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Total p -differentials on schemes over $\mathbf{Z}/p^2$

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ABSTRACT

For a scheme X defined over the length 2 p -typical Witt vectors $W_2(\mathbf{k})$ of a characteristic p field, we introduce total p -differentials which interpolate between Frobenius-twisted differentials and Buium's p -differentials. They form a sheaf over the reduction X_0 , and behave as if they were the sheaf of differentials of X over a deeper base below $W_2(\mathbf{k})$. This allows us to construct the analogues of Gauss–Manin connections and Kodaira–Spencer classes as in the Katz–Oda formalism. We make connections to Frobenius lifts, Borger–Weiland's birthing formalism, and Deligne–Illusie classes.

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1. Introduction

In a series of papers, Alexandru Buium introduced a notion of arithmetic derivations, extending the analogy between function fields like $\mathbf{C}[[t]]$ and mixed characteristic local fields like $\mathbf{Z}_p^{\text{unr}}$ (see the textbook [5]). For example, there is an arithmetic derivation operator on $\mathbf{Z}_p^{\text{unr}}$, called a p -derivation, defined by

$$\delta x = \frac{\varphi(x) - x^p}{p},$$

where φ is the Frobenius. While δ does not obey the usual Leibniz rule and is not even additive, it has many formal properties in common with derivations. Just as the usual notion of derivation measures non-constancy of an object over a base, this arithmetic derivation measures how much the Frobenius action on an object over $\mathbf{Z}_p^{\text{unr}}$ differs from $x \mapsto x^p$: the arithmetic analogue of triviality of a family is the existence of a lift of Frobenius. This is consistent with Borger’s approach to the field of one element [1], where one views lifts of Frobenius as a necessary condition for descent to $\mathbf{F}_1$, just as a vector field on a family extending a vector field on the base is a necessary condition for descent from $\mathbf{C}[[t]]$ to $\mathbf{C}$.

More generally, Buium [4] classified all arithmetic analogues of derivations on a local domain R of characteristic zero. By an “arithmetic analogue of a derivation,” we mean an operation $\theta: R \rightarrow R$ on a ring R , which has a sum rule, a product rule, and which is appropriately normalized to take certain values on 0 and 1. Buium proved that for R , a local ring of characteristic zero, there can only exist four types of operations up to equivalence: derivations, difference operations, p -derivations, and p -difference operations where p is some prime. These various notions afford analogous definitions of differentials and jet spaces.

It turns out that operations θ satisfying the conditions imposed in [4] are intimately tied to ring schemes. Under this correspondence, usual derivations correspond to the ring scheme of dual numbers, and Buium’s p -derivations correspond to the ring of p -typical Witt vectors of length two. In fact, the ring scheme formalism of Borger–Weiland [6] provides a uniform framework for formulating differentials and jet spaces as well as working over a base.

One classical construction using differentials that has been missing its p -differential analogue is that of taking differentials on the total space. Specifically, if one has a scheme $\pi: X \rightarrow S$ smooth over a base, one may form the exact sequence

$$0 \longrightarrow \pi^* \Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow 0.$$

This construction is crucial to Katz–Oda’s construction of the Gauss–Manin connection [12]. The purpose of this paper is to construct an analogue of the restriction of this exact sequence to a closed fiber, at least for the case where the base is $\text{Spec } \mathbf{Z}/p^2$. In fact,

we will consider schemes defined over $\mathrm{Spec} \mathbf{Z}/p^2$ as families over a “deeper” base below $\mathbf{Z}/p^2$ and measure their non-triviality. In this sense, our work comes into contact with the philosophy of the field of one element. The geometric counterpart of our construction can be found in [7].

This paper accomplishes the goal by introducing a new class of arithmetic derivations from a ring modulo p^2 to a ring modulo p that interpolates between Buium’s p -derivations and ordinary Frobenius semi-linear derivations. We call these **total p -derivations**. They globalize on schemes X defined over $W_2(\mathbf{k})$, the p -typical Witt vectors length 2 of a field of characteristic p . In fact, their dual notion, total p -differentials form a sheaf $\Omega_X^{1,\mathrm{tot}}$ on X_0 , the reduction of X . They fit into an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_0} \xrightarrow{\alpha} \Omega_X^{1,\mathrm{tot}} \xrightarrow{\beta} F_{X_0}^* \Omega_{X_0}^1 \longrightarrow 0$$

where $F_{X_0}^* \Omega_{X_0}^1$ is the pullback by absolute Frobenius of the sheaf of differentials on X_0 . The appearance of $\mathcal{O}_{X_0}$ is a reflection of the fact that $\mathrm{Spec} W_2(\mathbf{k})$ should be thought of as an object with a one-dimensional cotangent space over a “deeper” scheme. Regarding p -differentials as analogous to differentials on a total space, we construct analogues of the Gauss–Manin connection and the Kodaira–Spencer class. The usual Gauss–Manin connection is a measure of the non-triviality of a local system arising as the cohomology of a family; our analogue measures the existence of lifts of Frobenius. This gives an alternative view of Deligne–Illusie’s deformation-theoretic approach [8] to studying lifts of Frobenius. The arithmetic analogue of the Kodaira–Spencer class via a different construction already appears with applications to Diophantine geometry in [2]. We also give a description of total p -differentials in terms of the biring formalism. Further discussion of the arithmetic analogue of Kodaira–Spencer classes is in [10].

