$, and $\mathtt{S} = \{\infty_1, \infty_4\}$, then we have

$\mathfrak{B}_\mathtt{S}^2 =$



(3.1.1)

When drawing in the xy-plane, points are placed on the x-axis at points of coordinates $(0,0), \dots, (g-1,0)$, and the diagram for a periodic semimeander is taken to be periodic in the x-direction of period g. The curves connecting the points can connect across the imaginary boundary lines at $x = -1/2$ and $x = g - 1/2$ (which are identified). See, for example, (3.1.1). An elementary calculation shows that $\#\mathfrak{B}_\mathtt{S}^r = \binom{d}{r}$.

A *standard presentation* of a semimeander is where all the arcs are monotonic in the x-direction (namely, it does not twist back and forth). Using the xy-plane picture, we define the *left* and *right end-nodes* of an arc, as follows.

- When the arc appears as one arc in the standard presentation, its left (resp., right) end-node is the left (resp., right) endpoint of the arc.
- When the arc appears in two parts linked through the imaginary boundary lines at $x = -1/2$ and $x = g - 1/2$, its left (resp., right) end-node is the right (resp., left) endpoint of the arc.

For $\mathfrak{a} \in \mathfrak{B}^r_{\mathtt{S}}$, we use $\ell(\mathfrak{a})$ to denote the *total span* of $\mathfrak{a}$, that is, the sum of the span of all curves over the band, where the span takes into account the periodicity at the imaginary boundary. For example, the last element of $\mathfrak{B}^2_{\mathtt{S}}$ in (3.1.1) has two arcs with spans 1 and 5, respectively, and hence its total span is 6. The second element of $\mathfrak{B}^2_{\mathtt{S}}$ in (3.1.1) has two arcs with spans 1 and 2, respectively, and hence its total span is 3.

Remark 3.2

We chose the graphic presentation of semimeanders because it is intuitive, but one might argue that it lacks rigorousness (if one holds the highest standard). We point out that there are more abstract definitions of semimeanders which make our argument rigorous. For example, one of the anonymous referees kindly suggested to us the following definition. As in Remark 2.18, we write $\widehat{\mathtt{S}}^c_\infty$ for the preimages of $\mathtt{S}^c_\infty$ under the projection $\mathbb{Z} \to \mathbb{Z}/g\mathbb{Z}$ (and view $\mathtt{S}^c_\infty$ as the subset of $\widehat{\mathtt{S}}^c_\infty \cap \{0, \dots, g-1\}$ consisting of the lifts of its elements). A periodic semimeander is a function $f : \widehat{\mathtt{S}}^c_\infty \to \widehat{\mathtt{S}}^c_\infty$ satisfying

- $f(x+g) = f(x) + g$,
- $f(f(x)) = x$,
- for every $x, y \in \widehat{\mathtt{S}}^c_\infty$ such that $x < y < f(x)$, we have $x < f(y) < f(x)$ and $f(y) \neq y$.

This definition is equivalent to the graphic definition. Indeed, for such a function f, the graph corresponding to f is given as follows. We draw an arc between the node at x and the node at $f(x)$ if $f(x) \neq x$, and we draw a semiline attached to a node at x if $f(x) = x$. With this definition, one can perform most of the combinatorics involved with semimeanders and links in this article, while keeping the argument rigorous. For example, the span of a semimeander $\mathfrak{a}$ (associated to a function f above) is given by $\ell(\mathfrak{a}) = \frac{1}{2}\sum_{x \in \mathtt{S}^c_\infty} |f(x) - x|$. One can check that all of our intuitive descriptions of operations involving periodic semimeanders can be translated to this language and therefore made rigorous.

3.3. Gram matrix

For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{B}^r_{\mathtt{S}}$, we consider the drawing $D(\mathfrak{a}, \mathfrak{b})$ obtained by taking the mirror image of $\mathfrak{b}$ reflected about the x-axis and then identifying the d nodes of $\mathfrak{b}$ with those of $\mathfrak{a}$ according to their labelings.

- We say that a loop (namely, a closed curve) in $D(\mathfrak{a},\mathfrak{b})$ is *contractible* if it can be continuously contracted to a point on the cylinder (ignoring all other curves and lines on the picture). We write $m_0(\mathfrak{a},\mathfrak{b})$ for the number of contractible loops in $D(\mathfrak{a},\mathfrak{b})$.
- We say that a loop in $D(\mathfrak{a},\mathfrak{b})$ is *noncontractible* if, ignoring other curves and lines on the picture, it can be continuously deformed into a loop wrapped around the cylinder. (Since all loops do not intersect themselves, the loop can only wrap the cylinder for one round.) We write $m_T(\mathfrak{a},\mathfrak{b})$ for the number of noncontractible loops in $D(\mathfrak{a},\mathfrak{b})$. This number can be nonzero only when $r = d/2$.
- We use $\mathtt{S}_\mathfrak{a}$ to denote the union of $\mathtt{S}$ with the nodes that are connected to some arc of $\mathfrak{a}$. So the band of $\mathtt{S}_\mathfrak{a}$ may be obtained from the band of $\mathfrak{a}$ by replacing the end-nodes of arcs in $\mathfrak{a}$ with plus signs. We define $\mathtt{S}_\mathfrak{b}$ similarly.
- Assume that $r < d/2$, that neither two semilines of $\mathfrak{a}$ nor two semilines of $\mathfrak{b}$ are connected together in $D(\mathfrak{a},\mathfrak{b})$. We define the *reduction* of $D(\mathfrak{a},\mathfrak{b})$ to be a link $\eta_{\mathtt{S}_\mathfrak{a},\mathtt{S}_\mathfrak{b}}$ from the band of $\mathtt{S}_\mathfrak{a}$ to the band of $\mathtt{S}_\mathfrak{b}$ such that each node $\tau_\mathfrak{a}$ of $\mathtt{S}_\mathfrak{a}$ (corresponding to an element of $\mathtt{S}^c_{\mathfrak{a},\infty}$) is linked to a node $\tau_\mathfrak{b}$ of $\mathtt{S}_\mathfrak{b}$ in the same way as the semiline at $\tau_\mathfrak{a}$ is linked to the semiline at $\tau_\mathfrak{b}$ in $D(\mathfrak{a},\mathfrak{b})$. In practice, this amounts to removing all the (contractible) loops in $D(\mathfrak{a},\mathfrak{b})$, and then continuously deforming the remaining curves into a link (with top and bottom extended by semilines). We put $m_v(\mathfrak{a},\mathfrak{b})$ to be the total displacement of $\eta_{\mathtt{S}_\mathfrak{a},\mathtt{S}_\mathfrak{b}}$.
- When $r = \frac{d}{2}$, $\mathtt{S}_\mathfrak{a} = \mathtt{S}_\mathfrak{b}$ contains all the Archimedean places. For consistency, we write $\eta_{\mathtt{S}_\mathfrak{a},\mathtt{S}_\mathfrak{b}}$ for the trivial link from the band of $\mathtt{S}_\mathfrak{a}$ to the band of $\mathtt{S}_\mathfrak{b}$ (as there are no nodes on the bands).

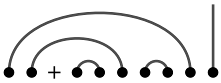
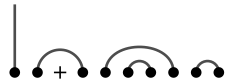
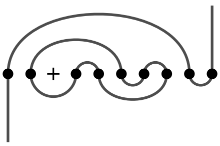
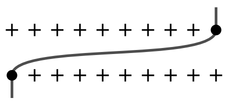
We define the *Gram product* to be the following pairing:

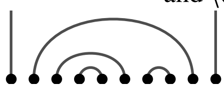
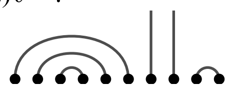
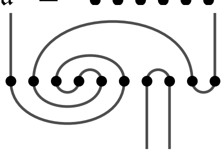
$$\langle\cdot\,|\,\cdot\rangle_{\mathtt{S}} : \mathfrak{B}^r_{\mathtt{S}} \times \mathfrak{B}^r_{\mathtt{S}} \longrightarrow \begin{cases} \overline{\mathbb{Q}}_\ell(v) & \text{if } r < d/2, \\ \overline{\mathbb{Q}}_\ell[T] & \text{if } r = d/2. \end{cases}$$

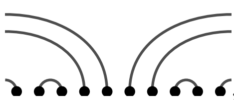
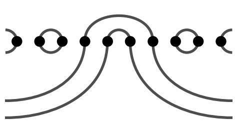
$$\langle\mathfrak{a}|\mathfrak{b}\rangle_{\mathtt{S}} = \begin{cases} 0 & \text{if in the diagram } D(\mathfrak{a},\mathfrak{b}), \text{two semilines} \\ & \text{of } \mathfrak{a} (\text{or of } \mathfrak{b}) \text{ are connected, then} \\ (-2)^{m_0(\mathfrak{a},\mathfrak{b})} v^{m_v(\mathfrak{a},\mathfrak{b})} & \text{otherwise if } r < d/2, \text{and} \\ (-2)^{m_0(\mathfrak{a},\mathfrak{b})} T^{m_T(\mathfrak{a},\mathfrak{b})} & \text{otherwise if } r = d/2. \end{cases}$$

Note that only one of $m_v(\mathfrak{a},\mathfrak{b})$ and $m_T(\mathfrak{a},\mathfrak{b})$ can be nonzero by definition. We use $\mathfrak{V}^r_{\mathtt{S}}$ to denote the $\overline{\mathbb{Q}}_\ell$-vector space with basis $\mathfrak{B}^r_{\mathtt{S}}$ and extend the Gram product linearly to all of $\mathfrak{V}^r_{\mathtt{S}}$.

Example 3.4
The following examples are copied from [24].

(1) $\mathfrak{a}$ = , $\mathfrak{b}$ = , $D(\mathfrak{a},\mathfrak{b})$ =
, the reduction of the link is $\eta_{\mathtt{S}_\mathfrak{a},\mathtt{S}_\mathfrak{b}}$ =
, and $\langle \mathfrak{a}|\mathfrak{b}\rangle_{\mathtt{S}} = (-2)v^{-9}$.

(2) $\mathfrak{a}$ = , $\mathfrak{b}$ = , $D(\mathfrak{a},\mathfrak{b})$ =
, and $\langle \mathfrak{a} \mid \mathfrak{b}\rangle_{\mathtt{S}} = 0$.

(3) $\mathfrak{a}$ = , $\mathfrak{b}$ = , and $D(\mathfrak{a},\mathfrak{b})$ =
, and $\langle \mathfrak{a}|\mathfrak{b}\rangle_{\mathtt{S}} = (-2)^3T^2$.

Remark 3.5
When $\mathtt{S} = \emptyset$, the vector space $\mathfrak{V}^r_{\mathtt{S}}$ is the *link representation* of the so-called *periodic Temperley–Lieb algebra* $\mathcal{E}TLP_N(T,-2)$ under the notation of [24]. (In particular, we specialize the theory to the case when the quantum variable $q = i$.) With respect to the bilinear form we introduced earlier, the representation is †-Hermitian with respect to the natural involution † on the Temperley–Lieb algebra. Since we will not use the structure of this representation, we simply refer to [24, Section 2.3] for further discussion. It seems that the mysterious relationship between this mathematical physics calculation and our Shimura variety calculation probably comes from some common representation theory feature. It might be an intriguing question to ask what quantization could mean for Shimura varieties (or its local analogues) so that the intersection matrix computed in a similar manner as ours would have a chance to match the quantized version of the Gram determinant in [24]

The following theorem is essentially the main theorem of [24] (which seems to have been known by [10] using a different argument).

THEOREM 3.6
Put $t_{d,r} = \sum_{i=0}^{r-1} \binom{d}{i}$. *Let* $\mathfrak{G}_{\mathtt{S}}^r$ *denote the Gram matrix* $(\langle \mathfrak{a} \mid \mathfrak{b} \rangle)_{\mathfrak{a},\mathfrak{b}\in\mathfrak{B}_{\mathtt{S}}^r}$. *Then its determinant is given as follows.*

(1) *When* d *is even,* $\det \mathfrak{G}_{\mathtt{S}}^{d/2} = \pm(T^2-4)^{t_{d,d/2}}$.

(2) *For* $r < d/2$, $\det \mathfrak{G}_{\mathtt{S}}^r = \pm(v^g - v^{-g})^{2t_{d,r}}$.

Proof
When $\mathtt{S} = \emptyset$ (so $d = g$), this is a special case of [24, Theorem 4.1]. Indeed, the parameter α in that theorem is T in our notation, and since its β is -2, its C_k are equal to ± 1 for all k. One easily simplifies the formula from [24] to the one stated in this theorem.

The general case requires little modification, but the method of the proof may be viewed as a toy model for the proof of Theorem 4.5 later. When $r = \frac{d}{2}$, we just simply ignore all points corresponding to $\mathtt{S}_\infty$. This verifies item (1). So we assume that $r < \frac{d}{2}$ from now on to prove item (2). We use $\langle \mathfrak{a}|\mathfrak{b}\rangle_d$ to denote the pairing computed by removing all points from $\mathtt{S}_\infty$ (and shrink the cylinder accordingly) and hence with displacements computed with respect to only the d nodes. Let $\mathfrak{G}_d^r$ denote the corresponding matrix. Then [24, Theorem 4.1] implies that $\det \mathfrak{G}_d^r = \pm(v^d - v^{-d})^{2t_{d,r}}$. We need to compare $\det \mathfrak{G}_d^r$ with $\det \mathfrak{G}_{\mathtt{S}}^r$, by showing that $\det \mathfrak{G}_{\mathtt{S}}^r$ can be obtained by replacing all v^d in the expression of $\det \mathfrak{G}_d^r$ by v^g.

By the definition of determinant, $\det \mathfrak{G}_{\mathtt{S}}^r$ is the sum over all permutations s of the set $\mathfrak{B}_{\mathtt{S}}^r$, of the product of the sign $\mathrm{sgn}(s)$ and, for every cycle $(\mathfrak{a}_1 \dots \mathfrak{a}_t)$ in s, the product

$$\langle \mathfrak{a}_1|\mathfrak{a}_2\rangle_{\mathtt{S}} \cdot \langle \mathfrak{a}_2|\mathfrak{a}_3\rangle_{\mathtt{S}} \cdots \langle \mathfrak{a}_t|\mathfrak{a}_1\rangle_{\mathtt{S}}. \tag{3.6.1}$$

The same applies to $\det \mathfrak{G}_d^r$, except that the product (3.6.1) is taken for the pairing $\langle\cdot|\cdot\rangle_d$. The product (3.6.1), if not zero, is equal to $(-2)^{m_0} v^{m_v}$, where $m_0 = m_0(\mathfrak{a}_1,\mathfrak{a}_2) + \cdots + m_0(\mathfrak{a}_t,\mathfrak{a}_1)$ is the sum of the total number of contractible loops in the diagrams $D(\mathfrak{a}_1,\mathfrak{a}_2), D(\mathfrak{a}_2,\mathfrak{a}_3), \dots, D(\mathfrak{a}_t,\mathfrak{a}_1)$, and $m_v = m_v(\mathfrak{a}_1,\mathfrak{a}_2) + \cdots + m_v(\mathfrak{a}_t,$
$\mathfrak{a}_1)$ is equal to the total displacement of the composition of the link

$$\eta_{\mathtt{S}_{\mathfrak{a}_t},\mathtt{S}_{\mathfrak{a}_1}} \circ \cdots \circ \eta_{\mathtt{S}_{\mathfrak{a}_2},\mathtt{S}_{\mathfrak{a}_3}} \circ \eta_{\mathtt{S}_{\mathfrak{a}_1},\mathtt{S}_{\mathfrak{a}_2}}, \tag{3.6.2}$$

by the additivity of total displacements as remarked in Section 2.15. Note that (3.6.2) is in fact a link from $\mathtt{S}_{\mathfrak{a}_1}$ to itself. So it must be an integer power n of the fundamental link $\eta_{\mathtt{S}_{\mathfrak{a}_1}}$ defined in Section 2.15. In particular, we have $m_v = ng$. Making the same observation for computing the standard Gram determinant $\det \mathfrak{G}^r_d$, the product (3.6.1) with $\langle\cdot|\cdot\rangle_d$ is instead equal to $(-2)^{m_0}v^{m'_v}$ with the same m_0 as above, and m'_v is the total displacement of (3.6.2) with all points corresponding to $\mathtt{S}_\infty$ removed. By the same discussion above, we have $m'_v = nd$ with the *same* n above. In conclusion, each term of $\det \mathfrak{G}^r_{\mathtt{S}}$ can be obtained from the corresponding term of $\det \mathfrak{G}^r_d$ via replacing v^d by v^g. Therefore, $\det \mathfrak{G}^r_{\mathtt{S}} = \pm(v^g - v^{-g})^{2t_{d,r}}$. □

Notation 3.7

Using the illustration of periodic semimeanders as in (3.1.1), we say that an arc δ *lies over* another arc δ' if the contractible closed loop in the picture given by adjoining δ with the equator contains δ' inside. For example, in the list of $\mathfrak{B}^2_{\mathtt{S}}$ in (3.1.1), the last five periodic semimeanders each has an arc lying over another.

In a periodic semimeander for $\mathtt{S}$, a *basic arc* is an arc δ which satisfies the following equivalent conditions:

- in the 2-dimensional picture, δ does not lie over any other arcs,
- in the 2-dimensional picture, the only points below δ are plus signs,
- δ is an arc which links some τ to τ^- (see Notation 2.16 for the notation).

For example, in the list of $\mathfrak{B}^2_{\mathtt{S}}$ in (3.1.1), each of the five periodic semimeanders in the first row has two basic arcs, and each of the five periodic semimeanders in the second row has one basic arc.

It is clear that every periodic semimeander has at least one basic arc, except the one with only semilines. Given a periodic semimeander $\mathfrak{a} \in \mathfrak{B}^r_{\mathtt{S}}$ for $\mathtt{S}$ with a basic arc δ linking two nodes $\tau, \tau^- \in \mathtt{S}^c_\infty$, we can delete the arc and replace its end-nodes by $+$ to get a periodic semimeander $\mathfrak{a}\backslash\delta \in \mathfrak{B}^{r-1}_{\mathtt{S}\cup\{\tau,\tau^-\}}$ for $\mathtt{S} \cup \{\tau, \tau^-\}$.

3.8. Goren–Oort cycles

We fix a pair $(\mathtt{S}, \mathtt{T})$ as before. For a periodic semimeander $\mathfrak{a}$ for $\mathtt{S}$ with r arcs, we define a pair $(\mathtt{S}_\mathfrak{a}, \mathtt{T}_\mathfrak{a})$ as follows: $\mathtt{S}_\mathfrak{a}$ is obtained by adjoining to $\mathtt{S}$ all end-nodes of the arcs of $\mathfrak{a}$ and $\mathtt{T}_\mathfrak{a}$ is obtained by adjoining to $\mathtt{T}$ all the *right* end-nodes (in the sense of Section 3.1) of the arcs of $\mathfrak{a}$.

We now construct the *Goren–Oort cycle* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a}$ associated to a periodic semimeander $\mathfrak{a}$ for $(\mathtt{S}, \mathtt{T})$.[14] Then the cycle will admit an r-step iterated $\mathbb{P}^1$-bundle morphism,

$$\pi_\mathfrak{a}\colon\ \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a} \to \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}).$$

The resulting correspondence

[14]It is expected that the Goren–Oort cycle $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a}$ is independent of the auxiliary choices for the definition of the unitary Shimura variety $\mathrm{Sh}_{K''_p}(G''_{\tilde{\mathtt{S}}})$. However, we do not know how to prove this.

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}) \xleftarrow{\pi_\mathfrak{a}} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a} \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \tag{3.8.1}$$

will be constructed using the unitary Shimura varieties via Construction 2.12 depending on a shift $\boldsymbol{t}_\mathfrak{a} = \boldsymbol{t}_{\emptyset,\mathfrak{a}} \in E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_\mathfrak{p}}^\times$, which is canonical up to $F^{\times,\mathrm{cl}}\backslash\mathbb{A}_F^{\infty,\times}/\mathcal{O}_{F_\mathfrak{p}}^\times$, as explained in Construction 2.12.

We proceed by induction on $r \geq 0$ to define the Goren–Oort cycle correspondence (3.8.1). For $r = 0$, we define $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a} := \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$. Suppose that $r \geq 1$ and that we have defined, for every $0 \leq t < r$ and every $\mathfrak{a} \in \mathfrak{B}_\mathtt{S}^{r-t}$, the Goren–Oort cycles $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})_\mathfrak{b}$ together with a correspondence

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a},\mathfrak{b}},\mathtt{T}_{\mathfrak{a},\mathfrak{b}}}) \xleftarrow{\pi_\mathfrak{b}} \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})_\mathfrak{b} \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})$$

with some shift $\boldsymbol{t}_\mathfrak{b}$ associated to all the periodic semimeanders $\mathfrak{b} \in \mathfrak{B}_{\mathtt{S}_\mathfrak{a}}^t$.

We now define the Goren–Oort cycle $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a}$ associated to every $\mathfrak{a} \in \mathfrak{B}_\mathtt{S}^r$. For this, we fix a basic δ as in Notation 3.7 with end-nodes τ and τ^-. Set $\mathtt{S}_\delta = \mathtt{S} \cup \{\tau, \tau^-\}$ and $\mathtt{T}_\delta = \mathtt{T} \cup \{\tau\}$. Then Proposition 2.31(1) implies that we have a natural correspondence

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta}) \xleftarrow{\pi_\delta} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\tau \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$$

with shift $\boldsymbol{t}_\delta$ (which we fix).

