CERTAIN LIOUVILLE PROPERTIES OF EIGENFUNCTIONS OF ELLIPTIC OPERATORS

ARI ARAPOSTATHIS, ANUP BISWAS, AND DEBDIP GANGULY

Abstract. We present certain Liouville properties of eigenfunctions of second-order elliptic operators with real coefficients, via an approach that is based on stochastic representations of positive solutions, and criticality theory of second-order elliptic operators. These extend results of Y. Pinchover to the case of *nonsymmetric* operators of Schrödinger type. In particular, we provide an answer to an open problem posed by Pinchover in [*Comm. Math. Phys.* **272** (2007), no. 1, 75–84, Problem 5]. In addition, we prove a lower bound on the decay of positive supersolutions of general second-order elliptic operators in any dimension, and discuss its implications to the Landis conjecture.

Contents

1. Introduction

The main objective of this paper is to establish Liouville properties of eigenfunctions of second-order elliptic operators. These type of results came to prominence after the paper of Pinchover [38] where he proved a very interesting property which can be stated as follows. Let $\mathcal{D}$ be a domain in $\mathbb{R}^d$ and let $P_i = -\mathrm{div}(Au) - V_i u$, $i = 1, 2$, be two nonnegative Schrödinger operators, with $V_i \in L^p_{\mathrm{loc}}(\mathcal{D})$ for $p > d/2$, and A locally non-degenerate in $\mathcal{D}$. Suppose that P_1 is critical in $\mathcal{D}$ with ground state Ψ_1^*, that the generalized principal eigenvalue of P_2 is nonnegative, and that there exists a subsolution Ψ, with $\Psi^+ \neq 0$, to $P_2 u = 0$ in $\mathcal{D}$ satisfying $\Psi^+ \leq C\Psi_1^*$ for some constant C. Then P_2 is also critical with ground state Ψ. In particular, the principal eigenvalue of P_2 equals 0, and $\Psi > 0$. In the same paper, Pinchover proposed two problems on the generalization of this result for (a) general non-symmetric second order elliptic operators and (b) quasilinear operators of p-Laplacian type. Later, a similar result for p-Laplacian operators was proved by

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Pinchover, Tertikas and Tintarev in [39]. However, the problem concerning general second-order elliptic operators remains open so far. The main goal of this paper is to address this problem for a large class of second-order elliptic operators.

Pinchover's approach was variational. He first established the existence of a *null sequence* for the quadratic form associated with P_i, and then using *criticality theory*, together with a bound on the positive part of the subsolution, he obtained the above mentioned Liouville-type result. Unfortunately, for general (nonsymmetric) operators the existence of such a null sequence is not possible, despite the fact that criticality theory is well developed for general operators. Moreover, in [38, Remark 4.1] Pinchover discussed the difficulty in obtaining the above Liouville-type results for general (nonsymmetric) second-order elliptic operators. In this paper we show that the above Liouville-type result holds for a fairly general class of second-order elliptic operators and potentials. Our approach differs significantly from variational arguments, and relies on stochastic representations of positive solutions studied in [5], and criticality theory of second-order elliptic operators. This allows us to bypass the use of a null sequence.

For $\mathcal{D} = \mathbb{R}^d$, it is known that the criticality of the operator is equivalent to the recurrence of the twisted process [5, 40]. An interesting observation in this paper is that criticality is also equivalent to the *strict right monotonicity* of the (generalized) principal eigenvalue. Let $\mathcal{L}$ be a second-order elliptic operator and $\lambda^*(V)$ denote the principal eigenvalue of the operator $\mathcal{L} + V$, with potential V. We say that $\lambda^*(V)$ is *strictly right monotone at* V, or *strictly monotone at* V *on the right*, if $\lambda^*(V) < \lambda^*(V + h)$ for any non-zero, nonnegative continuous function h that vanishes at infinity. This equivalence is established in Theorem 2.1. We also show that given two potential functions $V_i \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, $i = 1, 2$, if $V_1 - V_2$ has a fixed sign outside some compact subset of $\mathbb{R}^d$, then a result analogous to the one described in the preceding paragraph holds (see Theorems 2.2 and 2.3). In particular, if $P_1 = \mathcal{L}_1 + V_1$ is a small perturbation of $P_2 = \mathcal{L}_2 + V_2$, then, under suitable assumptions, the criticality of P_1 implies that of P_2. To further strengthen these results, we study the *strict monotonicity* of the principal eigenvalue, by which we mean that the principal eigenvalue is strictly left and right monotone at V (i.e., $\lambda^*(V - h) < \lambda^*(V) < \lambda^*(V + h)$). It is shown in Theorem 2.4 that if the principal eigenvalue corresponding to P_1 is strictly monotone, $\mathcal{L}_1 = \mathcal{L}_2$ outside a compact subset of $\mathbb{R}^d$, and $V_1 - V_2$ vanishes at infinity, then under analogous hypotheses, the principal eigenvalue of P_2 is also strictly monotone. These results can be further improved to $V_i \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, provided we impose a 'stability' assumption on $\mathcal{L}$. See Theorem 3.1 for more details.

The second part of this paper deals with the lower bound on the decay of positive supersolutions of general second-order elliptic operators in any dimension. The results obtained here extend those of Agmon [2], Carmona [16], Carmona and Simon [17]. Our proof is based on the stochastic representation of positive solutions (see Theorem 4.1). As a consequence, we prove the Landis conjecture for a large class of potentials. Landis' conjecture [28] can be loosely stated as follows: for a bounded potential V, if a solution u of $\Delta u + Vu = 0$ satisfies the estimate $|u(x)| \le C \exp(-c|x|^{1+})$, for some positive constants C and c, then u is identically 0. For a precise statement, we refer the reader to Section 4. This conjecture is open when u and V are real valued, while a counterexample was constructed by Meshkov [30] for complex-valued u and V. This conjecture was revisited recently in [18, 26], for dimension 2 and for $V \le 0$. Note that this conjecture trivially holds for $V \le 0$ due to the strong maximum principle. The main contribution in [18, 26] is the lower bound on the decay rate of solutions that may not vanish at infinity. In this direction, Kenig has conjectured in [27, Question 1] a lower bound on the decay of the eigenfunctions of the Schrödinger's equation. In Theorem 4.3 we validate the Landis conjecture for a large class of potentials and in any dimension $d \ge 2$. This class of potentials includes compactly supported functions. We also wish to bring to the attention of the reader a recent study of Liouville properties for nonlinear operators [9].

The paper is organized as follows. In Section 2 we briefly review some basic results from the criticality theory of second-order elliptic operators and state our main results. Section 3 is devoted to the proofs of Theorems 2.1 and 2.2 to 2.4. In Section 4 we establish a lower bound on the decay of positive supersolutions (Theorem 4.1), and discuss its implication to Landis conjecture.

Notation. The open ball of radius r around a point $x \in \mathbb{R}^d$ is denoted by $B_r(x)$, and B_r stands for $B_r(0)$. By $\mathcal{C}_0(\mathbb{R}^d)$ ($\mathcal{B}_0(\mathbb{R}^d)$) we denote the collection of all real valued continuous (Borel measurable) functions on $\mathbb{R}^d$ that vanish at infinity. By $\|\cdot\|_\infty$ we denote the L^∞ norm. Also $\kappa_1, \kappa_2, \dots$ are used as generic constants whose values might vary from place to place.

2. Preliminaries and main results

In this section we introduce our assumptions and state our main results. The conditions (A1)–(A3) on the coefficients of the operator that follow are used in most of the results of the paper, so we assume that they are in effect throughout unless otherwise mentioned. A notable exception to this is Theorem 2.2, where only (A3) is assumed.

(A1) *Local Lipschitz continuity:* The function $a = \left[a^{ij}\right] : \mathbb{R}^d \to \mathcal{S}_+^{d\times d}$, where $\mathcal{S}_+^{d\times d}$ denotes the set of real, symmetric positive definite matrices, is locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, we have

$$\|a(x) - a(y)\| \le C_R\,|x-y| \qquad \forall\, x, y \in B_R\,,$$

where $\|a\|^2 := \operatorname{trace}\left(aa^{\mathsf{T}}\right)$. The drift function $b : \mathbb{R}^d \to \mathbb{R}$ is a locally bounded Borel measurable function.

(A2) *Affine growth condition:* b and a satisfy a global growth condition of the form

$$\langle b(x), x\rangle^+ + \|a(x)\| \le C_0\bigl(1 + |x|^2\bigr) \qquad \forall\, x \in \mathbb{R}^d,$$

for some constant $C_0 > 0$.

(A3) *Nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \ge C_R^{-1}|\xi|^2 \qquad \forall\, x \in B_R\,,$$

and for all $\xi = (\xi_1, \dots, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$.

We define $\upsigma(x) = \sqrt{2}a^{1/2}(x)$. Then under (A1) and (A3), $\upsigma$ is also locally Lipschitz and has at most linear growth. We say that a is uniformly elliptic if (A3) holds for a positive $C = C_R$ which is independent of R.

Consider the Itô stochastic differential equation (SDE) given by

$$\mathrm{d}X_s \,=\, b(X_s)\,\mathrm{d}s + \sigma(X_s)\,\mathrm{d}W_s\,, \tag{2.1}$$

where W is a standard d-dimensional Wiener process defined on some complete, filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$. By a strong solution of (2.1) we mean an $\mathfrak{F}_t$–adapted process X_t which satisfies

$$X_t \,=\, X_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s\,, \quad t \ge 0, \quad \text{a.s.}\,,$$

where third term on the right hand side is an Itô stochastic integral. It is well known that given a complete, filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$ with a Wiener process W, there exists a unique strong solution of (2.1) [22, Theorem 2.8]. The process X is also strong Markov, and we denote its transition kernel by $P^t(x, \cdot)$. It also follows from the work in [15] that the transition probabilities of X have densities which are locally Hölder continuous. The *extended generator* $\mathcal{L}$ is given by

$$\mathcal{L}f(x) \,=\, a^{ij}(x)\,\partial_{ij}f(x) + b^i(x)\,\partial_i f(x)\,, \tag{2.2}$$

for $f \in C^2(\mathbb{R}^d)$. The operator $\mathcal{L}$ is the generator of a strongly-continuous semigroup on $C_b(\mathbb{R}^d)$, which is strong Feller. We let $\mathbb{P}_x$ denote the probability measure, and $\mathbb{E}_x$ the expectation operator on the canonical space of the process conditioned on $X_0 = x$.

The closure, boundary, and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\bar{A}$, ∂A, and A^c, respectively. We write $A \Subset B$ to indicate that $\bar{A} \subset B$. By $\uptau(D)$ we denote the first exit time of the process X from a domain $D \subset \mathbb{R}^d$, i.e.,

$$\uptau(D) \,\coloneqq\, \inf\,\{t : X_t \notin D\}\,.$$

The process X is said to be *recurrent* if for any bounded domain D we have $\mathbb{P}_x(\uptau(D^c) < \infty) = 1$ for all $x \in \bar{D}^c$. Otherwise the process is called *transient*. A recurrent process is said to be *positive recurrent* if $\mathbb{E}_x[\uptau(D^c)] < \infty$ for all $x \in \bar{D}^c$. It is known that for a non-degenerate diffusion the property of recurrence (or positive recurrence) is independent of domain D and x, i.e., if it holds for some domain D and some $x \in \bar{D}^c$, then it also holds for every bounded domain D, and all $x \in \bar{D}^c$ [6, Theorem 2.6.12 and Theorem 2.6.10]. By $\breve{\uptau}_r$ ($\uptau_r$) we denote the first hitting (exit) time of the ball B_r of radius r around 0, i.e., $\breve{\uptau}_r = \uptau(B_r^c)$ and $\uptau_r = \uptau(B_r)$.

In order to state the results in this paper, we review some basic definitions from criticality theory which have been introduced by various authors [1,31,32,41], and have been further developed by Y. Pinchover (see [35–37] and references therein). The reader should keep in mind that although the convention in criticality theory is to consider the eigenvalues of the operator $-\mathcal{L}$, we find it more convenient to work with the eigenvalues of $\mathcal{L}$.

Definition 2.1. Throughout the paper, $\mathcal{D} \subset \mathbb{R}^d$ denotes a domain, and $\mathfrak{D} \coloneqq \{\mathcal{D}_j\}_{j=1}^\infty$ a sequence of bounded subdomains with smooth boundaries, such that $\bar{\mathcal{D}}_j \subset \mathcal{D}_{j+1}$, and $\mathcal{D} = \cup_{j=1}^\infty \mathcal{D}_j$. We denote the cone of all positive solutions of the equation $\mathcal{L}u = 0$ in $\mathcal{D}$ by $\mathcal{C}_{\mathcal{L}}(\mathcal{D})$. We always assume that solutions u are in $\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{D})$, i.e., u is a strong solution, so that $\mathcal{L}u$ is defined pointwise almost everywhere.

Given a *potential* $V \in L^\infty_{\mathrm{loc}}(\mathcal{D})$, we introduce the operator

$$\mathcal{L}_V \,\coloneqq\, \mathcal{L} + V\,.$$

We say that $-\mathcal{L}_V$ is *nonnegative in* $\mathcal{D}$ (and denote it by $-\mathcal{L}_V \ge 0$ in $\mathcal{D}$), if $\mathcal{C}_{\mathcal{L}_V}(\mathcal{D}) \neq \varnothing$. The generalized principal eigenvalue of the operator $\mathcal{L}_V$ is defined by

$$\lambda^*(\mathcal{L}, V) \coloneqq \inf \left\{ \lambda \in \mathbb{R} : \mathcal{C}_{\mathcal{L}_V - \lambda}(\mathcal{D}) \neq \varnothing \right\}.$$

Note that $-\mathcal{L}_V$ is nonnegative in $\mathcal{D}$ if and only if $\lambda^*(\mathcal{L}, V) \le 0$.

It is clear that $-\mathcal{L}$ is always nonnegative, since $\mathbf{1} \in \mathcal{C}_{\mathcal{L}}(\mathcal{D})$, where $\mathbf{1}$ is the constant function on $\mathcal{D}$ having value 1 at every $x \in \mathcal{D}$. In the sequel we shall use the notation $\lambda^*(V)$ instead of $\lambda^*(\mathcal{L}, V)$, whenever this is not ambiguous. In most of the paper we deal with the case $\mathcal{D} = \mathbb{R}^d$. An exception to this is Theorem 2.2, where we address the question of Pinchover [38, Problem 5] for general domains $\mathcal{D}$.

Let us now recall the definitions of critical and subcritical operators and the ground state.

Definition 2.2 (Minimal growth at infinity)**.** A positive function $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{D})$ satisfying

$$\mathcal{L}_V u = 0 \quad \text{a.e. in } \mathcal{D},$$

is said to be a solution of minimal growth at infinity, if for any compact $K \subset \mathcal{D}$ and any positive function $v \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{D} \setminus K)$ which satisfies $\mathcal{L}_V v \le 0$ a.e. in $\mathcal{D} \setminus K$, there exist $\mathcal{D}_i \in \mathfrak{D}$, with $K \subset \mathcal{D}_i$, and a constant $\kappa > 0$ such that $\kappa u \le v$ in $\bar{\mathcal{D}}_i^c \cap \mathcal{D}$. A positive solution $u \in \mathcal{C}_{\mathcal{L}_V}(\mathcal{D})$ which has minimal growth at infinity in $\mathcal{D}$ is called the (*Agmon*) *ground state* of $\mathcal{L}_V$ in $\mathcal{D}$.

Remark 2.1. Definition 2.2 is equivalent to what is generally used in criticality theory. In criticality theory for an operator P, a function $u \in \mathcal{C}_P(\mathcal{D})$ is said to have a minimal growth at infinity in $\mathcal{D}$, if for any $K \Subset \mathcal{D}$, with a smooth boundary, and any positive supersolution $v \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{D} \setminus K)$ of $Pv = 0$ in $\mathcal{D} \setminus K$ such that $v \in \mathcal{C}((\mathcal{D} \setminus K) \cup \partial K)$, and $u \le v$ on ∂K, it holds that $u \le v$ in $\mathcal{D} \setminus K$.

