COMPUTING POLYNOMIAL CONFORMAL MODELS FOR LOW-DEGREE BLASCHKE PRODUCTS

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Abstract. For any finite Blaschke product B, there is an injective analytic map $\varphi : \mathbb{D} \to \mathbb{C}$ and a polynomial p of the same degree as B such that $B = p \circ \varphi$ on $\mathbb{D}$. Several proofs of this result have been given over the past several years, using fundamentally different methods. However, even for low-degree Blaschke products, no method has hitherto been developed to explicitly compute the polynomial p or the associated conformal map φ. In this paper, we show how these functions may be computed for a Blaschke product of degree at most three, as well as for Blaschke products of arbitrary degree whose zeros are equally spaced on a circle centered at the origin.

1. Introduction

For domains $D, E \subset \mathbb{C}$ and analytic functions $f : D \to \mathbb{C}$, $g : E \to \mathbb{C}$, we say that g on E is a conformal model for f on D if there is some analytic bijection $\varphi : D \to E$ such that $f = g \circ \varphi$ on D. By precomposing both sides of this equation by $\varphi^{-1} : E \to D$, it follows immediately that f on D is a conformal model for g on E. In this case, we say that the pairs (f, D) and (g, E) are conformally equivalent, and it is easy to see that this defines an equivalence relation on the set of pairs of the form (f, D).

There has been significant interest in recent years in the "Polynomial Conformal Modeling Question" (PCMQ), which asks whether a given pair (f, D) has any conformal model (g, E) for which the function g is a polynomial. A positive answer is known for the PCMQ when the domain D is the unit disk $\mathbb{D}$ and the function f is a finite Blaschke product. In this case, the polynomial conformal model may be assumed to have the same degree as f. In the following, we use the notation $\mathbb{D}_p = \{z \in \mathbb{C} : |p(z)| < 1\}$ for any polynomial $p \in \mathbb{C}[z]$.

Theorem 1. *For any finite Blaschke product B, there is a polynomial p of the same degree as B such that p on $\mathbb{D}_p$ is a conformal model for B on $\mathbb{D}$.*

Theorem 1 has seen several proofs in recent years using a variety of approaches. We mention

- the characterization of fingerprints of polynomial lemniscates obtained by Ebenfelt et. al. [2] in view of applications to computer vision, which has Theorem 1 as a corollary;
- the proof of the first author [5] using critical level curve configurations;
- the proof of the second author [9] using conformal welding.

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Theorem 1 was further generalized in [6] to arbitrary functions analytic on the closed unit disk. As far as we know, the most general version is a solution on the online mathematics forum *math.stackexchange.com* by Lowther and Speyer [3, 8] showing that the PCMQ has a positive answer as long as the domain D is bounded and the function f is analytic on the closure of D. This solution relies on approximation by polynomials interpolating certain derivative data, and can readily be generalized to meromorphic functions f, in which case polynomials need to be replaced by rational maps. Also of interest on this topic is the paper of the authors [7] which again brings the tools of conformal welding to bear on the PCMQ, also addressing the question of the degree of the polynomial conformal model in more detail.

Conspicuously absent in the aforementioned proofs of Theorem 1 is anything of a constructive nature. In this paper, we present a method for computing a polynomial conformal model as well as obtaining an algebraic formula for its associated conformal map in two cases : first for finite Blaschke products of degree at most three, in Section 2, and then for finite Blaschke products of arbitrary degree whose zeros are equally spaced on a circle centered at the origin, in Section 3.

The work in Sections 2 and 3 requires several lemmas, which are proved in Section 4.

2. The Polynomial Conformal Model for a finite Blaschke product of degree at most three

First, observe that if B is a degree one finite Blaschke product, then B itself is injective on $\mathbb{D}$, so that we may write B as $B = p \circ \varphi$, where the polynomial is $p(z) = z$ and the conformal map is $\varphi(z) = B(z)$.

If B has degree two, then by precomposing B with a disk automorphism sending 0 to the critical point of B, we may assume without loss of generality that the two zeros of B are symmetric with respect to the origin (this follows from Lemma 3). In other words, we may assume that B falls in the case treated in Section 3.

Consequently, it only remains to treat the case of a finite Blaschke product B with $\deg(B) = 3$. In this case, the derivative of B is a rational function with at most four zeros, two of which lie in $\mathbb{D}$. The critical points of B may hence be computed by means of the quartic formula (see for example [4]).