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2. Total p -differentials

2.1. Total p -derivations

Let $\mathbf{k}$ be a perfect field of characteristic $p > 2$. Let $R = W_2(\mathbf{k})$ be the p -typical Witt vectors [13] of length 2. Let $\varphi: R \rightarrow R$ be the homomorphism induced by Witt functoriality from the Frobenius on $\mathbf{k}$. Let A be a finitely generated flat R -algebra with

mod p reduction A_0 . For $r \in R$, we will write r for the element $r \cdot 1 \in A$. Let $C_p(X, Y)$ be the polynomial

$$C_p(X, Y) = \frac{X^p + Y^p - (X + Y)^p}{p}.$$

Definition 2.1.1. For an A_0 -module D_0 , a *total p -derivation* of A into D_0 is a map $\delta: A \rightarrow D_0$ such that

- (1) for $a, b \in A$, $\delta(a + b) = \delta a + \delta b + (C_p(a, b))\delta p$,
- (2) for $a, b \in A$ $\delta(ab) = (\delta a)b^p + a^p(\delta b)$, and
- (3) for $r \in R$, $\delta r = ((\varphi(r) - r^p)/p)(\delta p)$.

Here, we treat D_0 as an A -module via the natural quotient $A \rightarrow A_0$. If $\delta p = 0$, then the derivation is a *Frobenius semi-linear derivation*. If $\delta p = 1$, then it is a p -derivation. Both of these notions are central to [3]. Total p -derivations interpolate between them.

Example 2.2. For each $c \in \mathbf{Z}/p$ we may specify a total p -derivation $\theta_{p,c}: \mathbf{Z}/p^2 \rightarrow \mathbf{Z}/p$ by setting (following item 3):

$$\theta_{p,c}(r) = c \left(\frac{r - r^p}{p} \right).$$

So, $\theta_{p,c}(p) = c$. Observe that total p -derivations $\mathbf{Z}/p^2 \rightarrow \mathbf{Z}/p$ are determined by their value on p : if θ and θ' are two such derivations then $\theta - \theta'$ is a Frobenius semi-linear derivation, and the only Frobenius semi-linear derivation from $\mathbf{Z}/p^2$ to $\mathbf{Z}/p$ is zero.

2.3. Witt interpolation

Another view of total p -derivations can be given in terms of a construction interpolating between length 2 Witt vectors and a twisted infinitesimal thickening. Let D_0 be a $\mathbf{k}$ -algebra. For $c \in D_0$, let $U_c(D_0)$ be the ring whose underlying set is $D_0 \times D_0$ equipped with the operations

$$\begin{aligned} (x_0, x_1) + (y_0, y_1) &= (x_0 + y_0, x_1 + y_1 + cC_p(x_0, y_0)) \\ (x_0, x_1) \cdot (y_0, y_1) &= (x_0 y_0, x_0^p y_1 + y_0^p x_1). \end{aligned}$$

It is easily seen that $U_1(D_0) = W_2(D_0)$. On the other hand, $U_0(D_0)$ is an infinitesimal thickening of D_0 twisted by Frobenius. The ring $U_c(D_0)$ can be equipped with an R -algebra structure (using the fact that $(1, 0)$ is the multiplicative identity) by

$$r \cdot (1, 0) = \left(r, c \left(\frac{\varphi(r) - r^p}{p} \right) \right).$$

Observe that $p \cdot (1, 0) = (0, c)$. For any value of c , there is a homomorphism $U_c(B_0) \rightarrow B_0$ given by $(x_0, x_1) \mapsto x_0$. The kernel I satisfies $I^2 = 0$ and $pI = 0$. The operation $D_0 \mapsto U_c(D_0)$ can be seen as a *Witt interpolation* functor going from $\mathbf{k}$ -algebras to R -algebras.

The proof of the following is straightforward.

Lemma 2.4. *For $e \in A_0$, there is a homomorphism $h_e: U_c(D_0) \rightarrow U_{ce}(D_0)$ given by $h(x_0, x_1) = (x_0, ex_1)$.*

Proposition 2.5. *Given a homomorphism $f: A_0 \rightarrow D_0$ making D_0 into an A_0 -module, total p -derivations δ of A into D_0 with $\delta p = c \in D_0$ are in one-to-one correspondence with R -algebra homomorphisms*

$$(f, \delta): A \rightarrow U_c(D_0), \quad a \mapsto (f(\pi_0(a)), \delta a)$$

where $\pi_0: A \rightarrow A_0$ is reduction.