It is easy to see that this definition implies minimal growth at infinity according to Definition 2.2 for $P = \mathcal{L}_V$. To see the converse direction, define

$$\kappa_0 = \inf_{\mathcal{D} \cap K^c} \frac{v}{u}.$$

Since $u \le v$ on ∂K, we must have $\kappa_0 \le 1$ by continuity. We claim that $\kappa_0 = 1$. Arguing by contradiction, suppose that $\kappa_0 < 1$. Then $v - \kappa_0 u$ must be positive on $\mathcal{D} \setminus K$ by the strong maximum principle. Since $P(v - \kappa_0 u) \le 0$, then by Definition 2.2 there exist $\kappa \in (0, 1 - \kappa_0)$ and $\mathcal{D}_i \in \mathfrak{D}$, such that $\kappa u \le v - \kappa_0 u$ in $\bar{\mathcal{D}}_i^c \cap \mathcal{D}$. Without loss of generality, suppose that $\mathcal{D}_i \supset K$. Applying the strong maximum principle in $\mathcal{D}_i \cap K^c$ to

$$\mathcal{L}\Phi - V^- \Phi \le 0, \quad \Phi = v - (\kappa_0 + \kappa) u,$$

we have $\kappa u \le v - \kappa_0 u$ in $\bar{\mathcal{D}}_i \cap K^c$, and therefore $(\kappa_0 + \kappa) u \le v$ in $\mathcal{D} \cap K^c$. But this contradicts the definition of κ_0. Hence $\kappa_0 = 1$.

Definition 2.3. The operator $\mathcal{L}_V$ is said to be *critical* in $\mathcal{D}$, if $\mathcal{L}_V$ admits a ground state in $\mathcal{D}$. The operator $\mathcal{L}_V$ is called *subcritical* in $\mathcal{D}$, if $-\mathcal{L}_V \ge 0$ in $\mathcal{D}$, but $\mathcal{L}_V$ does not admit a ground state solution.

Example 2.1. Let $\mathcal{L} = \Delta$ in $\mathbb{R}^d$, $d \ge 1$. It is well known that $\lambda^*(\mathcal{L}, 0) = 0$. Moreover, $\mathcal{L}$ is critical if and only if $d \le 2$.

Example 2.2. Let $\mathcal{D} = \mathbb{R}^d \setminus \{0\}$, $d \geq 3$, and consider the Hardy operator

$$\mathcal{L}_V := \Delta + \frac{(d-2)^2}{4}\frac{1}{|x|^2}\,.$$

Then it is well known that $\mathcal{L}_V$ is critical, and the corresponding ground state is $|x|^{\frac{2-d}{2}}$.

Remark 2.2. For $\mathcal{D} = \mathbb{R}^d$ one can also define the (generalized) principal eigenvalue in the sense of Berestycki and Rossi [14] (see also [33]) by

$$\hat{\lambda}^*(\mathcal{L}, V) \,:=\, \inf\left\{\lambda \in \mathbb{R} : \exists\, \varphi \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\ \varphi > 0,\ \mathcal{L}_V\varphi - \lambda\varphi \leq 0 \text{ a.e. in } \mathbb{R}^d\right\}\,.$$

For $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, it is known from [14, Theorem 1.4] that there exists a (generalized) positive eigenfunction corresponding to $\hat{\lambda}^*(\mathcal{L}, V)$, whenever this is finite. Thus $\hat{\lambda}^*(\mathcal{L}, V) = \lambda^*(\mathcal{L}, V)$.

Remark 2.3. Let $P = \mathcal{L}_V$ in $\mathcal{D}$. It is well known that the operator P is critical in $\mathcal{D}$, if and only if the equation $Pu = 0$ in $\mathcal{D}$ has a unique (up to a multiplicative constant) positive supersolution (see [35,36]). In particular, P is critical in $\mathcal{D}$ if and only if P does not admit a positive Green's function in $\mathcal{D}$. However, there exists a sign-changing Green's function for a P which is critical in $\mathcal{D}$ (see [20]). In addition, in the critical case, we have $\dim \mathcal{C}_P(\mathcal{D}) = 1$, and the unique positive solution (up to a multiplicative positive constant) is a ground state of P in $\mathcal{D}$.

On the other hand, P is subcritical in $\mathcal{D}$ if and only if P admits a unique positive minimal Green's function $G^{\mathcal{D}}_P(x,y)$ in $\mathcal{D}$. Moreover, for any fixed $y \in \mathcal{D}$, the function $G^{\mathcal{D}}_P(\cdot,y)$ is a positive solution of minimal growth in a neighborhood of infinity in $\mathcal{D}$, i.e., in $\mathcal{D} \setminus K$ for some compact set K (see [19]).

For an eigenpair (Ψ, λ) of $\mathcal{L}_V$ in $\mathbb{R}^d$, i.e., a solution of

$$\mathcal{L}_V\Psi \,=\, \lambda\Psi, \quad \Psi > 0 \quad \text{in } \mathbb{R}^d\,,$$

the *twisted process corresponding to* (Ψ, λ) is defined by the SDE

$$\mathrm{d}Y_s \,=\, b(Y_s)\,\mathrm{d}s + 2a(Y_s)\nabla\psi(Y_s)\,\mathrm{d}s + \upsigma(Y_s)\,\mathrm{d}W_s\,, \tag{2.3}$$

with $\psi = \log\Psi$. The process Y also goes by the name of Doob's h-transformation in the literature. Since $\psi \in \mathscr{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, it follows that ψ is locally bounded (in fact, it is locally Hölder continuous), and therefore (2.3) has a unique strong solution up to its explosion time. In what follows, we use the notation $(\Psi^*, \lambda^*(V))$ to denote a principal eigenpair.

Let us introduce one more definition which is related to the criticality of an operator. By $\mathcal{C}^+_0(\mathbb{R}^d)$ we denote the collection of all nonnegative, non-zero, real valued continuous functions on $\mathbb{R}^d$ that vanish at infinity. We fix $\mathcal{L}$, and dropping the dependence on $\mathcal{L}$ in the notation, as mentioned earlier, we let $\lambda^*(V)$ denote the principal eigenvalue of $\mathcal{L} + V$.

Definition 2.4. $\lambda^*(V)$ is said to be *strictly monotone at* V if for all $h \in \mathcal{C}^+_0(\mathbb{R}^d)$ we have $\lambda^*(V-h) < \lambda^*(V) < \lambda^*(V+h)$. Also, $\lambda^*(V)$ is said to be *strictly monotone at* V *on the right* if for all $h \in \mathcal{C}^+_0(\mathbb{R}^d)$ we have $\lambda^*(V) < \lambda^*(V+h)$.

It is known from [5, Theorem 2.2] that if $\lambda^*(V-h) < \lambda^*(V)$ for some $h \in \mathcal{C}^+_0(\mathbb{R}^d)$, then $\lambda^*(V-h) < \lambda^*(V) < \lambda^*(V+h)$ for all $h \in \mathcal{C}^+_0(\mathbb{R}^d)$. This assertion also follows using the fact that $V \mapsto \lambda^*(V)$ is a convex function (see for instance, [14]).

Example 2.3. For $\mathcal{L} = \Delta$ in $\mathbb{R}^2$ and $V = 0$, it is known that $\lambda^*(V)$ strictly monotone at V on the right, but not strictly monotone at V.

Throughout the paper, with the exception of Theorem 2.2, we consider potential functions V that are Borel measurable and bounded from below. We also assume that $\lambda^*(V)$ is finite. Let us begin with the following equivalence between the strict right monotonicity of the principal eigenvalue and the criticality of the operator [5]. See also [40, Theorems 4.3.3 and 7.3.6] for similar results concerning operators with regular coefficients.

Theorem 2.1. *Let $\mathcal{D} = \mathbb{R}^d$. The following are equivalent.*

(a) *A function $\Psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{D})$ is a ground state for $\mathcal{L}_V - \lambda$, with $\lambda \in \mathbb{R}$.*
(b) *The twisted process corresponding to the eigenpair (Ψ, λ) is recurrent.*
(c) *$\lambda^*(V)$ is strictly monotone at V on the right.*
(d) *For any $r > 0$, the eigenpair (Ψ, λ) satisfies*

$$\Psi(x) \;=\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V(X_s)-\lambda)\,\mathrm{d}s}\,\Psi(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr]\,,\quad x \in B_r^c\,, \tag{2.4}$$

where, as defined earlier, $\breve{\tau}_r$ denotes the first hitting time to the ball B_r.

We often exploit the above equivalence between strict monotonicity and criticality. To state our next result we need some additional notation. Let

$$\mathcal{L}_k f \;=\; a^{ij}_k(x)\,\partial_{ij} f(x) + b^i_k(x)\,\partial_i f(x)\,,\quad k = 1, 2\,.$$

We assume that (a_k, b_k), $k = 1, 2$, satisfies (A1)–(A3). We say that $\mathcal{L}_1$ is a *small perturbation* [35] of $\mathcal{L}_2$ if $\|a_1(x)-a_2(x)\|+|b_1(x)-b_2(x)| = 0$ outside some compact set. The first main result of this section is the following theorem which gives a partial answer (see also Theorem 2.3) to the open question posed by Y. Pinchover in [38, Problem 5]. Simplifying the notation, in the sequel we sometimes denote by λ^*_k (instead of $\lambda^*(\mathcal{L}_k, V_k)$) the principal eigenvalue of the operator $\mathcal{L}_k + V_k$, $k = 1, 2$.

Theorem 2.2. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^d$, $d \ge 1$. Consider two Schrödinger operators defined on $\mathcal{D}$ of the form*

$$P_k \;:=\; \mathcal{L}_k + V_k\,,\quad k = 1, 2,$$

where a_k, $k = 1, 2$, are continuous and satisfy (A3), $b_k, V_k \in L^\infty_{\mathrm{loc}}(\mathcal{D})$, and $V_2 \ge V_1$ outside a compact set in $\mathcal{D}$. In addition, assume that $\mathcal{L}_1$ is a small perturbation of $\mathcal{L}_2$ in $\mathcal{D}$, and

(1) *The operator $P_1 - \lambda^*_1$ is critical in $\mathcal{D}$. Denote by Ψ^*_1 its ground state.*
(2) *$\lambda^*_2 \le \lambda^*_1$ and there exists $\Psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{D})$, with $\Psi^+ \ne 0$, satisfying*

$$\mathcal{L}_2\Psi + V_2\Psi \;\ge\; \lambda^*_1\Psi\,, \tag{2.5}$$

and

$$\Psi^+(x) \;\le\; C\,\Psi^*_1(x)\quad \text{for all } x \in \mathcal{D}\,, \tag{2.6}$$

for some constant $C > 0$.

Then the operator $P_2 - \lambda^*_2$ is critical in $\mathcal{D}$, $\lambda^*_1 = \lambda^*_2$, and Ψ is its ground state.

Remark 2.4. One can not expect any pair V_1, V_2 to satisfy the hypotheses of Theorem 2.2, even if we restrict V_1 and V_2 to have compact support, and consider the same second-order operator $\mathcal{L} = \mathcal{L}_1 = \mathcal{L}_2$. To see this, let us take $V_2 \gneqq V_1$, both of them compactly supported, and suppose that $\mathcal{L} + V_1 - \lambda^*_1$ is critical. Then it is not possible to have $\lambda_1^* = \lambda_2^*$ and $\Psi_2^* \le C\Psi_1^*$, for some constant C. Indeed, if $\tilde{\mathcal{L}}$ denotes the generator of the twisted process corresponding to the eigenpair (Ψ_1^*, λ_1^*), and $\lambda_1^* = \lambda_2^*$, then for $\Phi = \frac{\Psi_2^*}{\Psi_1^*}$ we have

$$\tilde{\mathcal{L}}\Phi \,=\, (V_1 - V_2)\Phi \,\ge\, 0\,.$$

Since the twisted process Y_s corresponding to (Ψ_1^*, λ_1^*) is recurrent by Theorem 2.1, and $\Phi(Y_s)$ is a bounded submartingale, Φ must be constant. This implies $(V_1 - V_2)\Psi_1^* = 0$, which is a contradiction.

More precisely, one can find a relation between V_1 and V_2 as follows. Suppose $\mathcal{D} = \mathbb{R}^d$ and the operators $\mathcal{L} + V_i$, $V_i \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, are critical in $\mathbb{R}^d$ with principal eigenfunctions Ψ_i^*, $i = 1, 2$. Then by Theorem 2.1 we know that for any $r > 0$ we have

$$\Psi_i^*(x) \,=\, \mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r} V_i(X_s)\,\mathrm{d}s}\,\Psi_i^*(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr]\,,\quad x \in B_r^c\,. \tag{2.7}$$

Now if (2.6) holds, i.e., $\Psi_2^* \le C\Psi_1^*$ in $\mathbb{R}^d$, then by (2.7), for every $r > 0$ we can find a constant C_r such that

$$\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r} V_2(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr] \,\le\, C_r\,\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r} V_1(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr]\,,\quad x \in B_r^c\,.$$

This in particular, provides a necessary condition on the potentials for Liouville type theorems like Theorem 2.2 to hold.

For the rest of the results in this section we let $\mathcal{D} = \mathbb{R}^d$.

Theorem 2.3. *Consider two Schrödinger operators defined on $\mathbb{R}^d$ of the form*

$$P_k \,\coloneqq\, \mathcal{L}_k + V_k\,,\quad k = 1, 2,$$

whose coefficients satisfy (A1)–(A3), and $V_k \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Let

$$\tilde{V}(x) \,\coloneqq\, \max\{V_1(x), V_2(x)\}\,.$$

Suppose that there exists a positive $\tilde{\Phi} \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ and a compact set K such that

$$\mathcal{L}_1 = \mathcal{L}_2, \quad \mathcal{L}_2\tilde{\Phi} + \tilde{V}\tilde{\Phi} \,\le\, \lambda_1^*\tilde{\Phi} \quad \text{in } K^c\,. \tag{2.8}$$

In addition, assume that

1. *The operator $P_1 - \lambda_1^*$ is critical in $\mathbb{R}^d$. Denote by Ψ_1^* its ground state.*
2. *$\lambda_2^* \le \lambda_1^*$, and there exists subsolution $\Psi \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, which may be sign-changing but $\Psi^+ \ne 0$, that satisfies*

$$\mathcal{L}_2\Psi + V_2\Psi \,\ge\, \lambda_1^*\Psi\,, \tag{2.9}$$

and for some constant $C > 0$,

$$\Psi^+(x) \,\le\, C\,\Psi_1^*(x) \quad \text{for all } x \in \mathbb{R}^d\,.$$

Then the operator $P_2 - \lambda_2^*$ is critical in $\mathbb{R}^d$, $\lambda_1^* = \lambda_2^*$, and Ψ is its ground state.

It should be noted that the second display in (2.8) is an assumption on the operators; compare to [38, Theorem 1.7]. However, there is a large family of elliptic operators for which (2.8) holds, as the following examples show.

Example 2.4. Note that (2.8) is satisfied if $V_1 - V_2$ has a fixed sign outside a compact set K. If $\tilde{V} = V_1$ in K^c we can choose $\tilde{\Phi} = \Psi_1^*$. On the other hand, if $\tilde{V} = V_2$ in K^c, we know from [14, Theorem 1.4] that there exists a positive Φ_1 satisfying

$$\mathcal{L}_2\Phi_1 + V_2\Phi_1 \;=\; \lambda_1^*\Phi_1 \quad \text{in } \mathbb{R}^d\,.$$

Hence we can take $\tilde{\Phi} = \Phi_1$ in K^c.

Example 2.5. Let us now give an example where the sign of $V_1 - V_2$ may not be fixed outside some compact set. Consider $P_1 = \Delta + V_1$ in $\mathbb{R}^d$, $d \ge 3$, such that V_1 has compact support and P_1 is critical in $\mathbb{R}^d$ with $\lambda^* = 0$. Now let

$$\mathcal{L}_2 \;\coloneqq\; \Delta + b^i\partial_i\,,$$

where the vector field b has compact support in $\mathbb{R}^d$. Let a nonnegative $\tilde{W}$ be a *small perturbation* (see [36, Definition 2.1, Example 2.2]) with respect to the operator $-\Delta$. Then there exists a positive ε such that $-\Delta\Psi - \varepsilon\tilde{W}\Psi = 0$ has a positive solution Ψ in B_r^c, $r > 0$ [36, Lemma 2.4]. Therefore, if we choose a potential V_2 which decays faster than $\tilde{W}$ at infinity, i.e., for every $\delta > 0$ there exists a compact K_δ such that $|V_2(x)| \le \delta\tilde{W}(x)$ for $x \in K_\delta^c$, it is easy to see that, by choosing $\delta < \varepsilon$, we have

$$\mathcal{L}_2\Psi + \tilde{V}\Psi \;\le\; \Delta\Psi + |V_2|\Psi \;\le\; 0 \quad \text{on the complement of a compact set in } \mathbb{R}^d\,.$$

Therefore (2.8) holds.

