Fair Enough: Guaranteeing Approximate Maximin Shares

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We consider the problem of fairly allocating indivisible goods, focusing on a recently introduced notion of fairness called *maximin share guarantee*: each player's value for his allocation should be at least as high as what he can guarantee by dividing the items into as many bundles as there are players and receiving his least desirable bundle. Assuming additive valuation functions, we show that such allocations may not exist, but allocations guaranteeing each player 2/3 of the above value always exist. These theoretical results have direct practical implications.

8

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1 INTRODUCTION

We are interested in the *fair* allocation of *indivisible* goods; however, to explain the intricacies of this problem, we start from discussing the easier case of *divisible* goods. In the latter setting, known as *cake cutting*, we need to divide a heterogeneous cake between players with different valuation functions (i.e., different players may have different values for the same piece of cake).

When there are only two players, the *Cut and Choose* protocol provides a compelling method for dividing a cake—and will play an important conceptual role later on. Under this protocol, player 1 cuts the cake into two pieces that he values equally, and player 2 subsequently chooses the piece that he prefers, giving the other piece to player 1. The resulting allocation is fair in a precise, formal sense known as *envy-freeness*: Each player (weakly) prefers his own allocation to the allocation of the other player. Envy-free cake divisions exist for any number of players; today we know exactly how many cuts are needed to achieve such allocations in the worst case (Alon 1987) and how to constructively find them (Brams and Taylor 1995) (although subtle complexity questions remain open (Procaccia 2009, 2013)). It is interesting to note that in the standard cake-cutting setting, envy-freeness implies another natural fairness property called *proportionality*: each player in the

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set of players $\mathcal{N}$ receives a piece of cake whose value is at least $1/|\mathcal{N}|$ of the player's value for the entire cake.

Cake cutting is a nice metaphor for real-world problems such as land division; the study of cake cutting distills insights about fairness that are useful in related settings, such as the allocation of computational resources (Ghodsi et al. 2011; Parkes et al. 2015; Kash et al. 2014; Procaccia 2013). However, typical real-world situations where fairness is a chief concern, notably divorce settlements and the division of an estate between heirs, involve *indivisible goods* (e.g., houses, cars, and works of art), which in general preclude envy-free, or even proportional, allocations. As a simple example, if there are several players and only one indivisible item to be allocated, the allocation cannot possibly be proportional or envy free. Foreshadowing the approach that we take in the following, we note that no allocation can be even approximately (in a multiplicative sense) fair according to these notions, because some players receive an empty allocation of zero value.

So how can we divide an estate without lawyers? Potentially using an intriguing alternative to classical fairness notions, recently presented by Budish (2011) (building on concepts introduced by Moulin (1990)). Imagine that player 1 partitioned the items into $|\mathcal{N}|$ bundles, and each player in $\mathcal{N} \setminus \{1\}$ *adversarially* chose a bundle before player 1. A smart player would partition the bundles to maximize his minimum value for any bundle. For the same reason we intuitively view the Cut and Choose protocol as fair to player 1, even before specifying fairness axioms, the allocation that leaves player 1 with his least desired bundle seems fair to player 1—as he is the one who divided the items in the first place. Budish calls the value that player 1 can guarantee in this way his *maximin share* (MMS) guarantee.[1] But an allocation based on the division of player 1 may make another player regret the fact that he was not the one to divide the items. The question is this: can we allocate the items in a way that *all* players receive a bundle worth at least as much as their MMS guarantee? This question was recently addressed by Bouveret and Lemaître (2014), and although they were able to answer it for special cases (which we list in Section 1.3), they left the general question open.

1.1 Model, Conceptual Contribution, and Technical Results

Denote the set of players by $\mathcal{N}$ and the set of indivisible items to be allocated by $\mathcal{G}$. Furthermore, for notational convenience, let $n = |\mathcal{N}|$ and $m = |\mathcal{G}|$. Each player i is endowed with a valuation function $\mathtt{val}_i : 2^{\mathcal{G}} \to \mathbb{R}^+$. We simplify notation by writing $\mathtt{val}_i(j)$ instead of $\mathtt{val}_i(\{j\})$ for an item $j \in \mathcal{G}$. We assume that the valuation functions are *additive*:

$$\forall S \subseteq \mathcal{G},\ \mathtt{val}_i(S) = \sum_{j \in S} \mathtt{val}_i(j).$$

This assumption is also made in most of the related work on fair division of indivisible goods (see Section 1.3), including the work of Bouveret and Lemaître (2014) that studies the MMS guarantee in the same setting. And more importantly, people find it difficult to specify combinatorial preferences, which is why some deployed implementations of fair division methods (see Section 1.2) rely on additive valuation functions. Finally, our positive result does not hold under larger classes of valuation functions, such as subadditive and superadditive functions.

For a set $S \subseteq \mathcal{G}$, let $\Pi_k(S)$ be the set of k-partitions of S. Define the *k-maximin share (k-MMS) guarantee* of player $i \in \mathcal{N}$ as

$$\mathtt{MMS}_i(k, S) = \max_{(T_1, \ldots, T_k) \in \Pi_k(S)} \min_{j \in [k]} \mathtt{val}_i(T_j),$$

[1]This term should not be confused with the terminology of the systems literature, where max-min fairness simply refers to maximizing the value any player receives (Demers et al. 1989) rather than an axiomatic notion of fairness.

where $[k] = \{1, \ldots, k\}$; we call a partition that realizes this value player i's *k-maximin partition of S*. The valuation function used to determine a player's MMS guarantee will be clear from the context. An *allocation* $(A_1, \ldots, A_n) \in \Pi_n(\mathcal{G})$ allocates the subset of items A_i to each player i. We say that $(A_1, \ldots, A_n)$ is an *MMS allocation* if and only if

$$\forall i \in \mathcal{N},\ \mathtt{val}_i(A_i) \geq \mathtt{MMS}_i(n, \mathcal{G}).$$

Our first result is negative.

THEOREM 2.1. *For any set of players $\mathcal{N}$ such that $n \geq 3$, there exist a set of items $\mathcal{G}$ of size $m \leq 3n + 4$, and (additive) valuation functions, that do not admit an MMS allocation.*

We find this theorem surprising because extensive automated experiments by several groups of researchers (including us) have failed to find a counterexample. Indeed, the counterexamples rely on very precise constructions. In Section 2, we first provide explicit counterexamples for the cases of three and four players (the latter illustrates the key ideas), and then we give the full proof.

Although it seems that MMS allocations almost always exist, we wish to relax this fairness notion to guarantee existence. Fortunately, unlike other fairness notions such as envy-freeness, the MMS guarantee supports a multiplicative notion of approximation. Our main question is this:

> *Is there a value $\gamma > 0$ such that we can always find an allocation $A_1, \ldots, A_n$ that satisfies $\mathtt{val}_i(A_i) \geq \gamma \cdot \mathtt{MMS}_i(n, \mathcal{G})$ for all i?*

We answer this question in the positive for

$$\gamma = \gamma_n \triangleq \frac{2\lfloor n \rfloor_{odd}}{3\lfloor n \rfloor_{odd} - 1},$$

where $\lfloor n \rfloor_{odd}$ is the largest odd number that is smaller or equal to n. Note that γ_n is always greater than $2/3$, and it is equal to $3/4$ for the important cases of three and four players. More precisely, we prove the following theorem in Section 3.

THEOREM 3.1. *There always exists an allocation $A_1, \ldots, A_n$ such that for all $i \in \mathcal{N}$, $\mathtt{val}_i(A_i) \geq \gamma_n \mathtt{MMS}_i(n, \mathcal{G})$. Moreover, for every $\varepsilon > 0$, an allocation $A_1, \ldots, A_n$ such that for all $i \in \mathcal{N}$, $\mathtt{val}_i(A_i) \geq (1 - \varepsilon)\gamma_n \mathtt{MMS}_i(n, \mathcal{G})$ can be computed in polynomial time in n and m.*

1.2 Practical Applications of Our Results

The theory of fair division has been extensively studied, as shown, for example, by the books by Moulin (2003) and Brams and Taylor (1996). Despite the abundance of extremely clever fair division algorithms, very few have been implemented. The work of Budish (2011) is a rare example; his method is currently used for MBA course allocation at the Wharton School of the University of Pennsylvania. Another example is the *adjusted winner* method (Brams and Taylor 1996), which assumes that there are exactly two players (with additive valuation functions). Adjusted winner has been patented by NYU and licensed to Fair Outcomes Inc.

Two of us (Kurokawa and Procaccia) are involved in an effort to change this situation by building a fair-division Web site called *Spliddit* (Goldman and Procaccia 2014), available at `http://www.spliddit.org`. Quoting from the Web site:

> *Spliddit is a not-for-profit academic endeavor. Its mission is twofold:*
>
> — *To provide easy access to carefully designed fair division methods, thereby making the world a bit fairer.*
> — *To communicate to the public the beauty and value of theoretical research in computer science, mathematics, and economics, from an unusual perspective.*

Since its launch in November 2014, Spliddit has attracted more than 110,000 users (as of September 24, 2017) and has received significant press coverage.

Spliddit contains implementations of existing mechanisms for the division of rent, credit, fare, and chores. However, for the fifth application—dividing indivisible goods—we were unable to find satisfactory methods for more than two players, despite discussions with leading experts on fair division (we survey some existing methods in Section 1.3). This provided strong motivation for the theoretical work reported here.

The approach that we ultimately implemented relies heavily on Theorem 3.1. We consider three "levels" of fairness: envy-freeness, proportionality, and (approximate) MMS guarantee. It is easy to verify that each of these fairness notions implies the ones following it. Users specify their valuation functions by distributing a fixed pool of points between the items. We then find an allocation that maximizes social welfare—$\sum_{i \in \mathcal{N}} \mathtt{val}_i(A_i)$—subject to the strongest feasible fairness constraint (using an integer linear programming formulation, which is solved via CPLEX). For MMS, we maximize the value of γ for which the γ-MMS guarantee is feasible. By Theorem 3.1, achieving $\gamma > 2/3$ of the MMS guarantee is always feasible, so the theorem ensures an outcome that is, well, fair enough. By providing rigorous fairness guarantees that are easy to explain, it justifies Spliddit's tagline: "provably fair solutions."