Lemma 2.6. *Let $g: A \rightarrow B$ be formally smooth. Let D_0 be a B_0 -algebra, with an A_0 -algebra structure induced by g . Then a total p -derivation δ_A of A into D_0 has a lift δ_B of B into D_0 (that is $\delta_A = \delta_B \circ g$). If, in addition, g is formally unramified, this lift is unique.*

Proof. Let $c = \delta_A(p)$. Then, the total p -derivativation δ_A induces a homomorphism $(f, \delta_A): A \rightarrow U_c(D_0)$. Consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{h} & D_0 \\ \uparrow g & \searrow & \uparrow \\ A & \longrightarrow & U_c(D_0). \end{array}$$

The top horizontal arrow h is the B_0 -algebra structure, and the right vertical arrow is reduction. The dotted arrow exists by formal smoothness (and is unique if f is formally unramified). It must be of the form (h, δ_B) for a total p -derivation δ_B . $\square$

The construction of U_c is functorial: if $h: B_0 \rightarrow C_0$ is a A_0 -algebra homomorphism, then there is an induced homomorphism $h: U_c(B_0) \rightarrow U_{h(c)}(C_0)$.

2.7. Total p -differentials

The module of *total p -differentials* $\Omega_A^{1, \text{tot}}$ is the A_0 -module generated by the symbols $d^{\text{tot}}x$ for $x \in A$ subject to the relations

$$(1) \text{ for } a, b \in A, d^{\text{tot}}(a + b) = d^{\text{tot}}a + d^{\text{tot}}b + (C_p(a, b))d^{\text{tot}}p,$$

- (2) for $a, b \in A$, $d^{\text{tot}}(ab) = (d^{\text{tot}}a)b^p + a^p(d^{\text{tot}}b)$, and
- (3) for $r \in R$, $d^{\text{tot}}r = ((\varphi(r) - r^p)/p)(d^{\text{tot}}p)$.

Here, we treat $\Omega_A^{1,\text{tot}}$ as an A -module with the A -action factoring through the reduction $A \rightarrow A_0$. There is a natural total p -derivation $\delta: A \rightarrow \Omega_A^{1,\text{tot}}$ that satisfies the following universality property: for any total p -derivation $\delta': A \rightarrow D_0$, there is a unique A_0 -module homomorphism $g: \Omega_A^{1,\text{tot}} \rightarrow D_0$ such that $\delta' = g \circ \delta$.

Example 2.8. For $A = R[x_1, \dots, x_n]$, a free polynomial algebra, $\Omega_A^{1,\text{tot}}$ is a free A_0 -module of rank $n + 1$, generated by $d^{\text{tot}}p, d^{\text{tot}}x_1, \dots, d^{\text{tot}}x_n$.

There are pullbacks for total p -differentials: given an R -algebra homomorphism $f: A \rightarrow B$, there is a well-defined map of B_0 -modules,

$$\Omega_A^{1,\text{tot}} \otimes_{A_0} B_0 \rightarrow \Omega_B^{1,\text{tot}}$$

taking $d^{\text{tot}}a \otimes b_0 \mapsto b_0 d^{\text{tot}}(f(a))$: it can be verified to be linear in the first coordinate and to respect the tensor product over A_0 .

For an A_0 -module M_0 , we will consider $F_{A_0}^* M_0$ to be the Frobenius tensor product of M : $F_{A_0}^* M_0$ is $M_0 \otimes_{A_0} A_0$ where A_0 is given an A_0 -module structure twisted by Frobenius (for $r \in A_0, x \in A_0$, $r \cdot x = r^p x$). The quotient of $\Omega_A^{1,\text{tot}}$ by $d^{\text{tot}}p$ is $F_{A_0}^* \Omega_{A_0}^1$. There is a natural A_0 -module homomorphism $\alpha: A_0 \rightarrow \Omega_A^{1,\text{tot}}$ given by $\alpha(x) = x d^{\text{tot}}p$. A left-inverse of α is a homomorphism $h: \Omega_A^{1,\text{tot}} \rightarrow A_0$ with $h(d^{\text{tot}}p) = 1$.