There are several choices for a small perturbation $\tilde{W}$ (see [36, Example 2.2]). For instance, we could take any nonnegative $\tilde{W}$ which is locally Hölder continuous and satisfies

$$(1+|x|)^2\tilde{W}(x) \le \varphi(|x|) \quad \forall\, x \in \mathbb{R}^d\,, \quad \text{and} \quad \int_{r_0}^{\infty} \frac{1}{r}\varphi(r)\,\mathrm{d}r \;<\; \infty\,, \quad r_0 > 0\,.$$

Example 2.6. We define for $k = 1, 2$,

$$P_k \;\coloneqq\; \Delta + b_k^i\partial_i + V_k \quad \text{in } \mathbb{R}^d\,,\ d \ge 3\,,$$

with the vector fields b_k smooth, and satisfying

$$|b_k(x)| \;\le\; \frac{C}{(1+|x|)^{1+\varepsilon}}\,,$$

for some constants $C > 0$ and $\varepsilon > 0$. It is known that the operator $\mathcal{L}_k \coloneqq \Delta + b_k^i\partial_i$ is subcritical; this follows from the fact that $\mathbf{1}$ is a positive solution, together with the above decay estimate on b_k. Hence there exists a minimal Green's function for $\mathcal{L}_k$. Also, the Green functions $G_{-\Delta}$ and $G_{-\mathcal{L}_k}$ are comparable [3, Theorem 1]. Therefore, a small perturbation of Δ is also a small perturbation of $\mathcal{L}_k$ [36]. Let $|b_1(x) - b_2(x)| = 0$ outside a compact set. As earlier, we suppose that P_1 is critical and $\lambda_1^* = 0$. Assume that $\tilde{V} = \max\{V_1, V_2\}$ decays faster than $(1+|x|)^{-2-\epsilon}$ at infinity. In particular, we may choose $V_2(x) = (1+|x|)^{-2-\varepsilon}$. Then $\tilde{V}$ satisfies the estimate above, and hence, as before, there exists a positive supersolution Ψ to $\mathcal{L}_2\Psi + \tilde{V}\Psi \le 0$ on the complement of some compact set in $\mathbb{R}^d$. Thus (2.8) holds.

Remark 2.5. Recall that the criticality of $\mathcal{L}_V - \lambda^*$ is equivalent to the strict monotonicity of λ^* at V on the right by Theorem 2.1. However, strict right monotonicity does not necessarily imply strict monotonicity of λ^*. Later, in Theorem 2.4, we show that if λ^* is strictly monotone at V, then we do not require (2.8). Also observe that if $V \in \mathcal{B}_0(\mathbb{R}^d)$ and λ^* is not strictly monotone at V, then $\lambda^*(V) \le 0$. Indeed, since λ^* is not strictly monotone at V and $\lambda^*(V) \ge \lambda^*(-V^-)$ it is obvious that $\lambda^*(V) \le 0$. In addition, the following hold. If X is not positive recurrent and $\lambda^*(0) = 0$, then $\lambda^*(V) = 0$, otherwise $\lambda^*(-V^-) \le \lambda^*(V) < 0 = \lambda^*(0)$. However, this implies that X is geometrically ergodic [5, Theorem 2.7], and therefore, positive recurrent. If a is bounded and uniformly elliptic, and $|x|^{-1}\langle b(x), x\rangle \to 0$ as $|x| \to \infty$, then $\lambda^*(V) = 0$ for any Lipschitz $V \in \mathcal{C}_0(\mathbb{R}^d)$, since by [25, Proposition 6.2] $\lambda^*(V) \ge 0$. Therefore assuming that $\lambda_1^* = 0$ in Examples 2.5 and 2.6 is not very restrictive.

In [12], Berestycki, Caffarelli and Nirenberg asked the following question. Is it true that if there exists a bounded, sign-changing solution Ψ to $\Delta\Psi + V\Psi = 0$ in $\mathbb{R}^d$, for some locally bounded potential V, then necessarily $\lambda^*(V) > 0$? This question has been resolved in [10, 12, 21], and the answer is "yes" if and only if $d = 1, 2$. Applying Theorems 2.1 and 2.3 we can extend the sufficiency part of this answer to a more general class of elliptic operators. Noting that the Brownian motion is recurrent for $d = 1, 2$, and transient for higher dimensions, we focus on elliptic operators $\mathcal{L}$ satisfying (A1)–(A3) which are generators of a recurrent process. Using Theorems 2.1 and 2.3 we obtain the following two corollaries.

Corollary 2.1. *Suppose the solution of (2.1) is recurrent, and V is a locally bounded function which does not change sign outside some compact set in $\mathbb{R}^d$. Then the existence of a bounded, sign-changing solution $\Psi \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ to $\mathcal{L}\Psi + V\Psi = 0$ implies that $\lambda^*(V) > 0$.*

Proof. Since $(\mathbf{1}, 0)$ is an eigenpair of $\mathcal{L}$ and the corresponding twisted process is given by X, it follows by Theorem 2.1 that $\mathcal{L}$ is a critical operator with principal eigenvalue 0. Moreover, $\mathcal{C}_{\mathcal{L}} = \{c\mathbf{1} : c \in (0,\infty)\}$. We apply Theorem 2.3, with $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, $V_1 = 0$, $\lambda_1^* = 0$, $V_2 = V$, and $\Psi_1^* = c\mathbf{1}$. Suppose $\lambda^*(V) \le 0$. If V is positive outside a compact set, then (2.8) holds with $\tilde{\Phi} = \Psi_2^*$, the principal eigenfunction corresponding to $\lambda^*(V)$. On the other hand, if V is negative outside a compact set, then (2.8) holds for some positive $\tilde{\Phi}$ by [14, Theorem 1.4] (see also Example 2.4). It then follows from Theorem 2.3 that Ψ is a ground state, and therefore cannot be sign-changing. This contradicts the hypothesis that $\lambda^*(V) \le 0$, and completes the proof. □

Corollary 2.2. *Let the process (2.1) be recurrent and $V \le 0$ be a bounded function. Then there does not exist any nonconstant bounded solution u to*

$$\mathcal{L}u + Vu = 0\,. \tag{2.10}$$

Proof. Since $\lambda^*(V) \le 0$, Corollary 2.1 implies that any bounded solution u to (2.10) cannot be sign-changing. So without loss of generality we assume that $0 \le u < C$. Then $C - u$ is a positive supersolution of $\mathcal{L}u = 0$. Since $\mathcal{L}$ is critical by hypothesis, it has a unique supersolution (up to a multiplicative constant). Hence u must be constant. □

The conclusion of Corollary 2.2 might not hold if $V \not\le 0$. For instance, in dimension $d = 2$ we know that the standard Wiener process is recurrent. But $u(x, y) = \sin(x)\sin(y)$ satisfies $\Delta u + 2u = 0$. Corollary 2.2 is also comparable to [38, Theorem 1.7]. Note that for $V = 0$ the operator in (2.10) is critical in the sense of Pinchover (see Theorem 2.1 above). Therefore Corollary 2.2 provides a Liouville property for the perturbed operator.

As shown in Theorem 2.1, criticality is equivalent to the strict right monotonicity of the principal eigenvalue λ^*. However, if we assume strict monotonicity of $\lambda^*(\mathcal{L}_1, V_1)$ at V_1, then Theorem 2.3 holds for a bigger class of potentials without assuming (2.8). This is the subject of our next result. Also note that the the theorem which follows provides sufficient conditions for strict monotonicity of the principal eigenvalue of the perturbed problem.

Theorem 2.4. *Let $\mathcal{L}_1$ be a small perturbation of $\mathcal{L}_2$, $V_i \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$, $i = 1, 2$, and $V_1 - V_2 \in \mathcal{B}_0(\mathbb{R}^d)$. Let λ^*_i denote the principal eigenvalue of $\mathcal{L}_i + V_i$, $i = 1, 2$, and suppose that λ^*_1 is strictly monotone at V_1. Suppose also that $\lambda^*_2 \le \lambda^*_1$, and that there exists $\Psi \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, which may be sign-changing but $\Psi^+ \ne 0$, that satisfies*

$$\mathcal{L}_2\Psi + V_2\Psi \;\ge\; \lambda^*_1\Psi\,, \tag{2.11}$$

and

$$\Psi^+(x) \le C\,\Psi^*_1(x) \quad \text{for all } x \in \mathbb{R}^d\,, \tag{2.12}$$

for some constant $C > 0$. Then λ^*_2 is strictly monotone at V_2, and $\Psi = \Psi^*_2$ (up to a multiplicative constant), where Ψ^*_2 is the principal eigenfunction of $\mathcal{L}_2 + V_2$.

Remark 2.6. Strict monotonicity sometimes implies an interesting spectral property. To explain this, we restrict ourselves to symmetric operators. In particular, we consider a second-order elliptic operator in $\mathcal{D}$ in divergence form given by

$$\mathcal{L}u \;=\; \operatorname{div}\bigl(A\,\nabla u\bigr)\,,$$

where $A\colon \mathbb{R}^d \to \mathcal{S}^{d\times d}_+$ is locally non-degenerate. The assumptions on the coefficients are the same as before. Let $\mathrm{d}\nu = \rho(x)\mathrm{d}x$, where $\rho(x)$ is a positive measurable function on $\mathcal{D}$. The operator $\mathcal{L}$ is self-adjoint in the space $L^2(\mathcal{D}, \mathrm{d}\nu)$ (in the sense of the Friedrichs extension).

Let $V \in L^\infty(\mathcal{D})$, and $\sigma(\mathcal{L}_V)$ denote the $L^2(\mathcal{D}, \mathrm{d}\nu)$-spectrum of the Friedrichs extension of $\mathcal{L}_V$, which is also denoted as $\mathcal{L}_V$, abusing the notation in the interest of simplicity. We next show that if $\lambda^*(V)$ is strictly monotone at V, then it must be an isolated eigenvalue in $\sigma(\mathcal{L}_V)$. Indeed, by Persson's formula (see [34] or [19, Proposition 4.2]) the supremum of the essential spectrum $\sigma_{\mathsf{ess}}(\mathcal{L}_V)$ is given by

$$\lambda_\infty(V) \;\coloneqq\; \inf\,\{\lambda : \exists\, K \Subset \mathcal{D}\,,\; \mathcal{C}_{\mathcal{L}_V - \lambda}(\mathcal{D}\setminus K) \ne \varnothing\}\,.$$

In addition, $\lambda^*(V)$ is the supremum of $\sigma(\mathcal{L}_V)$. It is clear that $\lambda_\infty(V) \le \lambda^*(V)$. We claim that $\lambda_\infty(V) < \lambda^*(V)$. Arguing by contradiction, let us assume $\lambda_\infty(V) = \lambda^*(V)$. It is known that

$$\sigma_{\mathsf{ess}}(\mathcal{L}_V) \;=\; \sigma_{\mathsf{ess}}(\mathcal{L}_V - h) \tag{2.13}$$

for any $h \in \mathcal{C}_0(\mathcal{D})$. Using (2.13), we have $\lambda_\infty(V) = \lambda_\infty(V - h)$ for all $h \in \mathcal{C}^+_0(\mathcal{D})$. By hypothesis, $\lambda^*(V)$ is strictly monotone at V and therefore, we have

$$\lambda^*(V-h) \;<\; \lambda^*(V) \;=\; \lambda_\infty(V) \;=\; \lambda_\infty(V-h) \;\le\; \lambda^*(V-h)\,.$$

Thus we arrive at a contradiction, which implies that $\lambda_\infty(V) < \lambda^*(V)$ (for a more general related result see [5, Theorem 2.5]). Since the Friedrichs extension is a self-adjoint operator, its spectrum can be written as $\sigma(\mathcal{L}_V) = \sigma_{\mathsf{ess}}(\mathcal{L}_V) \cup \sigma_{\mathsf{dis}}(\mathcal{L}_V)$, with $\sigma_{\mathsf{ess}}(\mathcal{L}_V) \cap \sigma_{\mathsf{dis}}(\mathcal{L}_V) = \varnothing$, where $\sigma_{\mathsf{dis}}(\mathcal{L}_V)$ is the discrete spectrum. On the other hand, [11, Theorem 1.1] shows that $\lambda^*(V) \in \sigma(\mathcal{L}_V)$. Therefore, $\lambda^*(V)$ is an isolated eigenvalue in $\sigma(\mathcal{L}_V)$.

Remark 2.7. Let $P_i = \mathcal{L} + V_i$, $i = 1, 2$, be self-adjoint operators, and $\lambda_{\infty,i}$ denote the supremum of the essential spectrum of P_i. If $V_1 - V_2 \in \mathcal{C}_0(\mathbb{R}^d)$, then it is known that $\sigma_{\mathsf{ess}}(P_1) = \sigma_{\mathsf{ess}}(P_2)$, which in turn implies that $\lambda_{\infty,1} = \lambda_{\infty,2}$. Suppose that the hypotheses of Theorem 2.4 hold. Then using Theorem 2.4 and Remark 2.6 we deduce that $\lambda^*_2 > \lambda^*_{\infty,2}$, and that the corresponding operator $P_2 - \lambda^*_2$ is critical. In particular, Theorem 2.4 provides a necessary condition for the spectral gap of the operator P_2.

Example 2.7. Let $\mathcal{D} = \mathbb{R}^d \setminus \{0\}$, where $d \geq 3$, and consider the Hardy operator

$$\mathcal{L}_V := \Delta + \frac{(d-2)^2}{4} \frac{1}{|x|^2}\,.$$

Then it is well known (see [19]) that for this operator we have $\lambda^*(V) = \lambda_\infty(V)$. Hence $\lambda^*(V)$ cannot be strictly monotone at V, although it is strictly right monotone.

There is a large class of operators for which the strict monotonicity property holds. The following example suggests that the assumptions in Theorems 2.2 and 2.4 hold for a large class of operators.

Example 2.8. Suppose that the solution of (2.1) is recurrent. Consider two functions $\tilde{V}_i \in \mathcal{C}^+_0(\mathbb{R}^d)$, $i = 1, 2$, which are compactly supported. Then as shown in [5, Theorem 2.7], the map $\beta \mapsto \Lambda^i_\beta := \lambda^*(\beta \tilde{V}_i)$ is strictly monotone in $[0, \infty)$, and $\Lambda^i_0 = 0$, for $i = 1, 2$. Since $\beta \mapsto \Lambda^i_\beta$ is an increasing, convex function [14], we have $\lim_{\beta\to\infty} \Lambda^i_\beta = \infty$. Therefore, for any $\beta_1 > 0$, we can find $\beta_2 > 0$ such that $\Lambda^1_{\beta_1} = \Lambda^2_{\beta_2}$. Thus by defining $V_i := \beta_i \tilde{V}_i$, $i = 1, 2$, we note that $\lambda^*_1 = \lambda^*(V_1) = \lambda^*(V_2) = \lambda^*_2$, and V_i has compact support. On the other hand, $\mathcal{L} + V_i - \lambda^*_i$ is critical by Theorem 2.1. In fact, the corresponding twisted processes are geometrically ergodic by [5, Theorem 2.7]. Thus, if Ψ^*_i, $i = 1, 2$, are the principal eigenfunctions, then they have a stochastic representation by Theorem 2.1. Hence, if we choose r large enough such that support$(V_i) \subset B_r$, we have

$$\Psi^*_i(x) \,=\, \mathbb{E}_x\Bigl[e^{-\lambda^*_i \breve{\tau}_r} \Psi^*_i(X_{\breve{\tau}_r}) \mathbb{1}_{\{\breve{\tau}_r < \infty\}}\Bigr], \quad x \in B^c_r\,.$$

Since $\lambda^*_1 = \lambda^*_2$, it is easy to see from the above that $\Psi^*_2 \leq C\,\Psi^*_1$ for some $C > 0$.

3. Proofs of Theorems 2.1 to 2.4

Before we proceed with the proofs of the results in Section 2, let us recall the Itô–Krylov formula [29, p. 122] for generalized derivatives. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary and $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Let $\uptau = \uptau(\mathcal{D})$. Then for any $\varphi \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, we have

$$\mathbb{E}_x\Bigl[e^{\int_0^{T\wedge\uptau} V(X_s)\,\mathrm{d}s}\,\varphi(X_{T\wedge\uptau})\Bigr] - \varphi(x) \,=\, \mathbb{E}_x\biggl[\int_0^{T\wedge\uptau} e^{\int_0^t V(X_s)\,\mathrm{d}s}\,\mathcal{L}_V\varphi(X_t)\,\mathrm{d}t\biggr] \tag{3.1}$$

for all $x \in \mathcal{D}$ and $T > 0$. We start with the proof of Theorem 2.1.

Proof of Theorem 2.1. The equivalence between (b), (c) and (d) is established in [5]. Since the twisted process corresponding to an eigenpair (Ψ, λ) with $\lambda > \lambda^*(V)$ is transient by [5, Theorem 2.1 (c)], part (b) together with [5, Corollary 2.1] imply that $\lambda = \lambda^*(V)$.