1.3 Related Work

1.3.1 Prior Work. Motivated by the problem of allocating courses to students, Budish (2011) studies a solution concept that he calls *approximate competitive equilibrium from equal incomes* (CEEI). Budish shows the existence of an approximate CEEI (with certain approximation parameters), even when the preferences of players are unrestricted (so they may correspond to any combinatorial valuation functions). Roughly speaking, an approximate CEEI guarantees that $\mathtt{val}_i(A_i) \geq \mathsf{MMS}_i(n+1, \mathcal{G})$—that is, each of the n players receives its $(n+1)$-MMS guarantee. However, this result takes advantage of an approximation error in the items that are allocated (some items might be in excess demand or excess supply). The approximation error grows with the overall number of items, and with the number of items demanded by each player, but not with the number of players or the number of copies of each item. Therefore, as the two latter parameters go to infinity, the error goes to zero. A large economy, in this sense, is plausible in the context of MBA course allocation, because there are many MBA students and many seats in each course, but relatively few courses that are offered, and even fewer courses a single student can take. But Budish's results do not provide practical guarantees when there are, say, three or four players, and (very possibly) only one copy of each item—which is the setting in which we are interested.

Like us, Bouveret and Lemaître (2014) focus on the division of indivisible goods between players with additive valuations. They study a hierarchy of fairness properties, of which the MMS guarantee is the weakest (it is easy to see that allocations satisfying the other properties may not exist). Among other results, they show that MMS allocations exist in the following cases: (i) valuations for items are 0 or 1, (ii) the values different players assign to items form identical multisets, and (iii) $m \leq n+3$. They also present results from extensive simulations using different distributions over item values; MMS allocations exist in each and every trial.

Also related is the work of Lipton et al. (2004). Among other results, they give a polynomial time algorithm that computes approximately envy-free allocations, where the approximation is *additive*. Specifically, they let α be the largest possible increase in value a player can have from adding one item to his bundle and produce an allocation such that $\mathtt{val}_i(A_i) \geq \mathtt{val}_i(A_j) - \alpha$ for all $i, j \in \mathcal{N}$. This interesting result may not be very practical in and of itself; for example, if one of the items is extremely valuable, the players would not be guaranteed anything. In contrast, assuming that items have positive values, an MMS allocation (or any multiplicative approximation thereof) gives some player a bundle worth zero (if and) only if *any* allocation gives some player a bundle worth zero.

Hill (1987) shows that when valuations are additive, indivisible items can be allocated in a way that a certain value is guaranteed to each player, and Markakis and Psomas (2011) refine this guarantee and construct a polynomial time algorithm that achieves it. However, the guaranteed value is defined using an unwieldy function that depends on the number of players as well as on the value of the most valuable item, and even for three players the function's value quickly goes down to zero as the most valuable item becomes more valuable.

When there are exactly two players, practical methods for dividing indivisible goods are available. For example, recent work by Brams et al. (2014) gives a method satisfying several desirable properties, including envy-freeness; its main shortcoming is that it may not allocate all items (it generates a "contested pile" of unallocated items). The adjusted winner method (Brams and Taylor 1996), mentioned earlier, is another practical method (which is routinely being used, as discussed in Section 1.2)—but it implicitly assumes that the items are divisible and would typically require splitting one of the items. In any case, for more than two players, one encounters a great many paradoxes when contemplating standard fairness notions (Brams et al. 2003). Moreover, generalizing these practical two-player protocols is impossible. For example, adjusted winner can be interpreted as a special case of the *egalitarian equivalent* (Pazner and Schmeidler 1978) rule (for two players and additive valuation functions), but the latter method strongly relies on divisibility and may end up splitting all goods.

From an algorithmic viewpoint, our work is related to works on the problem of allocating indivisible goods to maximize the minimum value any player has for his bundle (under additive valuation functions)—also known as the *Santa Claus* problem (Bezáková and Dani 2005; Bansal and Sviridenko 2006; Asadpour and Saberi 2007). Woeginger (1997) studies the special case of players with identical valuations and presents a polynomial time approximation scheme (PTAS) that we leverage in the following.

Somewhat further afield, recent years have seen quite a bit of computational work on cake cutting (see Procaccia (2013) for an overview). One question that received some attention from the theoretical computer science community is the complexity of proportional and envy-free cake cutting in a concrete complexity model (Magdon-Ismail et al. 2003; Edmonds and Pruhs 2006a, 2006b; Woeginger and Sgall 2007; Procaccia 2009).

1.3.2 Subsequent Work. Since the publication of the earliest version of our results (Procaccia and Wang 2014), several works have followed up on ours.

The preliminary version of Theorem 3.1 (Procaccia and Wang 2014) achieves a $2/3 - \varepsilon$ approximation of the MMS guarantee in polynomial time *only in m*—that is, computational efficiency requires a constant number of players. The main result of Amanatidis et al. (2015) improves the running time of that algorithm: they achieve a $2/3 - \varepsilon$ fraction of the MMS guarantee in polynomial time for any number of players. They do this by modifying one of the steps of the original (unintuitive) algorithm of Procaccia and Wang (2014). The current proof of Theorem 3.1 is completely different from the original one, and, in particular, immediately leads to an (arguably) intuitive, polynomial time algorithm. Among other results, Amanatidis et al. (2015) also show that a $7/8$-MMS allocation can be guaranteed for three players, improving on our bound of $3/4$ for this case.

Independently of our work on the current proof of Theorem 3.1, Barman and Krishna Murthy (2017) provide an almost identical (slightly weaker) bound through a relatively simple, polynomial time algorithm. The two algorithms are technically related in that both build on the envy-cycle elimination procedure of Lipton et al. (2004). One key difference is that Barman and

Krishna Murthy effectively leverage a reduction, due to Bouveret and Lemaître (2014), to the case where all players have the same ordinal preferences over the items. Barman and Krishna Murthy also achieve a 1/10 approximation of the MMS guarantee when the valuation functions are submodular.

A brand new work by Ghodsi et al. (2017) provides even stronger bounds. Most importantly, they design an algorithm that gives a 3/4-approximation to the MMS guarantee for additive valuations. Like Barman and Krishna Murthy (2017), they also examine more general combinatorial valuations, achieving approximation ratios of 1/3 for submodular, 1/5 for XOS, and $\Theta(\log m)$ for subadditive.

In a slightly different direction, Amanatidis et al. (2016) design *truthful* approximations algorithms for the MMS guarantee. For the so-called cardinal model, where players report their value for each item, they provide a truthful algorithm that achieves a $\Theta(m)$-approximation of the MMS guarantee. For the case of two players (where an MMS allocation always exists), they are able to give a truthful 1/2-approximation of the MMS guarantee and prove that no truthful algorithm can yield a better ratio.

In our recent work with Caragiannis et al. (2016), we advocate the *max Nash welfare* solution, which maximizes the product of utilities, as a method for allocating indivisible goods. We show that this solution, which is clearly Pareto efficient, satisfies an approximate envy-freeness property and also provides a $\Theta(1/\sqrt{n})$ approximation of the MMS guarantee in theory, and a much better approximation in practice. The new solution has been deployed on Spliddit since May 2016.

1.4 Open Problems

One obvious question remains open. Theorem 2.1 does not provide an upper bound on the the constant $\gamma > 0$ such that γ-MMS allocations always exist, and our constructions in Section 2 provide very weak upper bounds. Our lower bound, given by Theorem 3.1, is 2/3; as noted earlier, it was improved to 3/4 by Ghodsi et al. (2017). Further narrowing this gap is, in our view, an important challenge.

As noted earlier, Budish (2011) introduced a different notion of MMS approximation. In its ideal form, we would ask for an allocation such that $\mathtt{val}_i(A_i) \geq \mathtt{MMS}_i(n+1, \mathcal{G})$. We have designed an algorithm that achieves this guarantee for the case of three players (it is already nontrivial). Proving or disproving the existence of such allocations for a general number of players remains an open problem; a positive result would provide a compelling alternative to Theorem 3.1.

2 NONEXISTENCE OF EXACT MMS ALLOCATIONS

In this section, we will show that, in general, MMS allocations are not guaranteed to exist (even under our assumption of additive valuation functions). But to give some context for this result, let us briefly discuss a case where they do exist. As pointed out by Bouveret and Lemaître (2014), when there are two players we can achieve an MMS allocation—essentially via an indivisible analog of the Cut and Choose protocol. First, let player 1 divide the items according to a 2-maximin partition S_1, S_2 of his (i.e., the partition that maximizes $\min_{j \in [2]} \mathtt{val}_1(S_j)$). Allocate to player 2 his preferred subset, and give the other subset to player 1. Player 1 clearly achieves his MMS guarantee, but what about player 2? By the additivity of $\mathtt{val}_2$, there exists $j \in [2]$ such that $\mathtt{val}_2(S_j) \geq \mathtt{val}_2(\mathcal{G})/2$. In addition, in any partition S'_1, S'_2 there exists $k \in [2]$ such that $\mathtt{val}_2(S'_k) \leq \mathtt{val}_2(\mathcal{G})/2$, hence $\mathtt{MMS}_2(2, \mathcal{G}) \leq \mathtt{val}_2(\mathcal{G})/2$. It follows that there exists $j \in [2]$ such that $\mathtt{val}_2(S_j) \geq \mathtt{MMS}_2(2, \mathcal{G})$.

In contrast, MMS allocations may not exist when the number of players is at least three.

Theorem 2.1. *For any set of players $\mathcal{N}$ such that $n \geq 3$, there exist a set of items $\mathcal{G}$ of size $m \leq 3n + 4$, and (additive) valuation functions, that do not admit an MMS allocation.*

The case of $n = 3$ is handled separately in Section 2.1. A single construction works for any $n \geq 4$, but because it is rather complex, we first illustrate the main ideas in Section 2.2 for the special case of $n = 4$ and then provide the full construction in Section 2.3.

2.1 Proof of Theorem 2.1 for $n = 3$

Let the set of items be $\mathcal{G} = \{(i, j) \mid i \in [3], j \in [4]\}$ (note that $m = 12 < 3n + 4$). The valuation functions of the three players are defined using the following two matrices:

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 17 & 25 & 12 & 1 \\ 2 & 22 & 3 & 28 \\ 11 & 0 & 21 & 23 \end{bmatrix},$$

in conjunction with the three matrices:

$$E^{(1)} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad E^{(3)} = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

For each item $(i, j) \in \mathcal{G}$, we let

$$\mathtt{val}_k(\{(i, j)\}) = 10^6 \cdot S_{i,j} + 10^3 \cdot T_{i,j} + E^{(k)}_{i,j}.$$

Our first goal is to compute the MMS guarantee of each player. To this end, we will find it convenient to label each element of T with three of nine possible labels $(1, 2, 3, \alpha, \beta, \gamma, +, -, *)$:

$$\begin{bmatrix} {}^{\alpha}17^1_+ & {}^{\alpha}25^1_- & {}^{\beta}12^1_+ & {}^{\gamma}1^1_* \\ {}^{\alpha}2^2_- & {}^{\beta}22^2_* & {}^{\gamma}3^2_+ & {}^{\gamma}28^2_- \\ {}^{\alpha}11^3_* & {}^{\beta}0^3_- & {}^{\beta}21^3_* & {}^{\gamma}23^3_+ \end{bmatrix}.$$

T has the following Sudoku-like property: for each label, there are exactly four elements with that label, and the sum of these four elements is exactly 55. Moreover, any four elements whose sum is 55 must have the same label.