Let $\mathfrak{a}\backslash\delta$ denote the periodic semimeander for $\mathtt{S}_\delta$ obtained by removing the arc δ from $\mathfrak{a}$ and replacing the nodes at τ and τ^- by plus signs. So the induction hypothesis gives a correspondence,

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}) \xleftarrow{\pi_{\mathfrak{a}\backslash\delta}} \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta})_{\mathfrak{a}\backslash\delta} \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta}) \tag{3.8.2}$$

with shift $\boldsymbol{t}_{\mathfrak{a}\backslash\delta}$, where $\pi_{\mathfrak{a}\backslash\delta}$ is an $(r-1)$-step iterated $\mathbb{P}^1$-bundle. We define the *Goren–Oort cycle* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a}$ to be

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a} := \pi_\delta^{-1}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta})_{\mathfrak{a}\backslash\delta}\big);$$

namely, it fits into the following commutative diagram where the square is Cartesian:

$$\begin{array}{ccccc}
\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\mathfrak{a} & \hookrightarrow & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_\delta & \hookrightarrow & \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}) \\
\downarrow & & \downarrow \pi_\delta & & \\
\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta})_{\mathfrak{a}\backslash\delta} & \hookrightarrow & \mathrm{Sh}_{K_p}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta}) & & \\
\downarrow \pi_{\mathfrak{a}\backslash\delta} & & & & \\
\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}) & & & &
\end{array}$$

The induced correspondence

$$\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \xleftarrow{\pi_{\mathfrak{a}}:=\pi_{\mathfrak{a}\backslash\delta}\circ\pi_{\delta}} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}} \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$$

has shift

$$\boldsymbol{t}_{\mathfrak{a}} := \boldsymbol{t}_{\delta} \cdot \boldsymbol{t}_{\mathfrak{a}\backslash\delta}.$$

This completes the inductive construction of the Goren–Oort cycles. Using Theorem 2.32(1), it is easy to see inductively that such a definition of $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}$ does not depend on the choice of the basic arc δ. (We point out that a key feature of our construction is that *the dimension of fibers of* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}$ *over* $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$ *is the same as the codimension of* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}$ *in* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$, which is r.)

We fix a regular multiweight $(\underline{k}, w)$. Recall that $\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}$ denotes the automorphic ℓ-adic local system on $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$. The same construction above also gives rise to a natural isomorphism,

$$\pi^{\sharp}_{\mathfrak{a}}\colon \pi^{*}_{\mathfrak{a}}(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \xrightarrow{\cong} \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}|_{\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}}.$$

Remark 3.9
It was pointed out to us by X. Zhu that the union of all Goren–Oort cycles associated to periodic semimeanders with r arcs is exactly the closure of certain *Newton strata* of the unitary Shimura variety, transported to the quaternionic side. In the case of Hilbert modular varieties, the union of all codimension r generalized Goren–Oort cycles are exactly the closed Newton stratum associated to the Newton polygon with slopes $\frac{r}{g}$ and $\frac{g-r}{g}$, both with multiplicity g. So maybe the name "Goren–Oort" is slightly misleading, as it usually refers to the stratification given by the p-torsion subgroup of the universal Abelian varieties.

Example 3.10
Let F be of degree 6 over $\mathbb{Q}$ and $\mathtt{S} = \mathtt{T} = \emptyset$. Then $\mathrm{Sh}_K(G_{\emptyset,\emptyset})$ is (the special fiber of) the Hilbert modular variety for F. Let $\tau_0, \ldots, \tau_5$ denote the embeddings of $\mathcal{O}_F$ into $\mathbb{Z}_p^{\mathrm{ur}}$ so that $\tau_i = \tau_{i \pmod 6}$ and $\tau_{i+1} = \sigma\tau_i$. We have a universal Abelian variety A over $\mathrm{Sh}_K(G_{\emptyset,\emptyset})$ equipped with an $\mathcal{O}_F$-action.

We consider the periodic semimeander $\mathfrak{a} =$ . For each $\overline{\mathbb{F}}_p$-point $x \in \mathrm{Sh}_K(G_{\emptyset,\emptyset})$, the covariant Dieudonné module $\mathcal{D}_x$ of the universal Abelian variety A_x at x decomposes as $\mathcal{D}_x = \bigoplus_{i=0}^{5} \mathcal{D}_{x,i}$, where $\mathcal{O}_F$ acts on the ith factor via τ_i. Let $V_i\colon \mathcal{D}_{x,i+1} \to \mathcal{D}_{x,i}$ denote the Verschiebung map for $i \in \mathbb{Z}/5\mathbb{Z}$. Then $x \in \mathrm{Sh}_K(G_{\emptyset,\emptyset})_{\mathfrak{a}}$ if and only if

$$V_1 \circ V_2(\mathcal{D}_{x,3}) \subseteq p\mathcal{D}_{x,1}, \qquad V_4 \circ V_5(\mathcal{D}_{x,0}) \subseteq p\mathcal{D}_{x,4}, \qquad \text{and}$$
$$V_0 \circ V_1 \circ V_2 \circ V_3(\mathcal{D}_{x,4}) \subseteq p^2\mathcal{D}_{x,0}.$$

In fact, these inclusions are forced to be equalities. In this case, $\mathrm{Sh}_{K_p}(G_{\emptyset,\emptyset})_{\mathfrak{a}}$ is just a three-step iterated $\mathbb{P}^1$-bundle over the discrete Shimura variety $\mathrm{Sh}_K(G_{\Sigma_\infty,\{\tau_2,\tau_3,\tau_5\}})$. Moreover, one can prove that each geometric connected component is isomorphic to the product of $\mathbb{P}^1$ (corresponding to the arc linking τ_4 and τ_5) with the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-p) \oplus \mathcal{O}_{\mathbb{P}^1}(p))$ over $\mathbb{P}^1$ (which corresponds to the two arcs over $\tau_0, \tau_1, \tau_2, \tau_3$. More canonically, the second factor is the projective bundle attached to a rank 2 bundle E over $\mathbb{P}^1$ which sits inside an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(p) \to E \to \mathcal{O}_{\mathbb{P}^1}(-p) \to 0$; this exact sequence splits (noncanonically).

4. Cohomology of Goren–Oort cycles

Let $\mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})$ be the special fiber of a quaternionic Shimura variety as in Section 2.1. Using Gysin maps, the cohomology of the Goren–Oort cycles gives rise to part of the cohomology of the big Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})$.

4.1. Generalities on Gysin maps

We recall first some generalities on Gysin maps. Let ℓ be a fixed prime number, and let k be an algebraically closed field of characteristic different from ℓ.

Consider a closed immersion $i : Y \hookrightarrow X$ of smooth varieties over k of codimension r. The functor of direct image i_* has a right adjoint, denoted by $i^!$. For an ℓ-adic étale sheaf $\mathcal{F}$ on X, $i^!\mathcal{F}$ is the sheaf of sections of $\mathcal{F}$ with support in Y. This is a left exact functor, and let $R^q i^!$ denote its qth derived functor. Then by relative cohomological purity (see [1, XVI, Théorème 3.7]), we have $R^q i^! \overline{\mathbb{Q}}_\ell = 0$ for $q \neq 2r$, and a canonical isomorphism $R^{2r} i^! \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \overline{\mathbb{Q}}_\ell(-r)$. Explicitly, the inverse isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\cong} R^{2r} i^! \overline{\mathbb{Q}}_\ell(r)$ is given by the fundamental class of Y in X: $\mathrm{cl}_X(Y) \in H^{2r}_{\mathrm{et},Y}(X, \overline{\mathbb{Q}}_\ell(r)) \cong H^0_{\mathrm{et}}(Y, R^{2r} i^! \overline{\mathbb{Q}}_\ell)$. Now, for any lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X, we define the Gysin map as the composition

$$\mathrm{Gysin}\colon H^q_{\mathrm{et}}(Y, i^*\mathcal{F}) \xrightarrow{\cup \mathrm{cl}_X(Y)} H^{q+2r}_{\mathrm{et},Y}\big(X, \mathcal{F}(r)\big) \to H^{q+2r}_{\mathrm{et}}\big(X, \mathcal{F}(r)\big), \tag{4.1.1}$$

where the second map is the canonical morphism from cohomology supported in Y to the usual cohomology group. If $i_Z\colon Z \hookrightarrow X$ is another closed immersion of smooth varieties such that Y intersects with Z transversally, then one has $i_Z^* \mathrm{cl}_X(Y) = \mathrm{cl}_Z(Y \cap Z)$. It follows that the following diagram is commutative:

$$\begin{array}{ccc} H^q_{\mathrm{et}}(Y, i^*\mathcal{F}) & \xrightarrow{\mathrm{Gysin}} & H^{q+2r}_{\mathrm{et}}\big(X, \mathcal{F}(r)\big) \\ \downarrow{\scriptstyle \mathrm{Restr.}} & & \downarrow{\scriptstyle \mathrm{Restr.}} \\ H^q_{\mathrm{et}}(Y \cap Z, i^*_{Y\cap Z}\mathcal{F}) & \xrightarrow{\mathrm{Gysin}} & H^{q+2r}_{\mathrm{et}}\big(Z, i_Z^*\mathcal{F}(r)\big) \end{array} \tag{4.1.2}$$

where the vertical maps are given by natural restrictions, and $i_{Y\cap Z}\colon Y\cap Z\hookrightarrow X$ is the natural embedding.

4.2. *Étale cohomology of iterated $\mathbb{P}^1$-bundles*

We continue to assume that k is an algebraically closed field of characteristic different from ℓ. Let $\pi\colon X\to Y$ be an r-step iterated $\mathbb{P}^1$-bundle of proper and smooth k-varieties; that is, π admits a factorization

$$\pi\colon X_0:=X\xrightarrow{\pi_1}X_1\xrightarrow{\pi_2}X_2\to\cdots\xrightarrow{\pi_r}X_r:=Y, \tag{4.2.1}$$

where each $\pi_i\colon X_{i-1}\to X_i$ is a $\mathbb{P}^1$-bundle for $1\le i\le r$. Then the trace map

$$\mathrm{Tr}_\pi\colon R^{2r}\pi_*\big(\overline{\mathbb{Q}}_\ell(r)\big)\xrightarrow{\cong}\overline{\mathbb{Q}}_\ell$$

is an isomorphism. We denote by $\mathrm{cl}_\pi\in H^0(Y,R^{2r}\pi_*\overline{\mathbb{Q}}_\ell(r))$ with $\mathrm{Tr}_\pi(\mathrm{cl}_\pi)=1$, and call it the *fundamental class of the fibration* π. For any $\overline{\mathbb{Q}}_\ell$-lisse sheaf $\mathcal{F}$ on Y and any integer $q\ge 0$, the isomorphism Tr_π induces a map,

$$\pi_!\colon H^q_{\mathrm{et}}\big(X,\pi^*\mathcal{F}(r)\big)\to H^{q-2r}_{\mathrm{et}}\big(Y,\mathcal{F}\otimes R^{2r}\pi_*\big(\overline{\mathbb{Q}}_\ell(r)\big)\big)\xrightarrow{\mathrm{Tr}_\pi}H^{q-2r}_{\mathrm{et}}(Y,\mathcal{F}), \tag{4.2.2}$$

where the first morphism comes from the Leray spectral sequence $E_2^{a,b}=H^a_{\mathrm{et}}(Y,R^b\pi_*\pi^*\mathcal{F}(r))\Rightarrow H^{a+b}_{\mathrm{et}}(X,\pi^*\mathcal{F}(r))$. Explicitly, $\pi_!$ admits the following description. Put $\pi_{[0,i]}:=\pi_i\circ\pi_{i-1}\circ\cdots\circ\pi_1$ for $1\le i\le r$. Let $\mathcal{O}_{\pi_i}(1)$ be the tautological quotient line bundle of the $\mathbb{P}^1$-bundle π_i, and let $c_1(\mathcal{O}_{\pi_i}(1))\in H^2_{\mathrm{et}}(X_{i-1},\overline{\mathbb{Q}}_\ell(1))$ be its first Chern class. Put $\xi_i=\pi^*_{[0,i-1]}c_1(\mathcal{O}_{\pi_i}(1))\in H^2_{\mathrm{et}}(X,\overline{\mathbb{Q}}_\ell(1))$. By induction on r, one deduces easily from [11, VII, Corollaire 2.2.6] a decomposition:

$$H^q_{\mathrm{et}}\big(X,\pi^*\mathcal{F}(r)\big)\cong\bigoplus_{0\le j\le r}\Big(\bigoplus_{1\le i_1<\cdots<i_j\le r}\pi^*H^{q-2j}_{\mathrm{et}}\big(Y,\mathcal{F}(r-j)\big)\cup\xi_{i_1}\cup\cdots\cup\xi_{i_j}\Big).$$

Then, for an element $x=\sum_j\sum_{1\le i_1<\cdots<i_j\le r}\pi^*(y_{i_1,\dots,i_j})\cup\xi_{i_1}\cup\cdots\cup\xi_{i_j}$, one has

$$\pi_!(x)=y_{1,\dots,r}. \tag{4.2.3}$$

In particular, the fundamental class cl_π is the image of $\xi_1\cup\cdots\cup\xi_r$ in $H^0_{\mathrm{et}}(X,R^{2r}\pi_*\overline{\mathbb{Q}}_\ell(r))$.

4.3. *Gysin and restriction maps*

We keep the notation of Section 3.8. The pair of morphisms $(\pi_{\mathfrak{a}},\pi^\sharp_{\mathfrak{a}})$ induces the following sequence of natural homomorphisms, whose composition we denote by $\mathrm{Gys}_{\mathfrak{a}}$:

$$
\begin{aligned}
&H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}}\big)\\
&\quad\xrightarrow{\pi_{\mathfrak a}^*,\cong} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a,\overline{\mathbb F}_p},\pi_{\mathfrak a}^*(\mathcal L^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}})\big)\\
&\quad\xrightarrow{\pi_{\mathfrak a}^\sharp,\cong} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a,\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}|_{\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a}}\big)\\
&\quad\xrightarrow{\text{Gysin},(4.1.1)} H^{d}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}(r)\big).
\end{aligned}
$$

We can also consider the dual picture, defining the morphism $\mathrm{Res}_{\mathfrak a}$ to be the composition of the following homomorphisms:

$$
\begin{aligned}
\mathrm{Res}_{\mathfrak a}\colon\ &H^{d}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}(r)\big)\\
&\xrightarrow{\text{Restriction}} H^{d}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a,\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}|_{\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a}}(r)\big)\\
&\xrightarrow{(\pi_{\mathfrak a}^\sharp)^{-1},\cong} H^{d}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a,\overline{\mathbb F}_p},\pi_{\mathfrak a}^*\mathcal L^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}}(r)\big)\\
&\xrightarrow{\pi_{\mathfrak a,!},(4.2.2)} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}}\big).
\end{aligned}
$$

It is clear from the construction that both morphisms $\mathrm{Gys}_{\mathfrak a}$ and $\mathrm{Res}_{\mathfrak a}$ are equivariant for the prime-to-p Hecke action of $G(\mathbb A^{\infty,p})$.

The following theorem is the key to proving our main result. We defer its proof to the next section.

THEOREM 4.4

Fix $\pi\in\mathcal A_{(\underline{k},w)}$, and fix a choice of system of shifts $\boldsymbol t_{\mathfrak a}$ of the correspondences $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}})\xleftarrow{\pi_{\mathfrak a}}\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak a}\hookrightarrow\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ as in Section 3.8. For $\mathfrak a,\mathfrak b\in\mathfrak B^r_{\mathtt S}$, we have the following description of the composition:

$$
\begin{aligned}
H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak b},\mathtt{T}_{\mathfrak b}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak b},\mathtt{T}_{\mathfrak b}}\big)&\xrightarrow{\mathrm{Gys}_{\mathfrak b}} H^{d}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}(r)\big)\\
&\xrightarrow{\mathrm{Res}_{\mathfrak a}} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak a},\mathtt{T}_{\mathfrak a}}\big).
\end{aligned}
$$

(1) *When $\langle\mathfrak a|\mathfrak b\rangle=0$, the π-isotypical component of the composed map $\mathrm{Res}_{\mathfrak a}\circ\mathrm{Gys}_{\mathfrak b}$ factors through the π-isotypical component of the cohomology group $H^{d-2(r+1)}_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb F}_p},\mathcal L^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'})(-1)$ of some quaternionic Shimura variety of dimension $d-2(r+1)$ with $\#\mathtt{T}'=\#\mathtt{T}+(r+1)$ and $\mathtt{S}'$ having the same set of finite places as $\mathtt{S}$.*

(2) *When $r<\frac d2$ and $\langle\mathfrak a|\mathfrak b\rangle=(-2)^{m_0(\mathfrak a,\mathfrak b)}v^{m_v(\mathfrak a,\mathfrak b)}$, we can define the induced link $\eta_{\mathtt{S}_{\mathfrak a},\mathtt{S}_{\mathfrak b}}\colon\mathtt{S}_{\mathfrak a}\to\mathtt{S}_{\mathfrak b}$ as in Section 3.3. Then there exists a normalized link morphism in the sense of Section 2.28,*

$$\eta^{\star}_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(z)}\colon H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}}\big) \to H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}\big),$$

associated to $\eta_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}}}$ *with shift* $\boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}_{\mathfrak{b}}^{-1}$ *and indentation degree*

$$z=\begin{cases}\ell(\mathfrak{a})-\ell(\mathfrak{b}) & \text{if } \mathfrak{p} \text{ splits in } E/F,\\ 0 & \text{if } \mathfrak{p} \text{ is inert in } E/F.\end{cases}$$

Moreover, we have an equality:

$$\mathrm{Res}_{\mathfrak{a}}\circ\mathrm{Gys}_{\mathfrak{b}}=(-2)^{m_0(\mathfrak{a},\mathfrak{b})}\cdot p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}\cdot\eta^{\star}_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(z)}.$$

(3) *When* $r=\frac{d}{2}$ *and* $\langle\mathfrak{a}|\mathfrak{b}\rangle=(-2)^{m_0(\mathfrak{a},\mathfrak{b})}T^{m_T(\mathfrak{a},\mathfrak{b})}$*, we have*

$$\mathrm{Res}_{\mathfrak{a}}\circ\mathrm{Gys}_{\mathfrak{b}}=(-2)^{m_0(\mathfrak{a},\mathfrak{b})}\cdot p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}\cdot(T_{\mathfrak{p}}/p^{g/2})^{m_T(\mathfrak{a},\mathfrak{b})}\circ\eta^{\star}_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(z)},$$

where $\eta_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}}}$ *is the trivial link from* $\mathtt{S}_{\mathfrak{a}}$ *to* $\mathtt{S}_{\mathfrak{b}}$ *and*

$$\eta^{\star}_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(z)}\colon H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big) \longrightarrow H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p},\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)$$

is the associated normalized link morphism with shift $\boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}_{\mathfrak{b}}^{-1}\varpi_{\bar{\mathfrak{q}}}^{-m_T(\mathfrak{a},\mathfrak{b})}$ *and indentation degree* $z=\ell(\mathfrak{a})-\ell(\mathfrak{b})-m_T(\mathfrak{a},\mathfrak{b})g$.

We now assume Theorem 4.4 and deduce the following main theorem of this article.

THEOREM 4.5

Fix a positive integer $r\leq\frac{d}{2}$.

(1) *For each periodic semimeander* $\mathfrak{a}\in\mathfrak{B}^r_{\mathtt{S}}$*, the Goren–Oort cycle* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}$ *of the Shimura variety* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ *is a subvariety of codimension* r*, stable under the action of the tame Hecke action of* $G(\mathbb{A}^{\infty,p})$*. Moreover, it admits a natural* $G(\mathbb{A}^{\infty,p})$*-equivariant* r*-step iterated* $\mathbb{P}^1$*-bundle morphism,*

$$\pi_{\mathfrak{a}}:\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}\to\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}),$$

to another quaternionic Shimura variety (in characteristic p*).*

(2) *We fix a cuspidal automorphic representation* $\pi\in\mathcal{A}_{(\underline{k},w)}$ *appearing in the cohomology of* $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ *so that its associated Galois representation* ρ_π *is unramified at* p*. Let* α_π *and* β_π *denote the (generalized) eigenvalues of*

$\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_{p^g})$. *Suppose that* α_π/β_π *is not a* $2n$*th root of unity for any* $n \leq d$ *so that* $\alpha_\pi^{2i}\beta_\pi^{2(d-i)}$ *are distinct from each other for* $1 \leq i \leq d$. *Then the action of* $\mathrm{Frob}_{p^{2g}}$ *on the generalized eigenspace of* $H^d_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathrm{S,T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathrm{S,T}}(r))[\pi]$ *with eigenvalue* $\alpha_\pi^{2(d-r)}\beta_\pi^{2r}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathrm{T}}p^{-2gr}$ *is semisimple (so that the generalized eigenspace is a genuine eigenspace), and the direct sum of the Gysin morphisms,*

$$\bigoplus_{\mathfrak{a}\in\mathfrak{B}^r_{\mathrm{S}}} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathrm{S,T}})_{\mathfrak{a},\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathrm{S,T}}\big)[\pi] \xrightarrow{\sum_{\mathfrak{a}}\mathrm{Gys}_{\mathfrak{a}}} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathrm{S,T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathrm{S,T}}(r)\big)[\pi], \quad (4.5.1)$$

induces an isomorphism *on the* $\mathrm{Frob}_{p^{2g}}$*-eigenspaces with eigenvalue* $\alpha_\pi^{2(d-r)}\beta_\pi^{2r}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathrm{T}}p^{-2gr}$.

(2′) *Keep the notation in (2) but assume that* $r = \frac{d}{2}$ *(so* d *is even) and* $(\underline{k}, w) = \underline{2}$. *Suppose that* α_π/β_π *is not a* $2n$*th root of unity for* $n \leq \frac{d}{2}$. *Then the* $\mathrm{Frob}_{p^{2g}}$*-invariant subspace of* $H^d_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathrm{S,T}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(\frac{d}{2}))[\pi]$ *is generated by the cycle classes of* $\mathrm{Sh}_{K_p}(G_{\mathrm{S,T}})_{\mathfrak{a}}$ *for* $\mathfrak{a} \in \mathfrak{B}^{d/2}_{\mathrm{S}}$.

