

A CR Embedding Problem for an Algebraic Levi Non-degenerate Hypersurface into a Hyperquadric



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Abstract We give a survey on the current developments of the embeddability problem of a Levi non-degenerate hypersurface into its model, i.e., hyperquadrics. We also formulate and study a local sums-of-squares problem, and make connections with the embeddability problem.

Keywords Hyperquadrics · Spheres · CR embedding

1 Introduction

Let M be a smooth real hypersurface. We say that M is a Levi non-degenerate hypersurface in $\mathbb{C}^{n+1}$ at $p \in M$ with signature $\ell \leq n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|_\ell^2 + o(|z|^2 + |zu|) = 0 \quad (1)$$

Dedicated to Kang-Tae Kim on the occasion of his 60th birthday

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Here, we write $u = \operatorname{Re}(w)$, $v = \operatorname{Im}(w)$ and $|z|_\ell^2 = \langle z, \bar{z} \rangle_\ell = -\sum_{j \leq \ell} |z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2$. When $\ell = 0$, we regard $\sum_{j \leq \ell} |z_j|^2 = 0$ and p is a strongly pseudoconvex point of M .

The compact model of Levi non-degenerate hypersurfaces with signature ℓ is the boundary of the generalized ℓ -ball:

$$\mathbb{B}_\ell^{n+1} := \left\{ [z_0, \dots, z_{n+1}] \in \mathbb{P}^{n+1} : -\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 < 0 \right\}.$$

The boundary $\partial \mathbb{B}_\ell^{n+1}$ is locally holomorphically equivalent to the hyperquadric $\mathbb{H}_\ell^{n+1} \subset \mathbb{C}^{n+1}$ of signature ℓ defined by $\Im z_{n+1} = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2$, where $(z_1, \dots, z_{n+1})$ is the coordinates of $\mathbb{C}^{n+1}$.

When $\ell = 0$, it reduces to the standard sphere (compact model) or the Heisenberg hypersurface (non-compact model).

A long-standing question in Complex Analysis is to understand when a (compact) Levi non-degenerate hypersurface of signature $\ell \geq 0$ can be holomorphically embedded into its model, namely, the generalized ball with the same signature. Along these lines, there have been much work done in the past 40 years and we refer to a recent paper of Huang-Xiao [11] for a detailed discussion on history and references. In this article, we summarize some of the work done in Huang-Zaitsev [13], Huang-Li-Xiao [10] and Huang-Xiao [11] and also connect this study with others such as writing a positive real algebraic functions as the sum of norm square of holomorphic functions.

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We start by mentioning a famous theorem of Fornæss obtained in 1976, which states as follows:

Theorem (Fornaess [6]) *Any compact strongly pseudoconvex hypersurface M can be smoothly embedded into a certain compact strongly convex hypersurface of a larger dimension.*

It is not clear from any known proof of the Fornæss embedding Theorem whether, when M is real analytic (or real algebraic), we can embed M into a compact real analytic (or real algebraic) strongly convex hypersurface of larger dimension or not. It indicates, however, that some type of embedding theorem into manifolds of a more special class is possible.

On the other hand, Forstneric [7] proved in 1986 (See also [5]) the following theorem:

Theorem (Forstneric [7]) *A generic real analytic strongly pseudoconvex hypersurface can not be locally (transversally) holomorphically embedded into a hyperquadric (or a generalized sphere) of any dimension.*

A more recent result by Forstneric [8] states even more: Most real-analytic hypersurfaces do not admit a transversal holomorphic embedding even into a merely algebraic hypersurface of higher dimension. Forstneric's argument is based on a Baire category theory, which roughly says that the set of real analytic strongly pseudoconvex hypersurfaces is much larger than the set of holomorphic maps sending them into spheres. It cannot, however, be used to produce any specific examples. In 2006, Zaistev [21] constructed explicitly for the first time a germ of a real analytic strongly pseudoconvex hypersurface which can not be holomorphically embedded into a sphere of any dimension.

The situation in the algebraic category is very different. Webster proved in 1970's the following well-known theorem, which is in sharp contrast with the above mentioned result of Forstneric.

Theorem (Webster [19]) *Every real-algebraic Levi-nondegenerate hypersurface $M \subset \mathbb{C}^n$ admits a transversal holomorphic embedding into a non-degenerate hyperquadric (likely with a much bigger signature) in complex space $\mathbb{C}^N$ of sufficiently large dimension N .*

In Webster's theorem, the embedding dimension N depends on both the source dimension n and the degree of the defining function of M . Indeed, in a recent paper of Kossovskiy-Xiao [15], the following non-embeddability result has been established: For any integers $N > n \geq 1$, there exists $\mu = \mu(n, N)$ such that a Zariski generic real-algebraic hypersurface $M \subset \mathbb{C}^{n+1}$ of any degree $k \geq \mu$ is not transversally holomorphically embeddable into a hyperquadric in $\mathbb{C}^N$.