This observation facilitates a straightforward computation of MMS guarantees. Player 1 can divide the 12 items into three subsets: a subset consisting of the four elements labeled with 1 (the first row), a subset consisting of the four elements labeled by 2 (the second row), and a subset consisting of the four elements labeled by 3 (the third row). For each subset, the sum of its four elements in S, T, and $E^{(1)}$ is 4, 55, and 0, respectively. Hence, $\mathsf{MMS}_1(3, \mathcal{G}) = 4 \cdot 10^6 + 55 \cdot 10^3 + 0 = 4{,}055{,}000$. Player 2's maximin partition is obtained by dividing the items into three subsets according to the labels α, β, and γ, and player 3's maximin partition corresponds to the labels $+, -,$ and $*$; all MMS guarantees are 4,055,000.

We next characterize MMS allocations of $\mathcal{G}$, with the goal of showing that no such allocations exist. First note that a valid MMS allocation of $\mathcal{G}$ must allocate at least 4 items to each player. Indeed, for any bundle $X \subseteq \mathcal{G}$ such that $|X| = 3$ and each player $i = 1, 2, 3$, $\mathtt{val}_i(X) \leq 3 \cdot 10^6 + 76 * 10^3 + 3 < 4{,}055{,}000$. Because there are 12 items, each player must receive exactly 4 items.

We now claim that in an MMS allocation each player must receive four items with the same label. Indeed, as noted earlier, the only bundles whose values in T add up to 55 consist of four items with identical labels. Suppose that a player is allocated four items with different labels. Since the sum of all the elements in T is $165 = 55 \times 3$, there must be a player with four items whose sum in T is less than 55. This player's value is at most $4 \cdot 10^6 + 54 \cdot 10^3 + 3 < 4{,}055{,}000$.

It is easy to verify that there are only three ways to divide $\mathcal{G}$ into three subsets such that the items in each subset have identical labels:

(1) Dividing according to the labels 1,2,3.
(2) Dividing according to the labels α, β, γ.
(3) Dividing according to the labels $+, -$ and $*$.

All three ways will fail to give some player his MMS guarantee of 4,055,000. Indeed, in case (1), there is a player $i_1 \in \{2, 3\}$ who is allocated items labeled by 2 or 3. The sum of the corresponding elements in $E^{(i_1)}$ is -1, hence the value i_1 obtains is $4 \cdot 10^6 + 55 \cdot 10^3 - 1 = 4{,}054{,}999 < 4{,}055{,}000$. In case (2), a player $i_2 \in \{1, 3\}$ must be allocated a subset of items labeled with β or γ; in case (3), a player $i_3 \in \{1, 2\}$ must be allocated a subset of items labeled with $-$ or $*$. By the same reasoning as in case (1), in cases (2) and (3), player i_l, $l = 2, 3$, ends up with value at most 4,054,999. We conclude that it is impossible to satisfy the MMS guarantees of all three players. □

2.2 Proof of Theorem 2.1 for $n = 4$

Because the construction for $n \geq 4$ is somewhat intricate, we start by explicitly providing the special case of $n = 4$ as mentioned previously. To this end, let us define the following two matrices, where ε is a very small positive constant ($\varepsilon = 1/16$ will suffice):

$$S = \begin{bmatrix} \frac{7}{8} & 0 & 0 & \frac{1}{8} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{8} \end{bmatrix}, \quad T = \begin{bmatrix} 0 & \varepsilon^4 & 0 & -\varepsilon^4 \\ \varepsilon^3 & 0 & -\varepsilon^3 + \varepsilon^2 & -\varepsilon^2 \\ 0 & -\varepsilon^4 + \varepsilon & 0 & \varepsilon^4 - \varepsilon \\ -\varepsilon^3 & -\varepsilon & \varepsilon^3 - \varepsilon^2 & \varepsilon^2 + \varepsilon \end{bmatrix}.$$

Let $M = S + T$. Crucially, the rows and columns of M sum to 1. Let $\mathcal{G}$ contain goods that correspond to the nonzero elements of M—that is, for every entry $M_{i,j} > 0$, we have a good (i, j); note that $m = 14 < 3n + 4$.

Next, partition the four players into $P = \{1, 2\}$ and $Q = \{3, 4\}$. Define the valuations of the players in P as follows where $0 < \tilde{\varepsilon} \ll \varepsilon$ ($\tilde{\varepsilon} = 1/64$ will suffice):

$$M + \begin{bmatrix} 0 & 0 & 0 & -\tilde{\varepsilon} \\ 0 & 0 & 0 & -\tilde{\varepsilon} \\ 0 & 0 & 0 & -\tilde{\varepsilon} \\ 0 & 0 & 0 & 3\tilde{\varepsilon} \end{bmatrix}.$$

In other words, the values of the rightmost column are perturbed. For example, for $i \in P$, $\mathtt{val}_i(\{(1, 4)\}) = 1/8 - \varepsilon^4 - \tilde{\varepsilon}$. Similarly, for players in Q, the values of the bottom row are perturbed:

$$M + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\tilde{\varepsilon} & -\tilde{\varepsilon} & -\tilde{\varepsilon} & 3\tilde{\varepsilon} \end{bmatrix}.$$

It is easy to verify that the MMS guarantee of all players is 1 by partitioning the items based off their rows (for players in Q) or columns (for players in P). Moreover, our construction ensures the unique MMS partition of the players in P (where every subset has value 1) corresponds to the columns of M, and the unique MMS partition of the players in Q corresponds to the rows of M. If we divide the goods by columns, one of the two players in Q will end up with a bundle of goods worth at most $1 - \tilde{\varepsilon}$—which is less than his MMS guarantee of 1. Similarly, if we divide the goods by rows, one of the players in P will receive a bundle worth only $1 - \tilde{\varepsilon}$. Any other division will certainly fail assuming that $\tilde{\varepsilon}$ is sufficiently small. □

2.3 Proof of Theorem 2.1 for $n \geq 4$

With the illustrative example of $n = 4$ under our belt, we are now ready for the general case where $n \geq 4$. The crux of the argument is proving the existence of a matrix $M \in \mathbb{R}^{n\times n}$ with the following properties:

(1) All entries are nonnegative (i.e., $\forall i, j : M_{i,j} \geq 0$).
(2) All entries of the last row and column are positive (i.e., $\forall i : M_{i,n}, M_{n,i} > 0$).
(3) All rows and columns sum to 1 (i.e., $M\vec{1} = M^T\vec{1} = \vec{1}$).
(4) Define M^+ as the set of all positive entries in M. Then if we wish to partition M^+ into n subsets that sum to exactly 1, then our partition must correspond to the rows of M or the columns of M.

To begin, let $S \in \mathbb{R}^{n\times n}$ be the following matrix.

$$\begin{bmatrix} \frac{2^{n-1}-1}{2^{n-1}} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2^{n-1}} \\ 0 & \frac{2^{n-2}-1}{2^{n-2}} & 0 & \cdots & 0 & 0 & \frac{1}{2^{n-2}} \\ 0 & 0 & \frac{2^{n-3}-1}{2^{n-3}} & \cdots & 0 & 0 & \frac{1}{2^{n-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2^{n-1}} & \frac{1}{2^{n-2}} & \frac{1}{2^{n-3}} & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{2^{n-1}} \end{bmatrix} \text{For instance, } S_{i,j} = \begin{cases} \frac{2^{n-i}-1}{2^{n-i}} & \text{if } i = j \neq n \\ \frac{1}{2^{n-j}} & \text{if } i = n \text{ and } j \neq n \\ \frac{1}{2^{n-i}} & \text{if } j = n \text{ and } i \neq n \\ \frac{1}{2^{n-1}} & \text{if } i = j = n \\ 0 & \text{otherwise.} \end{cases}$$

Now for $\varepsilon \approx 0$ where $\varepsilon > 0$, and for all $i \in [n-2]$, let $r_i = \varepsilon^{2n-2i-2}$, and $c_i = \varepsilon^{2n-2i-3}$. Specifically, this implies

$$0 < r_1 \ll c_1 \ll r_2 \ll c_2 \ll \cdots \ll r_{n-2} \ll c_{n-2} = \varepsilon \approx 0.$$

Furthermore, let $T \in \mathbb{R}^{n\times n}$ be the matrix given by (where we will define x, y, z and the u_i, and v_i in the following):

$$\begin{bmatrix} 0 & v_1 & 0 & \cdots & 0 & 0 & -r_1 \\ u_1 & 0 & v_2 & \cdots & 0 & 0 & -r_2 \\ 0 & u_2 & 0 & \cdots & 0 & 0 & -r_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & v_{n-2} & -r_{n-2} \\ 0 & 0 & 0 & \cdots & u_{n-2} & 0 & -y \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-2} & -x & z \end{bmatrix} \text{For instance, } T_{i,j} = \begin{cases} u_j & \text{if } i = j+1 \text{ and } j \leq n-2 \\ v_i & \text{if } j = i+1 \text{ and } i \leq n-2 \\ -r_i & \text{if } j = n \text{ and } i \leq n-2 \\ -c_j & \text{if } i = n \text{ and } j \leq n-2 \\ -x & \text{if } i = n \text{ and } j = n-1 \\ -y & \text{if } i = n-1 \text{ and } j = n \\ z & \text{if } i = j = n \\ 0 & \text{otherwise.} \end{cases}$$

Note the only nonzero entries are on the first diagonals above and below the main diagonal, and the last row and column.

Now assign positive values to the u_i, v_i, x, y, and z such that all rows and columns sum to zero. A bit of arithmetic then gives

$$
\begin{aligned}
u_i &= \left(\sum_{j \le i, j \equiv i \pmod 2} c_j \right) - \left(\sum_{j \le i, j \not\equiv i \pmod 2} r_j \right) \approx c_i \\
v_i &= \left(\sum_{j \le i, j \equiv i \pmod 2} r_j \right) - \left(\sum_{j \le i, j \not\equiv i \pmod 2} c_j \right) \approx r_i \\
x &= v_{n-2} \approx r_{n-2} \\
y &= u_{n-2} \approx c_{n-2} \\
z &= \left(\sum_{j \le n-2, j \equiv n \pmod 2} r_j + c_j \right) \approx c_{n-2}.
\end{aligned}
$$

Now define $M = S + T$ and $M^+ = \{(i,j) \mid M_{i,j} \neq 0\}$. Moreover, for a set $X \subseteq M^+$, let $\sum X = \sum_{(i,j) \in X} M_{i,j}$. Then we see for sufficiently small ε that the following properties hold.