We will relate left-inverses of α to lifts of the absolute Frobenius. Here, a lift of absolute Frobenius is a ring homomorphism $\varphi: A \rightarrow A$ such that φ restricts to the Frobenius on R and makes the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A \\ \pi_0 \downarrow & & \downarrow \pi_0 \\ A_0 & \xrightarrow{x \mapsto x^p} & A_0 \end{array}$$

Remark 2.9. In much of the literature, one lifts the relative Frobenius which is a $\mathbf{k}$ -linear homomorphism $A_0 \rightarrow A_0^{(p)}$. Here, $A_0^{(p)} = A_0 \otimes_{\mathbf{k}} \mathbf{k}$ (with the $\mathbf{k}$ -algebra structure on $\mathbf{k}$ being given by $r \cdot x = r^p x$) so for $c \in \mathbf{k}$, $cx \otimes 1 = x \otimes c^p$. Lifts of relative Frobenius are in bijective correspondence with lifts of absolute Frobenius. Indeed, one has a commutative diagram

$$\begin{array}{ccccc}
 & & F_{A_0} & & \\
 & \swarrow & & \searrow & \\
 A_0 & \xleftarrow{F} & A_0^{(p)} & \xleftarrow{G} & A_0 \\
 & \uparrow & & \uparrow & \\
 \mathbf{k} & \xleftarrow{x^p \leftarrow x} & \mathbf{k} & &
 \end{array}$$

where F is the relative Frobenius given by $x \otimes c \mapsto cx^p$ and G is the structure homomorphism given by $x \mapsto x \otimes 1$. Similarly, we may form $A^{(\varphi)} = A \otimes_R R$ where R is given an R -algebra structure by $r \cdot x = \varphi(r)x$. A lift of relative Frobenius is an R -homomorphism $\Phi: A \rightarrow A^{(\varphi)}$ reducing to F . Throughout this paper, except in remarks, we will use absolute Frobenius which will be denoted by subscripted F .

Lemma 2.10. *Left-inverses $h: \Omega_A^{1,\text{tot}} \rightarrow A_0$ of α are in bijective correspondence with lifts of Frobenius on A where the correspondence is given by*

$$h \mapsto [\varphi_h: x \mapsto x^p + ph(d^{\text{tot}}x)].$$

Proof. Let h be a left-inverse. Define $\varphi_h: A \rightarrow A$ by $\varphi_h(x) = x^p + ph(d^{\text{tot}}x)$. We observe that

$$\begin{aligned}
 \varphi_h(x+y) &= (x+y)^p + ph(d^{\text{tot}}(x+y)) \\
 &= (x+y)^p + ph(d^{\text{tot}}x + d^{\text{tot}}y + C_p(x,y)d^{\text{tot}}p) \\
 &= x^p + y^p + ph(d^{\text{tot}}x + d^{\text{tot}}y) \\
 &= \varphi_h(x) + \varphi_h(y)
 \end{aligned}$$

and that

$$\begin{aligned}
 \varphi_h(xy) &= (xy)^p + ph(d^{\text{tot}}(xy)) \\
 &= (xy)^p + ph(x^p d^{\text{tot}}y + y^p d^{\text{tot}}x) \\
 &= (x^p + ph(d^{\text{tot}}x))(y^p + ph(d^{\text{tot}}y)) \\
 &= \varphi_h(x)\varphi_h(y).
 \end{aligned}$$

If $\varphi: A \rightarrow A$ is a lift of Frobenius, then we define $h: A \rightarrow A_0$ by $h(x) = (\varphi(x) - x^p)/p$. This makes sense since the reduction to $\mathbf{k}$ of $\varphi(x) - x^p$ is 0 and A has no p -torsion because A is a flat R -algebra. We claim that h factors through $d^{\text{tot}}: A \rightarrow \Omega_A^{1,\text{tot}}$. This follows from reversing the above arguments and noting that φ restricts to the usual Frobenius on R . $\square$

Lemma 2.11. *Let A be a smooth R -algebra. Then, there is an exact sequence of A_0 -modules,*

$$0 \longrightarrow A_0 \xrightarrow{\alpha} \Omega_A^{1,\text{tot}} \xrightarrow{\beta} F_{A_0}^* \Omega_{A_0}^1 \longrightarrow 0.$$

Proof. We only need to show that α is an injection. Because we are in the affine setting, there is a lift of Frobenius on A by smoothness, and therefore a left-inverse of α . $\square$

2.12. Globalization

The above constructions globalize by gluing. Indeed, given a scheme X over R , can produce a sheaf $\Omega_{X_0}^{1,\text{tot}}$ on the reduction X_0 over $\mathbf{k}$. We cover X by affines $U_i = \text{Spec } A_i$ and construct $\Omega_{A_i}^{1,\text{tot}}$ over $(A_i)_0$. To verify that these glue together, following [5], we must verify that given an R -algebra A , $f \in A$ with nonzero reduction $f_0 \in A_0$, we have a natural isomorphism of $(A_0)_{f_0}$ -modules:

$$\Omega_{A_f}^{1,\text{tot}} \cong \Omega_A^{1,\text{tot}} \otimes_{A_0} (A_0)_{f_0}.$$