Moreover, in Webster's theorem, it is crucial that the signature of the target hyperquadric is allowed to be bigger than that of the source manifold. Webster observed in the degree two case, one can always embed a real ellipsoid into the sphere in a complex space of one dimension higher.

Example of Webster [18–20]: Let $M \subset \mathbb{C}^{n+1}$ be a non-spherical ellipsoid, i.e., defined in real coordinates $z_j = x_j + iy_j$, $j = 1, \dots, n+1$, by $\sum_{j=1}^{n+1} (\frac{x_j^2}{a_j^2} + \frac{y_j^2}{b_j^2}) = 1$. After a simple scaling (if necessary), the ellipsoid M can be described by the equation of the form:

$$\sum_{j=1}^{n+1} (A_j z_j^2 + A_j \overline{z_j^2}) + \sum_{j=1}^{n+1} |z_j|^2 = 1, \quad 0 \leq A_1 \leq \dots \leq A_{n+1} < 1.$$

Let $G : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+2}$ be given by $G = \left(z_1, \dots, z_{n+1}, i(1 - 2 \sum_{j=1}^{n+1} A_j z_j^2) \right)$. Then it holomorphically embeds M into the Heisenberg hypersurface $\mathbb{H}_0^{n+2} \subset \mathbb{C}^{n+2}$.

With all these said, the following had been remained to be a long-standing folklore open question in the field:

Question *Is every compact real-algebraic strongly pseudoconvex real hypersurface in $\mathbb{C}^n$ holomorphically embeddable into a sphere of sufficiently large dimension?*

The question is closely related to write a positive polynomial in a certain sense to a sum of squares of the absolute values of holomorphic functions. (See the end of Sect. 2.)

2 Algebraic Perturbation of Kohn-Nirenberg Hypersurfaces

The above open problem has been answered in the negative through the work in Huang-Zaitsev [13], Huang-Li-Xiao [10] (See also [4, 14], etc) and finally in Huang-Xiao [11]. Indeed, we constructed a very concrete example based on small perturbations of the famous Kohn-Nirenberg hypersurfaces (see [16]), which we describe below.

Consider the following family of compact real-algebraic strongly pseudoconvex real hypersurfaces:

$$M_\epsilon = \{(z, w) \in \mathbb{C}^2 : \epsilon_0(|z|^8 + c\operatorname{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1 = 0\}. \quad (2)$$

Here, $0 < \epsilon < 1$, $2 < c < \frac{16}{7}$, and $\epsilon_0 > 0$ is a sufficiently small number such that M_ϵ is smooth for all $0 \leq \epsilon < 1$.

One can compute to verify that, for any $0 < \epsilon < 1$, M_ϵ is strongly pseudoconvex. M_ϵ is a small algebraic deformation of the famous Kohn-Nirenberg domain:

The Kohn-Nirenberg Domain:

$$M_{KN} = \{(z, w) \in \mathbb{C}^2 : \epsilon_0(|z|^8 + c\operatorname{Re}|z|^2 z^6) + |w|^2 + |z|^{10} - 1 = 0\},$$

with $2 < c < \frac{16}{7}$.

It is a compact real algebraic pseudo-convex hypersurface with exactly one weakly pseudo-convex point at $p^* = (z, w) = (0, 1)$. The significant nature of M_{KN} is that any local complex analytic curve in a neighborhood of p^* must stay in both sides of M_{KN} near p^* .

The theorem we proved in [11], based on the work in [10, 13], is as follows:

Main Theorem (Huang-Xiao [11]) *Let M_ϵ be defined as above with $\epsilon, \epsilon_0 > 0$ sufficiently small. For any $N \in \mathbb{N}$, a smooth CR map from an open piece of M_ϵ into the standard sphere $S^{2N-1} = \partial\mathbb{B}^N \subset \mathbb{C}^N$ must be a constant. Hence, there is no smooth CR embedding from any open piece of M_ϵ into S^{2N-1} .*

The theorem answers, in the negative, the above mentioned question.

As observed in [10], M_ϵ can be holomorphically transversally embedded into the generalized sphere in $\mathbb{C}^6$ with one negative Levi eigenvalue as follows:

Observe that $\operatorname{Re}(|z|^2 z^6) = \frac{1}{4}(|z^7 + z|^2 - |z^7 - z|^2)$. Thus the map

$$F(z, w) = (\sqrt{\varepsilon_0}z^4, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7 + z), w, z^5, \sqrt{\varepsilon}z, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7 - z))$$

holomorphically embeds M_ϵ into the generalized sphere in $\mathbb{C}^6$ defined by $\mathbb{S}^{11} = \{(Z_1, \dots, Z_6) \in \mathbb{C}^6 : \sum_{j=1}^5 |Z_j|^2 - |Z_6|^2 = 1\}$. Indeed, M_ϵ can be holomorphically embedded into $\mathbb{S}^9 = \{(Z_1, \dots, Z_5) \in \mathbb{C}^5 : \sum_{j=1}^4 |Z_j|^2 - |Z_5|^2 = 1\}$. See Remark 2.4.