[P1] $M_{i,j} \geq 0$ and if $S_{i,j} \neq 0$ or $T_{i,j} \neq 0$, then $M_{i,j} > 0$.
[P2] $M_{i,j} \approx S_{i,j}$.
[P3] All rows and columns sum to 1 (i.e., $M\vec{1} = M^T\vec{1} = \vec{1}$).
[P4] $\forall i \in [n-1]$ if we have $X \subseteq M^+$ subject to $(i,i) \in X$ and $\sum X = 1$, then exactly one of the following is true:
  (a) $(i,n) \in X$.
  (b) $(n,i) \in X$.
  (c) $(1,n), (2,n), \ldots, (i-1,n), (n,n) \in X$.
  (d) $(n,1), (n,2), \ldots, (n,i-1), (n,n) \in X$.
  (e) $\exists j,k < i$ subject to $(j,n), (n,k) \in X$.
  This is easy to see when we take note that $M \approx S$ by [P2].
[P5] If $X \subseteq M^+$ subject to $\sum X = r_i$, then $X = \{(i,i-1), (i,i+1)\}$.
[P6] If $X \subseteq M^+$ subject to $\sum X = c_i$, then $X = \{(i-1,i), (i+1,i)\}$.
[P7] If $X \subseteq M^+$ subject to $\sum X = x$, then $X = \{(n-2,n-1)\}$.
[P8] If $X \subseteq M^+$ subject to $\sum X = y$, then $X = \{(n-1,n-2)\}$.

We now make a key observation with respect to M.

Lemma 2.2. *Suppose that $X_1, \ldots, X_n$ is a partition of M^+ such that $\sum X_i = 1$ for all i. Then for sufficiently small ε, the partition must correspond to the rows of M or the columns of M.*

Proof. Let us first consider the subset in the partition that includes $(1,1)$. Without loss of generality, assume that this is X_1. We wish to prove that X_1 is either:

(1) the first row $= \{(1,1), (1,2), (1,n)\}$, or
(2) the first column $= \{(1,1), (2,1), (n,1)\}$.

By [P4], we see that exactly one of $(n,1)$, $(1,n)$, and (n,n) must be part of X_1:

(1) Suppose that $(n,n) \in X_1$. Then $\sum X_1 \geq M_{1,1} + M_{n,n} = 1 + z > 1$. This is therefore impossible.
(2) Suppose that $(1,n) \in X_1$. As $M_{1,1} + M_{1,n} = 1 - r_1$, we see that by [P5] we must have $(1,2) \in X_1$. Then X_1 corresponds to the first row.

(3) Suppose that $(n, 1) \in X_1$. As $M_{1,1} + M_{n,1} = 1 - c_1$, we see that by [P6] we must have $(2, 1) \in X_1$. Then X_1 corresponds to the first column.

Now suppose that we wish to find a partition as in the lemma's statement such that the first $i - 1$ rows are in the partition where $i \in \{2, \dots, n\}$. Then we claim that row i must be in the partition as well. Importantly, this implies that if the first row is to be in the partition, then the partition must be the rows.

We first consider the case where $i \leq n - 1$. Let X_i denote the subset in the partition that includes (i, i). By [P6], we see that we must have one of the following:

(1) $(i, n) \in X_i$.
If $i \leq n - 2$, we find that $M_{i,i} + M_{i,n} = -r_i$, and so by [P5] we have $(i, i - 1), (i, i + 1) \in X_i$. We therefore find that $X_i = \{(i, i - 1), (i, i), (i, i + 1), (i, n)\}$. Furthermore, if $i = n - 1$, we find that $M_{i,i} + M_{i,n} = -y$, and so by [P8] we have $(n - 1, n - 2) \in X_i$. Thus, $X_i = \{(n - 1, n - 2), (n - 1, n - 1), (n - 1, n)\}$. In either case, X_i is the i^{th} row.

(2) $(n, i) \in X_i$.
If $i \leq n - 2$, we find that $M_{i,i} + M_{n,i} = -c_i$, and so by [P6] we have $(i - 1, i) \in X_i$. But $(i - 1, i)$ is in a previous row, which by our assumption is already assigned to a subset in the partition. Furthermore, if $i = n - 1$, we have $M_{i,i} + M_{n,i} = -x$, and so by [P7] we have $(n - 2, n - 1) \in X_i$. Similarly to before, this element is in a previous row and thus is already assigned to a subset in the partition.

(3) $(1, n), (2, n), \dots, (i - 1, n), (n, n) \in X_i$.
As $(1, n)$ is in a previous row, it is already assigned to a subset in the partition.

(4) $(n, 1), (n, 2), \dots, (n, i - 1), (n, n) \in X_i$.
This is impossible because

$$\begin{aligned}\sum X_i &\geq M_{i,i} + M_{n,1} + M_{n,2} + \cdots + M_{n,i-1} + M_{n,n} \\ &= 1 - r_1 - r_2 - \cdots - r_{i-1} + z \\ &= 1 + r_i + r_{i+1} + \cdots + r_{n+2} + y \\ &> 1.\end{aligned}$$

(5) $\exists j, k < i$ subject to $M_{j,n}, M_{n,k} \in X_i$.
As (j, n) is in a previous row, it is already assigned to a subset in the partition.

Next, suppose that $i = n$. In this case, since we are only allowed n subsets in this partition, all remaining entries (i.e., the last row) must be in the last set. By [P3], we know that this last row sums to 1. We therefore have shown that if the first row is in the partition, then the partition simply corresponds to the rows. A similar argument gives an analogous result for columns. As the first row or first column must be a subset in the partition (namely as X_1), we are done. □

To show the $n \geq 4$ counterexample, we now consider our construction through the lens of MMS allocations. We first show that there exists a set of $5n - 6$ such goods for $n \geq 4$.

Partition the n players into two groups P and Q such that $|P|, |Q| \geq 2$, and let $\mathcal{G} = M^+$. Note that there are $|M^+| = 5n - 6$ such goods. For $k \in P$, we define

$$\mathtt{val}_k(\{(i, j)\}) = \begin{cases} M_{i,j} & \text{if } j < n \\ M_{i,j} - \tilde{\varepsilon} & \text{if } j = n \text{ and } i < n \\ M_{i,j} + (n - 1)\tilde{\varepsilon} & \text{if } j = n \text{ and } i = n, \end{cases}$$

and similarly for $k \in Q$, let

$$\mathtt{val}_k(\{(i,j)\}) = \begin{cases} M_{i,j} & \text{if } i < n \\ M_{i,j} - \tilde{\varepsilon} & \text{if } i = n \text{ and } j < n \\ M_{i,j} + (n-1)\tilde{\varepsilon} & \text{if } i = n \text{ and } j = n, \end{cases}$$

where $\tilde{\varepsilon} > 0$ and is small enough to ensure all $\mathtt{val}_k(\{(i,j)\}) \geq 0$. In other words, the player valuations are defined by the entries of M aside from perturbations on the last column for players in P and on the last row for players in Q.

As all players in P (respectively, Q) can partition the goods into columns (respectively, rows) such that the value of each subset in the partition is exactly 1, the MMS guarantee of all players in P (respectively, Q) must be 1.

Next, let us consider an allocation of the goods. Lemma 2.2 tells us that if the $\mathtt{val}_k(\{(i,j)\})$ were exactly equal to the $M_{i,j}$ there are only two ways to allocate the goods such that every subset in the partition has value 1 (i.e., we get an MMS allocation): via the rows or via the columns. But note that the alteration to the value of a good (i,j) from $M_{i,j}$ is at most $(n-1)\tilde{\varepsilon}$ and indeed no subset of goods can have its total value altered by more than $(n-1)\tilde{\varepsilon}$ for any player. Therefore, we claim that if we wish to have any hope of achieving an MMS allocation, we must still partition according to the rows or columns (assuming that $\tilde{\varepsilon}$ is sufficiently small). To see this, define

$$\gamma = \max_{(X_1,\ldots,X_n)\in\mathcal{X}} \min_{i\in\mathcal{N}} \sum X_i,$$

where $\mathcal{X}$ is the set of partitions of M^+ excluding the rows and the columns. Importantly, via Lemma 2.2 and the finite nature of $\mathcal{X}$, we know that $\gamma < 1$. Now suppose that $\tilde{\varepsilon} < \frac{1-\gamma}{n-1}$. Then for any allocation that did not correspond to the rows or columns, some player must have value at most $\gamma + (n-1)\tilde{\varepsilon} < 1$. This proves the claim.

Now note that if we split via rows, the players of P will believe that only the last row is worth at least 1 and all other rows are worth strictly less than 1. As there are at least two players in P, not all players can receive their MMS guarantee. A similar issue occurs when we split via the columns for the players in Q. Therefore, there exists no MMS allocation in this setting.

We have just shown the result for $5n-6$ goods (for $n \geq 4$) and now set our sights on $3n+4$ goods. Let $\tilde{n} = \lceil (n+4)/2 \rceil \geq 4$. We know that we can find $5\tilde{n}-6$ goods that do not admit an MMS allocation for $\tilde{n}$ players. Take this set of goods, and let there be n players such that $\lfloor n/2 \rfloor$ players are in group P and the remaining $\lceil n/2 \rceil$ are in group Q. Finally, add $n-\tilde{n}$ goods each of value 1 to all players. Note that the number of goods is

$$m = (5\tilde{n}-6) + (n-\tilde{n}) = 4\tilde{n} + n - 6 = 4\lceil (n+4)/2 \rceil + n - 6 \leq 3n+4.$$

Further observe that

$$n - \tilde{n} = n - \lceil (n+4)/2 \rceil = \lfloor n/2 \rfloor - 2.$$

Thus, we have that of the players who did not receive any of the new $n-\tilde{n}$ items of value 1, there must be at least $|P| - (n-\tilde{n}) \geq 2$ players in P. Similarly, there must be $|Q| - (n-\tilde{n}) \geq 2$ players in Q. As we still must have at least two players in both P and Q when we allocate the original $5\tilde{n}-6$ goods, no MMS allocation exists. □

3 EXISTENCE AND COMPUTATION OF APPROXIMATE MMS ALLOCATIONS

To circumvent Theorem 2.1, we introduce a new notion of *approximate* MMS guarantee: rather than asking for an allocation $A_1, \ldots, A_n$ such that $\mathtt{val}_i(A_i) \geq \mathtt{MMS}_i(n,\mathcal{G})$ for all $i \in \mathcal{N}$, we look for γ-approximate MMS allocations such that $\mathtt{val}_i(A_i) \geq \gamma \cdot \mathtt{MMS}_i(n,\mathcal{G})$ for some $\gamma > 0$.

To this end, recall that for all $k \in \mathbb{N}$, we denote

$$\gamma_k = \frac{2\lfloor k \rfloor_{odd}}{3\lfloor k \rfloor_{odd} - 1}.$$

Our main result is that γ_n-approximate MMS allocations always exist.