This is a consequence of the following straightforward computation for $a \in A$ and n a positive integer:

$$d^{\text{tot}} \left(\frac{a}{f^n} \right) = \frac{f_0^{pn} d^{\text{tot}} a - a_0^p d^{\text{tot}} (f^n)}{f_0^{2pn}}.$$

Recall that on an open affine $U_0 = \text{Spec } A_0 \subset X_0$, if the quasicoherent sheaf $\mathcal{F}$ corresponds to a module M , the pullback by absolute Frobenius $F_{X_0}^* \mathcal{F}$ on U corresponds to $F_{A_0}^* M$. Consequently, for X smooth over R , the short exact sequence of Lemma 2.11 globalizes to a sequence of sheaves of $\mathcal{O}_{X_0}$ -modules, the *fundamental exact sequence of total p -differentials*:

$$0 \longrightarrow \mathcal{O}_{X_0} \xrightarrow{\alpha} \Omega_X^{1,\text{tot}} \xrightarrow{\beta} F_{X_0}^* \Omega_{X_0}^1 \longrightarrow 0 \quad (2.12.1)$$

This exact sequence is an imprecise analogue of the first exact sequence of Kähler differentials [11, Prop 8.3A], albeit with the differentials twisted by Frobenius. If R were $\mathbf{k}[\varepsilon]/\varepsilon^2$ instead of $W_2(\mathbf{k})$, there would be a sequence of $X \rightarrow \text{Spec } R \rightarrow \text{Spec } \mathbf{k}$. In that situation, the sheaf $\Omega_{R/\mathbf{k}}^1$ would pull back to X_0 as the structure sheaf $\mathcal{O}_{X_0}$.

The existence of the locally split short exact sequence immediately yields the following.

Proposition 2.13. *If X is smooth over R , then $\Omega_X^{1,\text{tot}}$ is locally free.*

Given a morphism $f: X \rightarrow Y$ over $\text{Spec } R$, there is a morphism of sheaves on X_0 , $f^* \Omega_Y^{1,\text{tot}} \rightarrow \Omega_X^{1,\text{tot}}$.

Lemma 2.14. *Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over R . Then f induces an injection $f^* \Omega_Y^{1,\text{tot}} \rightarrow \Omega_X^{1,\text{tot}}$. If, in addition, f is étale, then f is an isomorphism.*

Proof. We have a commutative diagram of locally free sheaves with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}_{X_0} & \xrightarrow{\alpha} & \Omega_X^{1,\text{tot}} & \xrightarrow{\beta} & F_{X_0}^* \Omega_{X_0}^1 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & f^* \mathcal{O}_{Y_0} & \xrightarrow{\alpha} & f^* \Omega_Y^{1,\text{tot}} & \xrightarrow{\beta} & f^* F_{Y_0}^* \Omega_{Y_0}^1 \longrightarrow 0.
 \end{array}$$

The left vertical arrow is an isomorphism by the definition of the pullback of sheaves while the right vertical arrow is injective if f is smooth (and an isomorphism if f is étale). $\square$

By globalizing Lemma 2.10, we obtain the following:

Proposition 2.15. *Splittings of the fundamental exact sequence (2.12.1) are in bijective correspondence with lifts of Frobenius. The set of lifts of Frobenius is a torsor over $H^0(X_0, \mathcal{H}\text{om}(F_{X_0}^* \Omega_{X_0}^1, \mathcal{O}_{X_0}))$.*

Proof. A splitting of the fundamental exact sequence locally gives lifts of Frobenius which glue into a global lift. Any two splittings differ by a global section of $\mathcal{H}\text{om}(F_{X_0}^* \Omega_{X_0}^1, \mathcal{O}_{X_0})$. $\square$

2.16. Total jet spaces

Total p -differentials were originally developed by the authors to recontextualize some arguments of Buium [3]. Specifically, given a scheme X over $W(\mathbf{k})$, Buium introduced the p -jet space X^1 . Its reduction to $\mathbf{k}$, X_0^1 is a torsor for the Frobenius tangent space (that is, the pullback of the tangent space of X_0 by absolute Frobenius) and is related to the Greenberg transform. In [3, Prop 1.10], Buium considers the projective completion of X_0^1 (considered as an affine torsor). By making use of the fundamental exact sequence, one can see that the projectivization of $\Omega_X^{1,\text{tot}}$ is manifestly a projective completion of the dual to $F_{X_0}^* \Omega_{X_0}^1$. This gives a useful functorial understanding of such completions.