Let us show that (b) $\Rightarrow$ (a). Suppose that $v \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ is a positive function which satisfies $\mathcal{L}v + (V-\lambda)v \le 0$ a.e. in $B^c_{r_1}$, with $r_1 > 0$. Recall that $\uptau_R$ denotes the first exit time from the ball B_R. Then by the Itô–Krylov formula in (3.1) we have

$$\begin{aligned} v(x) \;&\ge\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve\uptau_r\wedge\uptau_R\wedge T}(V(X_s)-\lambda)\,\mathrm{d}s}\, v(X_{\breve\uptau_r\wedge\uptau_R\wedge T})\Bigr] \\ &\ge\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve\uptau_r}(V(X_s)-\lambda)\,\mathrm{d}s}\, v(X_{\breve\uptau_r})\,\mathbb{1}_{\{\breve\uptau_r<\uptau_R\wedge T\}}\Bigr]\,, \quad x\in B^c_r\cap B_R\,,\ r>r_1\,. \end{aligned}$$

Now letting first $T\to\infty$, and then $R\to\infty$, and using Fatou's lemma we obtain

$$v(x) \;\ge\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve\uptau_r}(V(X_s)-\lambda)\,\mathrm{d}s}\, v(X_{\breve\uptau_r})\mathbb{1}_{\{\breve\uptau_r<\infty\}}\Bigr], \quad x\in B^c_r\,,\ r>r_1\,.$$

Hence (a) follows by applying (2.4).

Next we show that (a) $\Rightarrow$ (b). By Corollary 3.2, which appears later in this section, there exists a ball $\mathcal{B}$, a constant $\delta \ge 0$, and a positive solution $\Psi^* \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ to $\mathcal{L}\Psi^* + (V + \delta\mathbb{1}_{\mathcal{B}} - \lambda)\Psi^* = 0$, such that $\lambda = \lambda^*(V+\delta\mathbb{1}_{\mathcal{B}})$, and

$$\Psi^*(x) \;=\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve\uptau_r}(V(X_s)-\lambda)\,\mathrm{d}s}\, \Psi^*(X_{\breve\uptau_r})\mathbb{1}_{\{\breve\uptau_r<\infty\}}\Bigr] \tag{3.2}$$

for $B_r \supset \mathcal{B}$. Let $\kappa_0 := \inf_{\mathbb{R}^d} \frac{\Psi^*}{\Psi}$. By the assumption of minimal growth, we have $\kappa_0 > 0$. Note that the value κ_0 is attained. If not, then the function $\Phi = \Psi^* - \kappa_0\Psi$ is positive and satisfies $\mathcal{L}\Phi + (V-\lambda^*)\Phi \le 0$. Consequently, the minimal growth of Ψ would imply that $\inf_{\mathbb{R}^d}\Bigl(\frac{\Psi^*}{\Psi} - \kappa_0\Bigr) > 0$, thus contradicting the definition of κ_0. Therefore, defining $\Phi = \Psi^* - \kappa_0\Psi$, it is easy to see that $\mathcal{L}\Phi - (V+\delta\mathbb{1}_{\mathcal{B}}-\lambda^*)^-\Phi \le 0$, and Φ attains its minimum value 0 in $\mathbb{R}^d$, which implies by the strong maximum principle that $\kappa_0\Psi^* = \Psi$. This of course implies that $\delta = 0$. Hence $\lambda = \lambda^*(V)$, and, in turn, (3.2) implies that the twisted process corresponding to the eigenpair (Ψ,λ) is recurrent. This completes the proof. □

We continue with the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $K \Subset \mathcal{D}$ be a compact set such that $V_2 - V_1 \ge 0$ and $\mathcal{L}_1 = \mathcal{L}_2$ in K^c. Since $\lambda^*_2 \le \lambda^*_1$, using Harnack's inequality, it follows that there exists a positive generalized eigenfunction Ψ^*_2 corresponding to the generalized eigenvalue λ^*_2, i.e.,

$$P_2\Psi^*_2 \;=\; \lambda^*_2\,\Psi^*_2 \quad \text{in } \mathcal{D}\,.$$

Thus we have

$$\mathcal{L}_1\Psi^*_2 + (V_1-\lambda^*_1)\Psi^*_2 \;\le\; \mathcal{L}_2\Psi^*_2 + (V_2-\lambda^*_2)\Psi^*_2 \;=\; 0 \quad \text{in } K^c\,. \tag{3.3}$$

By the minimal growth property of Ψ^*_1 and (3.3), we can find a positive constant κ and a set $\mathcal{D}_i \in \mathfrak{D}$, with $\mathcal{D}_i \supset K$, such that $\kappa\Psi^*_1 \le \Psi^*_2$ for all $x \in \mathcal{D}\setminus\mathcal{D}_i$. Let

$$\hat\kappa \;=\; \sup_{\mathcal{D}} \frac{\Psi}{\Psi^*_2} \;=\; \sup_{\mathcal{D}} \frac{\Psi^+}{\Psi^*_2}.$$

Then, using (2.6) and the bound $\kappa\Psi^*_1 \le \Psi^*_2$, we conclude that $\hat\kappa \in (0,\infty)$. Let us now define

$$\Phi(x) \;:=\; \hat\kappa\Psi^*_2(x) - \Psi(x) \quad \text{in } \mathcal{D}.$$

We claim that there exists $x_0 \in \mathcal{D}$ such that $\Phi(x_0) = 0$. If not, then $\Phi(x) > 0$ in $\mathcal{D}$. Then in K^c we have

$$\mathcal{L}_1\Phi + (V_1-\lambda^*_1)\Phi \;=\; \mathcal{L}_2\Phi + (V_1-\lambda^*_1)\Phi$$

$$
\begin{aligned}
&\leq \mathcal{L}_2\Phi + (V_2 - \lambda_1^*)\Phi \\
&= \bigl(\mathcal{L}_2 + V_2 - \lambda_1^*\bigr)(\hat{\kappa}\Psi_2^\star - \Psi) \\
&\leq \bigl(\mathcal{L}_2 + V_2 - \lambda_1^*\bigr)\hat{\kappa}\Psi_2^* \;=\; (\lambda_2^* - \lambda_1^*)\hat{\kappa}\,\Psi_2^* \;\leq\; 0\,.
\end{aligned}
$$

By the minimality of the growth of the ground state Ψ_1^*, there exist a positive constant κ_1 and a compact set $K_2 \supset K$ such that $\kappa_1\Psi_1^* \leq \Phi$ in K_2^c. Next, using (2.6), we obtain

$$
\hat{\kappa}\Psi_2^*(x) - \Psi(x) \;\geq\; \frac{\kappa_1}{C}\Psi(x) \;\Rightarrow\; \frac{\Psi(x)}{\Psi_2^*(x)} \;\leq\; \frac{\hat{\kappa}}{1 + {\kappa_1}/{C}} \;<\; \hat{\kappa} \qquad \forall\, x \in K_2^c\,.
$$

Thus the value $\hat{\kappa}$ is attained for some $x_0 \in K_2$. This shows that $\Phi(x_0) = 0$ at some $x_0 \in \mathcal{D}$.

On the other hand, Φ is nonnegative, and it satisfies

$$
\mathcal{L}_2\Phi + (V_2 - \lambda_1^*)\Phi \;\leq\; (\lambda_2^* - \lambda_1^*)\hat{\kappa}\Psi_2^* \;\leq\; 0 \quad \text{in } \mathcal{D}\,,
$$

which in turn implies that

$$
\mathcal{L}_2\Phi - (V_2 - \lambda_1^*)^-\Phi \;\leq\; -(V_2 - \lambda_1^*)^+\Phi \;\leq\; 0 \quad \text{in } \mathcal{D}\,.
$$

Thus by strong maximum principle we must have $\Phi \equiv 0$ in $\mathcal{D}$. This shows that $\hat{\kappa}\Psi_2^* = \Psi$, which implies by (2.5) that $\lambda_2^* = \lambda_1^*$.

To complete the proof it remains to show that Ψ_2^* is a ground state of $\mathcal{L}_2 + V_2 - \lambda_2^*$. Consider a compact set $\tilde{K}$, and let $v \in \mathscr{W}^{2,d}_{\mathrm{loc}}(\tilde{K}^c)$ be a positive supersolution of $\mathcal{L}_2 + V_2 - \lambda_2^*$, i.e.,

$$
\mathcal{L}_2 v + (V_2 - \lambda_2^*)v \;\leq\; 0 \quad \text{in } \tilde{K}^c\,.
$$

By hypothesis, we have

$$
\mathcal{L}_1 v + (V_1 - \lambda_1^*)v \;\leq\; \mathcal{L}_2 v + (V_2 - \lambda_2^*)v \;\leq\; 0 \qquad \text{on } K^c \cap \tilde{K}^c\,.
$$

Since Ψ_1^* has minimal growth at infinity, we can find a constant κ_2 and a compact set $\tilde{K}_2$ satisfying $\kappa_2\Psi_1^* \leq v$ in $\tilde{K}_2^c$. Combining this with (2.6) we have $\frac{\kappa_2}{C}\,\Psi_2^* \leq v$ in $\tilde{K}_2^c$. Therefore Ψ_2^* also has minimal growth at infinity, and hence is a ground state. This completes the proof. □

As an immediate corollary to Theorem 2.2, we have the following generalization of the result in [38, Corollary 1.8]

Corollary 3.1. *Let P_1 and P_2 be as in Theorem 2.2. Suppose that any Ψ which satisfies (2.5) and (2.6) cannot be a solution of $(\mathcal{L}_2 + V_2 - \lambda_1^*)\Psi = 0$ unless it is sign-changing. Then $\lambda_2^* > \lambda_1^*$.*

To prove Theorems 2.3 and 2.4 we need several lemmas which are stated next.

Lemma 3.1. *Suppose that $\lambda^*(V)$ is not strictly right monotone at V. Then for any ball $\mathcal{B}$ there exists a constant $\delta > 0$ such that $\lambda^*(V) = \lambda^*(V + \delta\mathbb{1}_{\mathcal{B}})$, and λ^* is strictly right monotone at $V + \delta\mathbb{1}_{\mathcal{B}}$.*

Proof. Let $F_\alpha(x) := V(x) - \lambda^*(V) - \alpha$, for $\alpha > 0$. It is evident that the Dirichlet eigenvalue of $-\mathcal{L} - F_\alpha$ on every ball B_n is positive. Thus by Proposition 6.2 and Theorem 6.1 in [13], for any $n \in \mathbb{N}$, the Dirichlet problem

$$
\mathcal{L}\varphi_{\alpha,n}(x) + F_\alpha(x)\,\varphi_{\alpha,n}(x) \;=\; -\mathbb{1}_{\mathcal{B}}(x) \quad \text{a.e. } x \in B_n\,, \quad \varphi_{\alpha,n} = 0 \;\text{ on } \partial B_n\,, \tag{3.4}
$$

has a unique solution $\varphi_{\alpha,n} \in \mathscr{W}^{2,p}_{\mathrm{loc}}(B_n) \cap C(\bar{B}_n)$, for any $p \geq 1$. In addition, by the *refined maximum principle* in [13, Theorem 1.1] $\varphi_{\alpha,n}$ is nonnegative. It is clear that $\varphi_{\alpha,n}$ cannot be identically equal to 0. Thus if we write (3.4) as

$$\mathcal{L}\varphi_{\alpha,n} - F^{-}_{\alpha}\,\varphi_{\alpha,n} \;=\; -F^{+}_{\alpha}\,\varphi_{\alpha,n} - \mathbb{1}_{\mathcal{B}}\,,$$

it follows by the strong maximum principle that $\varphi_{\alpha,n} > 0$ in B_n. By the Itô–Krylov formula in (3.1), since $\varphi_{\alpha,n} = 0$ on ∂B_n, we obtain from (3.4) that

$$\begin{aligned}\varphi_{\alpha,n}(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T F_\alpha(X_s)\,\mathrm{d}s}\,\varphi_{\alpha,n}(X_T)\,\mathbb{1}_{\{T\leq\uptau_n\}}\Bigr]&\\ +\,\mathbb{E}_x\biggl[\int_0^{T\wedge\uptau_n}\mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr]&\end{aligned} \tag{3.5}$$

for all $(T,x) \in \mathbb{R}_+ \times B_n$.

Now fix $\alpha > 0$. Let Ψ be a positive principal eigenfunction of $\mathcal{L} + V$ constructed canonically from Dirichlet eigensolutions. We can scale Ψ so that $\Psi \geq 1$ on $\mathcal{B}$. Let $\Psi_\alpha = \alpha^{-1}\Psi$. Then

$$\mathcal{L}\Psi_\alpha(x) + F_\alpha(x)\,\Psi_\alpha(x) \;\leq\; -\mathbb{1}_{\mathcal{B}}(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,.$$

Using the Itô–Krylov formula and Fatou's lemma, we obtain

$$\Psi_\alpha(x) \;\geq\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\mathfrak{t}} F_\alpha(X_s)\,\mathrm{d}s}\,\Psi_\alpha(X_{\mathfrak{t}})\Bigr] + \mathbb{E}_x\biggl[\int_0^{\mathfrak{t}}\mathrm{e}^{\int_0^{\mathfrak{t}} F_\alpha(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr]\,, \tag{3.6}$$

for any finite stopping time $\mathfrak{t}$, and any $\alpha > 0$. Also, Ψ being an eigenfunction, we have

$$\begin{aligned}\Psi_\alpha(x) \;&\geq\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{T\wedge\uptau_n} F_0(X_t)\,\mathrm{d}t}\,\Psi_\alpha(X_T)\,\mathbb{1}_{\{T\leq\uptau_n\}}\Bigr]\\ &\geq\; \alpha^{-1}\Bigl(\inf_{B_n}\,\Psi\Bigr)\,\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{T\wedge\uptau_n} F_0(X_t)\,\mathrm{d}t}\,\mathbb{1}_{\{T\leq\uptau_n\}}\Bigr]\,.\end{aligned} \tag{3.7}$$

Thus by (3.7) we have

$$\begin{aligned}&\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T F_\alpha(X_s)\,\mathrm{d}s}\,\varphi_{\alpha,n}(X_T)\,\mathbb{1}_{\{T\leq\uptau_n\}}\Bigr]\\ &\qquad\qquad\leq\; \mathrm{e}^{-\alpha T}\Bigl(\sup_{B_n}\,\varphi_{\alpha,n}\Bigr)\,\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T F_0(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\{T\leq\uptau_n\}}\Bigr]\,,\end{aligned}$$

and the right hand side tends to 0 as $T \to \infty$. Taking limits in (3.5) as $T \to \infty$, using monotone convergence for the second integral, we obtain

$$\varphi_{\alpha,n}(x) \;=\; \mathbb{E}_x\biggl[\int_0^{\uptau_n}\mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr]\,,$$

which implies by (3.6) that $\varphi_{\alpha,n} \leq \Psi_\alpha$ for all $n \in \mathbb{N}$. It therefore follows by the a priori estimates that $\{\varphi_{\alpha,n}\}$ is relatively weakly compact in $\mathscr{W}^{2,p}(B_n)$, for any $p \geq 1$ and $n \in \mathbb{N}$, and thus $\varphi_{\alpha,n}$ converges uniformly on compact sets along some sequence $n \to \infty$ to a nonnegative $\Phi_\alpha \in \mathscr{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, for any $p \geq 1$, which solves

$$\mathcal{L}\Phi_\alpha(x) + F_\alpha(x)\,\Phi_\alpha(x) \;=\; -\mathbb{1}_{\mathcal{B}}(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,.$$