Proof

Item (1) follows from the construction of Goren–Oort cycles in Section 3.8. Item (2′) is clearly a special case of item (2). We now focus on the proof of item (2). By Proposition 2.26, the Frobenius semisimplification of the morphism (4.5.1) is the same as

$$\bigoplus_{\mathfrak{a}\in\mathfrak{B}^r_{\mathrm{S}}} \rho_{\pi,\mathfrak{p}}^{\otimes(d-2r)} \otimes \big(\det\rho_{\pi,\mathfrak{p}}(1)\big)^{\otimes(\#\mathrm{T}+r)} \longrightarrow \rho_{\pi,\mathfrak{p}}^{\otimes d} \otimes \big(\det\rho_{\pi,\mathfrak{p}}(1)\big)^{\otimes\#\mathrm{T}}(r). \quad (4.5.2)$$

Thus the generalized eigenspace for the action of $\mathrm{Frob}_{p^{2g}}$ with eigenvalue

$$\alpha_\pi^{2(d-2r)}(\alpha_\pi\beta_\pi/p^g)^{2(\#\mathrm{T}+r)} = \alpha_\pi^{2(d-r)}\beta_\pi^{2r}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathrm{T}}p^{-2gr} \quad (4.5.3)$$

has dimension exactly equal to $\binom{d}{r}$ for both sides of (4.5.1). Thanks to the assumption on the ratio of Satake parameters, the generalized eigenspace on the left-hand side is a genuine eigenspace (since it is the direct sum of $\binom{d}{r}$-copies of 1-dimensional generalized eigenspace). Thus, the proof of (2) and (2′) will be finished if we show that (4.5.1) is injective on the corresponding generalized eigenspace.

We consider the composition of the Gysin morphisms (4.5.1) with the Restriction morphisms:

$$\bigoplus_{\mathfrak{b}\in\mathfrak{B}^r_{\mathrm{S}}} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathrm{S}_{\mathfrak{b}},\mathrm{T}_{\mathfrak{b}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathrm{S}_{\mathfrak{b}},\mathrm{T}_{\mathfrak{b}}}\big)[\pi]$$

$$\xrightarrow{\sum \mathrm{Gys}_{\mathfrak{b}}} H^d_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathscr{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)(r)[\pi]$$

$$\xrightarrow{\oplus \mathrm{Res}_{\mathfrak{a}}} \bigoplus_{\mathfrak{a}\in\mathfrak{B}^r_{\mathtt{S}}} H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p}, \mathscr{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}\big)[\pi]. \tag{4.5.4}$$

Here, we switched the first sum from over $\mathfrak{a}$ (as in (4.5.1)) to over $\mathfrak{b}$. Taking a basis of the generalized eigenspace for $\mathrm{Frob}_{p^{2g}}$ acting on (4.5.4) with the eigenvalue (4.5.3) and using the description in Proposition 2.26, we arrive at the following linear map

$$\bigoplus_{\mathfrak{b}\in\mathfrak{B}^r_{\mathtt{S}}} \overline{\mathbb{Q}}_\ell \to \bigoplus_{\mathfrak{a}\in\mathfrak{B}^r_{\mathtt{S}}} \overline{\mathbb{Q}}_\ell \tag{4.5.5}$$

of vector spaces, which is represented by a $\binom{d}{r} \times \binom{d}{r}$-matrix A with coefficients in $\overline{\mathbb{Q}}_\ell$. The proof of (2) will be finished if we can show that $\det(A)$ is nonzero.

We explain how this matrix A is related to the Gram matrix $\mathfrak{G}^r_{\mathtt{S}}$ for the periodic semimeanders (see Theorem 3.6). Let D be the diagonal matrix, whose $(\mathfrak{a},\mathfrak{a})$-entry with $\mathfrak{a}\in\mathfrak{B}^r_{\mathtt{S}}$ is $p^{-\ell(\mathfrak{a})/2}$. Let B be the product matrix DAD. Then dropping the auxiliary factors $p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}$ from the formulas in Theorem 4.4 gives the entries of B. We will prove that

$$\det B = \begin{cases} \det \mathfrak{G}^{d/2}_{\mathtt{S}}|_{T^2=T^{\mathfrak{n}}_p}, & \text{if } r = d/2,\\ \det \mathfrak{G}^{r}_{\mathtt{S}}|_{v^g=\eta^\star_{\mathrm{univ}}}, & \text{if } r < d/2,\end{cases}$$

where $|_{T^2=T^{\mathfrak{n}}_p}$ and $|_{v^g=\eta^\star_{\mathrm{univ}}}$ are formal substitutions, and $T^{\mathfrak{n}}_p$ and $\eta^\star_{\mathrm{univ}}$ are some formal symbols we define later.

We first compare the entries of B with the entries of $\mathfrak{G}^r_{\mathtt{S}}$ when $\langle\mathfrak{a}|\mathfrak{b}\rangle = 0$. In this case, by Theorem 4.4(1), the π-isotypical component of $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ factors through

$$H^{d-2(r+1)}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})_{\overline{\mathbb{F}}_p}, \mathscr{L}^{(\underline{k},w)}_{\mathtt{S}',\mathtt{T}'}\big)(-1)[\pi]$$

for some quaternionic Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathtt{S}',\mathtt{T}'})$ of dimension $d-2(r+1)$. By Proposition 2.26, the $\mathrm{Frob}_{p^{2g}}$-eigenvalues on this cohomology group are $\alpha_\pi^{2(d-j)} \times \beta_\pi^{2j}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathtt{T}}p^{-2gr}$ with $j = r+1,\dots,d-r-1$. Thanks to the assumption on the ratio of Satake parameters, we see that $\alpha_\pi^{2(d-j)}\beta_\pi^{2j}$ for $j=0,\dots,d$ are distinct. The above list of eigenvalues does not contain (4.5.3). Thus the $(\mathfrak{a},\mathfrak{b})$-entry of B is zero.

Next, we separate the discussion for $r < \frac{d}{2}$ and $r = \frac{d}{2}$. Suppose that $r < \frac{d}{2}$. A subtle point of our argument is that we can not directly identify the matrix B with $\mathfrak{G}^r_{\mathtt{S}}$ entry by entry, because there is no canonical choice of basis on each of the factors in (4.5.5). The proof resembles the proof of Theorem 3.6. The determinant of B is equal to the sum over all permutations s of the set $\mathfrak{B}^r_{\mathtt{S}}$, of the product of the sign $\mathrm{sgn}(s)$ and, for every cycle $(\mathfrak{a}_1\cdots\mathfrak{a}_t)$ of the permutation s, the product

$$p^{-2(\ell(\mathfrak{a}_1)+\cdots+\ell(\mathfrak{a}_t))}\cdot(\mathrm{Res}_{\mathfrak{a}_1}\circ\mathrm{Gys}_{\mathfrak{a}_2})\cdot(\mathrm{Res}_{\mathfrak{a}_2}\circ\mathrm{Gys}_{\mathfrak{a}_3})\cdots(\mathrm{Res}_{\mathfrak{a}_{t-1}}\circ\mathrm{Gys}_{\mathfrak{a}_t})(\mathrm{Res}_{\mathfrak{a}_t}\circ\mathrm{Gys}_{\mathfrak{a}_1}). \quad (4.5.6)$$

Let $m_0=m_0(\mathfrak{a}_1,\mathfrak{a}_2)+\cdots+m_0(\mathfrak{a}_t,\mathfrak{a}_1)$ be the sum of total number of contractible loops in the diagrams $D(\mathfrak{a}_1,\mathfrak{a}_2)$, $D(\mathfrak{a}_2,\mathfrak{a}_3)$, …, $D(\mathfrak{a}_t,\mathfrak{a}_1)$. Then, by Theorem 4.4(2), the expression (4.5.6) is equal to $(-2)^{m_0}$ times the following composition of link morphisms on the cohomology groups,

$$\eta^\star_{\mathtt{S}_{\mathfrak{a}_1},\mathtt{S}_{\mathfrak{a}_2},z(\mathfrak{a}_1,\mathfrak{a}_2)}\circ\eta^\star_{\mathtt{S}_{\mathfrak{a}_2},\mathtt{S}_{\mathfrak{a}_3},z(\mathfrak{a}_2,\mathfrak{a}_3)}\circ\cdots\circ\eta^\star_{\mathtt{S}_{\mathfrak{a}_t},\mathtt{S}_{\mathfrak{a}_1},z(\mathfrak{a}_t,\mathfrak{a}_1)}, \quad (4.5.7)$$

of shift $\prod_{i=1}^{t-1}(\boldsymbol{t}_{\mathfrak{a}_i}\boldsymbol{t}^{-1}_{\mathfrak{a}_{i+1}})\boldsymbol{t}_{\mathfrak{a}_t}\boldsymbol{t}^{-1}_{\mathfrak{a}_1}=\mathbf{1}$ and indentation degree

$$\sum_{i=1}^{t-1}z(\mathfrak{a}_i,\mathfrak{a}_{i+1})+z(\mathfrak{a}_t,\mathfrak{a}_1)$$
$$=\begin{cases}\sum_{i=1}^{t-1}(\ell(\mathfrak{a}_i)-\ell(\mathfrak{a}_{i+1}))+\ell(\mathfrak{a}_n)-\ell(\mathfrak{a}_1)=0, & \text{if } \mathfrak{p} \text{ splits in } E/F,\\ 0+\cdots+0=0, & \text{if } \mathfrak{p} \text{ is inert in } E/F.\end{cases}$$

So this composition (4.5.7) is the same link morphism associated to some nth power of the *fundamental link* $\eta_{\mathtt{S}_{\mathfrak{a}_1}}$ for $\mathtt{S}_{\mathfrak{a}_1}$, with trivial shift and indentation degree 0 (regardless of whether $\mathfrak{p}$ splits or not in E/F). The number n is equal to the total displacement $v(\eta_{\mathtt{S}_{\mathfrak{a}_t,\mathfrak{a}_1}}\circ\cdots\circ\eta_{\mathtt{S}_{\mathfrak{a}_1,\mathfrak{a}_2}})$ divided by $v(\eta_{\mathtt{S}_{\mathfrak{a}_1}})=g$.

Note that the action of the link morphism $(\eta^n_{\mathtt{S}_{\mathfrak{a}_1}})^\star_{(0)}$ on the 1-dimensional $\mathrm{Frob}_{p^{2g}}$-eigenspace

$$\big(H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}_1},\mathtt{T}_{\mathfrak{a}_1}}),\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}_1},\mathtt{T}_{\mathfrak{a}_1}}\big)[\pi]\big)^{\mathrm{Frob}_{p^{2g}}=\alpha_\pi^{2(d-2r)}(\alpha_\pi\beta_\pi/p^g)^{2(\#\mathtt{T}+r)}} \quad (4.5.8)$$

is just the multiplication by a scalar, which we denote by $\lambda_{\mathfrak{a}_1,n}$. We claim that $\lambda_{\mathfrak{a}_1,n}$ *does not depend on* $\mathfrak{a}_1\in\mathfrak{B}^r_{\mathtt{S}}$. Indeed, for $\mathfrak{a},\mathfrak{a}'\in\mathfrak{B}^r_{\mathtt{S}}$ with $\langle\mathfrak{a}|\mathfrak{a}'\rangle\neq 0$, Theorem 4.4(2) gives a normalized link morphism,

$$\eta^\star_{\mathtt{S}_\mathfrak{a},\mathtt{S}'_\mathfrak{a},(z)}\colon H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}'},\mathtt{T}_{\mathfrak{a}'}}),\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}'},\mathtt{T}_{\mathfrak{a}'}}\big)\to H^{d-2r}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}),\mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}}\big),$$

with some indentation degree z and some shift; then

$$(\eta^n_{\mathtt{S}_\mathfrak{a}})^\star_{(0)}=\eta^\star_{\mathtt{S}_\mathfrak{a},\mathtt{S}_{\mathfrak{a}'},(z)}\circ(\eta^n_{\mathtt{S}_{\mathfrak{a}'}})^\star_{(0)}\circ(\eta^\star_{\mathtt{S}_\mathfrak{a},\mathtt{S}_{\mathfrak{a}'},(z)})^{-1},$$

provided that one of $(\eta^n_{\mathtt{S}_\mathfrak{a}})^\star_{(0)}$ or $(\eta^n_{\mathtt{S}_{\mathfrak{a}'}})^\star_{(0)}$ exists. When this happens, we must have $\lambda_{\mathfrak{a},n}=\lambda_{\mathfrak{a}',n}$. For general $\mathfrak{a}$ and $\mathfrak{a}'$, we can always find a sequence $\mathfrak{a}_1=\mathfrak{a},\dots,\mathfrak{a}_t=\mathfrak{a}'\in\mathfrak{B}^r_{\mathtt{S}}$ such that $\langle\mathfrak{a}_i|\mathfrak{a}_{i+1}\rangle\neq 0$. So if for some n the link morphism $(\eta^n_{\mathtt{S}_\mathfrak{a}})^\star$ exists, then it does not depend on $\mathfrak{a}$. In the remainder of this proof, we put $\lambda_n=\lambda_{\mathfrak{a},n}$ as long as $(\eta^n_{\mathtt{S}_\mathfrak{a}})^\star_{(0)}$ exists for some $\mathfrak{a}\in\mathfrak{B}^r_{\mathtt{S}}$. The element λ_n is clearly multiplicative in n.

We can thus introduce the formal symbol $\eta^\star_{\mathrm{univ}}$ such that $(\eta^\star_{\mathrm{univ}})^n = \lambda_n$ whenever $(\eta^n_{\mathrm{S}_\mathfrak{a}})^\star_{(0)}$ exists for an integer n. Comparing this computation with $\det \mathfrak{G}^r_{\mathrm{S}}$ in the proof of Theorem 3.6, we see that $\det B$ is obtained by replacing every v^g in $\det \mathfrak{G}^r_{\mathrm{S}}$ by $\eta^\star_{\mathrm{univ}}$. By Theorem 3.6, this means that

$$\det B = \pm\big(\eta^\star_{\mathrm{univ}} - (\eta^\star_{\mathrm{univ}})^{-1}\big)^{2t_{d,r}}.$$

In particular, $(\eta^\star_{\mathrm{univ}})^2$ appears in the determinant, and hence $(\eta^2_{\mathrm{S}_\mathfrak{a}})^\star_{(0)}$ exists.

Finally, it follows from Proposition 2.27 that $(\eta^\star_{\mathrm{univ}})^{2(d-2r)} = \lambda_{2(d-2r)} = (\alpha_\pi/\beta_\pi)^{d-2r}$. Our assumption implies that $(\alpha_\pi/\beta_\pi)^{d-2r} \neq 1$; so $(\eta^\star_{\mathrm{univ}})^2 \neq 1$, and hence $\det B \neq 0$. This concludes (2).

We now treat the case of $r = \frac{d}{2}$. Similar to the discussion above, $\det B$ is equal to the sum over all permutations s of the set $\mathfrak{B}^r_{\mathrm{S}}$, of the product of the signature of s, and, for every cycle $(\mathfrak{a}_1 \cdots \mathfrak{a}_t)$ of the permutation s, the product (4.5.6). By Theorem 4.4(3), (4.5.6) in this case is of the form $(-2)^{m_0} \cdot (T_\mathfrak{p}/p^{g/2})^{m_T}$ times the link morphism from $(\mathrm{S}_{\mathfrak{a}_1}, \mathrm{T}_{\mathfrak{a}_1})$ to itself with shift $\prod_{i=1}^{t-1} (\boldsymbol{t}_{\mathfrak{a}_i} \boldsymbol{t}^{-1}_{\mathfrak{a}_{i+1}} \varpi_{\bar{\mathfrak{q}}}^{-m_T(\mathfrak{a}_i,\mathfrak{a}_{i+1})}) \boldsymbol{t}_{\mathfrak{a}_t} \boldsymbol{t}^{-1}_{\mathfrak{a}_1} \times \varpi_{\bar{\mathfrak{q}}}^{-m_T(\mathfrak{a}_n,\mathfrak{a}_1)} = \varpi_{\bar{\mathfrak{q}}}^{-m_T}$ and indentation degree

$$\sum_{i=1}^{t} \big(\ell(\mathfrak{a}_i) - \ell(\mathfrak{a}_{i-1}) - m_{T,i}\, g\big) = -m_T g.$$

Here $m_0 = m_0(\mathfrak{a}_1, \mathfrak{a}_2) + \cdots + m_0(\mathfrak{a}_t, \mathfrak{a}_1)$ (resp., $m_T = m_T(\mathfrak{a}_1, \mathfrak{a}_2) + \cdots + m_T(\mathfrak{a}_t, \mathfrak{a}_1)$) is the total number of contractible (resp., noncontractible) loops in $D(\mathfrak{a}_1, \mathfrak{a}_2), \ldots, D(\mathfrak{a}_t, \mathfrak{a}_1)$. By Example 2.22 and the uniqueness of link morphisms (Lemma 2.20), this link morphism is equal to the one associated to $S_\mathfrak{q}^{-m_T/2}$ with shift $\varpi_{\bar{\mathfrak{q}}}^{-m_T}$. This in particular says that m_T is even. By the second part of Example 2.22, we see that this link morphism is exactly $S_\mathfrak{p}^{-m_T/2}$. Therefore, (4.5.6) is given by

$$(-2)^{m_0} (T_\mathfrak{p}/p^{g/2})^{m_T} (S_\mathfrak{p})^{-m_T/2} = (-2)^{m_0} \big((\alpha_\pi + \beta_\pi)^2/(\alpha_\pi \beta_\pi)\big)^{m_T/2}.$$

Comparing this with the computation of $\det \mathfrak{G}^r_{\mathrm{S}}$, we see that $\det B$ is nothing but replacing every T^2 by $T^{\mathfrak{n}}_\mathfrak{p} := (\alpha_\pi + \beta_\pi)^2/\alpha_\pi\beta_\pi$. By Theorem 3.6, we see that

$$\det B = \pm\big((\alpha_\pi + \beta_\pi)^2/\alpha_\pi\beta_\pi - 4\big)^{t_{d,d/2}} = \pm\big((\alpha_\pi - \beta_\pi)^2/\alpha_\pi\beta_\pi\big)^{t_{d,d/2}}.$$

It is nonzero as long as $\alpha_\pi \neq \beta_\pi$.[15] This concludes the proof of Theorem 4.5. □

Before giving a more detailed discussion of the case $\alpha_\pi = \beta_\pi$, we first give some general remarks.

[15] Note that we still need α_π/β_π to avoid certain roots of unity to get (4.5.5).

Remark 4.6

(1) We discuss the possibility of generalizing this main theorem to the case when p is only assumed to be unramified (namely, $p\mathcal{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_h$). In this case, one can construct the twisted partial Frobenius $\mathfrak{F}''_{\mathfrak{p}_i^2}$ for each prime ideal $\mathfrak{p}_i$ as in [33, Section 3.22]. Roughly speaking, on the level of moduli space, this is to send the Abelian variety A to $A/\operatorname{Ker}_{\mathfrak{p}_i^2} \otimes_{\mathcal{O}_F} \mathfrak{p}_i$, where $\operatorname{Ker}_{\mathfrak{p}_i^2}$ is the $\mathfrak{p}_i$-component of the kernel of $\mathrm{Fr}^2 \colon A \to A^{(p^2)}$. Suppose that one can describe the action of each $\mathfrak{F}''_{\mathfrak{p}_i^2}$ on the cohomology of the unitary Shimura variety as in Proposition 2.27(3), or, more precisely, [34, Conjecture 5.18] holds true. Then the same argument above can generalize the theorem to the case when p is only assumed to be unramified, and every prime ideal $\mathfrak{p}_i$ behaves "in an independent way." More precisely, we fix $r_i \leq \frac{d_i}{2}$ for all i, where $d_i = \#(\mathtt{S}_\infty^c \cap \Sigma_{\mathfrak{p}_i})$ and $\Sigma_{\mathfrak{p}_i}$ is the subset of p-adic embeddings that induce the prime $\mathfrak{p}_i$. Then the Goren–Oort cycles would be parameterized by h-tuples whose ith component is a semimeander with d_i nodes and r_i arcs. Under the genericity condition, the eigenvalues of $\rho_\pi(\mathrm{Frob}_{\mathfrak{p}_i})$ avoid certain roots of unity, and the cohomology of the Goren–Oort cycles generate the subspace of the cohomology $H^d_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}})[\pi]$ where certain analogues of $\mathfrak{F}''_{\mathfrak{p}_i}$ act with appropriate eigenvalues determined by r_i and $\mathrm{Frob}_{\mathfrak{p}_i}$.

Without [34, Conjecture 5.18], we can only prove the analogous statement when $r_i = \frac{d_i}{2}$ for i, that is, in the case for Tate cycles.[16] Moreover, since we can not distinguish the actions of each $\mathfrak{F}''_{\mathfrak{p}_i}$, we would have to assume that the eigenvalues of $\rho_\pi(\mathrm{Frob}_{\mathfrak{p}_i})$ are "generic," so that all eigenvalues of $\mathrm{Frob}_p^{\mathfrak{g}}$ acting on $H^2_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}})$ are "as distinct as possible," where $\mathfrak{g}$ stands for the least common multiple of the inertia degrees of the $\mathfrak{p}_i$'s. For example, this excludes the case when both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have inertia degree 2 and $\mathrm{Frob}_{\mathfrak{p}_1}$ and $\mathrm{Frob}_{\mathfrak{p}_2}$ have the same set of eigenvalues (which would be okay if [34, Conjecture 5.18] is known).

(2) It would be interesting to know, when p is ramified in $F/\mathbb{Q}$, whether one can prove a similar result for the special fiber of the splitting model of the Hilbert modular variety of Pappas and Rapoport. The construction of the corresponding Goren–Oort divisors is discussed in [28].

(3) The construction of these Goren–Oort cycles uses the CM extension E of F. Even though we think these cycles should be independent of the choice of

[16] If $r_i < \frac{d_i}{2}$ for some i, then the determinant of the intersection matrix would involve the knowledge of different powers of the action of $\mathfrak{F}''_{\mathfrak{p}_i}$. But we only have the information of their product $S_p^{-1} \cdot F^2 := \prod_{i=1}^h \mathfrak{F}''_{\mathfrak{p}_i^2}$. On the other hand, the case $r_i = \frac{d_i}{2}$ is fine, because we only use the Hecke operators, whose action on the cohomology is known.