2.17. Birings

In [6], Borger and Weiland make use of the formalism of birings, which are ring objects in the category of rings, to put certain universal constructions in algebra on equal footing. One has two subcategories of rings $\mathcal{R}_1, \mathcal{R}_2$ and wishes to study functors $F: \mathcal{R}_1 \rightarrow \mathcal{R}_2$ between them. The example to keep in mind is the formation of Witt vectors from the category of $\mathbf{Z}/p$ -algebras to the category of $\mathbf{Z}/p^2$ -algebras. In certain cases, the functor will be representable by a biring. That is, there is a ring Q equipped with the data of coaddition $\Delta^+: Q \rightarrow Q \otimes Q$ and comultiplication $\Delta^\times: Q \rightarrow Q \otimes Q$ along with additive and multiplicative counits and additive antipode such that for $A \in \mathcal{R}_1$,

$$F(A) = \text{Hom}(Q, A)$$

where Hom denotes ring homomorphisms. The coaddition and comultiplication induce addition and multiplication on $F(A)$. One may further consider categories $\mathcal{R}_i$ of algebras over fields $\mathbf{k}_i$ and then incorporate the data for algebra structures.

We will describe the construction in terms of $\mathbf{Z}/p$ and $\mathbf{Z}/p^2$ knowing that they can be replaced by $\mathbf{k}$ and $W_2(\mathbf{k})$ for a characteristic p field k . For $c \in \mathbf{Z}/p$, we will consider the Witt interpolation functor U_c (see subsection 2.3) as a functor from $\mathbf{Z}/p$ -algebras to $\mathbf{Z}/p^2$ -algebras. We define

$$Q_c = (\mathbf{Z}/p)[e, \eta]$$

with coaddition and comultiplication

$$\begin{aligned}\Delta^+(e) &= e \otimes 1 + 1 \otimes e \\ \Delta^+(\eta) &= \eta \otimes 1 + 1 \otimes \eta - c \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} e^{p-j} \otimes e^j \\ \Delta^\times(e) &= e \otimes e \\ \Delta^\times(\eta) &= e^p \otimes \eta + \eta \otimes e^p\end{aligned}$$

with the other structures defined naturally. It is straightforward to verify the following:

Lemma 2.18. *For any $\mathbf{Z}/p$ -algebra D_0 , there is an isomorphism:*

$$\begin{aligned}\text{Hom}_{\mathbf{Z}/p}(Q_c, D_0) &\rightarrow U_c(D_0) \\ f &\mapsto (x_0, x_1) = (f(e), f(\eta)).\end{aligned}$$

Here, we take f to the vector in $U_c(D_0)$ whose components are $f(e)$ and $f(\eta)$. The coaddition and comultiplication induce addition and multiplication in $U_c(D_0)$. We may view the $\mathbf{Z}/p^2$ -algebra structure on $U_c(D_0)$ as being induced by the map

$$\begin{aligned}\gamma: \mathbf{Z}/p^2 &\rightarrow \text{Aut}_{\mathbf{Z}/p}(Q_c) \\ a &\mapsto \begin{bmatrix} e \mapsto a_0 e \\ \eta \mapsto a_0^p \eta + e^p \theta_{p,c}(a) \end{bmatrix}\end{aligned}$$

where $\theta_{p,c}$ is defined in Example 2.2. In other words, it is the map taking the vector $(f(e), f(\eta))$ to $(f(\gamma(a)(e)), f(\gamma(a)(\eta)))$.

3. Extension classes from the fundamental exact sequence

In this section, we will suppose that X is smooth over R . Recall that by Lemma 2.10 a splitting of the fundamental exact sequence, if it exists, induces a lift of Frobenius on X . As an extension class, the fundamental exact sequence gives an element

$$\kappa \in \mathrm{Ext}^1(F_{X_0}^* \Omega_{X_0}^1, \mathcal{O}_{X_0}) \cong H^1(X_0, F_{X_0}^* TX_0)$$

(where TX_0 is dual to $\Omega_{X_0}^1$) that is the obstruction to a lift of Frobenius. We might call this the arithmetic Kodaira–Spencer class. If we imagine $\mathrm{Spec} R$ to be a point with a vector in the “ p -direction”, this would be analogous to the Kodaira–Spencer map applied to this vector. The extension class is interpreted as a Čech class by picking a covering U_i of X such that there is a splitting $\sigma_i: F_{(U_i)_0}^* \Omega_{(U_i)_0}^1 \rightarrow \Omega_{U_i}^{1, \mathrm{tot}}$. Then, $s_{ij} = \sigma_i - \sigma_j$ gives a homomorphism $F_{(U_i \cap U_j)_0}^* \Omega_{(U_i \cap U_j)_0}^1 \rightarrow \mathcal{O}_{(U_i \cap U_j)_0}$. The cocycle $\{s_{ij}\}$ is a representative of κ in $H^1(X_0, \mathcal{H}\mathrm{om}(F_{X_0}^* \Omega_{X_0}^1, \mathcal{O}_{X_0})) \cong H^1(X_0, F_{X_0}^* TX_0)$.