To end this section, we make connections of the embeddability problem with a Hermitian version of Hilbert's 17th problem. The problem we introduce here, which is more of local nature, differs from the one studied in [2, 3], etc.

Let $p \in \mathbb{C}^n$ and U a simply connected open set containing p . Let $r(z, \bar{z})$ be a real analytic (real-valued) function in $z \in U$. Then there exist holomorphic functions $f_1, \dots, f_r, g_1, \dots, g_s$ and h defined in U with each $f_i(0) = 0$ and $g_j(0) = 0$ such that the decomposition holds:

$$r(z, \bar{z}) = \text{Im}(h) + \sum_{i=1}^s |f_i|^2 - \sum_{j=1}^t |g_j|^2. \quad (3)$$

Moreover, $f_1, \dots, f_s, g_1, \dots, g_t$ are $\mathbb{C}$ -linearly independent. See [17] for more details and also [2] which emphasizes more on the real polynomial setting.

In general, s, t may be infinite. When s, t are finite, then we say $r(z, \bar{z})$ has finite rank and signature (s, t) at p . Moreover, in the decomposition (3), h is uniquely determined up to a constant term. And $(f_1, \dots, f_s, g_1, \dots, g_t)$ is also uniquely determined up to a complex linear transformation in $U(s, t)$. Thus, the pair (s, t) is uniquely determined at p and does not depend on the choice of the decomposition in (3). See [17] for more details and the proof.

Lemma 2.1 *Let U be simply connected and $r(z, \bar{z})$ be finite rank as above. Let $p, q \in U$. Then the signatures of $r(z, \bar{z})$ at p and at q are identical. In particular, when $r(z, \bar{z})$ is a real polynomial, then it has the same signature at every point in $\mathbb{C}^n$.*

Proof This fact is contained in [17]. For self-containedness, we will sketch a proof based on the setup above. Assume the signature of $r(z, \bar{z})$ is (s, t) at p . Then there exist holomorphic functions $f_1, \dots, f_s, g_1, \dots, g_t$ and h such that (3) holds, where all $f_i(0) = 0$ and $g_j(0) = 0$. Moreover, $f_1, \dots, f_s, g_1, \dots, g_t$ are $\mathbb{C}$ -linearly independent. Note for a holomorphic function f in U , we have

$$\begin{aligned} |f(z)|^2 &= |f(z) - f(q) + f(q)|^2 = \\ &= |f(z) - f(q)|^2 + \text{Im} \left(2\sqrt{-1} f(z) \overline{f(q)} - \sqrt{-1} |f(q)|^2 \right). \end{aligned}$$

We apply this equation to each $|f_i(z)|^2$ and $|g_j(z)|^2$ and then (3) can be rewritten as,

$$r(z, \bar{z}) = \text{Im}(\phi(z)) + \sum_{i=1}^s |f_i(z) - f_i(q)|^2 - \sum_{j=1}^t |g_j(z) - g_j(q)|^2$$

for some holomorphic function $\phi(z)$ in U . Note $f_i(z) - f_i(q)$ and $g_j(z) - g_j(q)$ vanish at q . Moreover, the linear independence of $\{f_1, \dots, f_s, g_1, \dots, g_t\}$ is equivalent to that of $\{f_1(z) - f_1(q), \dots, f_s(z) - f_s(q), g_1(z) - g_1(q), \dots, g_t(z) - g_t(q)\}$. By the definition of signature, $r(z, \bar{z})$ is also of signature (s, t) at q and we thus have proved the first assertion in the lemma. The second part of the lemma also follows easily. $\square$

Let $\Lambda(U)$ be the set of real analytic functions of finite rank in U . Note $\Lambda(U)$ forms a ring. We pass this notion to the germ level at p . Let $R(z, \bar{z})$ be a germ of real analytic function at p . We say $R(z, \bar{z})$ is of finite rank at p if there is a neighborhood V of p such that $R(z, \bar{z}) \in \Lambda(V)$. Note the signature (s, t) of $R(z, \bar{z})$ does not depend on the choice of V . We will simply say $R(z, \bar{z})$ is finite rank and of signature (s, t) at p . Write $\Lambda(p)$ for the set of germs of real-analytic functions with finite rank at p . Note $\Lambda(p)$ is also a ring.

Recall in a simply connected domain, the signature of $R(z, \bar{z})$ does not depend on the choice of the base point. Assume $R(z, \bar{z})$ is real analytic over a connected domain U , and $p, q \in U$, $p \neq q$. We can choose a path $\gamma \in U$ connecting p, q and a small tube neighborhood V of γ that is simply connected. Then the signature of $R(z, \bar{z})$ is identical at p, q . This yields the following remark (one can also see [17]).