It is clear by the strong maximum principle that $\Phi_\alpha > 0$. Since, as we have already shown, $\varphi_{\alpha,n} \leq \Psi_\alpha$ for all $n \in \mathbb{N}$, it follows that $\Phi_\alpha \leq \Psi_\alpha$. Using (3.6) with $\mathfrak{t} = T$

and a slightly smaller α, then by (3.5) and dominated convergence, we obtain

$$\Phi_\alpha(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T F_\alpha(X_s)\,\mathrm{d}s}\,\Phi_\alpha(X_T)\Bigr] + \mathbb{E}_x\biggl[\int_0^T \mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr] \tag{3.8}$$

for all $T > 0$ and $x \in \mathbb{R}^d$. Since (3.5) also holds with T replaced by $T \wedge \breve{\mathfrak{t}}_r$, then again dominating this by (3.6) with $\mathfrak{t} = \breve{\mathfrak{t}}_r$ and choosing a slightly smaller α, we similarly obtain

$$\begin{aligned}\Phi_\alpha(x) \;=\; &\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\mathfrak{t}}_r} F_\alpha(X_s)\,\mathrm{d}t}\,\Phi_\alpha(X_{\breve{\mathfrak{t}}_r})\mathbb{1}_{\{\breve{\mathfrak{t}}_r<\infty\}}\Bigr]\\ &+ \mathbb{E}_x\biggl[\int_0^{\breve{\mathfrak{t}}_r} \mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr]\end{aligned} \tag{3.9}$$

for all $x \in B_r^c$ and $r > 0$. Using the bound $\Phi_\alpha \le \Psi_\alpha$ we have

$$\begin{aligned}\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T F_\alpha(X_s)\,\mathrm{d}s}\,\Phi_\alpha(X_T)\Bigr] \;&\le\; \mathrm{e}^{-\alpha T}\Bigl[\mathrm{e}^{\int_0^T F_0(X_s)\,\mathrm{d}s}\,\Psi_\alpha(X_T)\Bigr]\\ &\le\; \mathrm{e}^{-\alpha T}\Psi_\alpha(x) \;\xrightarrow[T\to\infty]{}\; 0\,.\end{aligned}$$

Thus by (3.8), we obtain

$$\Phi_\alpha(x) \;=\; \mathbb{E}_x\biggl[\int_0^\infty \mathrm{e}^{\int_0^t F_\alpha(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr]\,.$$

Since λ^* is not strictly right monotone at V, the twisted process is transient, and by [5, Lemma 2.7] we have

$$\mathbb{E}_0\biggl[\int_0^\infty \mathrm{e}^{\int_0^t F_0(X_s)\,\mathrm{d}s}\,\mathbb{1}_{\mathcal{B}}(X_t)\,\mathrm{d}t\biggr] \;<\; \infty\,.$$

It follows that $\Phi_\alpha(0)$ is bounded uniformly over $\alpha \in (0,1)$, and therefore is uniformly locally bounded by the superharmonic Harnack inequality [7]. Thus letting $\alpha \searrow 0$, we obtain a positive Φ as a limit of Φ_α, which solves

$$\mathcal{L}\Phi(x) + F_0(x)\,\Phi(x) \;=\; -\mathbb{1}_{\mathcal{B}}(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,.$$

Write this as

$$\mathcal{L}\Phi(x) + \bigl(V(x) + \Phi^{-1}(x)\mathbb{1}_{\mathcal{B}}(x)\bigr)\,\Phi(x) \;=\; \lambda^*(V)\Phi(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,.$$

On the other hand, taking limits in (3.9) as $\alpha \searrow 0$, choosing $r > 0$ such that $B_r \supset \mathcal{B}$, we obtain

$$\Phi(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\mathfrak{t}}_r} F_0(X_s)\,\mathrm{d}t}\,\Phi(X_{\breve{\mathfrak{t}}_r})\mathbb{1}_{\{\breve{\mathfrak{t}}_r<\infty\}}\Bigr]\,.$$

This shows that Φ has a stochastic representation, which implies that λ^* is strictly monotone at $V + \Phi^{-1}\mathbb{1}_{\mathcal{B}}$ on the right. Then the monotonicity property of $\delta \mapsto \lambda^*(V + \delta\mathbb{1}_{\mathcal{B}})$ implies that $\lim_{\delta\to\infty} \lambda^*(V + \delta\mathbb{1}_{\mathcal{B}}) > \lambda^*(V)$. So we define $\delta_0 = \inf\{\delta > 0 : \lambda^*(V + \delta\mathbb{1}_{\mathcal{B}}) > \lambda^*(V)\}$, then λ^* is strictly monotone at $V + \delta_0\mathbb{1}_{\mathcal{B}}$ on the right by [5, Corollary 2.4]. □

Corollary 3.2. *For any $\lambda > \lambda^*(V)$ and ball $\mathcal{B}$, there exists a constant δ such that $\lambda = \lambda^*(V + \delta\mathbb{1}_{\mathcal{B}})$ and λ^* is strictly right monotone at $V + \delta\mathbb{1}_{\mathcal{B}}$.*

Proof. By Lemma 3.1 there exists $\delta^* \geq 0$ such that $\lambda^*(V + \delta^* \mathbb{1}_{\mathcal{B}}) = \lambda^*(V)$, and λ^* is strictly right monotone at $V + \delta^* \mathbb{1}_{\mathcal{B}}$. Recall that the map $\delta \mapsto \lambda^*(V + \delta \mathbb{1}_{\mathcal{B}})$ is non-decreasing and convex. Since $\lambda^*(V + \delta \mathbb{1}_{\mathcal{B}}) > \lambda^*(V)$ for any $\delta > \delta^*$ by strict right monotonicity, it follows that this map is strictly increasing and convex on $[\delta^*, \infty)$, and hence its image is $[\lambda^*(V), \infty)$. □

Using Lemma 3.1 we can show the following.

Lemma 3.2. *Let $\mathcal{D} = \mathbb{R}^d$. Assume the hypotheses of Theorem 2.2 and (A1)–(A2), and in addition, suppose that $V_2 - V_1 \in \mathcal{B}_0(\mathbb{R}^d)$, and $\mathcal{L}_1 = \mathcal{L}_2$ outside a compact set K. Then $\lambda_1^* = \lambda_2^*$.*

Proof. Suppose that $\lambda_2^* < \lambda_1^*$. By Lemma 3.1 there exists $\delta \geq 0$, such that $\lambda_2^*(V_2 + \delta \mathbb{1}_{\mathcal{B}})$ is strictly monotone at $V_2 + \delta \mathbb{1}_{\mathcal{B}}$ on the right, and $\lambda_2^*(V_2 + \delta \mathbb{1}_{\mathcal{B}}) = \lambda_2^*$. Let Φ_δ denote the ground state corresponding to $\lambda_2^*(V_2 + \delta \mathbb{1}_{\mathcal{B}})$. Then

$$\mathcal{L}_1 \Phi_\delta + (V_1 - \lambda_1^*)\Phi_\delta \;=\; (\lambda_2^* - V_2 - \delta \mathbb{1}_{\mathcal{B}} + V_1 - \lambda_1^*)\Phi_\delta$$

outside the compact set K. Hence by the minimal growth property of Ψ_1^* we have $\Psi_1^* \leq \kappa_1 \Phi_\delta$. Note that the choice of $\mathcal{B}$ is arbitrary. This means we can select $\mathcal{B}$ so that $\Psi > 0$ on $\mathcal{B}$. Therefore

$$\mathcal{L}_2 \Psi + (V_2 + \delta \mathbb{1}_{\mathcal{B}})\Psi \;\geq\; \lambda_1^* \Psi.$$

Moreover, $\frac{\Psi}{\Phi_\delta} \leq \frac{\Psi^+}{\Phi_\delta}$ is bounded above by (2.6).

By $\tilde{\mathcal{L}}$ we denote the generator of the twisted process (2.3) corresponding to $(\Phi_\delta, \lambda^*(V_2 + \delta \mathbb{1}_{\mathcal{B}}))$ and $\mathcal{L}_2$. Therefore

$$\tilde{\mathcal{L}} f \;=\; \mathcal{L}_2 f + 2\langle a_2(x) \nabla \varphi_\delta, \nabla f \rangle\,, \quad \text{for } f \in \mathcal{C}^2(\mathbb{R}^d)\,,$$

where $\varphi_\delta = \log \Phi_\delta$. Since the twisted process (2.3) corresponding to $(\Phi_\delta, \lambda^*(V_2 + \delta \mathbb{1}_{\mathcal{B}}))$ is recurrent by Theorem 2.1, it exists for all time. Moreover, we note that for $\widehat{\Phi} = \frac{\Psi}{\Phi_\delta}$ we obtain from (2.5) that

$$\tilde{\mathcal{L}} \widehat{\Phi} - (\lambda_1^* - \lambda_2^*)\widehat{\Phi} \;\geq\; 0\,.$$

Now since $\widehat{\Phi}$ is bounded above, by applying the Itô–Krylov formula to the above equation, we obtain

$$\widehat{\Phi}(x) \;\leq\; \tilde{\mathbb{E}}_x\bigl[e^{-(\lambda_1^* - \lambda_2^*)T} \widehat{\Phi}^+(\hat{Y}_T)\bigr] \;\leq\; \|\widehat{\Phi}^+\|_\infty e^{-(\lambda_1^* - \lambda_2^*)T}\,, \quad \forall\, T > 0\,.$$

Letting $T \to \infty$ in this inequality, it follows that $\Phi(x) \leq 0$ for all x, which contradicts the fact that $\Psi^+ \neq 0$. Hence we have $\lambda_1^* = \lambda_2^*$. □

Note that the generators $\mathcal{L}_1$ and $\mathcal{L}_2$ agree outside the compact set K. Therefore, the processes associated to these generators must agree up to the hitting time $\breve{\tau}(K)$.

Lemma 3.3. *Let the assumptions of Theorem 2.3 hold, and $r > 0$ be large enough so that $K \subset B_r$. Then we have*

$$\Psi(x) \;\leq\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_2(X_s) - \lambda_1^*)\,\mathrm{d}s}\, \Psi^+(X_{\breve{\tau}_r}) \mathbb{1}_{\{\breve{\tau}_r < \infty\}}\Bigr]\,. \tag{3.10}$$

Proof. Choose $R > r$ and $x \in B_R \setminus B_r$. Applying the Itô–Krylov formula to (2.11) we obtain

$$\begin{aligned} \Psi(x) \;\leq\; & \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_2(X_s) - \lambda_1^*)\,\mathrm{d}s}\, \Psi(X_{\breve{\tau}_r}) \mathbb{1}_{\{\breve{\tau}_r < \tau_R \wedge T\}}\Bigr] \\ & + \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\tau_R} (V_2(X_s) - \lambda_1^*)\,\mathrm{d}s}\, \Psi(X_{\tau_R}) \mathbb{1}_{\{\tau_R < \breve{\tau}_r \wedge T\}}\Bigr] \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&\qquad\qquad + \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi(X_T)\mathbb{1}_{\{T\le\breve{\tau}_r\wedge\tau_R\}}\Big]\\
&\le \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi^+(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\tau_R\wedge T\}}\Big]\\
&\qquad + \underbrace{\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_R} (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi^+(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\wedge T\}}\Big]}_{\mathcal{J}_1}\\
&\qquad\qquad + \underbrace{\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi(X_T)\mathbb{1}_{\{T\le\breve{\tau}_r\wedge\tau_R\}}\Big]}_{\mathcal{J}_2}\,.
\end{aligned}
$$

We first show that $\mathcal{J}_2$ tends 0 as $T\to\infty$. By (Ψ_R,λ_R) we denote the principal eigenpair of $\mathcal{L}_2+V_2$ in B_R with Dirichlet boundary condition. It is known that λ_R is strictly increasing to λ_2^* as $R\to\infty$. An application of the Itô–Krylov formula shows that

$$
\begin{aligned}
\Psi_{R+1}(x) \;=\; &\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_2(X_s)-\lambda_{R+1})\,\mathrm{d}s}\,\Psi_{R+1}(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\tau_R\wedge T\}}\Big]\\
&+ \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_R} (V_2(X_s)-\lambda_{R+1})\,\mathrm{d}s}\,\Psi_{R+1}(X_{\breve{\tau}_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\wedge T\}}\Big]\\
&+ \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T (V_2(X_s)-\lambda_{R+1})\,\mathrm{d}s}\,\Psi_{R+1}(X_T)\mathbb{1}_{\{T\le\breve{\tau}_r\wedge\tau_R\}}\Big]
\end{aligned}
\tag{3.12}
$$

for $x\in B_R\setminus B_r$. Since $\lambda_R<\lambda_2^*\le\lambda_1^*$ and $\Psi_{R+1}>0$ in B_{R+1}, we deduce that

$$
\begin{aligned}
\mathcal{J}_2 \;&=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^T (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi(X_T)\mathbb{1}_{\{T\le\breve{\tau}_r\wedge\tau_R\}}\Big]\\
&\le\; \frac{1}{\min_{B_R}\Psi_{R+1}}\max_{B_R}|\Psi|\;\mathbb{E}_x\Big[\mathrm{e}^{\int_0^T (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_{R+1}(X_T)\mathbb{1}_{\{T\le\breve{\tau}_r\wedge\tau_R\}}\Big]\\
&\le\; \frac{\mathrm{e}^{(\lambda_{R+1}-\lambda_1^*)T}}{\min_{B_R}\Psi_{R+1}}\Big(\max_{B_R}|\Psi|\Big)\,\Psi_{R+1}(x)\to 0,\quad \text{as } T\to\infty,
\end{aligned}
$$

where in the last inequality we used (3.12).

Therefore letting $T\to\infty$ in (3.11) and using the monotone convergence theorem, we obtain

$$
\begin{aligned}
\Psi(x) \;\le\; &\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi^+(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\tau_R\}}\Big]\\
&+ \underbrace{\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_R} (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi^+(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Big]}_{\mathcal{J}_3}\,.
\end{aligned}
\tag{3.13}
$$

We next show that $\limsup\mathcal{J}_3\le 0$ as $R\to\infty$. Recall that $P_1-\lambda_1^*$ is critical and therefore, by Theorem 2.1, we have

$$
\Psi_1^*(x) \;=\; \mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_1(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Big],\quad x\in B_r^c\,. \tag{3.14}
$$

Since $\Psi^+\le C\Psi_1^*$ by (2.12), we see using (3.14) that

$$
\begin{aligned}
\mathcal{J}_3 \;&\le\; C\,\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_R} (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Big]\\
&=\; C\,\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\tau_R} (V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\\
&\qquad\qquad \mathbb{E}_{X_{\tau_R}}\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r} (V_1(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Big]\Big]
\end{aligned}
\tag{3.15}
$$

$$\leq\, C\,\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r}(\tilde{V}(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\breve{\tau}_r<\infty\}}\Psi_1^*(X_{\breve{\tau}_r})\Big]\,,$$

where in the third line we used strong Markov property. On the other hand, using (2.8) we note that

$$\mathbb{E}_x\Big[\mathrm{e}^{\int_0^{\breve{\tau}_r}(\tilde{V}(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Big]\;<\;\infty,\quad \text{for } |x|>r\,.$$

Therefore, since $\tau_R\to\infty$ a.s. as $R\to\infty$, applying the dominated convergence theorem to (3.15) we have

$$\limsup_{R\to\infty}\;\mathcal{J}_3\;\leq\;0\,. \tag{3.16}$$

Hence, (3.10) follows from (3.13) and (3.16) by applying the monotone convergence theorem. □

We are now ready to present the proofs of Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Without loss of generality, we may assume that the compact K is large enough so that there exists a ball $\mathcal{B}\subset K$ satisfying $\Psi>0$ in $\mathcal{B}$. Using Lemma 3.1, we deduce that there exists $\delta\geq 0$ such that $\lambda^*(V_2+\delta\mathbb{1}_{\mathcal{B}})=\lambda_2^*$, and λ^* is strictly monotone at $V_2+\delta\mathbb{1}_{\mathcal{B}}$ on the right. Let Φ_δ be the ground state of the operator $\mathcal{L}+V_2+\delta\mathbb{1}_{\mathcal{B}}-\lambda_2^*$. Then, for any $r>0$, we have from Theorem 2.1 that

$$\Phi_\delta(x)\;=\;\mathbb{E}_x\Big[e^{\int_0^{\breve{\tau}_r}(V_2(X_s)+\delta\mathbb{1}_{\mathcal{B}}(X_s)-\lambda_2^*)\,\mathrm{d}s}\,\Phi_\delta(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Big],\quad x\in B_r^c\,. \tag{3.17}$$

Fix $r>0$ large enough so that $K\subset B_r$. Since $\lambda_2^*\leq\lambda_1^*$ we obtain from Lemma 3.3 and (3.17) that $\Psi\leq\kappa_1\Phi_\delta$. Define $\widehat{\Phi}=\frac{\Psi}{\Phi_\delta}$. Let $\tilde{\mathcal{L}}$ be the generator of the twisted process corresponding to $(\Phi_\delta,\lambda_2^*)$ and $\mathcal{L}_2$. Since $\mathcal{L}_2\Psi+(V_2+\delta\mathbb{1}_{\mathcal{B}}-\lambda_1^*)\Psi\geq 0$, we have

$$\tilde{\mathcal{L}}\widehat{\Phi}+(\lambda_2^*-\lambda_1^*)\widehat{\Phi}\;\geq\;0\,. \tag{3.18}$$