Let z_1 and z_2 be the two critical points of B in $\mathbb{D}$, and set $k_1 = B(z_1)$ and $k_2 = B(z_2)$. Note that if p on $\mathbb{D}_p$ is a conformal model for B on $\mathbb{D}$, then p must have k_1 and k_2 as critical values as well. The converse also holds, as shown in the following lemma.

Lemma 2. *Let B be a degree three Blaschke product whose critical points in $\mathbb{D}$ are z_1 and z_2. If $p \in \mathbb{C}[z]$ is any degree three polynomial whose critical values are $B(z_1)$ and $B(z_2)$, then p on $\mathbb{D}_p$ is a conformal model for B on $\mathbb{D}$.*

Proof. By Theorem 1, there is a degree three polynomial $\widehat{p}$ such that $\widehat{p}$ on $\mathbb{D}_{\widehat{p}}$ is a conformal model for B on $\mathbb{D}$. As previously mentioned, it follows that the critical values of $\widehat{p}$ are precisely $B(z_1)$ and $B(z_2)$. On the other hand, there is exactly one degree three polynomial with any two given critical values, modulo precomposition with a linear map (see [1]). Since p and $\widehat{p}$ have the same critical values, it immediately follows that $(p, \mathbb{D}_p)$ is conformally equivalent to $(\widehat{p}, \mathbb{D}_{\widehat{p}})$. Finally, by the transitivity of conformal equivalence, we conclude that p on $\mathbb{D}_p$ is a conformal model for B on $\mathbb{D}$.

□

Having proved Lemma 2, it remains to show how to compute a degree three polynomial with two prescribed critical values.

If p is the desired polynomial, then for any $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, the polynomial $\widehat{p}(z) = p(\alpha z + \beta)$ has the same critical values as p.

Note that $\widehat{p}$ on $\mathbb{D}_{\widehat{p}}$ is a conformal model for p on $\mathbb{D}_p$, hence is also a conformal model for B on $\mathbb{D}$, again by transitivity. Moreover, one can easily check that by making an appropriate choice of α and β, we may choose $\widehat{p}$ to be of the form

$$\widehat{p}(z) = z^3 + cz + d$$

for some $c, d \in \mathbb{C}$. Let us simply replace p with this new conformal model, using the letter p to denote $\widehat{p}$. We now wish to compute c and d from the prescribed critical values of p, namely k_1 and k_2.

The critical points of p are the two square roots of $\dfrac{-c}{3}$. Let $\pm z_1$ denote these two roots. If $+z_1 = -z_1$, then $c = 0$, in which case $p(z) = z^3 + d$. The derivative of such a polynomial has a zero of order two, so the same must be true for B as well. By precomposing B with the appropriate disk automorphism, we may assume that the double critical point of B is at the origin, so that by Lemma 3, the zeros of B are equally spaced on a circle centered at the origin. This case being treated in Section 3, we shall henceforth assume that $+z_1 \neq -z_1$.

Substituting these roots back into p and setting the result equal to k_1 and k_2 respectively, we obtain the equations

$$k_1 = {z_1}^3 + cz_1 + d \text{ and } k_2 = (-z_1)^3 + c(-z_1) + d.$$

The solutions c and d to this system of equations are easily found to be

$$c = -3\left(\frac{k_2 - k_1}{4}\right)^{2/3} \text{ and } d = \frac{k_1 + k_2}{2},$$

for any choice of the third root in the equation for c. In order to simplify the notation, we leave this as simply c and d in what follows, keeping in mind that these quantities are computed in terms of k_1 and k_2. With these values of c and d, the polynomial $p(z) = z^3 + cz + d$ has critical values k_1 and k_2 and therefore is a conformal model for B on $\mathbb{D}$, by Lemma 2

Having found the polynomial conformal model p, we can now obtain a formula for the corresponding conformal map $\varphi : \mathbb{D} \to \mathbb{D}_p$ satisfying $B = p \circ \varphi$. In order to do so, we treat the equation $B = p \circ \varphi$ as a polynomial equation in the variable φ, with coefficients in the ring of rational functions in z. In other words, the function φ we are looking for is a solution to the equation

$$0 = \varphi^3 + c\varphi + (d - B).$$

The cubic formula (see again [4]) now implies that φ has the form

$$\varphi = U + V,$$

where

$$U = \sqrt[3]{-\frac{d-B}{2} + \sqrt{\frac{(d-B)^2}{4} + \frac{c^3}{27}}} \text{ and } V = \sqrt[3]{-\frac{d-B}{2} - \sqrt{\frac{(d-B)^2}{4} + \frac{c^3}{27}}}.$$

In the above formulas, the same choice must be made for the square roots in U and in V, while the cubic roots must and can be chosen to ensure that $UV = \dfrac{-c}{3}$.