E, we do not know how to prove this. This auxiliary CM extension is also responsible for avoiding $2n$th roots of unity as opposed to just nth roots of unity. We think these issues are purely technical, as our current technique relies very much on the PEL moduli interpretation.

(4) In the case of $r = d/2$ (namely, the case for Tate classes), the map (4.5.1) is injective as long as $\alpha_\pi \neq \beta_\pi$. We need α_π/β_π to avoid more roots of unity so that both sides of (4.5.1) have the same dimension.

(5) It is tempting to ask the following question: To what extent does this imply the semisimplicity of the Frobenius action on $H^d_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)[\pi]$? Unfortunately, our theorem is in its strongest form only when $\alpha_\pi \neq \beta_\pi$, where $\rho_\pi(\mathrm{Frob}_{p^g})$ is automatically semisimple. Thus if $\otimes \mathrm{Ind}^{\mathrm{Gal}_\mathbb{Q}}_{\mathrm{Gal}_F} \rho_\pi$ is irreducible as a representation of $\mathrm{Gal}_\mathbb{Q}$, then the $\mathrm{Gal}_\mathbb{Q}$ representation $H^d_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathrm{S},\mathrm{T}})_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\pi]$ is isomorphic to $\otimes \mathrm{Ind}^{\mathrm{Gal}_\mathbb{Q}}_{\mathrm{Gal}_F} \rho_\pi$ up to characters, so that Frob_{p^g} is semisimple. However, $\otimes \mathrm{Ind}^{\mathrm{Gal}_\mathbb{Q}}_{\mathrm{Gal}_F} \rho_\pi$ might be reducible (e.g., when π is CM). In this case, our theorem might provide some insight into the semisimplicity of $H^d_{\mathrm{et}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)[\pi]$ as a representation of $\mathrm{Gal}_\mathbb{Q}$. See also [26].

(6) It is also tempting to ask the following: In the case of $r = d/2$ (the Tate classes case), is the determinant of the intersection matrix related to the higher derivatives of the local L-function (to get a certain local version of the Beilinson–Bloch conjecture)? We think the answer might be negative. Note that the determinant is always a power of $(\alpha_\pi - \beta_\pi)$, but the higher derivatives of the local L-functions can involve factors of the form $\alpha^s - \beta^s$ for $s < d/2$. In the recent preprint [37] of Z. Yun and W. Zhang, they seem to suggest a new philosophy for higher derivatives of *global* L-functions. We do not know how to compare the determinant of our intersection matrix to their formulation.

Remark 4.7

It is a very interesting question to understand what happens when $\alpha_\pi = \beta_\pi$. We explain this in the quadratic case. Let F be a real quadratic field in which p is inert. Let $\pi \in \mathcal{A}_{(\underline{2},2)}$ be a cuspidal automorphic representation with trivial central character. Suppose that π appears in the cohomology of the quaternionic Shimura variety $X = \mathrm{Sh}_K(G_{\{v_1,v_2\},\emptyset})$, where v_1 and v_2 are two finite prime-to-p places of F (so that X is proper for simplicity). Suppose that $\alpha_\pi = \beta_\pi = \pm p$. For instance, when π comes from the base change of a usual modular form corresponding to an elliptic curve over $\mathbb{Q}$ which has supersingular (good) reduction at p, then the local Satake parameters of π at p are $\alpha_\pi = \beta_\pi = p$. An interesting related question is: Are there examples of π which do not come from base change? We consider this question as an analogue of Coleman's complete reducibility question (see [3, Remark 2, p. 232]).

In this case, $H^2_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))[\pi]$ is 4-dimensional, on which Frob_{p^2} acts trivially. More precisely, as pointed out by Prasanna, the action of Frob_p on this 4-dimensional subspace has two eigenvalues: α_π/p (with multiplicity 3) and $-\alpha_\pi/p$ (with multiplicity 1). There are two Goren–Oort cycles, both given by a collection of $\mathbb{P}^1$'s. The π-isotypical components of their cycle classes contribute nontrivially to the Frob_p-eigenspace with eigenvalue $-\alpha_\pi/p$. We claim that the π-isotypical component of their cycles classes does not contribute to the Frob_p-eigenspace with eigenvalue α_π/p. Indeed, the intersection matrix B given above is degenerate (with rank 1). Note that, for any cuspidal π, the π-isotypical component of the rational Néron–Severi group of X is orthogonal to the subspace of ample line bundles, and the Hodge index theorem implies that the intersection pairing on the π-isotypical component is nondegenerate. So the degeneracy of the intersection matrix means that the contribution from the Goren–Oort cycles is indeed a 1-dimensional subspace of $H^2_{\mathrm{et}}(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))[\pi]$, namely, the subspace with Frob_p-eigenvalue $-\alpha_\pi/p$.

We think this phenomenon is comparable to the case of Heegner points: when the rank of the elliptic curve is 1 ("generic rank"), the Heegner point gives a canonical generator of the Mordell–Weil group tensored with $\mathbb{Q}$; however, when the rank of the elliptic curve is strictly greater than 1 ("generic rank"), the Heegner point becomes torsion. In our case, the classes of the Goren–Oort cycles are similar to Heegner points. When the dimension of the corresponding Frobenius (generalized) eigenspace is "generic," the classes of the Goren–Oort cycles give a canonical basis, but when the dimension is strictly greater than the generic one, the contribution from the Goren–Oort cycles tends to degenerate.

5. Computation of the intersection matrix

The aim of this section is to establish Theorem 4.4 and hence to finish the proof of the main theorems. We keep the notation from the previous section.

Notation 5.1

For simplicity, we suppress the automorphic sheaf $\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}$, the level structure K_p, the change of base to $\overline{\mathbb{F}}_p$, and the subscript et from the notation of cohomology groups, as they are all fixed throughout this section. For example, we write

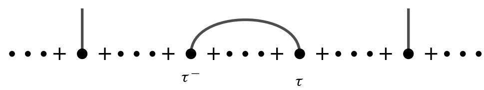
$$H^\star\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}\big)(r) \quad \text{for } H^\star_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a},\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}(r)|_{\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}}\big).$$

This should not cause any confusion because all the automorphic sheaves are compatible on the Goren–Oort cycles. As in Theorem 4.4, we fix a choice of system of shifts $\boldsymbol{t}_{\mathfrak{a}}$ of the correspondences $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \xleftarrow{\pi_{\mathfrak{a}}} \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}} \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ as in Section 3.8.

Before going into the intricate induction, we first handle a few simple but essential cases. The general case will be essentially reduced to these cases.

5.2. The case of $r = 1$ and $\mathfrak{a} = \mathfrak{b}$

This is the case where the corresponding periodic semimeanders are given as

$\mathfrak{a} = \mathfrak{b} =$ 

(or their shifts), linking τ with $\tau^- = \sigma^{-n_\tau}\tau$.

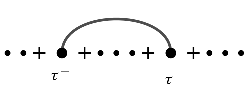
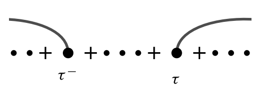
Unwinding the definition, we have the following commutative diagram:

$$\begin{array}{ccccc} H^{d-2}\big(\mathrm{Sh}(G_{\mathrm{S}_\mathfrak{a},\mathrm{T}_\mathfrak{a}})\big) & & \xrightarrow{\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{a}} & & H^{d-2}\big(\mathrm{Sh}(G_{\mathrm{S}_\mathfrak{a},\mathrm{T}_\mathfrak{a}})\big) \\ \downarrow \pi_\mathfrak{a}^* & & & & \uparrow \pi_{\mathfrak{a},!} \\ H^{d-2}\big(\mathrm{Sh}(G_{\mathrm{S},\mathrm{T}})_\mathfrak{a}\big) & \xrightarrow{\mathrm{Gysin}} & H^{d}\big(\mathrm{Sh}(G_{\mathrm{S},\mathrm{T}})\big)(1) & \xrightarrow{\mathrm{Restr.}} & H^{d}\big(\mathrm{Sh}(G_{\mathrm{S},\mathrm{T}})_\mathfrak{a}\big)(1) \end{array} \tag{5.2.1}$$

Recall that $\mathrm{Sh}(G_{\mathrm{S},\mathrm{T}})_\mathfrak{a}$ is a $\mathbb{P}^1$-bundle over $\mathrm{Sh}(G_{\mathrm{S}_\mathfrak{a},\mathrm{T}_\mathfrak{a}})$; hence $\pi_\mathfrak{a}^*$ and $\pi_{\mathfrak{a},!}$ are both isomorphisms. By the excessive intersection formula (see [7, Section 6.3]), the composition of the bottom line is given by the cup product with the first Chern class of the normal bundle of the embedding $\mathrm{Sh}(G_{\mathrm{S},\mathrm{T}})_\mathfrak{a} \hookrightarrow \mathrm{Sh}(G_{\mathrm{S},\mathrm{T}})$, which is isomorphic to $-2p^{n_\tau}$ times the universal quotient line bundle for the $\mathbb{P}^1$-bundle given by $\pi_\mathfrak{a}$, according to Proposition 2.31(2). Therefore, the morphism on the top row $\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{a}$ is nothing but the multiplication by $-2p^{n_\tau} = -2p^{\ell(\mathfrak{a})}$.

5.3. The case of $d = 2$ and $r = 1$ with $\mathfrak{a} \neq \mathfrak{b}$

This is the case where the corresponding periodic semimeanders are given as

$\mathfrak{a} =$  and $\mathfrak{b} =$ 

(or their simultaneous shifts). Let τ^- denote the left end-node of the arc of $\mathfrak{a}$, and let τ denote is the right end-node. We have $\tau^+ = \tau^-$. Here the meaning of "left" and "right" refers to the xy-plane presentation of $\mathfrak{a}$, as explained in Section 3.1.

Unwinding the definition, the morphism $\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{b}$ is the composition of the following commutative diagram from the upper-left to the lower-right (first rightward and then downward):

$$\begin{array}{ccccc}
H^0\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})\big) & \xrightarrow{\pi_{\mathfrak{b}}^*} & H^0\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau^-}\big) & \xrightarrow{\text{Gysin}} & H^2\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})\big)(1) \\
& \searrow & \big\downarrow{\scriptstyle \text{Restr.}} & & \big\downarrow{\scriptstyle \text{Restr.}} \\
& & H^0\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau^-,\tau\}}\big) & \xrightarrow{\text{Gysin}} & H^2\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau}\big)(1) \\
& & & \searrow^{\mathrm{Tr}_{\pi_{\mathfrak{a}}}} & \big\downarrow{\scriptstyle \pi_{\mathfrak{a},!}} \\
& & & & H^0\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})\big)
\end{array} \tag{5.3.1}$$

Here, the commutativity of the square follows from the diagram (4.1.2) and the fact that $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^-\}}$ is the transversal intersection of $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau^-}$ and $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau}$ in $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})$. The commutativity of the upper left triangle is obvious, and the commutativity of the lower right triangle follows from the fact that $\mathrm{Tr}_{\pi_{\mathfrak{a}}}$ is the trace map induced by the finite étale map (of 0-dimensional Shimura varieties)

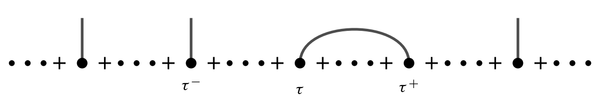
$$\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau^-,\tau\}} \hookrightarrow \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau} \xrightarrow{\pi_{\mathfrak{a}}} \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}),$$

and by the natural isomorphism between the pullback of the automorphic sheaf on $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$ with that on $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau}$.

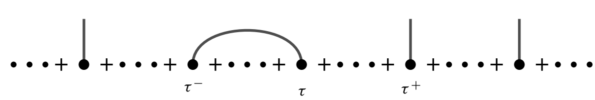
By Theorem 2.32(3), the diagonal composition from the upper left to the lower right, or, equivalently, the morphism $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$, is $T_{\mathfrak{p}} \circ (\eta^{\star}_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(n)})^{-1}$, where $\eta^{\star}_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(n)}$ is the link morphism associated to the trivial link $\eta_{\mathtt{S}_{\mathfrak{b}},\mathtt{S}_{\mathfrak{a}}} : \mathtt{S}_{\mathfrak{b}} \to \mathtt{S}_{\mathfrak{a}}$ with indentation degree $n = 2n_{\tau^-} = -(\ell(\mathfrak{a}) - \ell(\mathfrak{b}) - g)$ and shift $\varpi_{\bar{\mathfrak{q}}} \boldsymbol{t}_{\mathfrak{a}}^{-1} \boldsymbol{t}_{\mathfrak{b}}$. Thus the inverse $(\eta^{\star}_{\mathtt{S}_{\mathfrak{b}},\mathtt{S}_{\mathfrak{a}},(n)})^{-1} = (\eta_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}},(-n)})^{\star}$ is the link morphism associated to the link $\eta_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}}} = \eta^{-1}_{\mathtt{S}_{\mathfrak{b}},\mathtt{S}_{\mathfrak{a}}}$ with indention degree $\ell(\mathfrak{a}) - \ell(\mathfrak{b}) - g$ and shift $\varpi_{\bar{\mathfrak{q}}}^{-1} \boldsymbol{t}_{\mathfrak{a}} \boldsymbol{t}_{\mathfrak{b}}^{-1}$. This proves Theorem 4.4(3) for the given case.

5.4. *The case of* $r = 1$, $d > 2$, *and* $\langle \mathfrak{a}, \mathfrak{b} \rangle = v^{m_v}$

Assume that $m_v > 0$ first. In this situation, the corresponding periodic semimeanders, up to shifting, are given by

$\mathfrak{a} =$  and

(5.4.1)

$\mathfrak{b} =$  .

Note that the two arcs in $\mathfrak{a}$ and $\mathfrak{b}$ must be adjacent; otherwise, $\langle\mathfrak{a},\mathfrak{b}\rangle = 0$. Let τ denote the left end-node of the (unique) arc in $\mathfrak{a}$, as shown in the pictures above. Then τ^- is the left end-node of the arc in $\mathfrak{b}$, and τ^+ is the right end-node of the arc in $\mathfrak{a}$. So if $\tau = \sigma^{-n_{\tau^+}}\tau^+$ and $\tau^- = \sigma^{-n_\tau}\tau$, then $m_v = m_v(\mathfrak{a},\mathfrak{b}) = n_\tau + n_{\tau^+}$.

Unwinding the definition, the morphism $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ is the composition of the following commutative diagram from the upper left to the lower right:

$$\begin{array}{ccccc} H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})\big) & \xrightarrow{\pi_{\mathfrak{b}}^*} & H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_\tau\big) & \xrightarrow{\text{Gysin}} & H^d\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})\big)(1) \\ & \searrow^{\pi_{\mathfrak{b}}^*} & \downarrow{\scriptstyle \text{Restr.}} & & \downarrow{\scriptstyle \text{Restr.}} \\ & & H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^+\}}\big) & \xrightarrow{\text{Gysin}} & H^d\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau^+}\big)(1) \\ & & & \searrow_{\theta_! = (\theta^{-1})^*,\cong} & \downarrow{\scriptstyle \pi_{\mathfrak{a},!}} \\ & & & & H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})\big) \end{array} \tag{5.4.2}$$

Once again, the commutativity of the square follows from the diagram (4.1.2) and the fact that $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^+\}}$ is the transversal intersection of $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau^+}$ and $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_\tau$ in $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})$. The commutativity of the two triangles follows from the basic properties of star pullbacks and shriek pushforwards. By Theorem 2.32(2), the morphism

$$\theta\colon\ \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^+\}} \hookrightarrow \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau^+} \xrightarrow{\pi_{\mathfrak{a}}} \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$$

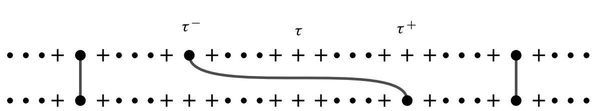
is an isomorphism, and the composition

$$\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \xleftarrow{\theta^{-1}} \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^+\}} \hookrightarrow \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_\tau \xrightarrow{\pi_{\mathfrak{b}}} \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})$$

is exactly the link morphism

$$\eta_{\mathfrak{a},\mathfrak{b},(z),\sharp}\colon\ \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \longrightarrow \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}}),$$

associated to the link $\eta_{\mathfrak{a},\mathfrak{b}}\colon \mathtt{S}_{\mathfrak{a}} \to \mathtt{S}_{\mathfrak{b}}$ given by



(5.4.3)

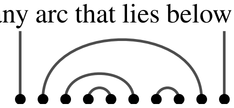
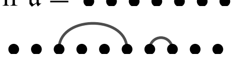
with shift $\boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}_{\mathfrak{b}}^{-1}$ and indentation degree z equal to $\ell(\mathfrak{a}) - \ell(\mathfrak{b})$ if $\mathfrak{p}$ splits in E/F and equal to 0 if $\mathfrak{p}$ is inert in E/F. Therefore, $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ is exactly $p^{v(\eta_{\mathfrak{a},\mathfrak{b}})/2}\eta^\star_{\mathfrak{a},\mathfrak{b},(z)} = p^{m_v/2}\eta^\star_{\mathfrak{a},\mathfrak{b},(z)}$ (note the normalization in (2.23.1)), verifying Theorem 4.4(2).

We now come to the case where m_v is negative. In this case, the picture of $\mathfrak{a}$ and $\mathfrak{b}$ in (5.4.1) are swapped. Then we have a commutative diagram similar to (5.4.2):

$$
\begin{array}{ccccc}
H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S}_\mathfrak{b},\mathtt{T}_\mathfrak{b}})\big) & \xrightarrow{\pi_\mathfrak{b}^*} & H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau^+}\big) & \xrightarrow{\text{Gysin}} & H^{d}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})\big)(1) \\
 & \searrow \cong & \downarrow \text{Restr.} & & \downarrow \text{Restr.} \\
 & & H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\{\tau,\tau^+\}}\big) & \xrightarrow{\text{Gysin}} & H^{d}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\tau}\big)(1) \\
 & & & \searrow & \downarrow \pi_{\mathfrak{a},!} \\
 & & & & H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})\big)
\end{array}
$$

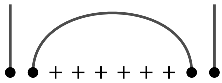
and the composed diagonal morphism gives $\mathrm{Res}_\mathfrak{a} \circ \mathrm{Gys}_\mathfrak{b}$. Let $\eta_{\mathfrak{b},\mathfrak{a}} \colon \mathtt{S}_\mathfrak{b} \to \mathtt{S}_\mathfrak{a}$ denote the inverse link of $\eta_{\mathfrak{a},\mathfrak{b}}$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are obtained by swapping with each other from the previous case, the link morphism $\eta_{\mathfrak{b},\mathfrak{a},(-z),\sharp} \colon \mathrm{Sh}(G_{\mathtt{S}_\mathfrak{b},\mathtt{T}_\mathfrak{b}}) \to \mathrm{Sh}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})$ with shift $\boldsymbol{t}_\mathfrak{b}\boldsymbol{t}_\mathfrak{a}^{-1}$ exists, where $z = \ell(\mathfrak{a}) - \ell(\mathfrak{b})$ if $\mathfrak{p}$ splits in E and $z = 0$ if $\mathfrak{p}$ is inert in E. Note also that $\eta_{\mathfrak{b},\mathfrak{a},(-z),\sharp}$ is finite flat of degree $p^{-m_v} = p^{v(\eta_{\mathfrak{b},\mathfrak{a}})}$ by Theorem 2.32. One sees easily that $\mathrm{Res}_\mathfrak{a} \circ \mathrm{Gys}_\mathfrak{b} = \mathrm{Tr}_{\eta_{\mathfrak{b},\mathfrak{a},(-z),\sharp}}$. By Lemma 2.29(3), this is exactly $p^{-m_v/2}(\eta^\star_{\mathfrak{b},\mathfrak{a},(-z)})^{-1} = p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}\eta^\star_{\mathfrak{a},\mathfrak{b},(z)}$. This proves Theorem 4.4(2) in this case.

5.5. *Decomposition of periodic semimeanders*

Before proceeding to the inductive proof, we discuss certain ways to "decompose" periodic semimeanders appearing in the induction. Let $\mathfrak{a} \in \mathfrak{B}^r_\mathtt{S}$ be a periodic semimeander. We call a subset Δ of r' arcs ($r' \le r$) in $\mathfrak{a}$ *saturated* if for each arc δ belonging to Δ any arc that lies below δ in the sense of Notation 3.7 belongs to Δ. For example, if $\mathfrak{a} =$ , the subset $\Delta =$  [17] is saturated, but  is not.

Now fix a saturated Δ. We use $\mathfrak{a}^\flat$ to denote the periodic semimeander for $\mathtt{S}$ given by all the arcs in Δ and then adjoining semilines to the rest of the nodes. Then $\mathtt{S}_{\mathfrak{a}^\flat}$ is the union of $\mathtt{S}$ and all nodes connected to an arc in Δ. We use $\mathfrak{a}_{\mathrm{res}} = \mathfrak{a} \backslash \Delta$ to denote the periodic semimeander for $\mathtt{S}_{\mathfrak{a}^\flat}$ obtained by removing all the arcs in Δ and replacing their end-nodes by plus signs. In the example above, $\mathfrak{a}^\flat =$  and

[17] Here Δ is only the set of the arcs, not including the nodes in the picture.

$\mathfrak{a}_{\mathrm{res}} =$ , where the plus signs indicates points corresponding to $\mathtt{S}_{\mathfrak{a}^\flat,\infty}$.