Remark 2.2 Let U be a connected open set in $\mathbb{C}^n$ and $R(z, \bar{z})$ real analytic in U with finite rank at some $p \in U$. Then $R(z, \bar{z})$ has finite rank and the same signature at every point in U .

Remark 2.3 Assume $r(z, \bar{z}) = 0$ defines a real hypersurface M in U . If $r(z, \bar{z})$ is of finite rank (s, t) with the decomposition in (3), then M admits a transversal holomorphic mapping into $\mathbb{H}_t^{s+t+1}$ given by $\Phi := (g_1, \dots, g_s, f_1, \dots, f_t, -h)$. Recall in Webster's ellipsoid example, the defining function of an ellipsoid is given by

$$\operatorname{Im}(2\sqrt{-1}A_j z_j^2 - \sqrt{-1}) + \sum_{j=1}^{n+1} |z_j|^2 = 0.$$

Clearly it is of signature $(n+1, 0)$.

Remark 2.4 Let $0 < \epsilon_0 \ll \epsilon \ll 1$. Let $\rho = \epsilon_0(|z|^8 + c\operatorname{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1$. Recall it is the defining function of the non-embeddable hypersurfaces in our main theorem. Note

$$\epsilon_0 c \operatorname{Re}(|z|^2 z^6) + \epsilon|z|^2 = -\frac{\epsilon_0^2 c^2}{4(\epsilon + \epsilon_0 c)} |z^7 - z|^2 + \frac{1}{\epsilon + \epsilon_0 c} \left| \frac{\epsilon_0 c}{2} (z^7 + z) + \epsilon z \right|^2.$$

We have ρ is of rank $(4, 1)$. Note also ρ is strongly plurisubharmonic.

When $R(z, \bar{z})$ is of signature $(s, 0)$ at p with $s > 0$ and finite, we say $R(z, \bar{z})$ is a (finite) sum of squares (modulo pure terms) at p . We will write $\mathcal{P}(p) \subset \Lambda(p)$ for

the set of (germs of) nonzero functions that are (finite) sums of squares (modulo pure terms) at p .

The following fact is a consequence of Remark 2.2.

Remark 2.5 Let U be a connected open set in $\mathbb{C}^n$ and $R(z, \bar{z})$ real analytic in U . Then $R(z, \bar{z}) \in \mathcal{P}(p)$ for some $p \in U$ if and only if $R(z, \bar{z}) \in \mathcal{P}(q)$ for every $q \in U$.

Let $r(z, \bar{z})$ be a germ of real analytic function at p with finite rank. Let $\mathcal{I}(r, p)$ be the ideal in $\Lambda(p)$ generated by $r(z, \bar{z})$. Assume $r(z, \bar{z})$ is strongly plurisubharmonic and $r(p, \bar{p}) = 0$. Then the local non-embeddability problem into spheres can be reduced to a real-algebraic geometry problem as follows.

Question Characterize the class of (strongly plurisubharmonic) real analytic functions $r(z, \bar{z})$ at p such that $\mathcal{I}(r, p) \cap \mathcal{P}(p) = \emptyset$.

Our main theorem yields the following result.

Corollary 2.6 Let ρ be in Remark 2.4 and $\rho(p, \bar{p}) = 0$. Then $\mathcal{I}(\rho, p) \cap \mathcal{P}(p) = \emptyset$.

Proof We will prove by contradiction. Suppose $\mathcal{I}(\rho, p) \cap \mathcal{P}(p) \neq \emptyset$. Then there exist some neighborhood V of p and a real analytic function $H(z, \bar{z})$ defined in V such that $H(z, \bar{z})\rho(z, \bar{z})$ is a sum of squares (modulo pure terms). That is, there exists $\mathbb{C}$ -linearly independent holomorphic functions $f_1, \dots, f_s$ and h such that

$$H(z, \bar{z})\rho(z, \bar{z}) = \operatorname{Im}(h) + \sum_{i=1}^s |f_i|^2.$$

Consequently, $\Phi := (f_1, \dots, f_s, -h)$ gives a map in V that sends the hypersurface $M := \{z \in V : \rho(z, \bar{z}) = 0\}$ into the Heisenberg surface $\mathbb{H}_0^{s+1}$, which is locally biholomorphic to the unit sphere in $\mathbb{C}^{s+1}$. By our main theorem, Φ is a constant map. Thus $H(z, \bar{z})\rho(z, \bar{z})$ is constant in V . Since $\rho(p, \bar{p}) = 0$, we have $H(z, \bar{z})\rho(z, \bar{z}) \equiv 0$. This is a contradiction. Thus $\mathcal{I}(\rho, p) \cap \mathcal{P}(p) = \emptyset$. This establishes the corollary. $\square$

The following signature stability result follows easily from Corollary 2.6.