THEOREM 3.1. *There always exists an allocation $A_1, \ldots, A_n$ such that for all $i \in \mathcal{N}$, $\mathtt{val}_i(A_i) \geq \gamma_n \mathsf{MMS}_i(n, \mathcal{G})$. Moreover, for every $\varepsilon > 0$, an allocation $A_1, \ldots, A_n$ such that for all $i \in \mathcal{N}$, $\mathtt{val}_i(A_i) \geq (1-\varepsilon)\gamma_n \mathsf{MMS}_i(n, \mathcal{G})$ can be computed in polynomial time in n and m.*

Paramount to the proof of Theorem 3.1 is Algorithm 1. The only part of the algorithm that is not elementary is Step 7(a), which says "repeat until no cycles exist." Intuitively, each time the bundles are rotated along a cycle, the total number of edges in the envy graph decreases, and therefore the cycle elimination process must terminate. This claim is formally established by Lipton et al. (2004), who use it to show that an algorithm that essentially coincides with steps 6 and 7 of Algorithm 1 achieves the α-envy-freeness property discussed in Section 1.3.

ALGORITHM 1

(1) If there is a player who believes any single item is worth γ_n of his MMS guarantee, give it to him and eliminate him and his item from all further consideration. Repeat until no such player exists.
(2) If only two players remain, let one of the two players produce a 2-MMS partition and have the other take his more preferred bundle. The remaining bundle is given to the player who produced the partition and the algorithm ends.
(3) In lexicographic order, give each remaining player their most favored item not already given away or eliminated (breaking ties between items lexicographically).
(4) In *reverse* lexicographic order, give each remaining player his most favored item not already given away or eliminated (breaking ties between items lexicographically).
(5) If a noneliminated player believes that his last received item (the one given to him in step 4), in addition to any two items not already given out or eliminated, is worth γ_n of his MMS guarantee, then
    (a) Exchange his current two items for these three items (his last received item remains with him).
    (b) Eliminate this player and his three items.
    (c) Have all other remaining players (the others who received items in steps 3 and 4) return their items.
    (d) Go to step 3.
(6) Create a directed *envy graph* $G = (V, E)$, where V represents the remaining players and there is an edge (i, j) if and only if i believes that his current bundle is worth strictly less than that of j.
(7) Loop through the following until all items have been allocated:
    (a) If there is a cycle in G, then eliminate it by having each player in the cycle give his bundle to the player before him in the cycle (and receive the bundle from the player after him). Update the edges so that (i, j) exists if and only if i believes that his current bundle is worth strictly less than that of j, as before. Repeat until no cycles exist.
    (b) As there is no cycle in G, there exists at least one player who has no incoming edges. Give one of the items not already given out or eliminated to one of these players.

We prove Theorem 3.1 in two steps. In Section 3.1, we show that Algorithm 1 produces a γ_n-approximate MMS allocation. The algorithm would clearly run in polynomial time, if it were given an oracle that can compute MMS partitions. In Section 3.2, we explain how to convert the algorithm to a polynomial time algorithm, at the cost of decreasing the MMS approximation ratio by ε.

3.1 Proof of Theorem 3.1: Existence

Fix the number of players n, and denote $\gamma = \gamma_n$. Assume, for the sake of contradiction, that the existence claim in Theorem 3.1 is false. In particular, we have a counterexample where player $i \in \mathcal{N}$ does not achieve the desired γ ratio of his MMS guarantee on the item set $\mathcal{G}$, when Algorithm 1 is executed on this instance. For notational convenience, further assume that i's MMS guarantee is 1 in this instance (normalizing if necessary) and that values always refer to those of player i unless otherwise specified.

Observation 3.2. *If i is eliminated at any point, then he achieves a γ fraction of his MMS guarantee (i.e., he receives a value of at least γ).*

Observation 3.3. *γ_k is a nonincreasing function of k and is in $(2/3, 3/4]$ for $k \geq 3$.*

Lemma 3.4. *If the set of players eliminated in step 1 of the algorithm in our counterexample (where i fails to achieve a value of γ) is nonempty but does not contain i, then there exists a counterexample where no players are eliminated in step 1.*

Proof. Let $\tilde{\mathcal{N}}$ be the set of players remaining after step 1 in our counterexample and $\tilde{\mathcal{G}}$ the set of items.

Now consider the execution of the algorithm on the set of players $\tilde{\mathcal{N}}$ and items $\tilde{\mathcal{G}}$. Observe that upon completion of step 1, the executions on this instance and the original (i.e., the execution with $\mathcal{N}$ and $\mathcal{G}$) are equivalent. In other words, the players in $\tilde{\mathcal{N}}$ are given the same items in both instances.

Finally, consider i's value in the altered instance. As each eliminated player took only one item and because a single item can only occupy a single bundle in an MMS partition, we have that

$$\mathsf{MMS}_i(|\tilde{\mathcal{N}}|, \tilde{\mathcal{G}}) \geq \mathsf{MMS}_i(|\mathcal{N}|, \mathcal{G}). \tag{1}$$

Recall that i's value on the altered instance is equal to his value on the original instance, which is less than $\gamma_{|\mathcal{N}|}\mathsf{MMS}_i(|\mathcal{N}|, \mathcal{G}) \leq \gamma_{|\tilde{\mathcal{N}}|}\mathsf{MMS}_i(|\tilde{\mathcal{N}}|, \tilde{\mathcal{G}})$, where the weak inequality follows from Equation (1) and Observation 3.3. Therefore, i does not achieve the desired $\gamma_{|\tilde{\mathcal{N}}|}$ ratio of his MMS guarantee on the altered instance. □

Observation 3.5. *If the algorithm terminates on step 2, then all players achieve the desired γ approximation.*

Importantly, Observation 3.2 and Lemma 3.4 suggest that we may safely assume that no player is eliminated in step 1 in our counterexample—and, in fact, that i is not eliminated at any point. Observation 3.5 further allows us to assume that the algorithm does not terminate at step 2, and so, in addition, $n \geq 3$. With this in mind, we introduce the following notation:

- For $j \in \mathcal{N}$, Φ_j denotes the set of the two most valuable items (in i's view) that j possesses at the beginning of step 6 (breaking ties arbitrarily). In particular, if j is not eliminated in step 5, he has exactly two items, and otherwise he has exactly three.
- For $j \in \mathcal{N}$, Ψ_j denotes the bundle containing Φ_j upon completion of the entire algorithm (it is easy to check that the algorithm will never separate them). Note that j may not receive this bundle upon algorithm completion due to step 7, but all players receive exactly one of these bundles.
- For $j \in \mathcal{N}$, v_j denotes $\mathtt{val}_i(\Phi_j)$.
- For $j \in \mathcal{N}$, V_j denotes $\mathtt{val}_i(\Psi_j)$.
- p denotes i's value for the item i received in the last iteration of step 3.

— q denotes i's value for the item i received in the last iteration of step 4. Note that $p + q = v_i$ and $p \geq q$.
— $\hat{i}$ denotes the index such that upon algorithm completion i receives bundle $\Psi_{\hat{i}}$. Note that we must have $V_{\hat{i}} < \gamma$.

With this notation, we are now ready for the following key observations and lemmas.

Observation 3.6. *$v_j \leq V_j$ (since $v_j = \mathtt{val}_i(\Phi_j) \leq \mathtt{val}_i(\Psi_j) = V_j$).*

Observation 3.7. *During steps 6 and 7, i's value is nondecreasing (since i only exchanges his bundle for one that he envies).*

Observation 3.8. *$p + q = v_i < \gamma$ (since by Observation 3.7, i must receive a value of at least v_i upon algorithm completion).*

Observation 3.9. *$q < \gamma/2$ (by Observation 3.8 and $q \leq p$).*

Lemma 3.10. *If $j \neq i$ is eliminated in step 5, then at most one of j's three items has value in $(q, p]$, and the others each have value at most q.*

Proof. When j is eliminated in step 5, he retains the item he received in step 4 and receives two others. The item he received in step 4 is clearly of value at most p, as otherwise i would have taken this in step 3. Similarly, the other two items each must be of value at most q, as i could have taken either in step 4. □

Corollary 3.11. *If $j \neq i$ is eliminated in step 5, then $V_j \leq p + 2q < \gamma + q$ (by Lemma 3.10 and Observation 3.8).*

Lemma 3.12. *If $\Phi_j \subsetneq \Psi_j$, then $v_j < \gamma$ and $V_j < \gamma + q$.*

Proof. If $\Phi_j \subsetneq \Psi_j$, then we must have one of two cases:

(1) *j is eliminated in step 5.*
Lemma 3.10 and Observation 3.8 shows us that $v_j \leq p + q < \gamma$, and Corollary 3.11 shows us that $V_j < \gamma + q$.
(2) *During steps 6 and 7, the bundle initially denoted by Φ_j and ending as Ψ_j received at least one item.*
Let us consider the last time this bundle received an item. i must not have envied whomever held the bundle at the time, and therefore its value to i before the addition of the new item must be less than γ. Thus, $v_j < \gamma$. Furthermore, the added item must have value at most q (as otherwise i would have selected this item in step 4), and therefore we have $V_j < \gamma + q$. □

Corollary 3.13. *If $v_j \geq \gamma$, then $\Psi_j = \Phi_j$ and thus $V_j = v_j$ as well (by Lemma 3.12).*

Lemma 3.14. *It holds that $v_j \leq V_j \leq \max(v_j, \gamma + q)$.*

Proof. $v_j \leq V_j$ is true by Observation 3.6. Regarding the second inequality, if $\Phi_j = \Psi_j$, then we clearly have $V_j = v_j \leq \max(v_j, \gamma + q)$. Otherwise, Lemma 3.12 applies and we see that $V_j < \gamma + q \leq \max(v_j, \gamma + q)$. □

Lemma 3.15. *If $v_j \leq \gamma + q$, we have $V_j \leq \gamma + q$.*

Proof. If $\Phi_j \subsetneq \Psi_j$, then Lemma 3.12 applies. Otherwise, $V_j = v_j$, which gives the result. □

For the following observations, recall that players choose in lexicographic order (increasing index) in step 3 and in reverse lexicographic order (decreasing index) in step 4.