Thus repeating the arguments in the proof of Lemma 3.2, we obtain $\lambda_1^*=\lambda_2^*$. But it then follows from (3.18) that $\{\widehat{\Phi}(Y_s)\}$ is a submartingale which is bounded above. This of course, implies that $\widehat{\Phi}(Y_s)$ converges almost surely as $s\to\infty$. Since Y_s is recurrent, $\widehat{\Phi}$ has to be constant, implying that $\Psi=\kappa_2\Phi_\delta$ for some positive $\kappa_2>0$. Using (2.9), we obtain $\delta=0$, and this completes the proof. □

Proof of Theorem 2.4. Let K be a compact set such that $\mathcal{L}_1\equiv\mathcal{L}_2$ in K^c. Let $h\in\mathcal{C}_0^+(\mathbb{R}^d)$ be a function with compact support. Then we know that

$$\beta\mapsto\Lambda_\beta\;=\;\lambda^*(V_1+\beta h)$$

is an increasing, convex function [14, Proposition 2.3]. In addition, it is strictly monotone at $\beta=0$. Let $\beta_c:=\inf\{\beta\in\mathbb{R}:\Lambda_\beta>\Lambda_{-\infty}\}$. It is then clear that $\beta_c<0$, and hence it follows from [5, Theorem 2.7] that for some $\beta<0$, close to 0, the twisted process corresponding to the eigenpair $(\Psi_\beta,\Lambda_\beta)$ and $\mathcal{L}_1$ is recurrent (in fact, geometrically ergodic), and $\Lambda_\beta<\lambda_1^*=\Lambda_0$. We also have

$$\mathcal{L}_1\Psi_\beta+(V_1+\beta h)\Psi_\beta\;=\;\Lambda_\beta\Psi_\beta\,. \tag{3.19}$$

Moreover, by Theorem 2.1, Ψ_β has a stochastic representation, i.e.,

$$\Psi_\beta(x)\;=\;\mathbb{E}_x\Big[e^{\int_0^{\breve{\tau}_r}(V_1(X_s)-\Lambda_\beta)\,\mathrm{d}s}\,\Psi_\beta(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Big]. \tag{3.20}$$

In (3.20) we use a radius r large enough so that the support of h and the set K lie in B_r. Also by Lemma 3.2 we have $\lambda_1^* = \lambda_2^*$. Let $\delta = \frac{1}{2}(\lambda_1^* - \Lambda_\beta) > 0$. It is clear that we can choose r large enough so that

$$V_2(x) - \lambda_2^* + \delta \;=\; V_2(x) - \lambda_1^* + \delta \;< V_1(x) - \Lambda_\beta \quad \text{for all } |x| \ge r\,.$$

For such a choice of r, we note from (3.20) that

$$\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V_2(X_s)-\lambda_2^*+\delta)\,\mathrm{d}s}\,\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr] \;<\; \infty, \quad |x| \ge r. \tag{3.21}$$

Using (3.21) and the arguments in [5, Theorem 2.2] (see for instance, (2.30) in [5]) it is easy to show that λ_2^* is strictly monotone at V_2.

Therefore, in order to complete the proof, it remains to show that Ψ is a positive multiple of Ψ_2^*. Since $V_1 + \beta h - \Lambda_\beta \ge V_1 - \lambda_1^*$ outside some compact set K_0, we obtain from (3.19) that

$$\mathcal{L}_1\Psi_\beta + (V_1 - \lambda_1^*)\Psi_\beta \;\le\; 0 \qquad \forall\, x \in K_0^c\,.$$

Therefore, by the minimal growth at infinity of Ψ_1^*, we can find a constant κ_β satisfying $\Psi_1^* \le \kappa_\beta\Psi_\beta$ in $\mathbb{R}^d$. Combining this with (2.12), we have $\Psi^+ \le C\kappa_\beta\Psi_\beta$. As earlier, we fix r large enough so that $V_2(x) - \lambda_2^* < V_1(x) - \Lambda_\beta$, $\mathcal{L}_1 \equiv \mathcal{L}_2$, and $h(x) = 0$ for $|x| \ge r$. We apply the Itô–Krylov formula to (2.11) to obtain

$$\begin{aligned}\Psi(x) \;\le\; &\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V_2(X_s)-\lambda_2^*)\,\mathrm{d}s}\,\Psi(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\uptau_R\}}\Bigr]\\ &+ \mathbb{E}_x\Bigl[e^{\int_0^{\uptau_R}(V_2(X_s)-\lambda_2^*)\,\mathrm{d}s}\,\Psi(X_{\uptau_R})\mathbb{1}_{\{\breve{\tau}_r>\uptau_R\}}\Bigr]\,.\end{aligned}$$

By the choice of r, we can estimate the second term as follows

$$\begin{aligned}&\mathbb{E}_x\Bigl[e^{\int_0^{\uptau_R}(V_2(X_s)-\lambda_2^*)\,\mathrm{d}s}\,\Psi^+(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r>\uptau_R\}}\Bigr]\\ &\qquad\le\; \kappa_2\,\mathbb{E}_x\Bigl[e^{\int_0^{\uptau_R}(V_1(X_s)-\Lambda_\beta)\,\mathrm{d}s}\,\Psi_\beta(X_{\uptau_R})\mathbb{1}_{\{\breve{\tau}_r>\uptau_R\}}\Bigr]\,.\end{aligned} \tag{3.22}$$

The right hand side of (3.22) tends to 0, as $R \to \infty$, by (3.20). Hence letting $R \to \infty$, we obtain

$$\Psi(x) \;\le\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V_2(X_s)-\lambda_2^*)\,\mathrm{d}s}\,\Psi(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr]\,.$$

Since Ψ_2^* also has a stochastic representation by Theorem 2.1, this implies that $\Psi \le \kappa_1\Psi_2^*$ for some $\kappa_1 > 0$. With $\Phi = \frac{\Psi}{\Psi_2^*}$ we have

$$\tilde{\mathcal{L}}\Phi \;\ge\; 0\,,$$

where $\tilde{\mathcal{L}}$ is the generator of twisted process Y corresponding to (Ψ_2^*, λ_2^*) and $\mathcal{L}_2$. Thus, $\{\Phi(Y_s)\}$ is a submartingale which is bounded from above. Since the twisted process Y is recurrent by Theorem 2.1, Φ must be constant. Since $\Psi^+ \ne 0$, this implies that Ψ is a positive function, which means of course, that it is a positive multiple of Ψ_2^*. $\square$

One interesting by-product of the proof of Theorem 2.4 is the corollary that follows. This result however might be known, but we could not locate it in the literature.

Corollary 3.3. *Let $\mathcal{L}$ be the operator in (2.2), and λ^* be the principal eigenvalue of $\mathcal{L}+V$, where V is a locally bounded function. In addition, suppose that $\mathcal{L}+V-\lambda^*$ is critical, and let Ψ_1^* denote the ground state. Then, there does not exist any non-zero solution $\Psi \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ of $\mathcal{L}\Psi + V\Psi = \lambda\Psi$, for $\lambda > \lambda^*$, with $|\Psi| \le \kappa\Psi_1^*$.*

We can improve the above results to a larger class of potentials if we impose a 'stability' condition of the underlying dynamics X. Let us assume the following

(H) There exists a lower-semicontinuous, inf-compact function $\ell\colon \mathbb{R}^d \to [0,\infty)$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}_x\Bigl[e^{\int_0^T \ell(X_s)\,\mathrm{d}s}\Bigr] \;<\; \infty \quad \text{for all } x\in\mathbb{R}^d\,.$$

By $\mathfrak{o}(\ell)$ we denote the collection of functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\limsup_{|x|\to\infty} \frac{|f(x)|}{\ell(x)} \;=\; 0\,.$$

We say that the elliptic operator $\mathcal{L}$ satisfies (H) if the process X with extended generator $\mathcal{L}$ satisfies (H). It is easy to see that under hypothesis (H), the process is recurrent. Therefore, if (H) holds for $\mathcal{L}_1$, it follows from [4, Lemma 2.3] that $\lambda_1^*(\ell)$ is finite. Moreover, there exists a positive eigenfunction $\varphi_1 \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p>1$, with $\inf_{\mathbb{R}^d}\varphi_1 > 0$, that satisfies

$$\mathcal{L}_1\varphi_1 + (\ell - \lambda_1^*(\ell))\varphi_1 \;=\; 0, \quad \text{in } \mathbb{R}^d\,.$$

If $\mathcal{L}_2$ is a small perturbation of $\mathcal{L}_1$, then $\mathcal{L}_2$ also satisfies (H). To see this, consider a ball $\mathcal{B} \subset \mathbb{R}^d$ such that $\mathcal{L}_1 = \mathcal{L}_2$ in $\mathcal{B}^c$. Let $\chi\colon \mathbb{R}^d \to [0,1]$ be a smooth function that vanishes in $\mathcal{B}$ and equals 1 outside a ball $B_r \supset \bar{\mathcal{B}}$. Define $\varphi_2 = (1-\chi) + \chi\varphi_1$. Note that $\varphi_2 = 1$ in $\mathcal{B}$, and $\varphi_2 \ge 1 \wedge \inf_{\mathbb{R}^d}\varphi_1 > 0$ on $\mathbb{R}^d$. Then, for some positive constants κ_1 and κ_2, we have

$$\begin{aligned}
\mathcal{L}_2\varphi_2 \;&=\; \mathcal{L}_2(1-\chi) + \chi\mathcal{L}_2\varphi_1 + \varphi_1\mathcal{L}_2\chi + 2\langle a_2\nabla\chi, \nabla\varphi_1\rangle \\
&=\; \mathcal{L}_1(1-\chi) + \chi\mathcal{L}_1\varphi_1 + \varphi_1\mathcal{L}_1\chi + 2\langle a_1\nabla\chi, \nabla\varphi_1\rangle \\
&=\; \mathcal{L}_1(1-\chi) + \chi(\lambda_1^*(\ell) - \ell)\varphi_1 + \varphi_1\mathcal{L}_1\chi + 2\langle a_1\nabla\chi, \nabla\varphi_1\rangle \\
&\le\; (\kappa_1 - \ell)\,\varphi_1 \\
&\le\; (\kappa_2 - \ell)\,\varphi_2 \quad \text{on } \mathbb{R}^d\,.
\end{aligned} \tag{3.23}$$

In (3.23), the first inequality arises from the fact that $\inf_{\mathbb{R}^d}\varphi_1 > 0$, while in the second inequality we use the fact that $\varphi_1 = \varphi_2$ on B_r^c, and $\inf_{\mathbb{R}^d}\varphi_2 > 0$. Equation (3.23) of course implies that

$$\limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}_x\Bigl[e^{\int_0^T \ell(X^2_s)\,\mathrm{d}s}\Bigr] \;<\; \kappa_2 \quad \text{for all } x\in\mathbb{R}^d\,,$$

where X^2 denotes the diffusion process with generator $\mathcal{L}_2$.

We have the following result.

Theorem 3.1. *Let all the assumptions of Theorem 2.4 hold, except we replace $V_1 - V_2 \in \mathcal{B}_0(\mathbb{R}^d)$ with $V_i \in \mathfrak{o}(\ell)$. Moreover, assume that (H) holds for $\mathcal{L}_1$. Then the conclusion of Theorem 2.4 also holds.*

Proof. By [5, Theorem 3.2] we know that λ^* is strictly monotone at both V_1 and V_2. Therefore, in order to complete the proof, we only need to show that $\lambda_1^* = \lambda_2^*$ and $\Psi_2^* = \Psi$. Since ℓ is inf-compact, (H) implies that the processes X^i, $i = 1, 2$, are recurrent. Moreover, there exists a positive $\widehat{V}^i \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p \geq 1$, such that

$$\mathcal{L}_i \widehat{V}^i + \ell \widehat{V}^i \;=\; \lambda_i^*(\ell)\widehat{V}^i \quad \text{in } \mathbb{R}^d,\ i = 1, 2\,, \tag{3.24}$$

and $\inf_{\mathbb{R}^d} \widehat{V}^i > 0$. Let B_r be a ball such that

$$\bigl|V_i(x) - \max\{\lambda_1^*(\ell), \lambda_2^*(\ell)\}\bigr| \;\leq\; \theta\bigl(\ell(x) - \max\{\lambda_1^*(\ell), \lambda_2^*(\ell)\}\bigr) \quad \forall\, x \in B_r^c\,, \tag{3.25}$$

and $\mathcal{L}_1 = \mathcal{L}_2$, $i = 1, 2$, on B_r^c, for some constant $\theta \in (0, 1)$. Recall that $\breve{\tau}_r$ denotes the first hitting time to B_r. Since both processes agree outside B_r, in what follows we use X to denote any one of these processes. Then applying the Itô–Krylov formula to (3.24), followed by Fatou's lemma, we obtain

$$\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(\ell(X_s)-\lambda_i^*(\ell))\,\mathrm{d}s}\,\widehat{V}^i(X_{\breve{\tau}_r})\Bigr] \;\leq\; \widehat{V}^i(x) \quad \text{for } x \in B_r^c\,. \tag{3.26}$$

We can choose B_r large enough so that $\Psi^+ \neq 0$ in B_r. Let $\mathcal{B} \Subset B_r$ be such that $\Psi > 0$ in $\mathcal{B}$. By Lemma 3.1 we can find $\delta \geq 0$ such that λ^* is strictly monotone on the right at $V_2 + \delta\mathbf{1}_{\mathcal{B}}$ and $\lambda_2^* = \lambda^*(V_2 + \delta\mathbf{1}_{\mathcal{B}})$. Let Ψ_δ be the corresponding principal eigenfunction. By Theorem 2.1 we have

$$\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V_2(X_s)-\lambda_2^*)\,\mathrm{d}s}\,\Psi_\delta(X_{\breve{\tau}_r})\Bigr] \;=\; \Psi_\delta(x) \quad \text{for } x \in B_r^c\,. \tag{3.27}$$

Since $\mathcal{L}_1 + V_1 - \lambda_1^*$ is critical by hypothesis, we have

$$\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V_1(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\breve{\tau}_r})\Bigr] \;=\; \Psi_1^*(x) \quad \text{for } x \in B_r^c\,. \tag{3.28}$$

It follows by (3.25), (3.26), and (3.28) that $\Psi_1^*(x) \leq \kappa(\widehat{V}^1(x))^\theta$ in $\mathbb{R}^d$, for some constant κ.

We claim that

$$\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}(V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr] \;\xrightarrow[R\to\infty]{}\; 0\,. \tag{3.29}$$

To prove the claim we define $\Gamma(R, m) = \{x \in \partial B_r : \Psi_1^*(x) \geq m\}$ for $m \geq 1$. Then

$$\begin{aligned}
&\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}(V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr]\\
&\qquad\leq\; m\,\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}\theta(\ell(X_s)-\lambda_1^*(\ell))\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr]\\
&\qquad\qquad+\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}(V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi_1^*(X_{\tau_R})\mathbb{1}_{\{x\in\Gamma(R,m)\}}\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr]\\
&\qquad\leq\; m\,\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}\theta(\ell(X_s)-\lambda_1^*(\ell))\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr]\\
&\qquad\qquad+\kappa_1 m^{1-1/\theta}\,\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}(\ell(X_s)-\lambda_1^*(\ell))\,\mathrm{d}s}\,\widehat{V}^1(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr]\\
&\qquad\leq\; m\,\mathbb{E}_x\Bigl[e^{\int_0^{\tau_R}\theta(\ell(X_s)-\lambda_1^*(\ell))\,\mathrm{d}s}\,\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr] + \kappa_1 m^{1-1/\theta}\widehat{V}^1(x)\,.
\end{aligned}$$

Then (3.29) follows by first letting $R \to \infty$, and then $m \to \infty$.

Applying the Itô–Krylov formula (3.1) to (2.11) we obtain

$$\Psi(x) \;\leq\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r\wedge\tau_R\wedge T}(V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi(X_{\breve{\tau}_r\wedge\tau_R\wedge T})\Bigr]\,,\quad T > 0\,. \tag{3.30}$$

Since $|V_2 - \lambda_2^*| \le \ell - \lambda_1^*(\ell)$ in B_r^c, and

$$\mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r\wedge\tau_R}(\ell(X_s)-\lambda_1^*(\ell))\,\mathrm{d}s}\Bigr] \;<\; \infty\,, \quad r<|x|<R\,,$$

for every fixed $R>r$, we have

$$\mathbb{E}_x\Bigl[e^{\int_0^T(V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi(X_T)\mathbb{1}_{\{T\le\breve{\tau}_r\wedge\tau_R\}}\Bigr] \;\xrightarrow[T\to\infty]{}\; 0\,.$$

Hence, first letting $T\to\infty$, and then $R\to\infty$ in (3.30), and using (2.12) and (3.29), we obtain

$$\Psi(x) \;\le\; \mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r}(V_2(X_s)-\lambda_1^*)\,\mathrm{d}s}\,\Psi(X_{\breve{\tau}_r})\Bigr]\,.$$

Combining this with (3.27) we have $\Psi\le C_1\Psi_\delta$. Now mimicking the arguments in the last part of the proof of Theorem 2.3, we obtain $\lambda_1^*=\lambda_2^*$, and $\Psi=\Psi_\delta$ with $\delta=0$. □

We next exhibit a family of operators for which (H) holds.