Corollary 2.7 (1) Let U be a connected open set in $\mathbb{C}^2$ with nonempty intersection with M_ϵ . Let $H(z, \bar{z})$ be a non-zero real analytic function over U . Assume the signature of $H(z, \bar{z})\rho(z, \bar{z})$ is (s, t) with s, t integers, then $s > 0, t > 0$ (at every point in U).

(2) Let $H(z, \bar{z})$ be a non-zero real polynomial over $\mathbb{C}^2$. Then the signature (s, t) of $H(z, \bar{z})\rho(z, \bar{z})$ satisfies $s > 0, t > 0$ (at every point in $\mathbb{C}^2$).

Proof Suppose $t = 0$. Then $H(z, \bar{z})\rho(z, \bar{z})$ is a sum of squares (modulo pure terms) at every point in U (See Remark 2.5). In particular, $H(z, \bar{z})\rho(z, \bar{z}) \in \mathcal{P}(p)$ for $p \in U \cap M_\epsilon$. It contradicts with the non-embeddability of any open piece of M_ϵ into the sphere (See the proof of Corollary 2.6). Thus we must have $t > 0$. We can similarly consider the signature of $-H(z, \bar{z})\rho(z, \bar{z})$ to conclude $s > 0$. This establishes part (1). Part (2) follows easily from part (1). $\square$

Remark 2.8 The assumption that $U \cap M_\epsilon \neq \emptyset$ in Corollary 2.7 cannot be dropped. Indeed, when $U \cap M_\epsilon = \emptyset$, we can take $H(z, \bar{z}) = \frac{\|z\|^2}{\rho(z, \bar{z})}$, then the conclusion in Corollary 2.7 fails.

The above discussion motivates the following question: Let $r(z, \bar{z})$ be a strongly plurisubharmonic real polynomial in $\mathbb{C}^n (n \geq 2)$, and assume $\{r(z, \bar{z}) = 0\}$ defines a real hypersurface passing through $p \in \mathbb{C}^n$. Assume $r(z, \bar{z})$ has signature (s, t) with $s > 0, t > 0$. Is it true that $\mathcal{I}(r, p) \cap \mathcal{P}(p) = \emptyset$?

Our result also has applications to the study of the holomorphic isometric embedding problems. Let U be a connected open subset in $\mathbb{C}^2$ and ω a $(1,1)$ -form defined in $V \subset U$ as follows:

$$\omega = \sqrt{-1} \partial \bar{\partial} \log(1 + R(z, \bar{z})), \quad z \in V, \quad (4)$$

where $R(z, \bar{z})$ is a (non-constant) real analytic function in U .

Corollary 2.9 *Let ω be as above which takes the form in (4). Let $(\mathbb{CP}^N, \omega_{FS})$ be the complex projective space equipped with the Fubini-Study metric. Assume there is a holomorphic map $F : (V, \omega) \rightarrow (\mathbb{CP}^N, \omega_{FS})$ satisfying*

$$F^*(\omega_{FS}) = \omega.$$

Then $R(z, \bar{z}) \in \mathcal{P}(q)$ for every $q \in U$.

Proof We will first prove $1 + R(z, \bar{z}) \in \mathcal{P}(p)$ for some $p \in V$. Write in the homogeneous coordinates $F = [F_0, \dots, F_N]$. Fix $p \in V$. Composing F with a self-isometry of $\mathbb{CP}^N$ if necessary, we can assume $F(p) = [1, 0, \dots, 0]$. Then by shrinking V if necessary, in the affine coordinates of $\mathbb{CP}^N$, we can write F as $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N)$, where $\hat{F}_i = \frac{F_i}{F_0}$. In particular, $\hat{F}_i(p) = 0$. By the metric-preserving assumption, we have

$$\partial \bar{\partial} \log(1 + R(z, \bar{z})) = \partial \bar{\partial} \log(1 + \|\hat{F}\|^2). \quad (5)$$

Assume in V that

$$1 + R(z, \bar{z}) = c(1 + \operatorname{Re}(2h) + \|f\|^2 - \|g\|^2). \quad (6)$$

Here c is a positive constant, $f = (f_1, \dots, f_s)$, $g = (g_1, \dots, g_t)$, and h are holomorphic maps satisfying $f(0) = 0, g(0) = 0, h(0) = 0$. Moreover, $f_1, \dots, f_s, g_1, \dots, g_t$ are $\mathbb{C}$ -linearly independent. Then (5) yields

$$\partial \bar{\partial} \log(1 + \operatorname{Re}(2h) + \|f\|^2 - \|g\|^2) = \partial \bar{\partial} \log(1 + \|\hat{F}\|^2) \quad (7)$$

By shrinking V if necessary, we assume $|h(z)| < 1$ for every $z \in V$. We will prove by contradiction that in the signature (s, t) we must have $t = 0$. Suppose $t > 0$.