These constraints still leave three possible solutions φ, corresponding to the three possible choices for the cubic roots in the expressions for U and V. Each of these solutions satisfies the equation $B = p \circ \varphi$ in $\mathbb{D}$, although in general only one is analytic in the disk. Indeed, in order to see this, suppose that the two critical points of B are distinct, which is the case of interest, and let $\varphi_1, \varphi_2 : \mathbb{D} \to \mathbb{C}$ be two analytic maps satisfying $B = p \circ \varphi_1 = p \circ \varphi_2$. According to Lemma 4, both φ_1 and φ_2 are injective on $\mathbb{D}$, and $\varphi_1(\mathbb{D}) = \varphi_2(\mathbb{D}) = \mathbb{D}_p$.

Now, we have $B \circ {\varphi_1}^{-1} = B \circ {\varphi_2}^{-1}$ on $\mathbb{D}_p$, which implies that $\psi = {\varphi_2}^{-1} \circ \varphi_1$ is a disk automorphism satisfying $B = B \circ \psi$ on $\mathbb{D}$. But then ψ must be the identity, in view of Lemma 6, so that $\varphi_1 = \varphi_2$ on $\mathbb{D}$. We conclude that at most one of the three solutions to the equation

$$0 = \varphi^3 + c\varphi + (d - B)$$

will be analytic.

In practice, determining which of the three choices of φ is analytic may be quite difficult, mostly due to the complicated nature of the formulas involved. We illustrate this by the following example.

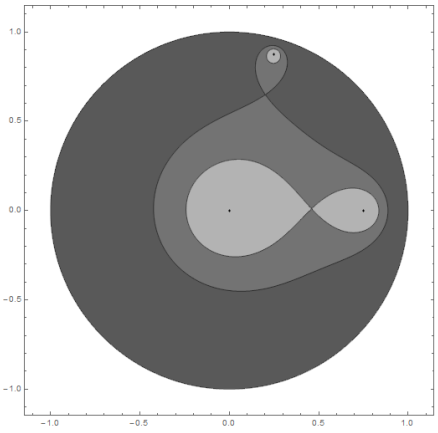

FIGURE 1. The critical level curves of B.

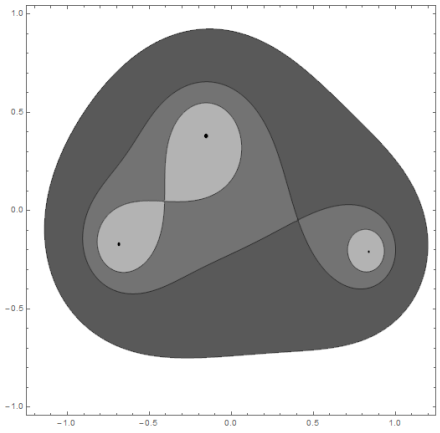

FIGURE 2. The critical level curves of p.

Example. Consider the finite Blaschke product

$$B(z) = z\frac{(z - 3/4)(z - (1/4 + 7i/8))}{(1 - (3/4)z)(1 - (1/4 - 7i/8)z)},$$

which has zeros at $0, 1/2$ and $1/4+7i/8$. The derivative of B is a degree four rational function, whose zeros can in principle be computed using the quartic formula. The closed forms for these critical points are too involved to display here, so we simply mention that those in the disk are approximately $z_1 = 0.2014 + 0.6494i$ and $z_2 = 0.4599 + 0.0103i$. The critical level curves of B are displayed in Figure 1. The outer boundary of the shaded region is the unit circle.

The corresponding critical values are $k_1 = B(z_1)$ and $k_2 = B(z_2)$. Recall that by Lemma 2, there is a unique conformal equivalence class of degree three polynomials with two prescribed critical values, and any degree three polynomial p having k_1 and k_2 as critical values is a polynomial conformal model on $\mathbb{D}_p$ for B on $\mathbb{D}$.

The previous work shows that the polynomial $p(z) = z^3 + cz + d$ (where $c = -3\left(\frac{k_2-k_1}{4}\right)^{2/3}$ and $d = \frac{k_1+k_2}{2}$) has critical values k_1 and k_2, so that p on $\mathbb{D}_p$ is a conformal model for B on $\mathbb{D}$. Furthermore, as discussed above, the conformal map $\varphi : \mathbb{D} \to \mathbb{D}_p$ satisfying the equation $B = p \circ \varphi$ is defined by one of the three algebraic formulas obtained by applying the cubic formulas to the equation $B = p \circ \varphi$ written in the form

$$\varphi^3 + c\varphi + (d - B) = 0$$

(again viewed as an equation in the unknown φ, with coefficients in the ring of rational functions in z). The complexity of these formulas makes it difficult to determine precisely which of the three solutions is conformal on the unit disk (and therefore is truly the one we are looking for).