Example 3.1. Let $\delta_1 I \le a(x) \le \delta_2 I$, for $\delta_1,\delta_2>0$ and $x\in\mathbb{R}^d$. Also $b(x) = b_1(x)+b_2(x)$ where $b_2\in L^\infty(\mathbb{R}^d)$, and

$$\langle b_1(x),x\rangle \;\le\; -\kappa|x|^\alpha \quad \text{on the complement of a compact set in } \mathbb{R}^d\,,$$

for some constant $\kappa>0$, and some $\alpha\in(1,2]$. Let ζ be a positive, twice differentiable function in $\mathbb{R}^d$ such that $\zeta(x)=\exp(\theta|x|^\alpha)$ for $|x|\ge1$. If we choose $\theta\in(0,1)$ small enough, then it is routine to check that there exists $R_0>0$ such that

$$\mathcal{L}\zeta(x) \;\le\; -\frac{\kappa\theta}{2}|x|^{2\alpha-2}\zeta(x) \qquad \text{for } |x|\ge R_0\,.$$

The above inequality is known as a (geometric) Foster–Lyapunov stability condition and ζ is generally referred to as a Lyapunov function. Therefore, if we choose a function ℓ which coincides with $\frac{\kappa\theta}{2}|x|^{2\alpha-2}$ outside a compact set, then using the above inequality and Itô's formula one can easily verify that (H) holds.

4. A lower bound on the decay of eigenfunctions

The main goal of this section is to exhibit a *sharp* lower bound on the decay of supersolutions, and also to use this estimate to prove several results for positive solutions.

Lemma 4.1. *Suppose that there exist positive constants M and η_0, and some $\beta\in[0,2]$ such that*

$$\bigl|\langle b(x),x\rangle\bigr| \;\le\; M|x|^\beta\,, \quad \text{and} \quad \langle\xi,a(x)\xi\rangle \;\ge\; \eta_0|\xi|^2 \quad \forall\,\xi\in\mathbb{R}^d \tag{4.1}$$

for all x outside some compact set in $\mathbb{R}^d$. Let $\alpha\ge\beta$ and K be any positive constants satisfying

$$K\alpha\eta_0 - M \;>\; 0\,, \quad \text{and} \quad \lim_{|x|\to\infty}\frac{1}{|x|^\alpha}\sum_{i=1}^d a^{ii}(x) \;=\; 0\,, \tag{4.2}$$

and define $\gamma\coloneqq\frac12K\alpha(K\alpha\eta_0-M)$, and $\mathcal{V}(x)\coloneqq\exp(-K|x|^\alpha)$. Then there exists $r_0>0$ such that for every $r\ge r_0$ we have

$$\mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\breve{\tau}_r}|X_s|^{2\alpha-2}\,\mathrm{d}s}\,\mathcal{V}(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr] \;\ge\; \mathcal{V}(x) \quad \textit{for } |x|\ge r. \tag{4.3}$$

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial x_i}\mathcal{V}(x) &= -K\alpha|x|^{\alpha-2}x_i\mathcal{V}\\ \frac{\partial^2}{\partial x_i\partial x_j}\mathcal{V}(x) &= (K\alpha)^2|x|^{2\alpha-4}x_ix_j\mathcal{V}(x) - K\alpha(\alpha-2)|x|^{\alpha-4}x_ix_j\mathcal{V}(x)\\ &\qquad - K\alpha|x|^{\alpha-2}\mathcal{V}(x)\delta_{ij}\end{aligned}$$

for $1 \le i, j \le d$, and $|x| \ge 1$. This implies that

$$\begin{aligned}\mathcal{L}\mathcal{V}(x) &= K\alpha|x|^{2\alpha-2}\biggl(K\alpha - \frac{\alpha-2}{|x|^\alpha}\biggr)\frac{\mathcal{V}(x)}{|x|^2}\langle x, ax\rangle\\ &\qquad - K\alpha|x|^{\alpha-2}\mathcal{V}(x)\sum_{i=1}^d a^{ii}(x) + \langle b(x), \nabla\mathcal{V}(x)\rangle\\ &\ge K\alpha|x|^{2\alpha-2}\biggl(K\alpha\eta_0 - M - \frac{\eta_0(\alpha-2)}{|x|^\alpha} - \frac{1}{|x|^\alpha}\sum_{i=1}^d a^{ii}(x)\biggr)\mathcal{V}(x)\,,\end{aligned}$$

which combined with (4.2) shows that there exists $r_0 \ge 1$, such that

$$\mathcal{L}\mathcal{V}(x) \ge \gamma|x|^{2\alpha-2}\mathcal{V}(x) \qquad \text{for } |x| \ge r_0\,. \tag{4.4}$$

Let $R > r \ge r_0$. Applying the Itô–Krylov formula to (4.4), we obtain

$$\begin{aligned}\mathcal{V}(x) &\le \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\breve\tau_r}|X_s|^{2\alpha-2}\,\mathrm{d}s}\,\mathcal{V}(X_{\breve\tau_r})\mathbb{1}_{\{\breve\tau_r<\tau_R\}}\Bigr]\\ &\qquad + \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\tau_R}|X_s|^{2\alpha-2}\,\mathrm{d}s}\,\mathcal{V}(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve\tau_r\}}\Bigr]\\ &\le \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\breve\tau_r}|X_s|^{2\alpha-2}\,\mathrm{d}s}\,\mathcal{V}(X_{\breve\tau_r})\mathbb{1}_{\{\breve\tau_r<\infty\}}\Bigr]\\ &\qquad + \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\tau_R}|X_s|^{2\alpha-2}\,\mathrm{d}s}\,\mathcal{V}(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve\tau_r\}}\Bigr]\,.\end{aligned} \tag{4.5}$$

for $r \le |x| \le R$. On the other hand,

$$\mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\tau_R}|X_s|^{2\alpha-2}\,\mathrm{d}s}\,\mathcal{V}(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve\tau_r\}}\Bigr] \le \mathbb{E}_x\Bigl[\mathcal{V}(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve\tau_r\}}\Bigr] \le e^{-KR^\alpha} \to 0\,,$$

as $R \to \infty$. Thus by letting $R \to \infty$ in (4.5), we obtain (4.3). □

The above result should be compared with Carmona [16], Carmona and Simon [17], where a weaker lower bound was obtained for Lévy processes. In these papers, the stationarity and independent increment property of the underlying process is used, and also the proof is much more complicated. For instance, see [16, Proposition 4.1] when X is a Brownian motion. We next use Lemma 4.1 to provide a quantitative estimate on the decay of positive supersolutions in the outer domain.

Theorem 4.1. *Assume (4.1), and let γ, α, and $\mathcal{V}$ be as in Lemma 4.1. Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set, and suppose $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{K}^c)$ is a nontrivial nonnegative function such that*

$$\mathcal{L}u + Vu = 0 \quad \text{in } \mathcal{K}^c\,, \tag{4.6}$$

where V is locally bounded, and $V(x) \ge -\gamma|x|^{2\alpha-2}$ for all $|x|$ sufficiently large. Then there exists a positive constant C, not depending on u, provided we fix $u(x_0) = 1$

at some $x_0 \in \mathcal{K}^c$*, and* $r > 0$ *such that*

$$u(x) \,\ge\, C\,\mathcal{V}(x) \quad \textit{for } |x| > r\,. \tag{4.7}$$

Proof. Let r_0 be as in Lemma 4.1. By the hypotheses of the theorem we may choose $r > r_0$ and sufficiently large, so that applying the Itô–Krylov formula to (4.6) we have

$$\begin{aligned} u(x) \,&\ge\, \mathbb{E}_x\Bigl[e^{\int_0^{\breve{\tau}_r} V(X_s)\,\mathrm{d}s}\, u(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\tau_R\}}\Bigr] \\ &\qquad\qquad + \mathbb{E}_x\Bigl[e^{\int_0^{\tau_R} V(X_s)\,\mathrm{d}s}\, u(X_{\tau_R})\mathbb{1}_{\{\tau_R<\breve{\tau}_r\}}\Bigr] \\ &\ge\, \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\breve{\tau}_r} |X_s|^{2\alpha-2}\,\mathrm{d}s}\, u(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\tau_R\}}\Bigr] \quad \text{for } |x| > r\,. \end{aligned} \tag{4.8}$$

By the Harnack inequality we have $\min_{|z|=r}\, u(z) \ge \kappa$ for some positive constant κ which does not depend on u. We let $R \to \infty$ in (4.8) and apply Fatou's lemma to obtain

$$\begin{aligned} u(x) \,&\ge\, \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\breve{\tau}_r} |X_s|^{2\alpha-2}\,\mathrm{d}s}\, u(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr] \\ &\ge\, \Bigl(\min_{|z|=r}\, u(z)\Bigr)\,\mathrm{e}^{Kr^\alpha}\; \mathbb{E}_x\Bigl[e^{-\gamma\int_0^{\breve{\tau}_r} |X_s|^{2\alpha-2}\,\mathrm{d}s}\, \mathcal{V}(X_{\breve{\tau}_r})\mathbb{1}_{\{\breve{\tau}_r<\infty\}}\Bigr] \\ &\ge\, \kappa\,\mathrm{e}^{Kr^\alpha}\,\mathcal{V}(x) \quad \text{for } |x| > r\,, \end{aligned}$$

by (4.3). Thus (4.7) follows. ☐

Remark 4.1. If $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathcal{K}^c)$ is a nontrivial nonnegative supersolution of (4.6) then it is necessarily positive on $\mathcal{K}^c$ by the strong maximum principle. Thus (4.7) is valid for nonnegative supersolutions; however the constant C depends, in general, on u.

As an immediate corollary to Lemma 4.1 and Theorem 4.1 we have the following.

Corollary 4.1. *Let* $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ *be a nontrivial nonnegative solution of*

$$\operatorname{trace}(a\,\nabla^2 u) + \langle b, \nabla u\rangle + V u \,=\, 0 \quad \textit{in } \mathcal{K}^c\,.$$

Here, we assume that $\sup_{\mathcal{K}^c}|b(x)| \le M$*,* $\sup_{\mathcal{K}^c}|V(x)| \le q^2$*, that* a *is bounded, and* $\langle\xi, a(x)\xi\rangle \ge \eta_0|\xi|^2$ *for all* $\xi \in \mathbb{R}^d$*. Then for every* $\varepsilon' > 0$ *there exist positive constants* $C_{\varepsilon'}$ *and* $R_{\varepsilon'}$ *such that*

$$u(x) \,\ge\, C_{\varepsilon'}\, \exp\biggl(-\biggl(\frac{q}{\sqrt{\eta_0}} + \frac{M}{\eta_0} + \varepsilon'\biggr)|x|\biggr)\,, \qquad |x| \ge R_{\varepsilon'}\,.$$

Proof. Let $K = \frac{q}{\sqrt{\eta_0}} + \frac{M}{\eta_0} + \varepsilon'$, $\alpha = 1$, and $\beta = 0$. Then the result follows from Theorem 4.1. ☐

Let us now discuss some important aspects of Theorem 4.1 and Corollary 4.1. When $a = I$, $b = 0$, and V is the potential function for the two body problem, a similar lower bound was obtained by Agmon [2]. In the context of Corollary 4.1, a lower bound was also obtained by Kenig, Silvestre and Wang [26, Theorem 1.5] for solutions which can be sign-changing; however it is assumed in [26] that $V \le 0$, a is the identity matrix, and $d = 2$. In contrast, Corollary 4.1 does not require these assumptions, but applies only to nonnegative solutions u. Note that the lower bound in [26, Theorem 1.5] is of the form $\mathrm{e}^{-CR(\log R)^2}$ in the radial direction R, whereas the lower bound in Corollary 4.1 is of the form e^{-CR}, and hence it is tighter. When $b = 0$, this bound is also sharper than the one conjectured by Kenig in [27, Question 1]. In fact, this bound is optimal in some sense. To see this, take $u(x) = e^{-|x|}$ in $\mathbb{R}^d$. Then $\Delta u + Vu = 0$ in $\{|x| > d\}$ where $V(x) = -1 + \frac{d-1}{|x|}$. Since we can take ε' arbitrarily small, the bound in Corollary 4.1 is very sharp.

We apply Corollary 4.1 to semi-linear or quasi-linear operators to find a lower bound on the decay of solutions.

Corollary 4.2. *Grant the hypotheses in Lemma 4.1.*

(a) *Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous function such that*

$$\limsup_{s\searrow 0}\ \frac{1}{s} f(s) < +\infty\,,$$

and $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ be a bounded, positive solution of $\mathcal{L}u = f(u)$. Then there exist constants $\gamma > 0$ and C_γ, depending on $\|u\|_\infty$, such that

$$u(x) \,\ge\, C_\gamma e^{-\gamma|x|} \quad \text{for all } |x| \text{ sufficiently large.} \tag{4.9}$$

(b) *Let $\mathbb{U}_1, \mathbb{U}_2$ be two compact metric spaces, and $V, b\colon \mathbb{R}^d \times \mathbb{U}_1 \times \mathbb{U}_2 \to \mathbb{R}^d$ be two continuous functions with $\|V\|_\infty < \infty$, and $b(\cdot, v_1, v_2)$ satisfying (A2) uniformly in $(v_1, v_2) \in \mathbb{U}_1 \times \mathbb{U}_2$. If $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ is a positive solution of*

$$\min_{v_1\in\mathbb{U}_1}\ \max_{v_2\in\mathbb{U}_2}\ \bigl[a^{ij}\partial_{ij}u + b^i(x, v_1, v_2)\partial_i u + V(x, v_1, v_2)u\bigr] \,=\, 0\,,$$

then it satisfies (4.9).

Proof. For part (a), note that since u is bounded, we have $f(u) \le Cu$ for some constant C (depending on $\|u\|_\infty$). Thus we obtain

$$\mathcal{L}u \,\le\, C\,u\,,$$

and the proof follows from Corollary 4.1. For part (b), observe that we can find measurable selectors $v_i^*\colon \mathbb{R}^d \to \mathbb{U}_i$ satisfying

$$a^{ij}\partial_{ij}u + b^i\bigl(x, v_1^*(x), v_2^*(x)\bigr)\partial_i u + V\bigl(x, v_1^*(x), v_2^*(x)\bigr)u \,=\, 0\,.$$

The rest follows as before using Corollary 4.1. □

In the rest of this section we discuss some connections of Theorem 4.1 and Corollary 4.1 with the Landis conjecture, and provide a partial answer to this conjecture. In 1960s, E. M. Landis conjectured (see [28]) that if u is a solution to $\Delta u + Vu = 0$, with $\|V\|_\infty \le q^2$, and there exist positive constants ε and C_ε such that $|u(x)| \le C_\varepsilon\, \mathrm{e}^{-(q+\varepsilon)|x|}$, then $u \equiv 0$. He also proposed a weaker version of this conjecture which states that if $|u(x)| \le C_k \mathrm{e}^{-k|x|}$ for any positive k, and some constant C_k, then $u \equiv 0$. This conjecture was disproved by Meshkov in [30] who constructed a non-zero solution to $\Delta u + Vu = 0$ which satisfies $|u(x)| \le C\mathrm{e}^{-c|x|^{4/3}}$ for some positive constants c and C. It is also shown in [30] that if for any $k > 0$, there exists a constant C_k satisfying $|u(x)| \le C_k \mathrm{e}^{-k|x|^{4/3}}$, then u is identically 0. The counterexample by Meshkov has V and u complex valued. Therefore the Landis conjecture remains open for real valued solutions and potentials. It is interesting to note that the Landis conjecture concerns the *unique continuation property* of u at infinity. In practice, Carleman type estimates are commonly used to treat such problems, but since a Carleman estimate does not distinguish between real and complex valued functions, it is hard to improve the results of Meshkov using such estimates. Landis' conjecture was recently revisited by Kenig, Silvestre and Wang

[26], and Davey, Kenig and Wang [18] for $V \le 0$ and $d = 2$. Note that if $V \le 0$ then Landis' conjecture follows from the strong maximum principle. The key contribution of [18, 26] is a lower bound on the decay of u. On the other hand, if we assume u to be nonnegative, then the Landis conjecture follows from Corollary 4.1. In Theorem 4.2 below we show that Landis' conjecture holds under the assumption that $\lambda^*(V) \le 0$. It should be observed that $\lambda^*(V) \le 0$ does not necessarily imply that $V \le 0$.