We next write

$$1 + \operatorname{Re}(2h) + \|f\|^2 - \|g\|^2 = (1 + h(z))(1 + \overline{h(z)})(1 + T(z, \bar{z}))$$

for some real analytic function $T(z, \bar{z})$ in V . Then it is easy to see that $T(z, \bar{z})$ has only mixed terms (in $z - p, \bar{z} - \bar{p}$) in its Taylor expansion at $(p, \bar{p})$.

Note $\partial\bar{\partial}\log(1 + h(z))(1 + \overline{h(z)}) = 0$. It follows from (7) that

$$\partial\bar{\partial}\log(1 + T(z, \bar{z})) = \partial\bar{\partial}\log(1 + \|\hat{F}\|^2).$$

Since both $T(z, \bar{z})$ and $\|\hat{F}\|^2$ have only mixed terms in their Taylor expansions. It follows that

$$1 + T(z, \bar{z}) = 1 + \|\hat{F}\|^2.$$

That is,

$$\frac{1 + \operatorname{Re}(2h) + \|f\|^2 - \|g\|^2}{|1 + h|^2} = 1 + \|\hat{F}\|^2.$$

This yields

$$\left\|\frac{f}{1+h}\right\|^2 - \left\|\frac{g}{1+h}\right\|^2 = \left|\frac{h}{1+h}\right|^2 + \|\hat{F}\|^2.$$

Note the functions $\{\frac{f_i}{1+h}, \frac{g_j}{1+h}\}_{1 \leq i \leq s, 1 \leq j \leq t}$ are still $\mathbb{C}$ -linearly independent. The left hand side of the above equation thus has signature (s, t) with $t > 0$, while the right hand side is a sum of squares. This is a contradiction. Hence $t = 0$. So $1 + R(z, \bar{z}) \in \mathcal{P}(p)$ and thus $R(z, \bar{z}) \in \mathcal{P}(p)$. By Remark 2.5, we have $R(z, \bar{z}) \in \mathcal{P}(q)$ for every $q \in U$. $\square$

Let ρ be as in Remark 2.4. Assume ω be a Kähler metric defined in $V \subset \mathbb{C}^2$ of the form:

$$\omega = \sqrt{-1}\partial\bar{\partial}\log(1 + H(z, \bar{z})\rho(z, \bar{z})), \quad (8)$$

where $H(z, \bar{z})$ is a (non-zero) global real analytic function (of finite rank) in $\mathbb{C}^2$. See the following example.

Example 2.10 Let $H(z, \bar{z}) \equiv \frac{1}{2}$, then

$$\omega = \sqrt{-1}\partial\bar{\partial}\log(2 + \rho(z, \bar{z}))$$

defines an algebraic $(1, 1)$ -form in $\mathbb{C}^2$ and in particular defines a (positive definite) metric in a sufficiently small neighborhood V of 0. Similarly, let $H(z, \bar{z}) \equiv \frac{1}{2} + P(z, \bar{z})$, where $P(z, \bar{z})$ is a homogeneous real polynomial of degree at least 3, then ω in (8) defines a (positive definite) metric in a sufficiently small neighborhood V of 0.

We have the following consequences of Corollary 2.7 and 2.9.

Corollary 2.11 *Let (V, ω) be as in (8) with $H(z, \bar{z})$ a (non-zero) global real analytic function (with finite rank) in $\mathbb{C}^2$. Then (V, ω) cannot be locally holomorphically isometrically embedded into $(\mathbb{P}^N, \omega_{FS})$ for any N .*

Example 2.12 Define $\omega = i\partial\bar{\partial}\log(2 + \rho)$ over $\mathbb{C}^2$. Then $(\mathbb{C}^2, \omega)$ can not be locally holomorphically isometrically embedded into $(\mathbb{P}^N, \omega_{FS})$ for any N .

3 Ideas of the Proof of the Main Theorem

Let F be a non-constant smooth CR map from an open piece of M_ϵ into $\mathbb{S}^{2N-1}$. By an algebraicity theorem due to the first author in [9], it follows that F is Nash-algebraic. A main step is to prove that F is rational—making use of a monodromy argument based on the invariant property of Segre foliations.

Let $M \subset U(\subset \mathbb{C}^n)$ be a closed real-analytic subset defined by a family of real-valued real analytic functions $\{\rho_\alpha(Z, \bar{Z})\}$, where Z is the coordinates of $\mathbb{C}^n$. Assume that the complexification $\rho_\alpha(Z, W)$ of $\rho_\alpha(Z, \bar{Z})$ is holomorphic over $U \times \text{conj}(U)$ with

$$\text{conj}(U) := \{W : \bar{W} \in U\}$$

for each α . Then the complexification $\mathcal{M}$ of M is the complex-analytic subset in $U \times \text{conj}(U)$ defined by $\rho_\alpha(Z, W) = 0$ for each α . Then for $W \in \mathbb{C}^n$, the Segre variety of M associated with the point W is defined by $Q_W := \{Z : (Z, \bar{W}) \in \mathcal{M}\}$.