For comparison sake, we also display the critical level curves of the polynomial conformal model p in Figure 2. The outer boundary of the shaded region is the set $\{z : |p(z)| = 1\}$. The function φ maps the lightest regions in Figure 1 to the corresponding lightest regions in Figure 2, and so forth.

3. The Polynomial Conformal Model for a finite Blaschke product with equally spaced zeros

Let B denote a degree $n \geq 2$ Blaschke product whose zeros are equally spaced on a circle centered at 0, i.e.

$$B(z) = \lambda \frac{z^n - c^n}{1 - \bar{c}^n z^n}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and some $c \in \mathbb{D}$.

Define $\varphi(z) = \dfrac{e^{i\pi/n} z}{\sqrt[n]{1 - \bar{c}^n z^n}}$. By Lemma 5, the map φ is analytic and injective on $\mathbb{D}$. A straightforward calculation shows that for $z \in \mathbb{D}$,

$$B(\varphi(z)) = \lambda \left(|c|^{2n} - 1\right) z^n - \lambda c^n.$$

It follows that the polynomial $p(z) = \lambda(|c|^{2n} - 1)z^n - \lambda c^n$ on the set $\varphi(\mathbb{D})$ is a polynomial conformal model for B on $\mathbb{D}$.

4. Proofs of the lemmas

Lemma 3. *If the critical points of a finite Blaschke product B of degree $n \geq 2$ are all at the origin, then the zeros of B are equally spaced on a circle centered at the origin.*

Proof. Let $\widehat{B}(z) = z^n$. Then B and $\widehat{B}$ have the same critical points in $\mathbb{D}$, so there exists a disk automorphism $\tau : \mathbb{D} \to \mathbb{D}$ such that $B = \tau \circ \widehat{B}$ (see e.g. [10]). This implies that $B(z) = 0$ if and only if $z^n = \tau^{-1}(0)$, so that the zeros of B are indeed equally spaced on a circle centered at 0.

□

Lemma 4. *Let B be a finite Blaschke product and let p be a polynomial with critical values all in $\mathbb{D}$. If $\deg(B) = \deg(p)$, then any analytic function $\varphi : \mathbb{D} \to \mathbb{C}$ such that $B = p \circ \varphi$ on $\mathbb{D}$ is injective on $\mathbb{D}$ and satisfies $\varphi(\mathbb{D}) = \mathbb{D}_p$.*

Proof. First note that since all the critical values of p lie in $\mathbb{D}$, the set $\mathbb{D}_p$ is connected (see e.g. [2, Proposition 2.1].) Also, we have $p \circ \varphi(\mathbb{D}) = B(\mathbb{D}) \subset \mathbb{D}$, from which we deduce that $\varphi(\mathbb{D}) \subset \mathbb{D}_p$. In order to show that $\varphi(\mathbb{D}) = \mathbb{D}_p$, it suffices to prove that $\varphi(\mathbb{D})$ is both open and closed in $\mathbb{D}_p$, by connectedness. Clearly $\varphi(\mathbb{D})$ is open in $\mathbb{D}_p$, by the open mapping theorem. Now, suppose that $\{w_n\} \subset \varphi(\mathbb{D})$ is such that $w_n \to w \in \mathbb{D}_p$. Then $|p(w)| < 1$, so $|p(w_n)| \not\to 1$, by continuity. For each n, let $z_n \in \mathbb{D}$ be chosen such that $w_n = \varphi(z_n)$. Passing to a subsequence if necessary, assume that $z_n \to z \in \overline{\mathbb{D}}$. Note that

$$|B(z)| = \lim_{n\to\infty} |B(z_n)| = \lim_{n\to\infty} |p(\varphi(z_n))| = \lim_{n\to\infty} |p(w_n)| < 1,$$

which shows that $|B(z)| < 1$ and thus $|z| < 1$. By continuity, we get $w = \varphi(z)$, so that $w \in \varphi(\mathbb{D})$. This shows that $\varphi(\mathbb{D})$ is closed in $\mathbb{D}_p$, so that $\varphi(\mathbb{D}) = \mathbb{D}_p$.