The Deligne–Illusie class [8, Sec. 2] (see also [9]), which is sometimes treated in the literature as analogous to the Kodaira–Spencer class, also obstructs the lift of Frobenius, and it is indeed equivalent to the above class by Theorem 3.2 below. We can construct the Deligne–Illusie class as a Čech class. Cover X with affine opens U_i which carry lifts of semi-linear Frobenius $F_{U_i}: U_i \rightarrow U_i$. One defines a homomorphism

$$c_{ij} = F_{U_i}^* - F_{U_j}^*: \mathcal{O}(U_i \cap U_j) \rightarrow pF_* \mathcal{O}(U_i \cap U_j)$$

where F is some lift of Frobenius on $U_i \cap U_j$. Now, if F is any lift of Frobenius on an open $U \subset X$, we have a canonical isomorphism $p: (F_{U_0})_* \mathcal{O}_{U_0} \rightarrow pF_* \mathcal{O}_U$ of sheaves on U_0 . Then, by [8, Sec. 2(c)(4)], $c_{ij}: \mathcal{O}(U_i \cap U_j) \rightarrow \mathcal{O}(U_i \cap U_j)$ factors through the differential d in the sense that there is a homomorphism

$$h_{ij}: \Omega_{X_0}^1(U_i \cap U_j) \rightarrow (F_{X_0})_* \mathcal{O}_{X_0}(U_i \cap U_j)$$

that fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(U_i \cap U_j) & \xrightarrow{c_{ij}} & pF_* \mathcal{O}_X(U_i \cap U_j) \\ \pi_0 \downarrow & & \uparrow p \\ \mathcal{O}_{X_0}(U_i \cap U_j) & & \\ d \downarrow & & \\ \Omega_{X_0}^1(U_i \cap U_j) & \xrightarrow{h_{ij}} & (F_{X_0})_* \mathcal{O}_{X_0}(U_i \cap U_j) \end{array}$$

By [8, Sec. 2(c)(5)], the collection $\{h_{ij}\}$ gives a cocycle and therefore an element h of

$$H^1(X_0, \mathcal{H}\text{om}(\Omega_{X_0}^1, (F_{X_0})_* \mathcal{O}_{X_0})) \cong H^1(X_0, \mathcal{H}\text{om}(F_{X_0}^* \Omega_{X_0}^1, \mathcal{O}_{X_0})).$$

Remark 3.1. The Deligne–Illusie class is also defined using lifts of relative Frobenius F and written as a element of $H^1(X_0^{(p)}, \mathcal{H}\text{om}(\Omega_{X_0^{(p)}}^1, F_* \mathcal{O}_{X_0}))$ where $X_0^{(p)}$ is the base-change by Frobenius, $X_0 \times_{\mathbf{k}} \mathbf{k}$. To see that this is equivalent, we relate the sheaves of differentials that appear. The absolute Frobenius on X_0 factors as

$$X_0 \xrightarrow{F} X_0^{(p)} \xrightarrow{G} X_0$$

where G is the isomorphism induced by the base-change by Frobenius on $\mathbf{k}$. From the exact sequence

$$0 \longrightarrow G^* \Omega_{X_0}^1 \longrightarrow \Omega_{X_0^{(p)}}^1 \longrightarrow \Omega_{X_0^{(p)}/X_0}^1 \longrightarrow 0$$

and the vanishing of $\Omega_{X_0^{(p)}/X_0}^1$, we have an isomorphism between $G^* \Omega_{X_0}^1$ and $\Omega_{X_0^{(p)}}^1$. Pulling each back by F , we get

$$F_{X_0}^* \Omega_{X_0}^1 \cong F_{X_0}^* \Omega_{X_0^{(p)}}^1.$$

Theorem 3.2. *We have the following relation between the arithmetic Kodaira–Spencer class and the Deligne–Illusie class:*

$$\kappa = -h.$$

Proof. The Frobenius lift $F_{U_i}^* : \mathcal{O}(U_i) \rightarrow \mathcal{O}(U_i)$ is equivalent to a splitting $h_i : \Omega_{U_i}^{1,\text{tot}} \rightarrow \mathcal{O}_{(U_i)_0}$ over U_i by

$$F_{U_i}^*(x) = x^p + h_i(d^{\text{tot}}x).$$

This splitting induces $\sigma_i : F_{U_i}^* \Omega_{(U_i)_0}^1 \rightarrow \Omega_{U_i}^{1,\text{tot}}$ by $\sigma_i(F_{(U_i)_0}^* dx_0) = d^{\text{tot}}x - \alpha(h_i(d^{\text{tot}}x))$ for any lift x of x_0 . Consequently, for $x \in \mathcal{O}(U_i \cap U_j)$, we have

$$\alpha(F_{U_i}^*(x) - F_{U_j}^*(x)) = \alpha(h_i(d^{\text{tot}}x) - h_j(d^{\text{tot}}x)) = \sigma_j(F_{(U_j)_0}^* dx_0) - \sigma_i(F_{(U_i)_0}^* dx_0)$$

giving the equality of the Deligne–Illusie and the extension classes. $\square$

4. Arithmetic Gauss–Manin homomorphism

There is an arithmetic Gauss–Manin homomorphism in our framework. It is the connecting homomorphism $H^0(X_0, F_{X_0}^* \Omega_{X_0}^1) \rightarrow H^1(X_0, \mathcal{O}_{X_0})$ of the long exact sequence attached to