Write $\mathcal{M}_\epsilon$ for the complexification of M_ϵ and write $\mathcal{M}'$ for the complexification of $\partial\mathbb{B}^N$.

Write Q_p^ϵ for the Segre variety of M_ϵ associated with the point p , and write Q'_q for the Segre variety of $\partial\mathbb{B}^N$ associated with the point q . For any $p \in \mathbb{C}^2$, write $p = (z_p, w_p)$ or $p = (\xi_p, \eta_p)$.

The following lemma from [13] can be proved by using the Thom transversality:

Let $U \subset \mathbb{C}^n$ be a simply connected open subset and $\mathcal{S} \subset U$ be a closed complex analytic subset of codimension one. Then for $p \in U \setminus \mathcal{S}$, the fundamental group $\pi_1(U \setminus \mathcal{S}, p)$ is generated by loops obtained by concatenating (Jordan) paths $\gamma_1, \gamma_2, \gamma_3$, where γ_1 connects p with a point arbitrary close to a smooth point $q_0 \in \mathcal{S}$, γ_2 is a loop around $\mathcal{S}$ near q_0 and γ_3 is γ_1 reversed.

Making use of the above lemma, we have the following:

Lemma (Huang-Zaitsev [13]) *Let M_ϵ be defined as before and let $p_0 \in M_\epsilon$. Let $\mathcal{S}$ be a complex analytic hyper-variety in $\mathbb{C}^2$ not containing p_0 . Let $\gamma \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ be obtained by concatenation of $\gamma_1, \gamma_2, \gamma_3$ as described in the Lemma, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S}$ with $w_{q_0} \neq 0$. Then γ can be slightly and homotopically perturbed to a loop $\tilde{\gamma} \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ such that there exists a null-homotopic loop $\lambda \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with $(\lambda, \tilde{\gamma})$ contained in the complexification $\mathcal{M}_\epsilon$ of M_ϵ . Also, for an element $\hat{\gamma} \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with a similar*

property described above, after a small perturbation to $\hat{\gamma}$ if needed, we can find a null-homotopic loop in $\hat{\lambda} \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ such that $(\hat{\gamma}, \hat{\lambda}) \subset \mathcal{M}_\epsilon$.

Notice that F is complex algebraic (possibly multi-valued). In particular, any branch of F can be holomorphically continued along a path not cutting a certain proper complex algebraic subset $\mathcal{S} \subset \mathbb{C}^2$. We need only to prove the single-valued property for F assuming that $\mathcal{S}$ is a hyper-complex analytic variety. Seeking a contradiction, suppose not. Then we can find a point $p_0 \in U \subset M_\epsilon$, $p_0 = (z_0, w_0)$ with $w_0 \neq 0$, a loop $\gamma \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ obtained by concatenation of $\gamma_1, \gamma_2, \gamma_3$ as in Lemma, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S}$, such that when we holomorphically continue F from a neighborhood of p_0 along γ one round, we will obtain another branch $F_2 (\neq F)$ of F near p_0 . Obviously, we can assume q_0 is a smooth point of some branching hypervariety $\mathcal{S}' \subset \mathcal{S}$ of F . We next proceed in two steps:

Case I: (See [10] for more details here): If we can find a loop γ as above such that the corresponding $\mathcal{S}' \neq \{w = 0\}$, by perturbing γ if necessary, we can make $w_{q_0} \neq 0$. By the Lemma, after slightly perturbing γ if necessary, there exists a null-homotopic loop λ in $\pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with $(\gamma, \bar{\lambda})$ contained in the complexification $\mathcal{M}_\epsilon$ of M_ϵ . We know that $(F, \bar{F}) := (F(\cdot), \overline{F(\cdot)})$ sends a neighborhood of $(p_0, \bar{p}_0)$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Applying the analytic continuation along the loop $(\gamma, \bar{\lambda})$ in $\mathcal{M}_\epsilon$ for $\sum_{j=1}^N (F_{(j)} \overline{F_{(j)}(\cdot)}) - 1$, one concludes by the uniqueness of analytic functions that $(F_2, \bar{F})$ also sends a neighborhood of $(p_0, \bar{p}_0)$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Consequently, we get $F_2(Q_p) \subset Q'_{F(p)}$ for $p \in M_\epsilon$ near p_0 . In particular, we have the following:

$$F_2(p) \in Q'_{F(p)}, \quad \forall p \in M_\epsilon, \quad p \approx p_0.$$

Now applying the holomorphic continuation along the loop $(\lambda, \bar{\gamma})$ in $\mathcal{M}_\epsilon$ for (F_2, F) , we get by uniqueness of analytic functions that $(F_2, \bar{F}_2)$ sends a neighborhood of (p_0, p_0) in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Hence, we also have

$$F_2(p) \in Q'_{F_2(p)}, \quad \forall p \in M_\epsilon, \quad p \approx p_0.$$

In particular, $F_2(p) \in \partial \mathbb{B}^N$. Combining this, and noting that for any $q \in \partial \mathbb{B}^N$, $\partial \mathbb{B}^N \cap Q'_q = q$, we get $F_2(p) = F(p)$ for any $p \in M_\epsilon$ near p_0 . Thus $F_2 \equiv F$ in a neighborhood of p_0 in $\mathbb{C}^2$, which is a contradiction.