Theorem 4.2. *Let $\mathcal{L}u + Vu = 0$ and suppose that the following hold.*

(i) *a is bounded and uniformly elliptic with ellipticity constant η_0, $\|b\|_\infty \le M$, $\|V\|_\infty \le q^2$, and $\lambda^*(V) \le 0$.*
(ii) *For some positive constants ε and C_ε, we have*
$$|u(x)| \;\le\; C_\varepsilon \,\exp\biggl(-\Bigl(\frac{q}{\sqrt{\eta_0}} + \frac{M}{\eta_0} + \varepsilon\Bigr)|x|\biggr)\,, \quad \forall\, x \in \mathbb{R}^d\,.$$

Then $u \equiv 0$.

Proof. Let Ψ be a positive function in $\mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying
$$\mathcal{L}\Psi + V\Psi \;=\; 0\,. \tag{4.10}$$
Existence of such Ψ follows, for example, from [14, Theorem 1.4] and the fact that $\lambda^*(V) \le 0$. By $\tilde{\mathcal{L}}$ we denote the twisted process corresponding to the eigenpair $(\Psi, 0)$ i.e.,
$$\tilde{\mathcal{L}}f \;=\; \mathcal{L}f + 2\langle a\nabla\psi, \nabla f\rangle\,, \quad f \in \mathcal{C}^2(\mathbb{R}^d)\,,$$
where $\psi = \log\Psi$. Let $\Phi = \frac{u}{\Psi}$. Then it is easy to check from (4.10) that
$$\tilde{\mathcal{L}}\Phi = 0\,. \tag{4.11}$$
On the other hand, by Corollary 4.1 we have
$$\Psi(x) \;\ge\; C_{\varepsilon'}\,\exp\biggl(-\Bigl(\frac{q}{\sqrt{\eta_0}} + \frac{M}{\eta_0} + \varepsilon'\Bigr)|x|\biggr) \quad \forall\, |x| > R_{\varepsilon'}\,.$$
for $\varepsilon' < \varepsilon$, and constants $C_{\varepsilon'}$ and $R_{\varepsilon'}$. This of course, implies that $\Phi(x) \to 0$ as $|x| \to \infty$. Therefore, applying the strong maximum principle to (4.11), we deduce that $\Phi \equiv 0$, which in turn implies that $u \equiv 0$. This completes the proof. □

Remark 4.2. Recall that W denotes the standard Brownian motion. Let $V \in \mathcal{C}_0(\mathbb{R}^d)$ be such that
$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^\infty \frac{1}{2}V^+(W_s)\,\mathrm{d}s}\Bigr] \;<\; \infty\,, \quad \forall\, x \in \mathbb{R}^d\,. \tag{4.12}$$
It is then known that $v(x) \coloneqq \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^\infty \frac{1}{2}V^+(W_s)\,\mathrm{d}s}\Bigr]$ is a positive solution to
$$\Delta v + V^+ v \;=\; 0\,.$$
This of course implies that $\Delta v + Vv \le 0$, and therefore, $\lambda^*(V) \le 0$. For $d \ge 3$, if we have
$$\sup_{x\in\mathbb{R}^d} \frac{2}{(d-2)\omega_d}\int_{\mathbb{R}^d} \frac{V^+(y)}{|x-y|^{d-2}} \;<\; 1\,,$$
with ω_d denoting the surface measure of the unit sphere in $\mathbb{R}^d$, then V also satisfies (4.12) by Khasminskii's lemma [42, Lemma B.1.2].

Recall that, as shown in Remark 2.5, if $V \in \mathcal{C}_0(\mathbb{R}^d)$ and λ^* is not strictly monotone at V, then $\lambda^*(V) \le 0$. This allows us to apply Theorem 4.2, to conclude that Landis' conjecture holds if $\lambda^*(\mathcal{L}, V)$ is not strictly monotone at $V \in \mathcal{C}_0(\mathbb{R}^d)$. This of course applies to the general class of operators $\mathcal{L}$ satisfying (A1)–(A3).

Though we have not been able to prove the Landis conjecture in its full generality, we can validate this conjecture for a large class of potentials, including compactly supported potentials. This can be done with the help of the Radon transformation and a support theorem from Helgason [23]. See also [8] which uses a similar approach, albeit for homogeneous elliptic equations.

Theorem 4.3. *Suppose* $\Delta u + \langle b, \nabla u\rangle + Vu = 0$ *in* $\mathcal{B}^c$*, where* $\mathcal{B}$ *is a bounded ball, and the following hold.*

(i) $\|b\|_\infty \le M$, $\|V\|_\infty \le q^2$. *There exists a ball* B_r*,* $r > 0$*, with* $\mathcal{B} \subset B_r$*, such that* b *and* V *are constant in* B_r^c*.*
(ii) *For some positive* $\varepsilon, C_\varepsilon$*, we have*

$$|u(x)| \,\le\, C_\varepsilon \, \exp\Bigl(-\bigl(q + M + \varepsilon\bigr)|x|\Bigr)\,, \quad \forall\, x \in \mathcal{B}^c\,.$$

Then $u \equiv 0$*.*

Proof. Without loss of generality we may assume $0 \in \mathcal{B}$, and $b(x) = b_0$, $V(x) = k$ for all $x \in B_r^c$. Also by standard regularity theory of elliptic PDE we may assume that u is smooth in $\mathcal{B}_1^c$. Let $(\omega, p) \in S^d \times \mathbb{R}$ where S^d is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$. We note that any hyperplane $\mathbb{R}^d$ can be identified by (ω, p) up to the equality $(-\omega, -p) = (\omega, p)$. Let ξ be a hyperplane in $\mathbb{R}^d$, i.e., for some (ω, p) we have $\xi = \{x \in \mathbb{R}^d \colon \langle x, \omega\rangle = p\}$. The Radon transformation of u is defined as

$$\breve{u}(\xi) \,:=\, \int_\xi u(y)\, S(\mathrm{d}y)\,, \quad \text{where } S(\mathrm{d}y) \text{ is the surface measure on } \xi\,.$$

We claim that if the hyperplane does not intersect $\mathcal{B}_1$, then we have

$$\breve{u}(\xi) \,=\, 0\,. \tag{4.13}$$

If (4.13) is true, then since u decays exponentially fast to 0 at infinity by (ii), then the support theorem [23, Theorem 1.2.6 and Corollary 1.2.8] implies that $u \equiv 0$ in B_r^c. This in turn, implies that $u \equiv 0$ by the unique continuation property of Hörmander [24, Theorem 2.4].

In order to complete the proof we need to prove (4.13). Note that if we define $v(x) = u(\mathcal{M}x)$, where $\mathcal{M}$ is any rotation matrix, then

$$\Delta v + \langle \mathcal{M}^{\mathsf{T}} b_0, \nabla v\rangle + kv \,=\, 0\,, \quad x \in \mathcal{B}_1^c\,.$$

Also a rotation does not change the norm of b. Therefore, without loss of generality, we may assume that $\xi = \xi(\kappa_0) := \{x \in \mathbb{R}^d \,:\, x_1 = \kappa_0,\, \kappa_0 > 0\}$. Define for $s \ge \kappa_0$,

$$w(s) \,:=\, \int_{\xi(s)} u(y)\, S(\mathrm{d}y) = \int_{\mathbb{R}^{d-1}} u(s, \bar{x})\, \mathrm{d}\bar{x}, \quad \text{where } \bar{x} = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}\,.$$

Note that w is smooth in $[\kappa_0, \infty)$ due to the smoothness of u and its decay at infinity. Moreover,

$$\frac{\mathrm{d}^2 w(s)}{\mathrm{d}s^2} \,=\, \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}^2 u}{\mathrm{d}s^2}(s, \bar{x})\, \mathrm{d}\bar{x} \tag{4.14}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^{d-1}} \sum_{i=2}^{d} \partial_{ii} u(s,\bar{x})\,\mathrm{d}\bar{x} - \int_{\mathbb{R}^{d-1}} \sum_{i=1}^{d} b_0^i \partial_i u(s,\bar{x})\,\mathrm{d}\bar{x} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - k\int_{\mathbb{R}^{d-1}} u(s,\bar{x})\,\mathrm{d}\bar{x} \\
&= -\int_{\mathbb{R}^{d-1}} b_0^1 \partial_1 u(s,\bar{x})\,\mathrm{d}\bar{x} - k\int_{\mathbb{R}^{d-1}} u(s,\bar{x})\,\mathrm{d}\bar{x} \\
&= -b_0^1 \frac{\mathrm{d}w(s)}{\mathrm{d}s} - kw(s)\,,
\end{aligned}
$$

where in the second equality we use the equation satisfied by u, and in the third equality we use the fundamental theorem of calculus. Thus we obtain from (4.14) a second-order ODE with constant coefficients, given by

$$
\frac{\mathrm{d}^2 w(s)}{\mathrm{d}s^2} + b_0^1 \frac{\mathrm{d}w(s)}{\mathrm{d}s} + kw(s) \;=\; 0 \quad \text{in } [\kappa_0,\infty)\,. \tag{4.15}
$$

We solve this ODE explicitly, and using the decay property of u in (ii) we show that $w(s) = 0$ in $[\kappa_0,\infty)$. In particular, $w(\kappa_0) = 0$ which proves (4.13). Denote by $\kappa_1 = M + q$. We first show that for $\varepsilon' < \varepsilon$ there exists a positive constant $C_{\varepsilon'}$ such that

$$
|w(s)| \;\le\; C_{\varepsilon'}\,\mathrm{e}^{-(\kappa_1+\varepsilon')s}\,, \quad \text{for } s \in [\kappa_0,\infty)\,. \tag{4.16}
$$

By (ii) and a choice of s_0, satisfying $\sqrt{s_0}(s_0-1) > 2$, we get

$$
\begin{aligned}
|w(s)| \;&\le\; \int_{\mathbb{R}^{d-1}} |u(s,\bar{x})|\,\mathrm{d}\bar{x} \\
&\le\; C_\varepsilon \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-\kappa_\varepsilon |x|}\,\mathrm{d}\bar{x} \quad [\text{for } \kappa_\varepsilon = \kappa_1 + \varepsilon] \\
&=\; C_\varepsilon s^{d-1} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-\kappa_\varepsilon s\sqrt{1+|\bar{x}|^2}}\,\mathrm{d}\bar{x} \\
&=\; \kappa_2 s^{d-1} \int_0^\infty \mathrm{e}^{-\kappa_\varepsilon s\sqrt{1+r^2}}\, r^{d-2}\mathrm{d}r \\
&=\; \kappa_2 s^{d-1} \left[\int_0^{s_0} \mathrm{e}^{-\kappa_\varepsilon s\sqrt{1+r^2}}\, r^{d-2}\mathrm{d}r + \int_{s_0}^\infty \mathrm{e}^{-\kappa_\varepsilon s\sqrt{1+r^2}}\, r^{d-2}\mathrm{d}r\right] \\
&\le\; \kappa_2 s^{d-1} \left[s_0^{d-1}\mathrm{e}^{-\kappa_\varepsilon s} + \int_{s_0}^\infty \mathrm{e}^{-\kappa_\varepsilon s(1+\sqrt{r})}\, r^{d-2}\mathrm{d}r\right] \\
&\le\; \kappa_3 s^{d-1}\mathrm{e}^{-\kappa_\varepsilon s} \left[1 + \int_{s_0}^\infty \mathrm{e}^{-\kappa_\varepsilon \kappa_0\sqrt{r}}\, r^{d-2}\mathrm{d}r\right],
\end{aligned}
$$

where in the third inequality we have used the fact that $\sqrt{1+r^2} > 1+\sqrt{r}$ for $r \ge s_0$. Equation (4.16) easily follows from the above estimate. To solve (4.15) we find the roots of the characteristic polynomial of the ODE which are given by

$$
r_1 = -\frac{1}{2}b_0^1 + \frac{1}{2}\sqrt{(b_0^1)^2 - 4k}\,, \quad \text{and} \quad r_2 = -\frac{1}{2}b_0^1 - \frac{1}{2}\sqrt{(b_0^1)^2 - 4k}\,.
$$

The solution of (4.15) can be written as

$$w(s) \,=\, c_1 \mathrm{e}^{r_1 s} + c_2 \mathrm{e}^{r_2 s}\,,$$

where the constants c_1 and c_2 are uniquely determined. Now if $(b_0^1)^2 - 4k < 0$, then the roots are complex and the decay of w is of order $\mathrm{e}^{-b_0^1 s}$ which is larger than the RHS of (4.16). Therefore, we must have $c_1 = c_2 = 0$. On the other hand, if $(b_0^1)^2 - 4k \geq 0$, and since $\bigl|-\frac{1}{2}b_0^1 \pm \frac{1}{2}\sqrt{(b_0^1)^2 - 4k}\bigr| \leq \kappa_1$, we conclude that $c_1 = c_2 = 0$. Hence we must have $w(s) = 0$ in $[\kappa_0, \infty)$. This completes the proof. □

As a concluding remark, we show that the Landis conjecture is true for solutions that satisfy a reverse Poincaré inequality. Specifically, consider an operator $\mathcal{L} + V$ with bounded coefficients and a the identity matrix. We say a solution u to $\mathcal{L}u + Vu = 0$ satisfies (G) if the following holds:

(G) There exist positive constants r and C, independent of x, such that

$$\int_{B_r(x)} |\nabla u(y)|^2 \,\mathrm{d}y \,\leq\, C \int_{B_r(x)} |u(y)|^2 \,\mathrm{d}y, \quad \forall\, |x| \gg 1\,.$$

Remark 4.3. Note that if $\int_{B_r(x)} |\nabla u(y)|^2 \,\mathrm{d}y = 0$ for some $x \in \mathbb{R}^d$, then it is shown by Hörmander [24, Theorem 2.4] that $u \equiv 0$.

The following (weaker) Landis' conjecture is true for solutions satisfying (G). Suppose that $\Delta u + \langle b, \nabla u\rangle + Vu = 0$ in $\mathcal{B}^c$, with $\mathcal{B}$ a bounded ball, $u \in \mathcal{C}^2(\mathcal{B}^c)$, and the following hold.

- $\|b\|_\infty \leq M$, $\|V\|_\infty \leq q^2$, and u satisfies (G).

Then for $\kappa = \bigl[2(M\sqrt{C} + q^2 + C)\bigr]^{1/2} + 1$, we have

$$\int_{B_r(x)} u^2(y)\,\mathrm{d}y \;\geq\; C_\kappa \exp(-\kappa\,|x|) \quad \text{for all } |x| \gg 1\,. \tag{4.17}$$

In particular, if for every $k \in \mathbb{N}$, we have $|u(x)| \leq C_k \mathrm{e}^{-k|x|}$ for some constant C_k, then $u \equiv 0$.

In order to prove this claim, we define

$$v(x) \,\coloneqq\, \int_{B_r(x)} u^2(y)\,\mathrm{d}y\,.$$

Since $u \in \mathcal{C}^2(\mathcal{B}^c)$, we have $v \in \mathcal{C}^2(\mathcal{B}_1^c)$ for some large ball $\mathcal{B}_1 \Supset \mathcal{B}$. A straightforward calculation shows that

$$\begin{aligned}
\Delta v(x) \,&=\, 2\int_{B_r(x)} u\Delta u\,\mathrm{d}y + 2\int_{B_r(x)} |\nabla u|^2\,\mathrm{d}y \\
&=\, -2\int_{B_r(x)} u\langle b, \nabla u\rangle\,\mathrm{d}y - 2\int_{B_r(x)} Vu^2\,\mathrm{d}y + 2\int_{B_r(x)} |\nabla u|^2\,\mathrm{d}y \\
&\leq\, 2M\biggl(\int_{B_r(x)} |u|^2\,\mathrm{d}y\biggr)^{1/2}\biggl(\int_{B_r(x)} |\nabla u|^2\,\mathrm{d}y\biggr)^{1/2} + 2q^2\int_{B_r(x)} u^2\,\mathrm{d}y \\
&\qquad + 2\int_{B_r(x)} |\nabla u|^2\,\mathrm{d}y \\
&\leq\, (2M\sqrt{C} + 2q^2 + 2C)v(x)\,.
\end{aligned}$$

Therefore (4.17) follows from Corollary 4.1.

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Department of ECE, The University of Texas at Austin, 2501 Speedway, EER 7.824, Austin, TX 78712, USA
Email address: ari@ece.utexas.edu

Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pune 411008, India
Email address: anup@iiserpune.ac.in

Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pashan, Pune 411008, India
Email address: debdip@iiserpune.ac.in