By the construction of the Goren–Oort cycles in Section 3.8, we have the following commutative diagram, where the middle square is Cartesian:

$$
\begin{array}{ccccc}
\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}} & \hookrightarrow & \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat} & \hookrightarrow & \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}}) \\
\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{a}^\flat}} & & \\
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}}} & \hookrightarrow & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}}) & & \\
\downarrow{\scriptstyle \pi_{\mathfrak{a}_{\mathrm{res}}}} & & & & \\
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) & & & &
\end{array}
\tag{5.5.1}
$$

Since the construction of this diagram comes from the unitary Shimura varieties, we point out that, the shift of the correspondence $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \xleftarrow{\pi_{\mathfrak{a}_{\mathrm{res}}}} \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}}} \hookrightarrow \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})$ is $\boldsymbol{t}_{\mathfrak{a}^\flat,\mathfrak{a}} = \boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}_{\mathfrak{a}^\flat}^{-1}$. From the commutative diagram, we can decompose the morphisms $\mathrm{Res}_{\mathfrak{a}}$ and $\mathrm{Gys}_{\mathfrak{a}}$ as follows:

$$
\begin{aligned}
\mathrm{Gys}_{\mathfrak{a}} \colon H^{d-2r}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})\big) &\xrightarrow{\pi_{\mathfrak{a}_{\mathrm{res}}}^*} H^{d-2r}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}}}\big) \\
&\xrightarrow{\mathrm{Gysin}} H^{d-2r'}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})\big)(r-r') \\
&\xrightarrow{\pi_{\mathfrak{a}^\flat}^*} H^{d-2r'}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat}\big)(r-r') \\
&\xrightarrow{\mathrm{Gysin}} H^{d}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})\big)(r)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Res}_{\mathfrak{a}} \colon H^{d}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})\big)(r) &\xrightarrow{\mathrm{Restr.}} H^{d}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat}\big)(r) \\
&\xrightarrow{\pi_{\mathfrak{a}^\flat,!}} H^{d-2r'}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})\big)(r-r') \\
&\xrightarrow{\mathrm{Restr.}} H^{d-2r'}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}}}\big)(r-r') \\
&\xrightarrow{\pi_{\mathfrak{a}_{\mathrm{res}},!}} H^{d-2r}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})\big).
\end{aligned}
$$

Here, to get the decomposition for $\mathrm{Res}_{\mathfrak{a}}$, we have used the fact that the trace map $\mathrm{Tr}_{\pi_{\mathfrak{a}}}$ can be factorized as

$$R^{2r}\pi_{\mathfrak{a}*}\overline{\mathbb{Q}}_\ell(r) \cong R^{2r-2r'}\pi_{\mathfrak{a}_{\mathrm{res}}*}(R^{2r'}\pi_{\mathfrak{a}^\flat *}\overline{\mathbb{Q}}_\ell)(r) \xrightarrow{\mathrm{Tr}_{\pi_{\mathfrak{a}^\flat}}} R^{2r-2r'}\pi_{\mathfrak{a}_{\mathrm{res}}*}(\overline{\mathbb{Q}}_\ell)(r-r') \xrightarrow{\mathrm{Tr}_{\pi_{\mathfrak{a}_{\mathrm{res}}}}} \overline{\mathbb{Q}}_\ell.$$

Summing up everything in short, we obtain thus

$$\mathrm{Gys}_{\mathfrak{a}} = \mathrm{Gys}_{\mathfrak{a}^\flat} \circ \mathrm{Gys}_{\mathfrak{a}_{\mathrm{res}}}, \qquad \mathrm{Res}_{\mathfrak{a}} = \mathrm{Res}_{\mathfrak{a}_{\mathrm{res}}} \circ \mathrm{Res}_{\mathfrak{a}^\flat}, \qquad \text{and} \qquad \boldsymbol{t}_{\mathfrak{a}} = \boldsymbol{t}_{\mathfrak{a}^\flat}\boldsymbol{t}_{\mathfrak{a}^\flat,\mathfrak{a}}.$$

We will apply this to appropriate Δ's to reduce the calculation to $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})$ and reduce the inductive proof essentially to the cases considered above.

5.6. *Decomposition of periodic semimeanders (continued)*

We will also encounter the following situation: assume that the set of arcs in a periodic semimeander $\mathfrak{a}$ is the disjoint union of two *saturated* subsets Δ and Δ'. Put $s = \#\Delta$ and $s' = \#\Delta'$ so that $r = s + s'$. We will show that Δ and Δ' "behave" independently.

We write $\mathfrak{a}^\flat$ (resp., $\mathfrak{a}^{\flat\prime}$) for the periodic semimeander for $\mathtt{S}$ given by all arcs in Δ (resp., Δ') and then adjoin semilines to the rest of the nodes. We put $\mathfrak{a}_{\mathrm{res}}$ (resp., $\mathfrak{a}'_{\mathrm{res}}$) for the periodic semimeander for $\mathtt{S}_{\mathfrak{a}^\flat}$ (resp., $\mathtt{S}_{\mathfrak{a}^{\flat\prime}}$) obtained by removing all arcs in Δ (resp., Δ') and replacing all their end-nodes by plus signs.

In this case, in view of the construction of the Goren–Oort cycle $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}}$ in Section 3.8, we could either go through the arcs in Δ first, or the arcs in Δ' first. So we have the following commutative *Cartesian* diagram:

$$\begin{array}{ccccc}
\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}}) & \hookleftarrow & \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat} & \xrightarrow{\pi_{\mathfrak{a}^\flat}} & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^{\flat\prime}} & \hookleftarrow & \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}} & \xrightarrow{\pi_{\Delta}} & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}}} \\
\downarrow \pi_{\mathfrak{a}^{\flat\prime}} & & \downarrow \pi_{\Delta'} & & \downarrow \pi_{\mathfrak{a}_{\mathrm{res}}} \\
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat\prime}},\mathtt{T}_{\mathfrak{a}^{\flat\prime}}}) & \hookleftarrow & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat\prime}},\mathtt{T}_{\mathfrak{a}^{\flat\prime}}})_{\mathfrak{a}'_{\mathrm{res}}} & \xrightarrow{\pi_{\mathfrak{a}'_{\mathrm{res}}}} & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})
\end{array} \tag{5.6.1}$$

where π_Δ and $\pi_{\Delta'}$ are the morphisms defined by the natural pullback of the upper-right and lower-left Cartesian squares, respectively. By Remark 2.13 the shifts satisfies the following equality:

$$\boldsymbol{t}_{\mathfrak{a}^\flat}\boldsymbol{t}_{\mathfrak{a}^\flat,\mathfrak{a}} = \boldsymbol{t}_{\mathfrak{a}} = \boldsymbol{t}_{\mathfrak{a}^{\flat\prime}}\boldsymbol{t}_{\mathfrak{a}^{\flat\prime},\mathfrak{a}} \quad \text{in } E^{\times,\mathrm{cl}}\backslash\mathbb{A}_E^{\infty,\times}/\mathcal{O}_{E_{\mathfrak{p}}}^\times. \tag{5.6.2}$$

This implies that both π_Δ and $\pi_{\Delta'}$ are iterated $\mathbb{P}^1$-bundles of relative dimensions s and s', respectively. We use $\pi_{\Delta,!}$ to denote the natural morphism

$$
\begin{aligned}
\pi_{\Delta,!}\colon\ & H^{\star}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a},\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}(s)\big) \\
& \xrightarrow{\cong} H^{\star}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}},\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}} \otimes R\pi_{\Delta *}\overline{\mathbb{Q}}_\ell(s)\big) \\
& \xrightarrow{\mathrm{Tr}\pi_\Delta} H^{\star-2s}_{\mathrm{et}}\big(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}},\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}}\big),
\end{aligned}
$$

where the last map is induced by the trace isomorphism $R^{2s}\pi_{\Delta,*}(\overline{\mathbb{Q}}_\ell(s)) \cong \overline{\mathbb{Q}}_\ell$.

As a consequence of the Cartesian property and Theorem 2.32(1), we have the following commutative diagram (which is placed into (5.6.1) vertically on the right):

$$
\begin{array}{ccccc}
H^{d-2s'}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat\prime}},\mathtt{T}_{\mathfrak{a}^{\flat\prime}}})_{\mathfrak{a}'_{\mathrm{res}}})(s) & \xrightarrow{\pi^*_{\Delta'}} & H^{d-2s'}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}})(s) & \xrightarrow{\text{Gysin}} & H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat})(s+s') \\
\downarrow{\scriptstyle \pi_{\mathfrak{a}'_{\mathrm{res}},!}} & & \downarrow{\scriptstyle \pi_{\Delta,!}} & & \downarrow{\scriptstyle \pi_{\mathfrak{a}^\flat,!}} \\
H^{d-2s-2s'}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})) & \xrightarrow{\pi^*_{\mathfrak{a}_{\mathrm{res}}}} & H^{d-2s-2s'}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\mathfrak{a}_{\mathrm{res}}}) & \xrightarrow{\text{Gysin}} & H^{d-2s}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}}))(s').
\end{array}
\tag{5.6.3}
$$

5.7. *Inductive proof of Theorem 4.4*

We now start the proof of Theorem 4.4 by induction on $d = \#\mathtt{S}^c_\infty$ or, equivalently, the dimension of the Shimura variety $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})$ (and also on r by keeping $d-2r$ fixed throughout the induction). The base cases $d = 0$ and $d = 1$ are trivial (as there is no nontrivial periodic semimeander).

We now assume that Theorem 4.4 holds for all Shimura varieties $\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})$ with $\#\mathtt{S}^c_\infty < d$. We now fix $\mathtt{S},\mathtt{T}$ so that $\#\mathtt{S}^c_\infty = d$. The case of $r = 0$ is clear. We henceforth assume that $r > 0$.

Let $\mathfrak{a},\mathfrak{b} \in \mathfrak{B}^r_{\mathtt{S}}$ be as in Theorem 4.4. We fix a *basic* arc $\delta_{\mathfrak{b}}$ of $\mathfrak{b}$, with right end-node $\tau \in \mathtt{S}^c_\infty$ (and left end-node $\tau^- \in \mathtt{S}^c_\infty$). As in Section 5.5, we use $\mathfrak{b}_{\mathrm{res}} \in \mathfrak{B}^{r-1}_{\mathtt{S}\cup\{\tau,\tau^-\}}$ to denote the periodic semimeander $\mathfrak{b}\backslash\delta_{\mathfrak{b}}$ obtained by removing $\delta_{\mathfrak{b}}$ from $\mathfrak{b}$ and replacing nodes τ,τ^- by plus signs. We will use $\delta_{\mathfrak{b}}$ itself to denote the corresponding $\mathfrak{b}^\flat$; that is, we also view $\delta_{\mathfrak{b}}$ as a periodic semimeander for $\mathtt{S}$ with only one arc $\delta_{\mathfrak{b}}$ (and $d-2$ semilines).

The basic idea is to factor the Gysin map $\mathrm{Gys}_{\mathfrak{b}}$ using $\delta_{\mathfrak{b}}$, in the sense of Section 5.5, and to factor the restriction map $\mathrm{Res}_{\mathfrak{a}}$ according to the following list of four cases.

(i) The two nodes τ,τ^- are both linked to semilines in $\mathfrak{a}$. This will force us to fall into the case (1) of Theorem 4.4.

(ii) There is a (basic) arc $\delta_{\mathfrak{a}}$ in $\mathfrak{a}$ linking τ^- to τ from left to right, so that $\delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{b}}$ form a contractible loop in $D(\mathfrak{a},\mathfrak{b})$. In other words, $\delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{b}}$ are the same (up to deformation of the arcs). We will reduce the proof of Theorem 4.4

to the case for $\mathtt{S}' = \mathtt{S} \cup \{\tau, \tau^-\}$, $\mathtt{T}' = \mathtt{T} \cup \{\tau\}$, $\mathfrak{a}_{\mathrm{res}} = \mathfrak{a}\backslash\delta_{\mathfrak{a}}$, and $\mathfrak{b}_{\mathrm{res}}$, and it hence follows from the inductive hypothesis. In particular, we will see that the contractible loop $\delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{b}}$ contributes a factor of $-2p^{\ell(\delta_{\mathfrak{b}})}$.

(iii) There is an arc $\delta_{\mathfrak{a}}$ in $\mathfrak{a}$ connecting τ to τ^- wrapped around the cylinder from right to left. In other words, $\delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{b}}$ together form a noncontractible loop in $D(\mathfrak{a}, \mathfrak{b})$. This can only happen if $r = d/2$. We will show that the composition $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ is essentially the $T_{\mathfrak{p}}$-operator composed with $\mathrm{Res}_{\mathfrak{a}\backslash\delta_{\mathfrak{a}}} \circ \mathrm{Gys}_{\mathfrak{b}\backslash\delta_{\mathfrak{b}}}$ for the Shimura variety with $\mathtt{S}' = \mathtt{S} \cup \{\tau, \tau^-\}$ and $\mathtt{T}' = \mathtt{T} \cup \{\tau\}$, up to some link morphism which we make explicit later.

(iv) Neither of the above happens. Then, in $\mathfrak{a}$, either τ is connected by an arc whose other end-node is not τ^-, and/or τ^- is connected by an arc whose other end-node is not τ. In either case, we will reduce to a case with the two nodes τ and τ^- removed, after composing with a certain link morphism.

We now treat each of the cases separately.

5.8. Case (i)

This is the case when τ and τ^- are connected to semilines in $\mathfrak{a}$. This implies that $\langle \mathfrak{a} | \mathfrak{b} \rangle = 0$. So we are in the situation of Theorem 4.4(1). We need to show that the π-isotypical component of $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ factors through the cohomology of a Shimura variety of smaller dimension. Let $\mathfrak{a}^*$ denote the periodic semimeander for $\mathtt{S}$ given by removing the two semilines of $\mathfrak{a}$ connected to τ and τ^- and reconnecting τ and τ^- by a (basic) arc. Note that this is possible because $\delta_{\mathfrak{b}}$ is a basic arc, so τ and τ^- are adjacent nodes in the band for $\mathtt{S}$. In particular, $\mathfrak{a}^* \in \mathfrak{B}_{\mathtt{S}}^{r+1}$.

By the discussion of Section 5.5, we see that the morphism $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ is the composition from the top left to the bottom right of the following commutative diagram by going first downward and then rightward:

$$\begin{array}{ccccc}
H^{d-2r}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})) & & & & \\
\downarrow{\scriptstyle \pi_{\delta_{\mathfrak{b}}}^* \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}} & & & & \\
H^{d-2}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\delta_{\mathfrak{b}}})(r-1) & \xrightarrow{\text{Restr.}} & H^{d-2}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^*})(r-1) & & \\
\downarrow{\scriptstyle \mathrm{Gysin}} & & \downarrow{\scriptstyle \mathrm{Gysin}} & & \\
H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}}))(r) & \xrightarrow{\text{Restr.}} & H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}})(r) & \xrightarrow{\pi_{\mathfrak{a},!}} & H^{d-2r}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})).
\end{array}$$

Here, the square is commutative because the corresponding morphisms on the Shimura varieties form a Cartesian square. The diagram implies that the π-component of $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ factors through the cohomology group

$$H^{d-2}\big(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^*}\big)(r-1)[\pi] \cong H^{d-2(r+1)}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^*},\mathtt{T}_{\mathfrak{a}^*}})\big)(-1)[\pi],$$

which is the π-isotypical component of the cohomology of a quaternionic Shimura variety of dimension $d-2(r+1)$. This means that the conclusion of Theorem 4.4(1) holds if we ever arrive in case (i) during the inductive proof.

5.9. Case (ii)

This is the case when there is a basic arc $\delta_\mathfrak{a}$ in $\mathfrak{a}$ linking τ^- to τ from left to right, and hence $\delta_\mathfrak{a}$ and $\delta_\mathfrak{b}$ are the same (up to deformation of the arcs). We write δ for the periodic semimeander for $\mathtt{S}$ with only one arc $\delta_\mathfrak{a}$. We write $\mathfrak{a}_{\mathrm{res}} = \mathfrak{a}\backslash\delta$ for the periodic semimeander for $\mathtt{S}_\delta$ obtained by removing $\delta_\mathfrak{a}$ from $\mathfrak{a}$ and replacing its end-nodes by plus signs.

Using the discussion of Section 5.5, the morphism $\mathrm{Res}_\mathfrak{a} \circ \mathrm{Gys}_\mathfrak{b}$ is the composition from the upper left to the upper right going all the way around: first downward to the bottom, then all the way to the right, and finally upward:

$$\begin{array}{ccccc}
H^{d-2r}(\mathrm{Sh}(G_{\mathtt{S}_\mathfrak{a},\mathtt{T}_\mathfrak{a}})) & & & & H^{d-2r}(\mathrm{Sh}(G_{\mathtt{S}_\mathfrak{b},\mathtt{T}_\mathfrak{b}})) \\
\Big\downarrow{\scriptstyle \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}} & & & & \Big\uparrow{\scriptstyle \mathrm{Res}_{\mathfrak{a}_{\mathrm{res}}}} \\
H^{d-2}(\mathrm{Sh}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta}))(r-1) & & \dashrightarrow & & H^{d-2}(\mathrm{Sh}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta}))(r-1) \\
\Big\downarrow{\scriptstyle \pi_\delta^*} & & & & \Big\uparrow{\scriptstyle \pi_{\delta,!}} \\
H^{d-2}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_\delta)(r-1) & \xrightarrow{\text{Gysin}} & H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}}))(r) & \xrightarrow{\text{Restr.}} & H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_\delta)(r)
\end{array}$$

As in Section 5.2, the composition of the bottom line is given by the excessive intersection formula, that is, to take the cup product with the first Chern class of the normal bundle of the embedding $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_\delta \hookrightarrow \mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})$, which is $-2p^{\ell(\delta)}$ times the class of the canonical quotient bundle for the $\mathbb{P}^1$-bundle given by π_δ, according to Proposition 2.31(2). Therefore, the dotted arrow in the middle is simply multiplication by $-2p^{\ell(\delta)}$. From this, we deduce that

$$\mathrm{Res}_\mathfrak{a} \circ \mathrm{Gys}_\mathfrak{b} = -2p^{\ell(\delta)} \cdot \mathrm{Res}_{\mathfrak{a}_{\mathrm{res}}} \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}, \tag{5.9.1}$$

where the latter morphism is constructed over the Shimura variety $\mathrm{Sh}(G_{\mathtt{S}_\delta,\mathtt{T}_\delta})$ of lower dimension. (Here we choose the shift $\boldsymbol{t}'_{\mathfrak{a}'}$ for a periodic semimeander $\mathfrak{a}'$ for $(\mathtt{S}_\delta, \mathtt{T}_\delta)$ to be $\boldsymbol{t}_{\delta,\tilde{\mathfrak{a}}'}$, where $\tilde{\mathfrak{a}}'$ is a periodic semimeander of $(\mathtt{S}, \mathtt{T})$ consisting of all the arcs and semilines of $\mathfrak{a}'$ together with the arc δ.)

We can now complete the induction in this case, since we have already known Theorem 4.4 for $\mathrm{Res}_{\mathfrak{a}_{\mathrm{res}}} \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}$ by the induction hypothesis.

(1) If $\langle \mathfrak{a}, \mathfrak{b} \rangle = 0$, then $\langle \mathfrak{a}_{\mathrm{res}}, \mathfrak{b}_{\mathrm{res}} \rangle = 0$ for simple combinatorics reasons. Then the π-isotypical component of $\mathrm{Res}_{\mathfrak{a}_{\mathrm{res}}} \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}$ factors through the cohomology of a lower-dimensional Shimura variety, so the same is true for $\mathrm{Res}_\mathfrak{a} \circ \mathrm{Gys}_\mathfrak{b}$.

(2) or (3) We have $\langle \mathfrak{a}|\mathfrak{b}\rangle = (-2)^{m_0} v^{m_v}$ or $(-2)^{m_0} T^{m_T}$. The picture $D(\mathfrak{a}_{\mathrm{res}}, \mathfrak{b}_{\mathrm{res}})$ is given by removing from $D(\mathfrak{a}, \mathfrak{b})$ the contractible loop consisting of $\delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{b}}$. So we have

$$\langle \mathfrak{a}_{\mathrm{res}}|\mathfrak{b}_{\mathrm{res}}\rangle = (-2)^{-1}\langle \mathfrak{a}|\mathfrak{b}\rangle = \begin{cases} (-2)^{m_0-1} v^{m_v} & \text{if } r < \frac{d}{2}, \\ (-2)^{m_0-1} T^{m_T} & \text{if } r = \frac{d}{2}. \end{cases}$$

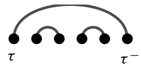
Since we have $\ell(\mathfrak{a}) - \ell(\mathfrak{a}_{\mathrm{res}}) = \ell(\mathfrak{b}) - \ell(\mathfrak{b}_{\mathrm{res}}) = \ell(\delta)$ and $\boldsymbol{t}'_{\mathfrak{a}_{\mathrm{res}}} \boldsymbol{t}'^{-1}_{\mathfrak{b}_{\mathrm{res}}} = \boldsymbol{t}_{\mathfrak{a}} \boldsymbol{t}_{\mathfrak{b}}^{-1}$, we see that $\eta_{\mathtt{S}_{\mathfrak{a}}, \mathtt{S}_{\mathfrak{b}}}$ gives the same link morphism as $\eta_{\mathtt{S}_{\delta,\mathfrak{a}_{\mathrm{res}}}, \mathtt{S}_{\delta,\mathfrak{b}_{\mathrm{res}}}}$ (with the same indentation and shift). By the inductive hypothesis and (5.9.1),

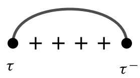
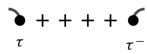
$$\begin{aligned} \mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}} &= -2p^{\ell(\delta)} \cdot \mathrm{Res}_{\mathfrak{a}_{\mathrm{res}}} \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}} \\ &= \begin{cases} -2p^{\ell(\delta)} \cdot (-2)^{m_0-1} \cdot p^{(\ell(\mathfrak{a}_{\mathrm{res}})+\ell(\mathfrak{b}_{\mathrm{res}}))/2} \eta^{\star}_{\mathtt{S}_{\delta,\mathfrak{a}_{\mathrm{res}}}, \mathtt{S}_{\delta,\mathfrak{b}_{\mathrm{res}}},(z)}, \\ \qquad \text{if } r < \frac{d}{2}, \\ -2p^{\ell(\delta)} \cdot (-2)^{m_0-1} \cdot p^{(\ell(\mathfrak{a}_{\mathrm{res}})+\ell(\mathfrak{b}_{\mathrm{res}}))/2} (T_{\mathfrak{p}}/p^{g/2})^{m_T} \eta^{\star}_{\mathtt{S}_{\delta,\mathfrak{a}_{\mathrm{res}}}, \mathtt{S}_{\delta,\mathfrak{b}_{\mathrm{res}}},(z)} \\ \qquad \text{if } r = \frac{d}{2}, \end{cases} \\ &= \begin{cases} (-2)^{m_0} \cdot p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2} \eta^{\star}_{\mathtt{S}_{\mathfrak{a}}, \mathtt{S}_{\mathfrak{b}},(z)}, \\ \qquad \text{if } r < \frac{d}{2}, \\ (-2)^{m_0} \cdot p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2} (T_{\mathfrak{p}}/p^{g/2})^{m_T} \eta^{\star}_{\mathtt{S}_{\mathfrak{a}}, \mathtt{S}_{\mathfrak{b}},(z)} \\ \qquad \text{if } r = \frac{d}{2}. \end{cases} \end{aligned}$$

5.10. Case (iii)

This is the case when there is an arc $\delta_{\mathfrak{a}}$ in $\mathfrak{a}$ connecting τ and τ^- wrapped around the cylinder from right to left, and hence $\delta_{\mathfrak{a}}$ and $\delta_{\mathfrak{b}}$ together form a noncontractible loop in $D(\mathfrak{a}, \mathfrak{b})$. We are forced to have $d = 2r$ in this case (and hence p splits in E/F). Moreover, the arc $\delta_{\mathfrak{a}}$ must lie over all other arcs of $\mathfrak{a}$ (if there is any). We now define a list of notations followed by an example.