Observation 3.16. *If $j \leq i$ is not eliminated, then the more valuable of Φ_j's two items (in i's view) i values less than γ (as otherwise i would have taken this item in step 1) and the other i values at most q (as otherwise i would have taken this item in step 4). This further implies $v_j < \gamma + q$.*

Observation 3.17. *If $j > i$ is not eliminated, then each of the two items in Φ_j must have value at most p to i (as otherwise i would have taken one of these items in step 3). This further implies $v_j \leq 2p$.*

Lemma 3.18. *For all $j \leq i$, we have $v_j \leq V_j \leq \gamma + q$.*

Proof. If j is eliminated in step 5, then Corollary 3.11 applies and we see that $v_j \leq \gamma + q$. Otherwise, by Observation 3.16, we still have that $v_j \leq \gamma + q$. Combining this with Lemma 3.14 gives the result. □

Lemma 3.19. *For all $j > i$, we have $v_j \leq V_j \leq \max(2p, \gamma + q)$.*

Proof. If j is eliminated in step 5, then Corollary 3.11 applies and we see that $v_j \leq \gamma + q$ as before. Otherwise, by Observation 3.17, we have that $v_j \leq 2p$. Combining this with Lemma 3.14 gives the result. □

Lemma 3.20. *It holds that $p < 1/2$.*

Proof. Assume for contradiction that $p \geq 1/2$. Let $S = \{j \mid v_j > 1\}$. We make the following observations for each $j \in S$:

— *$j > i$.*
If $j \leq i$, we have

$$\begin{aligned} v_j &\leq \gamma + q \text{ (by Lemma 3.18)} \\ &< \gamma + (\gamma - p) \text{ (since } p + q < \gamma) \\ &= 2\gamma - p \\ &\leq 2(3/4) - 1/2 \text{ (by Observation 3.3 and our assumption } p \geq 1/2) \\ &= 1. \end{aligned}$$

Thus, if $j \leq i$ we have $v_j \leq 1$, and therefore we cannot have that $j \in S$.

— *$\Psi_j = \Phi_j$.*
This follows from Corollary 3.13 and noting that $\gamma < 1$.

— *There are only two items in Ψ_j, and each has value at most p.*
Since $\Psi_j = \Phi_j$, we have that Ψ_j has only two items. Furthermore, as we know that $j > i$, the two items in Ψ_j each have value at most p by Observation 3.17.

Now let $T = \{j \mid v_j \in [\gamma, 1]\}$. Observe that for all $j \in T$, we have $\Psi_j = \Phi_j$, and therefore there are only two items in Ψ_j (by Corollary 3.13 and noting that $\gamma < 1$). Now consider the following algorithm (which we use only as a tool in our proof and not as a useful algorithm in and of itself).

---

(1) Let $P = \{A_1, \ldots, A_n\}$ be some MMS partition for i.
(2) Flag the item i receives in the last invocation of step 3 (which is worth p).[2]
(3) Flag the $2|S|$ items corresponding to the Ψ_j for $j \in S$.
(4) While $T \neq \emptyset$:
    (a) Remove some $t \in T$.
    (b) Denote the two items corresponding to Ψ_t by x and y.

[2]The concept of flagging can be thought of as inclusion in some *flag set*, but we find this approach intuitively clearer.

(c) Denote by G the bundle in P that x belongs to.
(d) Denote by H the bundle in P that y belongs to.
(e) Flag x and y.
(f) If $G \neq H$, replace G and H with $\{x, y\}$ and $(G \cup H) \setminus \{x, y\}$ in P. In other words, if $G \neq H$, we replace P with $(P \setminus \{G, H\}) \cup \{\{x, y\}, (G \cup H) \setminus \{x, y\}\}$.

We claim that an invariant of the loop in the algorithm (and therefore holds upon algorithm completion) is that for all $G \in P$:

— if there are zero flagged items in G, then i values G at least at 1;
— if there is exactly one flagged item in G, then i values the nonflagged items of G at least at $1 - p$.

As initially P is an MMS partition, we know that before we enter the loop for the first time, any bundle of P without any flagged items must have a value of at least i's MMS guarantee, which is 1. Furthermore, as all of the $2|S| + 1$ items initially flagged must have value at most p, any bundle with exactly one flagged item must have value at least $1 - p$ for the nonflagged items. Our invariant thus holds initially.

During a loop iteration, if we have that $G = H$, then since flagging x and y forces $G(= H)$ to have at least two flagged items, our invariant continues to hold vacuously. It therefore only remains to show that when we replace G, H with $\{x, y\}, (G \cup H) \setminus \{x, y\}$, our invariant still holds. As the set $\{x, y\}$ contains two flagged items, we need not show anything of this set. We focus now on the set $(G \cup H) \setminus \{x, y\}$.

During a loop iteration, we have the following cases:

— $(G \cup H) \setminus \{x, y\}$ has zero flagged items.
In this case, both G and H had zero flagged items before the flagging of x and y, and therefore they each have value at least 1. Thus, the nonflagged items in $(G \cup H) \setminus \{x, y\}$ have value at least

$$1 + 1 - \mathtt{val}_i(\{x, y\}) \geq 1 + 1 - 1 = 1$$

where we have used the fact that $\mathtt{val}_i(\{x, y\}) = v_t$ for some $t \in T$ and therefore by the definition of T is at most 1.
— $(G \cup H) \setminus \{x, y\}$ has exactly one flagged item.
In this case, exactly one of $G \setminus \{x\}$ and $H \setminus \{y\}$ has a flagged item (and exactly one flagged item). Then we have that the nonflagged items of $(G \cup H) \setminus \{x, y\}$ have value at least

$$1 + 1 - p - \mathtt{val}_i(\{x, y\}) \geq 1 + 1 - p - 1 = 1 - p.$$

— $(G \cup H) \setminus \{x, y\}$ has two or more flagged items.
In this case, we need not prove any property of the bundle.

This proves the loop invariant.

Once this algorithm completes, we have that for all $j \in S \cup T$, both of the two items in Ψ_j are flagged, as is i's first item received (which is worth p). If we let $k = |S \cup T|$, then we have that the total value of all nonflagged items is

$$\begin{aligned} -p + \sum_{j \notin S \cup T} V_j &= V_{\hat{i}} - p + \sum_{j \notin S \cup T \cup \{\hat{i}\}} V_j \\ &< \gamma - p + (n - k - 1)(\gamma + q) \text{ (by } V_{\hat{i}} < \gamma \text{ and Lemma 3.15)} \end{aligned}$$

$$
\begin{aligned}
&< \gamma - p + (n-k-1)(\gamma + (\gamma - p)) \text{ (since } p + q < \gamma) \\
&= (2n - 2k - 1)\gamma - (n-k)p.
\end{aligned}
$$

We will now contradict this statement by in fact demonstrating that the total value of all nonflagged items must simultaneously be at least $(2n - 2k - 1)\gamma - (n-k)p$—thus completing the proof.

Denote by α_j the number of bundles of the final partition with exactly j flagged items. Then the total value of nonflagged items must be at least $\alpha_0 + \alpha_1(1-p)$ due to the loop invariant. Importantly, by counting the number of flagged items, we also have that

$$
\begin{aligned}
2k+1 = \sum_{j \geq 1} j\alpha_j \geq \alpha_1 + 2\sum_{j\geq 2}\alpha_j &= \alpha_1 + 2(n - \alpha_0 - \alpha_1) \\
\Rightarrow \alpha_1 \geq 2n - 2k - 1 - 2\alpha_0.&
\end{aligned}
$$

Thus, to prove the desired contradiction, it suffices to show that the solution to the following optimization problem is at least 0:

$$
\begin{aligned}
\min_{\alpha_0, \alpha_1, p}\ & \alpha_0 + \alpha_1(1-p) - (2n-2k-1)\gamma + (n-k)p \\
\text{subject to } & \alpha_1 \geq 2n - 2k - 1 - 2\alpha_0 \\
& \alpha_0 \geq \max(0, n - 2k - 1).
\end{aligned}
$$

As $p < \gamma < 1$, we have that $1 - p > 0$, and so it is best to minimize α_1 under the constraint. In other words, the constraint should be tight at the optimal solution. We can therefore assume that $\alpha_1 = 2n - 2k - 1 - 2\alpha_0$, and with a bit of arithmetic we arrive at the following equivalent optimization problem:

$$
\begin{aligned}
\min_{\alpha_0, p}\ & \alpha_0(2p-1) + (2n-2k-1)(1-p-\gamma) + (n-k)p \\
\text{subject to } & \alpha_0 \geq \max(0, n-2k-1).
\end{aligned}
$$

As $p \geq 1/2$, we have that $2p - 1 \geq 0$, and so it is also best to minimize α_0—the number of bundles with zero flagged items. Thus, we have that $\alpha_0 = \max(0, n-2k-1)$, and therefore we have the further reduced optimization problem:

$$
\min_p \max(0, n-2k-1) \cdot (2p-1) + (2n-2k-1)(1-p-\gamma) + (n-k)p. \tag{2}
$$

With regard to the final variable of our optimization, p, we see that our objective is linear, and therefore we need only consider the extreme values of $1/2$ and γ. We are left with three cases to analyze—each of which is a matter of straightforward computation:

— $p = 1/2$.
The objective of (2) is

$$
\begin{aligned}
(2n-2k-1)(1/2-\gamma) + (n-k)(1/2) &= (2n-2k-1)\left(\frac{1}{2} - \frac{2\lfloor n \rfloor_{odd}}{3\lfloor n \rfloor_{odd} - 1}\right) + (n-k)/2 \\
&= \frac{n\lfloor n \rfloor_{odd} - 3n + \lfloor n \rfloor_{odd} + 1 - k(\lfloor n \rfloor_{odd} - 3)}{2(3\lfloor n \rfloor_{odd} - 1)}.
\end{aligned}
$$

As the denominator is always greater than 0, to show that this is at least 0 it suffices to show that the numerator itself is at least 0. The numerator is

$$\begin{aligned}
& n\lfloor n\rfloor_{odd} - 3n + \lfloor n\rfloor_{odd} + 1 - k(\lfloor n\rfloor_{odd} - 3) \\
& \geq n\lfloor n\rfloor_{odd} - 3n + \lfloor n\rfloor_{odd} + 1 \\
& \qquad - (n-1)(\lfloor n\rfloor_{odd} - 3) \text{ (since } \lfloor n\rfloor_{odd} \geq 3 \text{ and } k \leq n-1) \\
& = 2\lfloor n\rfloor_{odd} - 2 \\
& \geq 2(3) - 2 \\
& > 0.
\end{aligned}$$

— *$n \geq 2k + 1$ and $p = \gamma$.*
The objective of (2) is

$$\begin{aligned}
(n-2k-1)(2\gamma-1) + (2n-2k-1)(1-2\gamma) + (n-k)\gamma &= n - (n+k)\gamma \\
&= n - (n+k)\left(\frac{2\lfloor n\rfloor_{odd}}{3\lfloor n\rfloor_{odd} - 1}\right) \\
&= \frac{n\lfloor n\rfloor_{odd} - 2k\lfloor n\rfloor_{odd} - n}{3\lfloor n\rfloor_{odd} - 1}.
\end{aligned}$$