Suppose now that φ fails to be injective. Then for some distinct points $z_0, z_1 \in \mathbb{D}$, $\varphi(z_0) = \varphi(z_1)$. Let $w \in \mathbb{D}$ denote the image under p of this common value, so that $w = p(\varphi(z_0)) = p(\varphi(z_1))$. Let $\zeta_1, \zeta_2, \ldots, \zeta_k$ denote the preimages of w under p, ordered so that $\varphi(z_1) = \zeta_1$. Since $\varphi(\mathbb{D}) = \mathbb{D}_p$, for each $2 \leq j \leq k$, there exists $z_j \in \mathbb{D}$ such that $\varphi(z_j) = \zeta_j$. Note that then $B(z_j) = p(\varphi(z_j)) = p(\zeta_j) = w$ for each $0 \leq j \leq k$.

For any function f which is analytic at a point $\xi_0 \in \mathbb{C}$, we denote by $\mathrm{mult}_f(\xi_0)$ the multiplicity of ξ_0 as a solution to the equation $f(z) = f(\xi_0)$ (note of course that $\mathrm{mult}_f(\xi_0) \geq 1$). We shall use the well-known fact that multiplicity is multiplicative : if f is analytic at ξ_0, and g is analytic at $f(\xi_0)$, then $\mathrm{mult}_{g\circ f}(\xi_0) = \mathrm{mult}_f(\xi_0) \cdot \mathrm{mult}_g(f(\xi_0))$. In particular, if $h = g \circ f$, then $\mathrm{mult}_h(\xi_0) \geq \mathrm{mult}_g(f(\xi_0))$.

Now, note that $\deg(B) = \displaystyle\sum_{z\in B^{-1}(w)} \mathrm{mult}_B(z) \geq \sum_{j=0}^{k} \mathrm{mult}_B(z_j)$. Writing $B = p\circ\varphi$ therefore yields

$$\deg(B) \geq \sum_{j=0}^{k} \mathrm{mult}_{p\circ\varphi}(z_j) = \sum_{j=0}^{k} \mathrm{mult}_\varphi(z_j)\,\mathrm{mult}_p(\varphi(z_j)) > \sum_{j=1}^{k} \mathrm{mult}_p(\zeta_j) = \deg(p).$$

This contradicts the assumption that $\deg(p) = \deg(B)$. It follows that φ is injective, which completes the proof of the lemma.

□

Lemma 5. *For any $c \in \mathbb{D}$, the map $\varphi(z) = \dfrac{z}{\sqrt[n]{1-\overline{c}^n z^n}}$ is an injective analytic map on $\mathbb{D}$.*

Proof. Clearly, the function φ is analytic on the unit disk, since $1 - \overline{c}^n z^n$ is non-vanishing there.

Now suppose that $z, w \in \mathbb{D}$ are such that $\varphi(z) = \varphi(w)$. Then we have

$$z^n(1 - \overline{c}^n w^n) = w^n(1 - \overline{c}^n z^n)$$

so that $z^n = w^n$, and thus $z = e^{2\pi ik/n}w$ for some $k \in \{0, 1, \ldots, n-1\}$. Substituting back into the equation $\varphi(z) = \varphi(w)$ yields $e^{2\pi ik/n}w = w$, so that either $w = 0$, in which case $z = 0$, or $e^{2\pi ik/n} = 1$. In both cases, we get $z = w$. It follows that φ is injective.

□

Lemma 6. *If B is a degree three Blaschke product whose two critical points in $\mathbb{D}$ are distinct, and if $\psi : \mathbb{D} \to \mathbb{D}$ is a disk automorphism satisfying*

$$B \circ \psi = B \tag{1}$$

on $\mathbb{D}$, then ψ is the identity map.

Proof. It easily follows from Schwarz's lemma that an analytic function from $\mathbb{D}$ into $\mathbb{D}$ with two distinct fixed points is the identity. It thus suffices to find two distinct points in $\mathbb{D}$ which are fixed by ψ.

Let $z_1, z_2 \in \mathbb{D}$ denote the two distinct critical points of B. We claim that $B(z_1) \neq B(z_2)$. Indeed, if not, then the rational function $B(z) - B(z_1)$ would have zeros of order at least two at both z_1 and z_2, which is impossible since $B(z) - B(z_1)$ is a degree three rational function. [1]

Now, Equation (1) combined with the chain rule implies that ψ preserves the set of critical points of B. If $\psi(z_1) = z_2$, then again by Equation (1) we would have $B(z_1) = B(\psi(z_1)) = B(z_2)$, a contradiction. It follows that ψ fixes the distinct points z_1 and z_2, as required.

□

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