- Let $\delta_{\mathfrak{a},\bullet}$ (resp., $\delta_{\mathfrak{b},\bullet}$) denote the periodic semimeander of two nodes obtained from $\mathfrak{a}$ (resp., $\mathfrak{b}$) by keeping $\delta_{\mathfrak{a}}$ (resp., $\delta_{\mathfrak{b}}$) and its end-nodes and replacing the other nodes of $\mathfrak{a}$ by plus signs.
- Let $\mathfrak{a}^{\flat}_{\bullet} = \mathfrak{a} \backslash \delta_{\mathfrak{a}}$ denote the periodic semimeander for $\mathtt{S}_{\mathfrak{a}}$ given by removing the arc $\delta_{\mathfrak{a}}$ from $\mathfrak{a}$ and replacing the nodes at τ and τ^- by plus signs.
- Let $\mathfrak{a}^{\flat}$ denote the periodic semimeander for $\mathtt{S}$ given by removing the arc $\delta_{\mathfrak{a}}$ and adjoining two semilines attached to both τ and τ^-.
- Let $\mathfrak{a}^{\star}$ denote the semimeander for $\mathtt{S}$ obtained by replacing the arc $\delta_{\mathfrak{a}}$ in $\mathfrak{a}$ with $\delta_{\mathfrak{b}}$ instead.

For example, if $\mathfrak{a} =$  and $\mathfrak{b} =$ , and we choose $\delta_{\mathfrak{b}}$ to be the arc of $\mathfrak{b}$ linking the first and the last nodes (τ and τ^-, resp., in the pictures), then $\delta_{\mathfrak{a}}$ is the arc linking the first and the last nodes (but "over" all other arcs). In this case, we have

$\delta_{\mathfrak{a},\bullet} =$ , $\delta_{\mathfrak{b},\bullet} =$ , $\mathfrak{a}^{\flat}_{\bullet} =$ ,

$\mathfrak{a}^{\flat} =$ , $\mathfrak{a}^{\star} =$ , and $\mathfrak{b}_{\mathrm{res}} =$ .

Our goal is to prove an equality,

$$\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}} = T_{\mathfrak{p}} \circ \eta^{\star}_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{S}_{\mathfrak{a}}} \circ \mathrm{Res}_{\mathfrak{a}^{\flat}_{\bullet}} \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}, \tag{5.10.1}$$

where $\eta^{\star}_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{S}_{\mathfrak{a}}}$ is a certain link morphism associated to the trivial link $\eta_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{S}_{\mathfrak{a}}} : \mathtt{S}_{\mathfrak{a}^\star} \to \mathtt{S}_{\mathfrak{a}}$, which we specify later.

Using the discussion of Section 5.5, we see that the morphism $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ is the composition from the top left to the bottom left of the following diagram, by going first rightward to the end, then downward to the bottom, and finally to the left by the long arrow:

$$\begin{array}{ccccc}
H^0(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{b}},\mathtt{T}_{\mathfrak{b}}})) & & & & \\
\big\downarrow{\scriptstyle \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}} & & & & \\
H^{d-2}(\mathrm{Sh}(G_{\mathtt{S}_{\delta_{\mathfrak{b}}},\mathtt{T}_{\delta_{\mathfrak{b}}}}))\big(\tfrac{d}{2}-1\big) & \xrightarrow{\pi^*_{\delta_{\mathfrak{b}}}} & H^{d-2}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\delta_{\mathfrak{b}}})\big(\tfrac{d}{2}-1\big) & \xrightarrow{\mathrm{Gysin}} & H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}}))\big(\tfrac{d}{2}\big) \\
\big\downarrow{\scriptstyle \mathrm{Restr.}} & & \big\downarrow{\scriptstyle \mathrm{Restr.}} & & \big\downarrow{\scriptstyle \mathrm{Restr.}} \\
H^{d-2}(\mathrm{Sh}(G_{\mathtt{S}_{\delta_{\mathfrak{b}}},\mathtt{T}_{\delta_{\mathfrak{b}}}})_{\mathfrak{a}^{\flat}_{\bullet}})\big(\tfrac{d}{2}-1\big) & \xrightarrow{\pi^*_{\delta_{\mathfrak{b}}}} & H^{d-2}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\star})\big(\tfrac{d}{2}-1\big) & \xrightarrow{\mathrm{Gysin}} & H^{d}(\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat})\big(\tfrac{d}{2}\big) \\
\big\downarrow{\scriptstyle \pi_{\mathfrak{a}^{\flat}_{\bullet},!}} & & & & \big\downarrow{\scriptstyle \pi_{\mathfrak{a}^{\flat},!}} \\
H^0(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{T}_{\mathfrak{a}^\star}})) & \xrightarrow{\pi^*_{\delta_{\mathfrak{b},\bullet}}} & H^0(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\delta_{\mathfrak{b},\bullet}}) & \xrightarrow{\mathrm{Gysin}} & H^2(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}}))(1) \\
\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{Restr.}} \\
H^0(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})) & & \xleftarrow{\quad\pi_{\delta_{\mathfrak{a},\bullet},!}\quad} & & H^2(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\delta_{\mathfrak{a},\bullet}})(1)
\end{array}$$

(5.10.2)

The commutativity of the top left square in the diagram above follows from the commutative diagram of morphisms of varieties, and that of the top right square follows from (4.1.2) and the fact that $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\star}$ is the transversal intersection of $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\delta_\mathfrak{b}}$ and $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat}$ in $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})$. The middle rectangle of (5.10.2) is commutative by (5.6.3) (applied with our $\mathfrak{a}^\star$, $\mathfrak{a}^\flat$, and $\delta_\mathfrak{b}$ being the $\mathfrak{a}$, $\mathfrak{a}^\flat$, and $\mathfrak{a}^{\flat\prime}$ therein, respectively).

Now, for the bottom rectangle, we are simply working with the Shimura variety $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})$ and hence are reduced to the case of $d=2$. Using Section 5.3, we see that the dotted downward arrow on the left is exactly the operator $T_\mathfrak{p}$ times a link morphism $\eta^\star_{\mathtt{S}_\mathfrak{a},\mathtt{S}_{\mathfrak{a}^\star}}$ associated to the link $\eta_{\mathtt{S}_\mathfrak{a},\mathtt{S}_{\mathfrak{a}^\star}}:\mathtt{S}_\mathfrak{a}\to\mathtt{S}_{\mathfrak{a}^\star}$, with indentation degree $-2\ell(\delta_{\mathfrak{b},\bullet})$ and shift

$$\varpi_{\bar{\mathfrak{q}}}^{-1}\boldsymbol{t}^{-1}_{\mathfrak{a}^\flat,\mathfrak{a}^\star}\boldsymbol{t}_{\mathfrak{a}^\flat,\mathfrak{a}}=\varpi_{\bar{\mathfrak{q}}}^{-1}\boldsymbol{t}^{-1}_{\mathfrak{a}^\star}\boldsymbol{t}_{\mathfrak{a}}. \tag{5.10.3}$$

To sum up, the morphism $\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{b}$ is the same as the composition of the downward arrows on the left in (5.10.2). So we have proved (5.10.1).

We now complete the inductive proof of Theorem 4.4. The condition for case (iii) implies that we are in the setup of Theorem 4.4(3). Assume that we have $\langle\mathfrak{a}|\mathfrak{b}\rangle=(-2)^{m_0}T^{m_T}$. The picture $D(\mathfrak{a}^\flat_\bullet,\mathfrak{b}_{\mathrm{res}})$ is given by removing from $D(\mathfrak{a},\mathfrak{b})$ the noncontractible loop consisting of $\delta_\mathfrak{a}$ and $\delta_\mathfrak{b}$. So we have

$$\langle\mathfrak{a}^\flat_\bullet|\mathfrak{b}_{\mathrm{res}}\rangle=T^{-1}\langle\mathfrak{a}|\mathfrak{b}\rangle=(-2)^{m_0}T^{m_T-1}.$$

By the inductive hypothesis applied to the Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\delta_\mathfrak{b}},\mathtt{T}_{\delta_\mathfrak{b}}})$ of lower dimension (where the shift $\boldsymbol{t}'_{\mathfrak{a}'}$ for a periodic semimeander $\mathfrak{a}'$ for $(\mathtt{S}_{\delta_\mathfrak{b}},\mathtt{T}_{\delta_\mathfrak{b}})$ is taken to be $\boldsymbol{t}_{\delta_\mathfrak{b},\tilde{\mathfrak{a}}'}$, where $\tilde{\mathfrak{a}}'$ is a periodic semimeander of $(\mathtt{S},\mathtt{T})$ consisting of all the arcs and semilines of $\mathfrak{a}'$ together with the arc $\delta_\mathfrak{b}$),

$$\mathrm{Res}_{\mathfrak{a}^\flat_\bullet}\circ\mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}=(-2)^{m_0}\cdot p^{(\ell(\mathfrak{a}^\flat_\bullet)+\ell(\mathfrak{b}_{\mathrm{res}}))/2}(T_\mathfrak{p}/p^{g/2})^{m_T-1}\circ\eta^\star_{\mathtt{S}_{\mathfrak{a}^\flat_\bullet},\mathtt{S}_{\mathfrak{b}_{\mathrm{res}}},(z')}, \tag{5.10.4}$$

where $\eta^\star_{\mathtt{S}_{\mathfrak{a}^\flat_\bullet},\mathtt{S}_{\mathfrak{b}_{\mathrm{res}}},(z')}$ is the trivial link morphism with shift

$$\boldsymbol{t}'_{\mathfrak{a}^\flat_\bullet}\boldsymbol{t}'^{-1}_{\mathfrak{b}_{\mathrm{res}}}\varpi_{\bar{\mathfrak{q}}}^{-m_T+1}=\boldsymbol{t}_{\delta_\mathfrak{b},\mathfrak{a}^\star}\boldsymbol{t}^{-1}_{\delta_\mathfrak{b},\mathfrak{b}}\varpi_{\bar{\mathfrak{q}}}^{-m_T+1}=\boldsymbol{t}_{\mathfrak{a}^\star}\boldsymbol{t}^{-1}_{\mathfrak{b}}\varpi_{\bar{\mathfrak{q}}}^{-m_T+1} \tag{5.10.5}$$

and indentation degree $z=\ell(\mathfrak{a}^\flat_\bullet)-\ell(\mathfrak{b}_{\mathrm{res}})-(m_T-1)g$. Combining (5.10.1) and (5.10.4) with the numerical equalities

$$\ell(\mathfrak{a}^\flat_\bullet)=\ell(\mathfrak{a})-g+\ell(\delta_{\mathfrak{b},\bullet})\qquad\text{and}\qquad\ell(\mathfrak{b}_{\mathrm{res}})=\ell(\mathfrak{b})-\ell(\delta_{\mathfrak{b},\bullet}),$$

we deduce that

$$\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{b}=(-2)^{m_0}p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}(T_\mathfrak{p}/p^{g/2})^{m_T}\circ\eta^\star_{\mathtt{S}_\mathfrak{a},\mathtt{S}_{\mathfrak{a}^\star}}\circ\eta^\star_{\mathtt{S}_{\mathfrak{a}^\flat_\bullet},\mathtt{S}_{\mathfrak{b}_{\mathrm{res}}},(z')}.$$

The composition of the last two link morphisms is a link morphism $\mathrm{S}_{\mathfrak{a}} \to \mathrm{S}_{\mathfrak{a}^\star} = \mathrm{S}_{\mathfrak{a}^\flat_\bullet} \to \mathrm{S}_{\mathfrak{b}_{\mathrm{res}}} = \mathrm{S}_{\mathfrak{b}}$, whose indentation degree is

$$-2\ell(\delta_{\mathfrak{b},\bullet}) + z = \ell(\mathfrak{a}) - \ell(\mathfrak{b}) - m_T g$$

and whose shift is equal to the product of (5.10.3) and (5.10.5), or, explicitly,

$$\varpi_{\bar{\mathfrak{q}}}^{-1} \boldsymbol{t}_{\mathfrak{a}^\star}^{-1} \boldsymbol{t}_{\mathfrak{a}} \cdot \boldsymbol{t}_{\mathfrak{a}^\star} \boldsymbol{t}_{\mathfrak{b}}^{-1} \varpi_{\bar{\mathfrak{q}}}^{-m_T+1} = \boldsymbol{t}_{\mathfrak{a}} \boldsymbol{t}_{\mathfrak{b}}^{-1} \varpi_{\bar{\mathfrak{q}}}^{-m_T}.$$

This completes Theorem 4.4(3) in this case.

5.11. Case (iv)

Recall that $\delta_{\mathfrak{b}}$ is a basic arc of $\mathfrak{b}$ linking τ with τ^- from right to left. We are looking at the situation when at least one of τ and τ^- is linked to an arc in $\mathfrak{a}$ that is not connected to the other node. We start with a long list of combinatorics construction, followed by two examples.

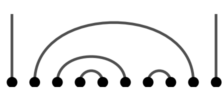
- Let $\mathfrak{a}^\circ$ be the periodic semimeander for $\mathrm{S}_{\delta_{\mathfrak{b}}}$ given by first replacing the nodes τ, τ^- by plus signs, then adjoining the basic arc $\delta_{\mathfrak{b}}$ to $\mathfrak{a}$ from *underneath* the band to connect to the arcs or links that are already linked to the nodes τ^-, τ, and finally continuously deforming the picture so that all arcs are above the band and all semilines are straight. Intuitively, one can view the last step as "pulling the strings to tighten the drawing."
- Let $\mathfrak{a}^\star$ denote the periodic semimeander for S modified from $\mathfrak{a}^\star$ by replacing the plus signs τ, τ^- by nodes and adjoining them by the arc $\delta_{\mathfrak{b}}$.
- Let $\mathfrak{a}^\flat$ denote the periodic semimeander for S that consists of two semilines at both τ and τ^-, all arcs in $\mathfrak{a}$ that do not intersect with these two semilines, and semilines at the nodes that are not connected to anything above. Let r' $(< r)$ denote the number of arcs in $\mathfrak{a}^\flat$ so that $\mathfrak{a}^\flat \in \mathfrak{B}_{\mathrm{S}}^{r'}$.
- Let $\mathfrak{a}^\flat_+$ denote the periodic semimeander for $\mathrm{S}_{\delta_{\mathfrak{b}}}$ obtained by removing the two semilines at both τ and τ^- from $\mathfrak{a}^\flat$ and replacing the nodes at τ, τ^- by plus signs.
- We use $\mathfrak{a}^\flat_\tau$ to denote the periodic semimeander for S given by replacing in $\mathfrak{a}^\flat$ the two semilines connected to τ and τ^- by $\delta_{\mathfrak{b}}$.
- We use $\delta_{\mathfrak{b}/\mathfrak{a}^\flat}$ to denote the periodic semimeander for $\mathrm{S}_{\mathfrak{a}^\flat}$ consisting of only one arc $\delta_{\mathfrak{b}}$ (and all semilines of $\mathfrak{a}^\flat_+$).
- We choose and fix an arc $\delta_{\mathfrak{a}}$ of $\mathfrak{a}$ such that
  - *Case (a)*: either τ is the left end-node of $\delta_{\mathfrak{a}}$, or
  - *Case (b)*: τ^- is the right end-node of $\delta_{\mathfrak{a}}$.

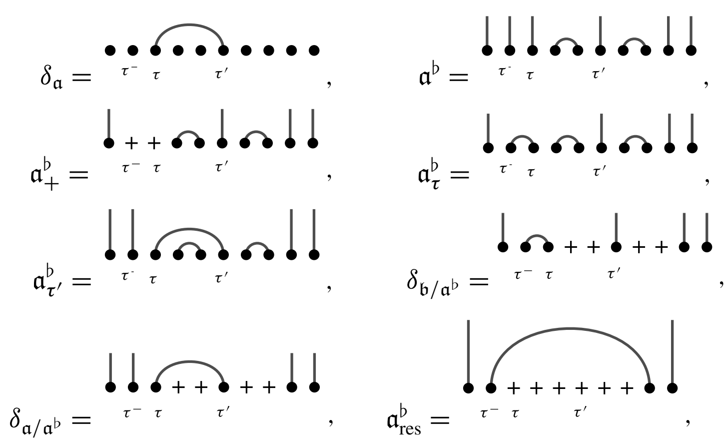
  Such an arc $\delta_{\mathfrak{a}}$ exists under the assumption of Case (iv) (there might be one or two such arcs). We use τ' to denote the right endpoint of $\delta_{\mathfrak{a}}$. Thus, τ' is neither τ nor τ^- in Case (a), and τ' is the same as τ^- in Case (b).

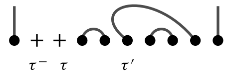
- We use $\mathfrak{a}^\flat_{\tau'}$ to denote the periodic semimeander for $\mathtt{S}$ given by deleting from $\mathfrak{a}^\flat$ the two semilines connected to the end-nodes of $\delta_\mathfrak{a}$ and then adjoining the arc $\delta_\mathfrak{a}$.
- We use $\delta_{\mathfrak{a}/\mathfrak{a}^\flat}$ to denote the periodic semimeander for $\mathtt{S}_{\mathfrak{a}^\flat}$ consisting of only one arc $\delta_\mathfrak{a}$ (and all semilines of $\mathfrak{a}^\flat_{\tau'}$).
- We use $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ to denote the link from $\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}}$ to $\mathtt{S}_{\mathfrak{a}^\flat_\tau}$ given by the reduction of $D(\delta_{\mathfrak{b}/\mathfrak{a}^\flat}, \delta_{\mathfrak{a}/\mathfrak{a}^\flat})$ as defined in Section 3.1.
- We use $\mathfrak{a}^\flat_{\text{res}}$ to denote the periodic semimeander for $\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}}$ given by deleting all arcs in $\mathfrak{a}$ that already appeared in $\mathfrak{a}^\flat_{\tau'}$, and changing their end-nodes to plus signs.
- We use $\mathfrak{a}^\circ_{\text{res}}$ to denote the periodic semimeander for $\mathtt{S}_{\mathfrak{a}^\circ}$ with nodes given by deleting all arcs in $\mathfrak{a}^\circ$ that already appeared in $\mathfrak{a}^\flat_+$, and changing their end-nodes to plus signs.
- We use $\eta_{\mathfrak{a},\mathfrak{a}^\star}$ to denote the link from $\mathtt{S}_\mathfrak{a}$ to $\mathtt{S}_{\mathfrak{a}^\star} = (\mathtt{S}_{\delta_\mathfrak{b}})_{\mathfrak{a}^\circ}$, which is the restriction of $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ to $\mathtt{S}_\mathfrak{a}$.
- We use $\mathfrak{b}_{\text{res}}$ to denote the semimeander for $\mathtt{S}_{\delta_\mathfrak{b}}$ obtained by deleting the arc $\delta_\mathfrak{b}$ and replacing the nodes τ, τ^- by plus signs.

We now give two examples. In both instances, $\mathfrak{b}$ has a basic arc connecting node 1 with 2 (starting with node 0 on the left). So node 1 is τ^- and node 2 is τ.

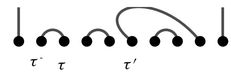
Example 1

We take $\mathfrak{a} =$ . Then the arc $\delta_\mathfrak{a}$ has to be the one connecting nodes 2 and 5 and τ' is node 5. We are in Case (a), and we have

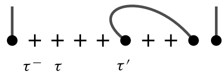


$\mathfrak{a}^\circ =$ 

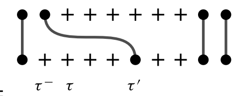
,[18]

$\mathfrak{a}^\star =$ 

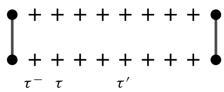
,

$\mathfrak{a}^\circ_{\mathrm{res}} =$ 

,

$\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau} =$ 

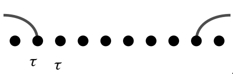
,

$\eta_{\mathfrak{a},\mathfrak{a}^\star} =$ 

.