As before, it suffices to show that the numerator is at least 0. When n is odd, we have that the numerator is

$$n^2 - 2kn - n = n(n - (2k+1)) \geq 0.$$

If, furthermore, n is even, we have that the numerator is

$$\begin{aligned}
n(n-1) - 2k(n-1) - n &= (n-1)(n-(2k+1)) - 1 \\
&\geq (n-1) - 1 \text{ (because } n \text{ is even } n - (2k+1) \geq 1) \\
&\geq 0.
\end{aligned}$$

— *$n \leq 2k$ and $p = \gamma$.*
The objective of (2) is

$$\begin{aligned}
(2n-2k-1)(1-2\gamma) + (n-k)\gamma &= (2n-2k-1) + (-3n+3k+2)\gamma \\
&= (2n-2k-1) + (-3n+3k+2)\left(\frac{2\lfloor n\rfloor_{odd}}{3\lfloor n\rfloor_{odd} - 1}\right) \\
&= \frac{\lfloor n\rfloor_{odd} - 2n + 2k + 1}{3\lfloor n\rfloor_{odd} - 1}.
\end{aligned}$$

As before, it suffices to show that the numerator is at least 0; it is at least

$$\begin{aligned}
(n-1) - 2n + 2k + 1 &= -n + 2k \\
&\geq 0.
\end{aligned}$$

□

Lemma 3.21. *It holds that $q > \frac{n}{n-1}(1-\gamma)$.*

Proof. Suppose for purposes of contradiction that $q \leq \frac{n}{n-1}(1-\gamma)$. For all $j \in \mathcal{N}$, we have

$$\begin{aligned}
V_j &\leq \max(2p, \gamma + q) \text{ (by Lemmas 3.18 and 3.19)} \\
&\leq \max\left(2(1/2), \gamma + \frac{n}{n-1}(1-\gamma)\right) \text{ (by Lemma 3.20 and our assumption on } q)
\end{aligned}$$

$$= \max\left(1, \frac{n-\gamma}{n-1}\right)$$
$$= \frac{n-\gamma}{n-1} \text{ (since } \gamma < 1\text{)}.$$

We then see that

$$\sum_{j=1}^{n} V_j = V_{\hat{i}} + \sum_{j \neq \hat{i}} V_j < \gamma + (n-1)\frac{n-\gamma}{n-1} = n.$$

That $\sum_{j=1}^{n} V_j < n$ clearly contradicts that i's MMS guarantee is 1. □

Lemma 3.22. *n is even.*

Proof. Suppose for purposes of contradiction that n is odd. By Lemma 3.21, we must have that

$$\begin{aligned} q &> \frac{n}{n-1}(1-\gamma) \\ &= \frac{n}{n-1}\left(1 - \frac{2n}{3n-1}\right) \text{ (by the definition of } \gamma\text{)} \\ &= \frac{n}{3n-1} \\ &= \gamma/2. \end{aligned}$$

This clearly contradicts Observation 3.9's statement that $q < \gamma/2$. □

Corollary 3.23. $\gamma = \frac{2(n-1)}{3(n-1)-1}$ *(by the definition of γ and Lemma 3.22).*

Lemma 3.24. *It holds that $\frac{n}{n-1}(1-\gamma) \geq 1/3$.*

Proof.

$$\begin{aligned} \frac{n}{n-1}(1-\gamma) &= \frac{n}{n-1}\left(1 - \frac{2(n-1)}{3(n-1)-1}\right) \text{ (by Corollary 3.23)} \\ &= \frac{1}{3}\frac{3n^2-6n}{3n^2-7n+4} \\ &\geq \frac{1}{3}\frac{3n^2-6n}{3n^2-7n+n} \text{ (since } n \geq 4 \text{ by Lemma 3.22 and } n \geq 3\text{)} \\ &= 1/3. \end{aligned}$$
□

Let us now take a moment to introduce the following notation:

— X: The set of players who are eliminated (in step 5).
— Y: The set of players $j \notin X$ and $j < i$ where $v_j \geq \gamma$.
— Z: The set of players $j \notin X$ and $j > i$ where $v_j \geq \gamma$.
— $x = |X|$, $y = |Y|$, and $z = |Z|$.

Observation 3.25. *$i \notin X \cup Y \cup Z$ (since i is not eliminated).*

Observation 3.26. *For all $j \notin Z$, we have $V_j \leq \gamma + q$ (by Corollary 3.11 and Lemmas 3.15 and 3.18).*

Observation 3.27. *For all $j \in Z$, we have $V_j = v_j \leq 2p < 1$ (by Corollary 3.13, Observation 3.17, and Lemma 3.20).*

Lemma 3.28. *$\hat{i} \notin X \cup Y \cup Z$.*

Proof. If $\hat{i} \in X$, then $\Psi_{\hat{i}}$ must go to the eliminated player $\hat{i}$ (who is thus not i). If $\hat{i} \in Y \cup Z$, then i would receive a value $= V_{\hat{i}} \geq v_{\hat{i}} \geq \gamma$. □

Lemma 3.29. *It holds that $Z = \emptyset$ (i.e., $z = 0$).*

Proof. Assume for purposes of contradiction that $z \geq 1$. Then we have the following:

$$\begin{aligned}
\sum_{j=1}^{n} V_j &= \sum_{j \in Z} V_j + V_{\hat{i}} + \sum_{j \in \mathcal{N} \setminus (Z \cup \{\hat{i}\})} V_j \\
&< \sum_{j \in Z} 1 + \gamma + \sum_{j \in \mathcal{N} \setminus (Z \cup \{\hat{i}\})} (\gamma + q) \text{ (by Observations 3.26 and 3.27 and } V_{\hat{i}} < \gamma) \\
&= z + \gamma + (n - z - 1)(\gamma + q) \\
&< z + \gamma + (n - z - 1)(\gamma + \gamma/2) \text{ (by Observation 3.9)} \\
&= (1 - 3\gamma/2)z + (3n - 1)(\gamma/2) \\
&< (1 - 3\gamma/2) + (3n - 1)(\gamma/2) \text{ (since } \gamma > 2/3 \text{ by Observation 3.3 and } z \geq 1) \\
&= \left(1 - \frac{3}{2} \cdot \frac{2(n-1)}{3(n-1) - 1}\right) + \frac{3n - 1}{2} \cdot \frac{2(n-1)}{3(n-1) - 1} \text{ (by Corollary 3.23)} \\
&= n.
\end{aligned}$$

That $\sum_{j=1}^{n} V_j < n$ clearly contradicts that i's MMS guarantee is 1. □

Lemma 3.30. *It holds that $x + y > n - 3$.*

Proof. Assume for purposes of contradiction that $x + y \leq n - 3$. Let us consider the $n - x - y - 1$ values V_j for $j \in \mathcal{N} \setminus (X \cup Y \cup \{\hat{i}\})$. As i was not eliminated in step 5, i must believe the two most valuable items not given out or eliminated at the beginning of step 6 sum to value $< \gamma - q$. This statement, along with the fact that $n - x - y - 1 \geq 2$ (since we are assuming $x + y \leq n - 3$), implies that i's value for the $n - x - y - 1$ largest items not given out or eliminated at the beginning of step 6 is at most $(n - x - y - 1)(\gamma - q)/2$. Simultaneously, we know that for all $j \in \mathcal{N} \setminus (X \cup Y \cup Z \cup \{\hat{i}\}) = \mathcal{N} \setminus (X \cup Y \cup \{\hat{i}\})$ (we have used Lemma 3.29 for the equality), the value of the bundle that at step 6 starts as Φ_j and upon algorithm completion becomes Ψ_j before it receives its last item is less than γ (as otherwise, i would envy this bundle). This yields

$$\sum_{j \in \mathcal{N} \setminus (X \cup Y \cup \{\hat{i}\})} V_j < (n - x - y - 1)(\gamma + (\gamma - q)/2).$$

Noting that $V_{\hat{i}} < \gamma$ and for all $j \in X \cup Y$ we have $V_j < \gamma + q$ by Corollary 3.11 and Lemma 3.18, we then get

$$\begin{aligned}
\sum_{j=1}^{n} V_j &= \sum_{j \in X \cup Y} V_j + V_{\hat{i}} + \sum_{j \in \mathcal{N} \setminus (X \cup Y \cup \{\hat{i}\})} V_j \\
&< (x + y)(\gamma + q) + \gamma + (n - x - y - 1)(\gamma + (\gamma - q)/2).
\end{aligned}$$

We claim this last quantity, a function that we will call V, is smaller than n for the relevant values of q—that is, $q \in (\frac{n}{n-1}(1 - \gamma), \gamma/2)$ (the relevant values are determined by Observation 3.9 and Lemma 3.21). Indeed, observe that V is a linear function in q. Moreover, note that since $\gamma \leq 3/4$ (by Observation 3.3), we have

$$\gamma/3 \leq (3/4)/3 = 1 - 3/4 \leq 1 - \gamma \leq \frac{n}{n-1}(1 - \gamma).$$

This implies that the domain $(\frac{n}{n-1}(1-\gamma), \gamma/2)$ is contained in $[\gamma/3, \gamma/2]$. Thus, to show the desired inequality $V < n$, it suffices to show the inequality for $q \in \{\gamma/3, \gamma/2\}$:

$$\begin{aligned}
V(\gamma/3) &= (x+y)(4\gamma/3) + \gamma + (n-x-y-1)(\gamma+\gamma/3) \\
&= (4\gamma/3)(n-1/4) \\
&\leq (4(3/4)/3)(n-1/4) \text{ (by Observation 3.3)} \\
&= n - 1/4 \\
&< n. \\
V(\gamma/2) &= (x+y)(3\gamma/2) + \gamma + (n-x-y-1)(\gamma+\gamma/4) \\
&= (\gamma/4)(5n+x+y-1) \\
&\leq (\gamma/4)(5n+(n-3)-1) \text{ (since we are assuming } x+y \leq n-3) \\
&= (\gamma/2)(3n-2) \\
&= \frac{3n-2}{2}\frac{2(n-1)}{3(n-1)-1} \text{ (by Corollary 3.23)} \\
&= n - \frac{n-2}{3n-4} \\
&< n.
\end{aligned}$$

We can therefore conclude that $\sum_{j=1}^{n} V_j < V < n$—contradicting that i's MMS guarantee is 1. □

Lemma 3.31. *It holds that $x + y \neq n - 2$.*

Proof. Assume for purposes of contradiction that $x + y = n - 2$. Consider the set $\mathcal{H}$ of items composed of the following:

— *The items in all of the Ψ_j for all $j \in X$ (equivalently, the items that go to the players in X).* There are $3x$ such items, and by Lemma 3.10 we know that i values all of these items at a value of at most q, except for at most x of them that may have value in $(q, p]$.
— *The items in all of the Ψ_j for all $j \in Y$.* By Corollary 3.13, there are $2y$ such items, but we will imagine as if the y largest items (in i's view) are in fact two inseparable items—giving us instead $3y$ such items. Note that each such pair of inseparable items are of value $< \gamma$ and the other y items have value at most q by Observation 3.16.
— *The two items in Φ_i (which i values at p and q).*
— *The item i values most (breaking ties arbitrarily) among those not eliminated nor given out at the beginning of step 6.* Let Δ denote i's value of this item. Note that $\Delta \leq q$, as otherwise i would have taken this item in step 4.