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \Omega_X^{1,\text{tot}} \longrightarrow F_{X_0}^* \Omega_{X_0}^1 \longrightarrow 0.$$

This is analogous to the Gauss–Manin connection for the first de Rham cohomology over a one-dimensional base $\pi: Y \rightarrow S$ arising from the exact sequence

$$0 \longrightarrow \pi^* \Omega_S^1 \longrightarrow \Omega_Y^1 \longrightarrow \Omega_{Y/S}^1 \longrightarrow 0.$$

Theorem 4.1. *The arithmetic Gauss–Manin homomorphism is given by cup product with the extension class κ .*

Proof. We compute the arithmetic Gauss–Manin homomorphism. Let $\tilde{\omega} \in H^0(X_0, F_{X_0}^* \Omega_{X_0}^1)$. We pick a covering U_i of X such that on U_i , $\tilde{\omega}_i$ lifts to $\omega_i \in \Omega_{U_i}^{1,\text{tot}}$. Then, the image of the Gauss–Manin homomorphism is the cocycle $g_{ij} = \omega_i - \omega_j$. Because $\tilde{\omega}_i = \tilde{\omega}_j$ on $U_i \cap U_j$, g_{ij} is valued in $\mathcal{O}_{X_0}$.

By refining the cover, we may suppose that we have a splitting of the fundamental exact sequence on each U_i , $\sigma_i: F_{(U_i)_0}^* \Omega_{(U_i)_0}^1 \rightarrow \Omega_{U_i}^{1,\text{tot}}$. Set $\omega_i = \sigma_i(\tilde{\omega}_i)$. Then the image of the Gauss–Manin map is given by the cocycle $g_{ij} = \sigma_i(\omega_i) - \sigma_j(\omega_j)$. On $U_i \cap U_j$ we have

$$\begin{aligned} g_{ij} &= \sigma_i(\tilde{\omega}_i) - \sigma_j(\tilde{\omega}_j) \\ &= \sigma_i(\tilde{\omega}_i) - \sigma_j(\tilde{\omega}_i) \\ &= (\sigma_i - \sigma_j)(\tilde{\omega}_i) \\ &= s_{ij}(\tilde{\omega}_i). \end{aligned}$$

Therefore, by unwinding the definition of the cup product, we see

$$\{g_{ij}\} = \{\tilde{\omega}_i\} \cup \{s_{ij}\}. \quad \square$$

References

- [1] James Borger, λ -rings and the field with one element, arXiv preprint, arXiv:0906.3146, 2009.
- [2] Alexandru Buium, Differential characters of abelian varieties over p -adic fields, *Invent. Math.* 122 (1) (1995) 309–340.
- [3] Alexandru Buium, Geometry of p -jets, *Duke Math. J.* 82 (2) (1996) 349–367.
- [4] Alexandru Buium, Arithmetic analogues of derivations, *J. Algebra* 198 (1) (1997) 290–299.
- [5] Alexandru Buium, *Arithmetic Differential Equations*, vol. 118, American Mathematical Soc., 2005.
- [6] James Borger, Ben Wieland, Plethystic algebra, *Adv. Math.* 194 (2) (2005) 246–283.
- [7] Taylor Dupuy, James Freitag, Aaron Royer, Order one differential equations on nonisotrivial algebraic curves, arXiv preprint, arXiv:1707.08714, 2017.
- [8] Pierre Deligne, Luc Illusie, Relèvements modulo p^2 et décomposition du complexe de de Rham, *Invent. Math.* 89 (2) (1987) 247–270.
- [9] Taylor Dupuy, Lifted torsors of lifts of the Frobenius for curves, preprint, available at <https://www.uvm.edu/~tdupuy/>.
- [10] Taylor Dupuy, David Zureick-Brown, Deligne–Illusie classes as arithmetic Kodaira–Spencer classes, arXiv preprint, arXiv:1804.00228, 2018.
- [11] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977.

- [12] Nicholas M. Katz, Tadao Oda, On the differentiation of de Rham cohomology classes with respect to parameters, *J. Math. Kyoto Univ.* 8 (1968) 199–213.
- [13] J.-P. Serre, *Corps locaux*, Deuxième édition, Publications de l'Université de Nancago, vol. VIII, Hermann, Paris, 1968.