Example 2

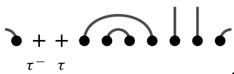
We take $\mathfrak{a} =$ . Then the arc $\delta_\mathfrak{a}$ has to be the one connecting nodes 1 and 8 through the imaginary boundary at $x = -1/2$ and $x = g - 1/2$. We are in Case (b), so $\tau' = \tau^-$ is the node 1. We have

$\delta_\mathfrak{a} =$ 

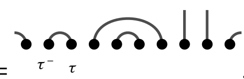
,

$\mathfrak{a}^\flat =$ 

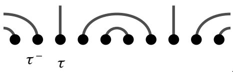
,

$\mathfrak{a}^\flat_+ =$ 

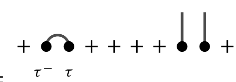
,

$\mathfrak{a}^\flat_\tau =$ 

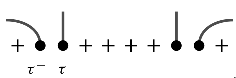
,

$\mathfrak{a}^\flat_{\tau'} =$ 

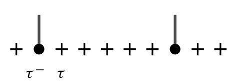
,

$\delta_{\mathfrak{b}/\mathfrak{a}^\flat} =$ 

,

$\delta_{\mathfrak{a}/\mathfrak{a}^\flat} =$ 

,

$\mathfrak{a}^\flat_{\mathrm{res}} =$ 

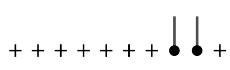
,

$\mathfrak{a}^\circ =$ 

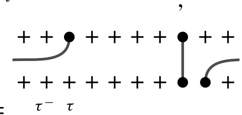
,

$\mathfrak{a}^\star =$ 

,

$\mathfrak{a}^\circ_{\mathrm{res}} =$ 

,

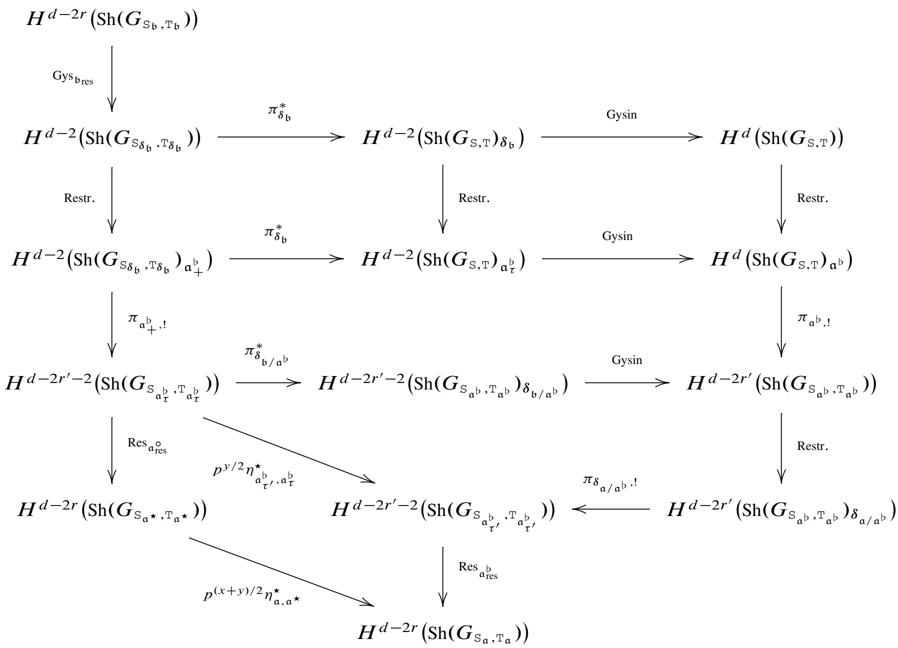
$\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau} = \eta_{\mathfrak{a},\mathfrak{a}^\star} =$ 

.[19]

[18]We give special shape to the arc linking nodes 5 and 8 here to remind the reader that this arc is obtained by "pulling the strings."

[19]When either τ or τ^- is connected to a semiline, a lot of the new periodic semimeanders constructed are either "simple" or "similar" to $\mathfrak{a}$.

Using the discussion in Section 5.5, we see that the morphism $\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}}$ is the composition of the following diagram from the top left to the bottom, first through $\mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}$ and then all the way to the right and then all the way downward, and finally through $\pi_{\delta_{\mathfrak{a}/\mathfrak{a}^\flat},!}$ and $\mathrm{Res}_{\mathfrak{a}^\flat_{\mathrm{res}}}$.



(5.11.1)

Here, the numbers x, y and the link morphisms $\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ and $\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}$ will be defined explicitly later. For simplicity, we have omitted the Tate twists from the notation, and each cohomology group $H^a(\star)$ should be understood as $H^a(\star)(b)$ with $a - 2b = d - 2r$; for instance, $H^{d-2r'-2}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}}))$ should be understood as $H^{d-2r'-2}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}}))(r - r' - 1)$.

We now explain the commutativity of this diagram. The commutativity of the top left square in the diagram above follows from the commutative diagram of morphisms of varieties, and that of the top right square follows from (4.1.2) and the fact that $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat_\tau}$ is the transversal intersection of $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\mathfrak{a}^\flat}$ and $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})_{\delta_{\mathfrak{b}}}$ in $\mathrm{Sh}(G_{\mathtt{S},\mathtt{T}})$. The commutativity of the middle rectangle follows from that of (5.6.3) (applied with our $\mathfrak{a}^\flat_\tau$ and $\mathfrak{a}^\flat$ being the $\mathfrak{a}$ and $\mathfrak{a}^\flat$ therein, respectively). The commutativity of the lower trapezoid will follow from applying Section 5.4 (applied to the Shimura variety $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})$ with our $\delta_{\mathfrak{a}/\mathfrak{a}^\flat}$ and $\delta_{\mathfrak{b}/\mathfrak{a}^\flat}$ being the $\mathfrak{a}$ and $\mathfrak{b}$ therein), once we have clarified the meaning of y and $\eta^{\star}_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ in Section 5.12 later. Finally, the commutativity of the bottom parallelogram will be justified in Section 5.13 and Lemma 5.14 later.

To sum up, the morphism $\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{b}$ will be the composition of (5.11.1) from the top left to the bottom by first going all the way down and through $\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}$. This gives the following equality,

$$\mathrm{Res}_\mathfrak{a}\circ\mathrm{Gys}_\mathfrak{b} = p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ\mathrm{Res}_{\mathfrak{a}^\star}\circ\mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}, \tag{5.11.2}$$

which we will use to complete the inductive proof of Theorem 4.4 in Case (iv), as we will explain in Section 5.15.

5.12. Link morphism $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$

Now, let us get to the details, starting with the link morphism associated with $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$. We distinguish the two cases:

Case (a). Suppose that the left node of $\delta_\mathfrak{a}$ is τ. Example 1 above falls into this case. All the curves in the link $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ are semilines, except for one that turns to the right, which we denote by ξ. The curve ξ sends τ^- to τ'. By Theorem 2.32(2), there exists a link morphism $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}$ with indentation degree $\ell(\delta_\mathfrak{a})-\ell(\delta_\mathfrak{b})$ and shift $\boldsymbol{t}_{\mathfrak{a}^\flat_{\tau'}}\boldsymbol{t}^{-1}_{\mathfrak{a}^\flat_\tau}$ (and also a link morphism on the local system as in Theorem 2.32(2)(d)), which fits into the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\delta_\mathfrak{a}} & \hookleftarrow & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\{\tau',\tau\}} & \hookrightarrow & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat},\mathtt{T}_{\mathfrak{a}^\flat}})_{\delta_\mathfrak{b}} \\
\Big\downarrow{\scriptstyle \pi_{\delta_\mathfrak{a}}} & \swarrow{\scriptstyle\cong} & & & \Big\downarrow{\scriptstyle \pi_{\delta_\mathfrak{b}}} \\
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}}) & & \xrightarrow{\quad\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}\quad} & & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})
\end{array}$$

By Theorem 2.32(2)(c), $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}$ is finite flat of degree p^y with $y := v(\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}) = \ell(\delta_\mathfrak{a}) + \ell(\delta_\mathfrak{b})$. Let $\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}\colon H^{d-2r'-2}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}}))\to H^{d-2r'-2}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}}))$ denote the induced link homomorphism on the cohomology groups. By the same argument as in Section 5.4, we see that the trapezoid in the diagram (5.11.1) is commutative.

Case (b). Suppose now that the right node of $\delta_\mathfrak{a}$ is τ^-. Example 2 above falls into this case. Then the only genuine curve in the link $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ is turning to the left with displacement $y=\ell(\delta_\mathfrak{a})+\ell(\delta_\mathfrak{b})$. Let $\eta_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'}}$ be the inverse link of $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$. Applying the discussion in *Case (a)* to $\eta_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'}}$, one gets a link morphism $\eta_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'},\sharp}\colon \mathrm{Sh}(G_{\mathtt{S}^\flat_\tau,\mathtt{T}^\flat_\tau})\to\mathrm{Sh}(G_{\mathtt{S}^\flat_{\tau'},\mathtt{T}^\flat_{\tau'}})$ of indentation degree $\ell(\delta_\mathfrak{b})-\ell(\delta_\mathfrak{a})$ and shift $\boldsymbol{t}_{\mathfrak{a}^\flat_\tau}\boldsymbol{t}^{-1}_{\mathfrak{a}^\flat_{\tau'}}$. By Lemma 2.29, we get a well-defined link morphism on the cohomology groups

$$\eta^{\star}_{\mathfrak{a}^{\flat}_{\tau'},\mathfrak{a}^{\flat}_{\tau}} = (\eta^{\star}_{\mathfrak{a}^{\flat}_{\tau},\mathfrak{a}^{\flat}_{\tau'}})^{-1} = p^{-y/2}\,\mathrm{Tr}_{\eta_{\mathfrak{a}^{\flat}_{\tau},\mathfrak{a}^{\flat}_{\tau'},\sharp}} \colon H^{d-2r'-2}_{\mathrm{et}}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau}}})\big) \to H^{d-2r'-2}_{\mathrm{et}}\big(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau'}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau'}}})\big) \quad (5.12.1)$$

of indentation degree $\ell(\mathfrak{a}) - \ell(\mathfrak{b})$ associated to the link $\eta_{\mathfrak{a}^{\flat}_{\tau'},\mathfrak{a}^{\flat}_{\tau}}$ and shift $\boldsymbol{t}_{\mathfrak{a}^{\flat}_{\tau'}}\boldsymbol{t}^{-1}_{\mathfrak{a}^{\flat}_{\tau}}$. Now the argument as in Section 5.4 proves the commutativity of the trapezoid in the diagram (5.11.1).

5.13. *Commutativity of the parallelogram in (5.11.1)*

We continue the discussion above by separating the two cases.

Case (a). Consider the $(r-r'-1)$-step iterated $\mathbb{P}^1$-bundle $\pi_{\mathfrak{a}^{\flat}_{\mathrm{res}}} \colon \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau'}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau'}}})_{\mathfrak{a}^{\flat}_{\mathrm{res}}} \to \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$. By applying repeatedly [33, Proposition 7.17] and Construction 2.12, one produces a commutative diagram:

$$\begin{array}{ccccccccccc}
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau'}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau'}}})_{\mathfrak{a}^{\flat}_{\mathrm{res}}} & \xrightarrow{\pi^{\flat}_1} & X_1 & \xrightarrow{\pi^{\flat}_2} & X_2 & \longrightarrow & \cdots & \longrightarrow & X_{r-r'-2} & \xrightarrow{\pi^{\flat}_{r-r'-1}} & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}) \\
\downarrow{\scriptstyle \eta_{\mathfrak{a}^{\flat}_{\tau'},\mathfrak{a}^{\flat}_{\tau},\sharp}} & & \downarrow{\scriptstyle \eta_{1,\sharp}} & & \downarrow{\scriptstyle \eta_{2,\sharp}} & & & & \downarrow{\scriptstyle \eta_{r-r'-2,\sharp}} & & \downarrow{\scriptstyle \eta_{\mathfrak{a},\mathfrak{a}^{\star},\sharp}} \\
\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau}}})_{\mathfrak{a}^{\circ}_{\mathrm{res}}} & \xrightarrow{\pi^{\circ}_1} & Y_1 & \xrightarrow{\pi^{\circ}_2} & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_{r-r'-2} & \xrightarrow{\pi^{\circ}_{r-r'-1}} & \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\star}},\mathtt{T}_{\mathfrak{a}^{\star}}})
\end{array} \quad (5.13.1)$$

where $\pi^{\flat}_i$ and π°_i are all $\mathbb{P}^1$-fibrations, the vertical arrows are link morphisms (associated to certain links), and the composition of the top (resp., bottom) horizontal arrows is $\pi_{\mathfrak{a}^{\flat}_{\mathrm{res}}}$ (resp., $\pi_{\mathfrak{a}^{\circ}_{\mathrm{res}}}$). There exist at the same time link morphisms $\eta^{\sharp}_i$ and $\eta^{\sharp}_{\mathfrak{a},\mathfrak{a}^{\circ}}$ on the étale local systems satisfying a similar commutative diagram. We explain now how to construct $\eta_{1,\sharp} \colon X_1 \to Y_1$; one chooses a basic arc $\delta_{\mathfrak{c}}$ in $\mathfrak{a}^{\flat}_{\mathrm{res}}$. Let $\mathfrak{a}^{\flat}_{\mathrm{res},1}$ be the periodic semimeander obtained by removing $\delta_{\mathfrak{c}}$ from $\mathfrak{a}^{\flat}_{\mathrm{res}}$ and replacing the end-nodes of $\delta_{\mathfrak{c}}$ by plus signs, and let $\mathfrak{a}^{\flat}_{\tau',1}$ be the periodic semimeander obtained by removing from $\mathfrak{a}^{\flat}_{\tau'}$ the semilines at the end-nodes of $\delta_{\mathfrak{c}}$ and adjoining $\delta_{\mathfrak{c}}$. Put $X_1 := \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau',1}}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau',1}})_{\mathfrak{a}^{\flat}_{\mathrm{res},1}}$, and denote by

$$\pi^{\flat}_1 \colon \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^{\flat}_{\tau'}},\mathtt{T}_{\mathfrak{a}^{\flat}_{\tau'}}})_{\mathfrak{a}^{\flat}_{\mathrm{res}}} \longrightarrow X_1$$

the $\mathbb{P}^1$-fibration given by the arc $\delta_{\mathfrak{c}}$. Let $\delta_{\mathfrak{c}^{\circ}}$ denote the arc $\eta_{\mathfrak{a}^{\flat}_{\tau'},\mathfrak{a}^{\flat}_{\tau}}(\delta_{\mathfrak{c}})$ obtained by extending $\delta_{\mathfrak{c}}$ using the curves of $\eta_{\mathfrak{a}^{\flat}_{\tau'},\mathfrak{a}^{\flat}_{\tau}}$ at the end-nodes of $\delta_{\mathfrak{c}}$. This $\delta_{\mathfrak{c}^{\circ}}$ is a basic arc in $\mathfrak{a}^{\circ}_{\mathrm{res}}$. We define periodic semimeanders $\mathfrak{a}^{\flat}_{\tau,1}$ and $\mathfrak{a}^{\circ}_{\mathrm{res},1}$ in the same way as $\mathfrak{a}^{\flat}_{\tau',1}$ and $\mathfrak{a}^{\flat}_{\mathrm{res},1}$ with $\delta_{\mathfrak{c}}$ replaced by $\delta_{\mathfrak{c}^\circ}$. Then we have a $\mathbb{P}^1$-fibration

$$\pi_1^\circ\colon \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})_{\mathfrak{a}^\circ_{\mathrm{res}}} \longrightarrow Y_1 := \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau,1}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau,1}}})_{\mathfrak{a}^\circ_{\mathrm{res},1}}.$$

If $\eta_1\colon \mathtt{S}_{\mathfrak{a}^\flat_{\tau',1}} \to \mathtt{S}_{\mathfrak{a}^\flat_{\tau,1}}$ denotes the link induced by $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$, then [33, Proposition 7.17] and Construction 2.12 implies the existence of the link morphism $\eta_{1,\sharp}$, which fits into the left commutative square of (5.13.1). This finishes the construction of X_1 and Y_1. The induced link η_1 has the same property as $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$; namely, all the curves of η_1 are semilines, except possibly one turning to the right. The rest of (5.13.1) can be constructed inductively in a similar way.

Since we require the diagram (5.13.1) to be commutative, by Remark 2.13, the link morphism $\eta_{\mathfrak{a},\mathfrak{a}^\star,\sharp}$ has shift

$$\boldsymbol{t}_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}}\cdot \boldsymbol{t}^{-1}_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\star}\cdot(\text{shift of }\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}) = \boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}^{-1}_{\mathfrak{a}^\star}.$$

Moreover, the indentation degree of $\eta_{\mathfrak{a},\mathfrak{a}^\star,\sharp}$ is $\ell(\delta_{\mathfrak{a}})-\ell(\delta_{\mathfrak{b}})+\ell(\mathfrak{a}^\flat_{\mathrm{res}})-\ell(\mathfrak{a}^\circ_{\mathrm{res}})$ if $\mathfrak{p}$ splits in E/F and degree 0 if $\mathfrak{p}$ is inert in E/F. Note that even though each $\eta_{i,\sharp}$ is not unique (since there are many ways to choose a basic arc of $\mathfrak{a}^\flat_{\mathrm{res}}$ for instance), the final link morphism $\eta_{\mathfrak{a},\mathfrak{a}^\star,\sharp}$ is uniquely determined by the uniqueness of link morphisms. By [33, Proposition 7.17(3)] and Construction 2.12, $\eta_{\mathfrak{a},\mathfrak{a}^\star,\sharp}$ is finite flat of degree $p^{v(\eta_{\mathfrak{a},\mathfrak{a}^\star})}$. We have thus the normalized link morphisms $\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ and $\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}$ on the corresponding cohomology groups as defined in (2.23.1) induced by $(\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp},\eta^\sharp_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau})$ and $(\eta^\sharp_{\mathfrak{a},\mathfrak{a}^\star},\eta^\sharp_{\mathfrak{a},\mathfrak{a}^\star})$, respectively.

Case (b). Suppose now that the right node of $\delta_{\mathfrak{a}}$ is τ^-. Applying the discussion in *Case (a)* to the inverse link $\eta_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'}}$, one gets a link morphism $\eta_{\mathfrak{a}^\star,\mathfrak{a},\sharp}$: $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{T}_{\mathfrak{a}^\star}})\to \mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}})$ associated to the inverse link of $\eta_{\mathfrak{a},\mathfrak{a}^\star}$ of indentation degree $\ell(\delta_{\mathfrak{b}})-\ell(\delta_{\mathfrak{a}})+\ell(\mathfrak{a}^\circ_{\mathrm{res}})-\ell(\mathfrak{a}^\flat_{\mathrm{res}})$, and shift $\boldsymbol{t}^{-1}_{\mathfrak{a}}\boldsymbol{t}_{\mathfrak{a}^\star}$. By Lemma 2.29, we get a well-defined link morphism on the cohomology groups

$$\eta^\star_{\mathfrak{a},\mathfrak{a}^\star} = (\eta^\star_{\mathfrak{a}^\star,\mathfrak{a}})^{-1} = p^{v(\eta_{\mathfrak{a},\mathfrak{a}^\star})/2}\,\mathrm{Tr}_{\eta_{\mathfrak{a}^\star,\mathfrak{a},\sharp}}\colon H^{d-2r}(\mathrm{Sh}_{G_{\mathtt{S},\mathtt{T}}}) \to H^{d-2r}(\mathrm{Sh}_{G_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{T}_{\mathfrak{a}^\star}}}) \quad (5.13.2)$$

of indentation degree $\ell(\delta_{\mathfrak{a}})-\ell(\delta_{\mathfrak{b}})+\ell(\mathfrak{a}^\flat_{\mathrm{res}})-\ell(\mathfrak{a}^\circ_{\mathrm{res}})$ and shift $\boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}^{-1}_{\mathfrak{a}^\star}$ associated to the link $\eta_{\mathfrak{a},\mathfrak{a}^\star}$.

LEMMA 5.14
Under the above notation, put $x = \ell(\mathfrak{a}^\flat_{\mathrm{res}}) - \ell(\mathfrak{a}^\circ_{\mathrm{res}})$ *and* $y = \ell(\delta_\mathfrak{a}) + \ell(\delta_\mathfrak{b})$. *Then in both Case (a) and Case (b) above, one has a commutative diagram of cohomology groups:*

$$\begin{array}{ccccc}
H^{d-2r'-2}_{\mathrm{et}}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})) & \xrightarrow{\mathrm{Rest.}} & H^{d-2r'-2}_{\mathrm{et}}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})_{\mathfrak{a}^\star_{\mathrm{res}}}) & \xrightarrow{\pi_{\mathfrak{a}^\circ_{\mathrm{res}},!}} & H^{d-2r}_{\mathrm{et}}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{T}_{\mathfrak{a}^\star}})) \\
\Big\downarrow {\scriptstyle p^{y/2}\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}} & & \Big\downarrow {\scriptstyle p^{y/2}\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}} & & \Big\downarrow {\scriptstyle p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}} \\
H^{d-2r'-2}_{\mathrm{et}}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}})) & \xrightarrow{\mathrm{Rest.}} & H^{d-2r'-2}_{\mathrm{et}}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}})_{\mathfrak{a}^\flat_{\mathrm{res}}}) & \xrightarrow{\pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}} & H^{d-2r}_{\mathrm{et}}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}},\mathtt{T}_{\mathfrak{a}}}))
\end{array}$$

Note that the composite of the top (resp., bottom) two horizontal morphisms above is exactly $\mathrm{Res}_{\mathfrak{a}^\circ_{\mathrm{res}}}$ *(resp.,* $\mathrm{Res}_{\mathfrak{a}^\flat_{\mathrm{res}}}$*). So this verifies the commutativity of the parallelogram in (5.11.1).*

Proof
The commutativity of the left square is evident. We check the commutativity of the right-hand side square case by case. Suppose first that we are in *Case (a)* (i.e., the left end-node of $\delta_\mathfrak{a}$ is τ). We distinguish three subcases:

Case (a1). τ^- is linked to a semiline in $\mathfrak{a}$. Then both $\mathfrak{a}^\flat_{\mathrm{res}}$ and $\mathfrak{a}^\circ_{\mathrm{res}}$ contain no arcs. It follows that $x = 0$, and $\pi_{\mathfrak{a}^\flat_{\mathrm{res}}}$ and $\pi_{\mathfrak{a}^\circ_{\mathrm{res}}}$ are isomorphisms. In this case, the commutativity of the right-hand side square is trivial.