Observe that any single item of $\mathcal{H}$ is of value $\leq p$ and any two items have value at most $\max(2p, \gamma) \leq \max(2(\gamma - q), \gamma) = 2(\gamma - q)$.

We are interested in the value of all items aside from these $3x + 3y + 2 + 1 = 3(n-1)$ items, which we will denote by r. In other words, $r = (\sum_{j \notin X \cup Y} V_j) - (p + q + \Delta)$. Now fix $A_1, \ldots, A_n$ to be some MMS partition for i. In each of the following four encompassing cases, we will demonstrate that $r \geq 2\gamma - p - q$:

(1) *There exists an A_j that contains no items in $\mathcal{H}$:*

$$\begin{aligned} r &\geq \mathtt{val}_i(A_j) \\ &\geq 1 \text{ (since } i\text{'s MMS value is 1)} \\ &= 2(2(3/4) - 1) \\ &\geq 2(2\gamma - 1) \text{ (by Observation 3.3)} \\ &= 2(\gamma - (1 - \gamma)) \\ &\geq 2\left(\gamma - \frac{n}{n-1}(1-\gamma)\right) \\ &\geq 2(\gamma - q) \text{ (by Lemma 3.21)} \\ &= 2\gamma - q - q \\ &\geq 2\gamma - p - q. \end{aligned}$$

(2) *There exists an A_j that contains exactly one item in $\mathcal{H}$.*
In this case, there must exist some other A_k with at most two items from $\mathcal{H}$ as $|\mathcal{H}| = 3(n-1)$. As observed previously, the single item in $A_j \cap \mathcal{H}$ must be of value $\leq p$, and the two items in $A_k \cap \mathcal{H}$ must be of value $\leq 2(\gamma - q)$. Thus, we have

$$\begin{aligned} r &\geq \mathtt{val}_i(A_j) + \mathtt{val}_i(A_k) - p - 2(\gamma - q) \\ &\geq 2 - p - 2(\gamma - q) \text{ (since } i\text{'s MMS value is 1)} \\ &= (2\gamma - p - q) + (-4\gamma + 3q + 2) \\ &\geq (2\gamma - p - q) + (-4(3/4) + 3(1/3) + 2) \text{ (since } q \geq 1/3 \text{ by Lemma 3.24)} \\ &= 2\gamma - p - q. \end{aligned}$$

(3) *$n \geq 6$ and all of the A_j contain at least two items in $\mathcal{H}$.*
In this case, there must be at least three A_j with exactly two items from $\mathcal{H}$ as $|\mathcal{H}| = 3(n-1)$. Without loss of generality, suppose that this is true of A_1, A_2, and A_3. In each of these three, we must have that the two items from $\mathcal{H}$ have value $\leq 2(\gamma - q)$ as mentioned previously. Thus, we have

$$\begin{aligned} r &\geq \mathtt{val}_i(A_1) + \mathtt{val}_i(A_2) + \mathtt{val}_i(A_3) - 3(2(\gamma - q)) \\ &\geq 3 - 3(2(\gamma - q)) \text{ (since } i\text{'s MMS value is 1)} \\ &= (2\gamma - p - q) + (-8\gamma + p + 7q + 3) \\ &\geq (2\gamma - p - q) + (-8\gamma + 8q + 3) \\ &\geq (2\gamma - p - q) + \left(-8\gamma + 8\frac{n}{n-1}(1-\gamma) + 3\right) \text{ (by Lemma 3.21)} \\ &= (2\gamma - p - q) \\ &\quad + \left(-8\frac{2(n-1)}{3(n-1)-1} + 8\frac{n}{n-1}\left(1 - \left(\frac{2(n-1)}{3(n-1)-1}\right)\right) + 3\right) \text{ (by Corollary 3.23)} \\ &= (2\gamma - p - q) + \frac{n^2 - 5n - 4}{(n-1)(3n-4)} \\ &> (2\gamma - p - q) + \frac{n^2 - 5n - n}{(n-1)(3n-4)} \text{ (since we are assuming } n \geq 6) \\ &= (2\gamma - p - q) + \frac{n(n-6)}{(n-1)(3n-4)} \\ &\geq 2\gamma - p - q \text{ (since we are assuming } n \geq 6). \end{aligned}$$

(4) *$n = 4$ and all of the A_j contain at least two items in $\mathcal{H}$.*
In this special case, there is one A_j with exactly three items in $\mathcal{H}$, and three A_j (without loss of generality, say A_1, A_2, and A_3) with exactly two items in $\mathcal{H}$ due to $|\mathcal{H}| = 3(n-1) = 9$. Furthermore, a tedious brute force computation (which we omit) demonstrates that in this case, the six most valuable items are of value at most $2\gamma + p + q$. Thus, we have

$$\begin{aligned} r &\geq \mathtt{val}_i(A_1) + \mathtt{val}_i(A_2) + \mathtt{val}_i(A_3) - (2\gamma + p + q) \\ &\geq 3 - (2\gamma + p + q) \text{ (since } i\text{'s MMS value is 1)} \\ &= 4(3/4) - (2\gamma + p + q) \\ &= 4\gamma - (2\gamma + p + q) \text{ (by the definition of } \gamma) \\ &= 2\gamma - p - q. \end{aligned}$$

As the four cases shown earlier encompass all possible scenarios, we do indeed find that $r \geq 2\gamma - p - q$. We therefore find

$$\begin{aligned} \sum_{j \notin X \cup Y} V_j &= r + p + q + \Delta \\ &\geq (2\gamma - p - q) + p + q + \Delta \\ &= 2\gamma + \Delta. \end{aligned}$$

However, we know regarding the two $j \notin X \cup Y$ that one of the V_j must go to i (i.e., $j = \hat{i}$) and is therefore of value $< \gamma$, whereas the other must have value $< \gamma + \Delta$. We thus simultaneously find that

$$\sum_{j \notin X \cup Y} V_j < \gamma + \gamma + \Delta = 2\gamma + \Delta.$$

This is a clear contradiction. □

Lemma 3.32. *It holds that $x + y \neq n - 1$.*

Proof. Assume for purposes of contradiction that $x + y = n - 1$. In this case, $\mathcal{N} \setminus \{X \cup Y\} = \{i\}$. Similarly to Lemma 3.31's proof, we introduce a set of interest under the name of $\mathcal{H}$. This is identical to before, except it does not include the one item whose value was denoted as Δ. For convenience, we have restated the rest of the set's contents here:

— *The items in all of the Ψ_j for all $j \in X$ (equivalently, the items that go to the players in X).*
There are $3x$ such items, and by Lemma 3.10 we know that i values all of these items at a value of at most q, except for at most x of them, which may have value in $(q, p]$.
— *The items in all of the Ψ_j for all $j \in Y$.*
By Corollary 3.13, there are $2y$ such items, but we will imagine as if the y largest items (in i's view) are in fact two inseparable items—giving us instead $3y$ such items. Note that each such pair of inseparable items are of value $< \gamma$, and the other y items have value at most q by Observation 3.16.
— *The two items in Φ_i (which i values at p and q).*

If we again let $A_1, \ldots, A_n$ be an MMS partition for i, we see that there exists some A_j that contains at most two of $\mathcal{H}$ since $|\mathcal{H}| = 3x + 3y + 2 = 3n - 1$. $\mathtt{val}_i(A_j \setminus \mathcal{H})$ must then be at least $\mathtt{val}_i(A_j) - \max(2p, \gamma) \geq 1 - \max(2p, \gamma)$. We therefore find that upon algorithm completion, i must receive a value of at least $1 - \max(2p, \gamma) + p + q$. If $2p \leq \gamma$, we have that

$$\begin{aligned}
&1 - \max(2p, \gamma) + p + q \\
&\quad = 1 - \gamma + p + q \\
&\quad \geq 1 - \gamma + q + q \text{ (since } q \leq p) \\
&\quad \geq 1 - 3/4 + 1/3 + 1/3 \text{ (by Observation 3.3 and Lemmas 3.21 and 3.24)} \\
&\quad = 11/12.
\end{aligned}$$

Furthermore, if $2p > \gamma$, we have that

$$\begin{aligned}
&1 - \max(2p, \gamma) + p + q \\
&\quad = 1 - 2p + p + q \\
&\quad = 1 - p + q \\
&\quad > 1 - (\gamma - q) + q \text{ (since } p + q < \gamma) \\
&\quad = 1 + 2q - \gamma \\
&\quad \geq 1 + 2(1/3) - 3/4 \text{ (by Observation 3.3 and Lemmas 3.21 and 3.24)} \\
&\quad = 11/12.
\end{aligned}$$

We therefore find that i must achieve a value of at least $11/12 \geq 3/4 \geq \gamma$. □

Note that the statements of Lemmas 3.30, 3.31, and 3.32 imply that $x + y = n$. However, as we know that $i \notin X \cup Y$ by Observation 3.25, we also see that $x + y < n$. This contradiction concludes the proof that Algorithm 1 must produce a γ-approximate MMS allocation. □

3.2 Proof of Theorem 3.1: Polynomial Time

Although Algorithm 1 seems rather innocent at first glance, it does make one computational leap by letting players compute their MMS guarantee, or an MMS partition. It is easy to see that this is NP-hard; in fact, even when there are two players with identical valuations, it is NP-hard to determine whether the the MMS guarantee is $\mathtt{val}_i(\mathcal{G})/2$—this can be shown via an immediate reduction from Partition.

Woeginger (1997) studied the problem of computing an MMS partition, albeit under a different name: scheduling jobs on identical machines to maximize the minimum completion time. He gave a PTAS and showed that no fully polynomial time approximation scheme (FPTAS) exists unless P = NP. Using our terminology, this means that given a constant $\varepsilon > 0$, we can compute a partition $A_1, \ldots, A_n$ of the set of items $\mathcal{G}$ so that $\min_{i \in \mathcal{N}} \mathtt{val}_i(A_i) \geq (1 - \varepsilon)\mathsf{MMS}_i(n, \mathcal{G})$ in polynomial time.

The modified algorithm is almost identical to Algorithm 1, but for two critical differences:

(1) When we need to compute a player's MMS guarantee, we instead compute a $1 - \varepsilon$ approximation via the PTAS.
(2) If two players remain in step 2, then we compute a $1 - \varepsilon$ approximation to an MMS partition via the PTAS.

The analysis of Section 3.1 goes through largely unchanged, giving each player a bundle of value $(1 - \varepsilon)\gamma\mathsf{MMS}_i(n, \mathcal{G})$. □

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