Case (a2). τ^- is the left end-node of an arc in $\mathfrak{a}$. Example 1 above falls into this case. It is easy to see that $x = y$, and that the link $\eta_{\mathfrak{a},\mathfrak{a}^\star}$ contains only semi-lines. By [33, Proposition 7.17(3)] and Construction 2.12, $\eta_{\mathfrak{a},\mathfrak{a}^\star,\sharp}$ is an isomorphism. Consider the commutative diagram (5.13.1). Both top and bottom rows are factorizations of $(r - r' - 1)$-step iterated $\mathbb{P}^1$-bundles as in (4.2.1). For each $1 \le i \le r - r' - 1$, let $\xi'_i \in H^2_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))$ (resp., $\xi_i \in H^2_{\mathrm{et}}(\mathrm{Sh}_{K_p}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))$) be the inverse image of the first Chern class of the tautological quotient line bundle of $\pi^\flat_i$ (resp., π°_i) as considered in Section 4.2. Note that the only curve in $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ links the *left* end-node of an arc of $\mathfrak{a}^\flat_{\mathrm{res}}$ to the *left* end-node of an arc of $\mathfrak{a}^\circ_{\mathrm{res}}$. Then by applying iteratively [33, Proposition 7.17(3)] and Construction 2.12, there exists a unique integer i_0 with $1 \le i_0 \le r - r' - 1$ such that $\eta^*_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}(\xi_{i_0}) = p^y \xi'_{i_0}$, and $\eta^*_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}(\xi_i) = \xi'_i$ for all $i \ne i_0$. Let

$$z = \sum_{1 \le j \le r-r'-1} \Big(\sum_{1 \le i_1 < \cdots < i_j \le r-r'-1} \pi^*_{\mathfrak{a}^\circ_{\mathrm{res}}}(z_{i_1,\dots,i_j}) \cup \xi_{i_1} \cup \cdots \cup \xi_{i_j} \Big)$$

be an element of $H^{d-2r'-2}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})_{\mathfrak{a}^\circ_{\mathrm{res}}})$ with $z_{i_1,\dots,i_j} \in H^{d-2r'-2-2j}(\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\star},\mathtt{T}_{\mathfrak{a}^\star}}))$. Then one has

$$\begin{aligned}
&\pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}\big(p^{y/2}\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}(z)\big)\\
&= \pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}\eta^*_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}(z)\\
&= p^y\pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}\Big(\sum\big(\eta^*_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}\big(\pi^*_{\mathfrak{a}^\circ_{\mathrm{res}}}(z_{i_1,\dots,i_j})\big)\cup\xi'_{i_1}\cup\cdots\cup\xi'_{i_j}\big)\Big)\\
&= p^y\pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}\Big(\sum\big(\pi^*_{\mathfrak{a}^\flat_{\mathrm{res}}}\big(\eta^*_{\mathfrak{a},\mathfrak{a}^\star,\sharp}(z_{i_1,\dots,i_j})\big)\cup\xi'_{i_1}\cup\cdots\cup\xi'_{i_j}\big)\Big)\\
&= p^y\eta^*_{\mathfrak{a},\mathfrak{a}^\star,\sharp}(z_{1,\dots,r-r'-1}) = p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\big(\pi_{\mathfrak{a}^\circ_{\mathrm{res}},!}(z)\big),
\end{aligned}$$

where the forth and fifth equalities use the formula (4.2.3). This shows the commutativity of the right square in the lemma.

Case (a3). τ^- is the right end-node of an arc in $\mathfrak{a}$. Then $x=-y$ and $\eta_{\mathfrak{a},\mathfrak{a}^\star}$ contains only semilines. Hence, $\eta_{\mathfrak{a},\mathfrak{a}^\star,\sharp}$ is an isomorphism as in *Case (a2)*. We want to show

$$\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ\pi_{\mathfrak{a}^\circ_{\mathrm{res}},!} = \pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}\circ\big(p^{y/2}\eta^\star_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}\big).$$

The argument is quite similar to that of *Case (a2)*. Let ξ_i,ξ'_i be as defined in *Case (a2)* for $1\le i\le r-r'-1$. Then by [33, Proposition 7.17(3)], we have $\eta^*_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau,\sharp}(\xi_i)=\xi'_i$ for all $1\le i\le r-r'-1$ (this differs from the situation of *Case (a2)* because the unique curve in $\eta_{\mathfrak{a}^\flat_{\tau'},\mathfrak{a}^\flat_\tau}$ links the *right* end-node of an arc of $\mathfrak{a}^\flat_{\mathrm{res}}$ to the *right* end-node of an arc of $\mathfrak{a}^\circ_{\mathrm{res}}$). Then the rest of the computation is the same as in *Case (a2)*.

Consider now *Case (b)* (i.e., the right end-node of $\delta_\mathfrak{a}$ is τ^-). Symmetrically, we have three subcases:

Case (b1). τ is linked to a semiline in $\mathfrak{a}$. Then as in *Case (a1)*, we have $x=0$, and $\pi_{\mathfrak{a}^\flat_{\mathrm{res}}}$ and $\pi_{\mathfrak{a}^\circ_{\mathrm{res}}}$ are both isomorphisms. The commutativity of the right hand side square is trivial.

Case (b2). τ is the left end-node of an arc in $\mathfrak{a}$. Then $x=-y$, and $\eta_{\mathfrak{a},\mathfrak{a}^\star}$ contains only semilines. Hence, $\eta_{\mathfrak{a}^\star,\mathfrak{a},\sharp}$ is an isomorphism as in *Case (a2)*. By (5.12.1) and (5.13.2), the desired commutativity is equivalent to

$$\mathrm{Tr}_{\eta_{\mathfrak{a}^\star,\mathfrak{a},\sharp}}\circ\pi_{\mathfrak{a}^\circ_{\mathrm{res}},!} = \pi_{\mathfrak{a}^\flat_{\mathrm{res}},!}\circ\mathrm{Tr}_{\eta_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'},\sharp}},$$

which is an easy consequence of the compatibility of trace maps with composition.

Case (b3). τ is the right end-node of an arc in $\mathfrak{a}$. Then $x = y$, and $\eta_{\mathfrak{a}^\star,\mathfrak{a},\sharp}$ is an isomorphism as in *Case (a2)*. The desired commutativity is equivalent to

$$\pi_{\mathfrak{a}^\flat_{\mathrm{res}},!} \circ p^{y/2}(\eta^\star_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'}})^{-1} = p^y(\eta^\star_{\mathfrak{a}^\star,\mathfrak{a}})^{-1} \circ \pi_{\mathfrak{a}^\circ_{\mathrm{res}},!}$$
$$\iff \quad \eta^\star_{\mathfrak{a}^\star,\mathfrak{a}} \circ \pi_{\mathfrak{a}^\flat_{\mathrm{res}},!} = \pi_{\mathfrak{a}^\circ_{\mathrm{res}},!} \circ (p^{y/2}\eta^\star_{\mathfrak{a}^\flat_\tau,\mathfrak{a}^\flat_{\tau'}}).$$

Thus the situation is exactly the same as *Case (a3)* above (except for switching the roles of $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_{\tau'}},\mathtt{T}_{\mathfrak{a}^\flat_{\tau'}}})$ and $\mathrm{Sh}(G_{\mathtt{S}_{\mathfrak{a}^\flat_\tau},\mathtt{T}_{\mathfrak{a}^\flat_\tau}})$), and we conclude by the same arguments. □

5.15. End of the proof in Case (iv)

We are now in position to complete the inductive proof of Theorem 2.32 in Case (iv). We have shown the commutativity of the diagram (5.11.1), from which we deduce (5.11.2):

$$\mathrm{Res}_{\mathfrak{a}} \circ \mathrm{Gys}_{\mathfrak{b}} = p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star} \circ \mathrm{Res}_{\mathfrak{a}^\circ} \circ \mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}},$$

where $\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}$ is the link homomorphism associated to the link $\eta_{\mathfrak{a},\mathfrak{a}^\star}: \mathtt{S}_{\mathfrak{a}} \to \mathtt{S}_{\mathfrak{a}^\star}$ with

- indentation $\ell(\delta_{\mathfrak{a}}) - \ell(\delta_{\mathfrak{b}}) + \ell(\mathfrak{a}^\flat_{\mathrm{res}}) - \ell(\mathfrak{a}^\circ_{\mathrm{res}})$ if p splits in E/F and trivial if p is inert in E/F,
- and shift $\boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}^{-1}_{\mathfrak{a}^\star}$.

Before proceeding, we point out the following equality of shifts which we will use later:

$$\boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}^{-1}_{\mathfrak{a}^\star} \cdot \boldsymbol{t}_{\delta_{\mathfrak{b}},\mathfrak{a}^\star}\boldsymbol{t}^{-1}_{\delta_{\mathfrak{b}},\mathfrak{b}} = \boldsymbol{t}_{\mathfrak{a}}\boldsymbol{t}^{-1}_{\mathfrak{b}}. \tag{5.15.1}$$

Also, we point out that our decomposition of periodic semimeanders gives numerical equalities of spans:

$$\begin{gathered}\ell(\mathfrak{a}) = \ell(\mathfrak{a}^\flat_+) + \ell(\delta_{\mathfrak{a}/\mathfrak{a}^\flat}) + \ell(\mathfrak{a}^\flat_{\mathrm{res}}), \qquad \ell(\mathfrak{a}^\circ) = \ell(\mathfrak{a}^\flat_+) + \ell(\mathfrak{a}^\circ_{\mathrm{res}}), \qquad \text{and} \\ \ell(\mathfrak{b}) = \ell(\delta_{\mathfrak{b}}) + \ell(\mathfrak{b}_{\mathrm{res}}).\end{gathered} \tag{5.15.2}$$

This (and the trivial equality $\ell(\delta_{\mathfrak{a}}) = \ell(\delta_{\mathfrak{a}/\mathfrak{a}^\flat})$) implies that the indentation degree of $\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}$ when p splits in E/F is equal to

$$\ell(\delta_{\mathfrak{a}}) - \ell(\delta_{\mathfrak{b}}) + \big(\ell(\mathfrak{a}^\flat_{\mathrm{res}}) - \ell(\mathfrak{a}^\circ_{\mathrm{res}})\big) = \ell(\mathfrak{a}) - \ell(\mathfrak{b}) - \big(\ell(\mathfrak{a}^\circ) - \ell(\mathfrak{b}_{\mathrm{res}})\big). \tag{5.15.3}$$

Similarly, (5.15.2) also implies that

$$\begin{aligned} x + y &= \ell(\mathfrak{a}^\flat_{\mathrm{res}}) - \ell(\mathfrak{a}^\circ_{\mathrm{res}}) + \ell(\delta_{\mathfrak{a}}) + \ell(\delta_{\mathfrak{b}}) \\ &= \ell(\mathfrak{a}) + \ell(\mathfrak{b}) - \big(\ell(\mathfrak{a}^\circ) + \ell(\mathfrak{b}_{\mathrm{res}})\big). \end{aligned} \tag{5.15.4}$$

Now we separate the discussion according to $\langle\mathfrak{a}|\mathfrak{b}\rangle$.

(1) If $\langle\mathfrak{a},\mathfrak{b}\rangle = 0$, then $\langle\mathfrak{a}^\circ|\mathfrak{b}_{\mathrm{res}}\rangle = 0$ for simple combinatorics reasons. Then the π-isotypical component of $\mathrm{Res}_{\mathfrak{a}^\circ}\circ\mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}$ factors through the cohomology of a lower-dimensional Shimura variety, so the same is true for $\mathrm{Res}_{\mathfrak{a}}\circ\mathrm{Gys}_{\mathfrak{b}}$.

(2) or (3) We have $\langle\mathfrak{a}|\mathfrak{b}\rangle = (-2)^{m_0}v^{m_v}$ or $(-2)^{m_0}T^{m_T}$. The picture $D(\mathfrak{a}^\circ,\mathfrak{b}_{\mathrm{res}})$ can be identified with the picture $D(\mathfrak{a},\mathfrak{b})$ after deforming some curves ("pulling strings"). In particular, we have $\langle\mathfrak{a}^\circ|\mathfrak{b}_{\mathrm{res}}\rangle = \langle\mathfrak{a}|\mathfrak{b}\rangle$. By the inductive hypothesis for the pair $(\mathtt{S}_{\delta_{\mathfrak{b}}},\mathtt{T}_{\delta_{\mathfrak{b}}})$[20] and (5.11.2), we have

$$\begin{aligned}
&\mathrm{Res}_{\mathfrak{a}}\circ\mathrm{Gys}_{\mathfrak{b}}\\
&\quad= p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ\mathrm{Res}_{\mathfrak{a}^\circ}\circ\mathrm{Gys}_{\mathfrak{b}_{\mathrm{res}}}\\
&\quad=\begin{cases} p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ(-2)^{m_0}\cdot p^{(\ell(\mathfrak{a}^\circ)+\ell(\mathfrak{b}_{\mathrm{res}}))/2}\eta^\star_{\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{a}^\circ,\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{b}_{\mathrm{res}}}, \\ \qquad \text{if } r<\frac{d}{2},\\ p^{(x+y)/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ(-2)^{m_0}\cdot p^{(\ell(\mathfrak{a}^\circ)+\ell(\mathfrak{b}_{\mathrm{res}}))/2}(T_{\mathfrak{p}}/p^{g/2})^{m_T}\eta^\star_{\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{a}^\circ,\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{b}_{\mathrm{res}}}\\ \qquad \text{if } r=\frac{d}{2},\end{cases}\\
&\overset{(5.15.4)}{=}\begin{cases}(-2)^{m_0}\cdot p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ\eta^\star_{\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{a}^\circ,\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{b}_{\mathrm{res}}},\\ \qquad\text{if } r<\frac{d}{2},\\ (-2)^{m_0}\cdot p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}(T_{\mathfrak{p}}/p^{g/2})^{m_T}\eta^\star_{\mathfrak{a},\mathfrak{a}^\star}\circ\eta^\star_{\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{a}^\circ,\mathtt{S}_{\delta_{\mathfrak{b}}},\mathfrak{b}_{\mathrm{res}}}\\ \qquad\text{if } r=\frac{d}{2}.\end{cases}
\end{aligned}$$

The composite of the two links is exactly $\eta^\star_{\mathtt{S}_{\mathfrak{a}},\mathtt{S}_{\mathfrak{b}}}$ of the needed indentation degree by (5.15.3) and of the required shift by (5.15.1).

This concludes the proof of Theorem 4.4.

Appendix. Cohomology of quaternionic Shimura varieties

We include the proof of Proposition 2.26 regarding the cohomology of our "slightly twisted" quaternionic Shimura varieties. It is based on comparing the cohomology with the known case when $\mathtt{T}=\emptyset$. This is certainly known to the experts, but we could not find the exact version in the literature.

A.1. Discrete Shimura varieties for $F^\times$

Consider a Deligne homomorphism for $T_{F,\mathtt{T}} := \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ given by

[20] Here, as before, the shift $\boldsymbol{t}'_{\mathfrak{a}'}$ for a periodic semimeander $\mathfrak{a}'$ for $(\mathtt{S}_{\delta_{\mathfrak{b}}},\mathtt{T}_{\delta_{\mathfrak{b}}})$ is taken to be $\boldsymbol{t}_{\delta_{\mathfrak{b}},\tilde{\mathfrak{a}}'}$, where $\tilde{\mathfrak{a}}'$ is a periodic semimeander of $(\mathtt{S},\mathtt{T})$ consisting of all the arcs and semilines of $\mathfrak{a}'$ together with the arc $\delta_{\mathfrak{b}}$.

$$\begin{aligned} h_{\mathtt{T}}\colon \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times &\longrightarrow T_F(\mathbb{R}) = (\mathbb{R}^\times)^{\mathtt{T}} \times (\mathbb{R}^\times)^{\Sigma_\infty - \mathtt{T}} \\ z &\longmapsto \big((|z|^2, \dots, |z|^2), (1, \dots, 1)\big). \end{aligned}$$

Under this choice of Deligne homomorphism, we can define a discrete Shimura variety $\mathcal{S}h_{K_{T,p}}(T_{F,\mathtt{T}})$ for $K_{T,p} = \mathcal{O}_{\mathfrak{p}}^\times$ whose complex points are given by

$$\mathcal{S}h_{K_{T,p}}(T_{F,\mathtt{T}})(\mathbb{C}) = F^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_{\mathfrak{p}}^\times.$$

It admits an integral canonical model with special fiber $\mathrm{Sh}_{K_{T,p}}(T_{F,\mathtt{T}})$ over $\mathbb{F}_{p^g}$ (in the sense of [33, Section 2.8]), which is determined by the Shimura reciprocity map

$$\mathrm{Rec}_{T,\mathtt{T},p}\colon \mathrm{Gal}_{\mathbb{F}_{p^g}} \longrightarrow F^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_{\mathfrak{p}}^\times.$$

Explicitly, $\mathrm{Rec}_{T,\mathtt{T},p}$ sends the geometric Frobenius Frob_{p^g} to the finite idele $(\underline{p}_F)^{\#\mathtt{T}}$.

Fix a prime number $\ell \neq p$. The algebraic representation $\rho^w_{T,\mathtt{T}}$ of $T_{F,\mathtt{T}} \times \mathbb{C} \cong \prod_{\tau \in \Sigma_\infty} \mathbb{G}_{m,\tau}$ sending x to $(x^{2-w}, \dots, x^{2-w})$ gives a lisse $\overline{\mathbb{Q}}_\ell$-étale sheaf $\mathcal{L}^w_{T,\mathtt{T}}$ of pure weight $2(w-2)\#\mathtt{T}$ on $\mathrm{Sh}_{K_{T,p}}(T_{F,\mathtt{T}})$.

A.2. Changing $\mathtt{T}$

We need to compare the Shimura varieties $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}})$ and $\mathrm{Sh}_{K_p}(G_{\mathtt{S},\emptyset})$. Note that the natural product morphism

$$\mathrm{pr}\colon G_{\mathtt{S},\emptyset} \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \to G_{\mathtt{S},\mathtt{T}}$$

is compatible with the Deligne homomorphism $h_{\mathtt{S},\emptyset} \times h_{\mathtt{T}}$ on the source and $h_{\mathtt{S},\mathtt{T}}$ on the target (i.e., $\mathrm{pr} \circ (h_{\mathtt{S},\emptyset} \times h_{\mathtt{T}}) = h_{\mathtt{S},\mathtt{T}}$). This gives a natural morphism of Shimura varieties,

$$\mathrm{pr}_\bullet\colon \mathrm{Sh}_{K_p}(G_{\mathtt{S},\emptyset}) \times \mathrm{Sh}_{K_{T,p}}(T_{F,\mathtt{T}}) \longrightarrow \mathrm{Sh}_{K_p}(G_{\mathtt{S},\mathtt{T}}). \tag{A.2.1}$$

Moreover, the product morphism is compatible with the algebraic representations in the sense that

$$\rho^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}} \circ \mathrm{pr} \cong \rho^{(\underline{k},w)}_{\mathtt{S},\emptyset} \boxtimes \rho^w_{T,\mathtt{T}}.$$

So we have a natural isomorphism of sheaves,

$$\mathrm{pr}^*_\bullet(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}) \cong \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\emptyset} \boxtimes \mathcal{L}^w_{T,\mathtt{T}}. \tag{A.2.2}$$

PROPOSITION A.3

Let $\pi \in \mathcal{A}_{(\underline{k},w)}$ be a cuspidal automorphic representation appearing in the cohomology of the Shimura variety $\mathcal{S}h_K(G_{\mathtt{S},\mathtt{T}})$. Then we have a canonical isomorphism,

$$H^i_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)[\pi]^{\mathrm{Fr}\text{-}s.s.} = \begin{cases} \rho_{\pi,\mathfrak{p}}^{\otimes d} \otimes [\det(\rho_{\pi,\mathfrak{p}})(1)]^{\otimes \#\mathtt{T}} & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

equivariant for the action of the geometric Frobenius Frob_{p^g}. *Here, the superscript* Fr-*s.s. means taking the semisimplification as* Frob_{p^g}*-modules.*

Proof
The proposition is known when $\mathtt{T} = \emptyset$ by [2, Section 3.2] (note that we have the tensor product instead of tensorial induction because $\rho_{\pi,\mathfrak{p}}$ is unramified at $\mathfrak{p}$). For general $\mathtt{T}$, the morphism (A.2.1) induces an isomorphism,

$$\begin{aligned} &H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big) \\ &\quad\cong H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p} \times \mathrm{Sh}_{K_{T,p}}(T_{F,\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathrm{pr}_\bullet^*(\mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}})\big)^{\mathbb{A}_F^{\infty,\times}} \\ &\quad\overset{(\mathrm{A.2.2})}{\cong} \big(H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\emptyset})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\emptyset}\big) \\ &\qquad \otimes H^0_{\mathrm{et}}\big(\mathrm{Sh}_{K_{T,p}}(T_{F,\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{w}_{T,\mathtt{T}}\big)\big)^{\mathbb{A}_F^{\infty,\times}}, \end{aligned}$$

where the superscript $\mathbb{A}_F^{\infty,\times}$ means to take the invariant part for the *antidiagonal action* of this group (i.e., $z \in \mathbb{A}_F^{\infty,\times}$ acts by (z, z^{-1})). So if ω_π denotes the central character of π, then we have

$$\begin{aligned} H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\mathtt{T}}\big)[\pi] &\cong H^\star_{\mathrm{et}}\big(\mathrm{Sh}_K(G_{\mathtt{S},\emptyset})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k},w)}_{\mathtt{S},\emptyset}\big)[\pi] \\ &\qquad \otimes H^0_{\mathrm{et}}\big(\mathrm{Sh}_{K_{T,p}}(T_{F,\mathtt{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}^{w}_{T,\mathtt{T}}\big)[\omega_\pi], \end{aligned}$$

where the last factor is the 1-dimensional subspace where $\mathbb{A}_F^{\infty,\times}$ acts through ω_π. By the Shimura reciprocity map $\mathrm{Rec}_{T,\mathtt{T},p}$ recalled in Section A.1 and the Eichler–Shimura relation (2.24.1), the geometric Frobenius Frob_{p^g} acts on this 1-dimensional space by multiplication by

$$\omega_\pi(\underline{p}_F)^{-\#\mathtt{T}} = \big(\det\big(\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_{p^g})\big)/p^g\big)^{\#\mathtt{T}}.$$

This concludes the proof of this proposition. □

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