Case II: Now, suppose $W := \{w = 0\}$ is the only branching locus of the algebraic extension of F . Since W is smooth and $\pi_1(\mathbb{C}^2 \setminus W) = \mathbb{Z}$, we get the cyclic branching property for F . Now, we notice that W cuts M_ϵ transversally at a certain point $p^* =: (z_0, 0)$. When we will continue along loops inside $T_{p^*}^{(1,0)} M_\epsilon$ near p^* , we recover all branches of $F(z, w)$. Since any loop inside $T_{p^*}^{(1,0)} M_\epsilon$ near p^* can be easily homotopically deformed into loops in M_ϵ near p^* , we conclude that we recover all branches of F near p^* by continuing any branch of F near p^* along loops inside $M_\epsilon \setminus W$ near p^* . Hence, we are now reduced to the local situation first encountered in

Huang-Zaitsev [13]. Hence, with an argument using the invariant property of Segre varieties, for $Z(\neq) \approx p^*$ and two branches F_1 and F_2 of F near $Z \in M \setminus W$, we have $F_1(Z), F_2(Z) \in Q'_{F_1(Z)} \cap Q'_{F_2(Z)}$. As above, we see that $F_1(Z) = F_2(Z)$. We thus conclude that F is single-valued.

Since F is algebraic, it is rational. Once we know that F is a rational map from M_ϵ into the sphere, by a theorem of Chiappari [1], we know that F extends to a holomorphic map from a neighborhood of D_ϵ and properly maps D_ϵ into the ball.

The next step is the application of the Kohn-Nirenberg condition:

Lemma *Let $p_0 = (0, 1) \in M_\epsilon$. There exists $\tilde{\epsilon} > 0$ such that for each $0 < \epsilon < \tilde{\epsilon}$, $Q_{p_0} \cap M_\epsilon$ is a real subvariety of dimension one.*

It suffices to show that there exists $q \in Q_{p_0}$ such that $q \in D_\epsilon$. Note that $Q_{p_0} = \{(z, w) : w = 1\}$. Set

$$\psi(z, \epsilon) = \varepsilon_0(|z|^8 + c\operatorname{Re}|z|^2 z^6) + |z|^{10} + \epsilon|z|^2, \quad 0 \leq \epsilon < 1.$$

Note $q = (\mu_0, 1) \in D_\epsilon$ if and only if $\psi(\mu_0, \epsilon) < 0$. Now Set $\phi(\lambda, \epsilon) = \varepsilon_0\lambda^8(1 - c) + \lambda^{10} + \epsilon\lambda^2, 0 \leq \epsilon < 1$. First we note there exists small $\lambda' > 0$, such that $\phi(\lambda', 0) < 0$. Consequently, we can find $\tilde{\epsilon} > 0$ such that for each $0 < \epsilon \leq \tilde{\epsilon}$, $\phi(\lambda', \epsilon) < 0$. Write $\mu_0 = \lambda' e^{i\frac{\pi}{6}}$. It is easy to see that $\psi(\mu_0, \epsilon) < 0$ if $0 < \epsilon \leq \tilde{\epsilon}$.

The next simple lemma we need is the following.

Lemma *Let $M := \{Z \in \mathbb{C}^n : \rho(Z, \bar{Z}) = 0\}, n \geq 2$, be a compact, connected, strongly pseudo-convex real-algebraic hypersurface. Assume that there exists $p \in M$ such that the associated Segre variety Q_p is irreducible and Q_p intersects M at infinitely many points. Let F be a holomorphic rational map sending M into the unit sphere $\mathbb{S}^{2N-1}$ in some $\mathbb{C}^N$. Then F is a constant map.*

Write $\mathcal{S}$ as the singular set of F , then it does not pass through M . Write Q'_q for the Segre variety of $\mathbb{S}^{2N-1}$ at q . For any $p \approx M$, $F(Q_p \setminus \mathcal{S}) \subset Q'_{F(p)}$. Note that $\mathcal{S} \cap Q_p$ is either empty or a Zariski closed subset of Q_p . Notice that Q_p is connected as it is irreducible. We conclude by unique continuation that if $\tilde{p} \in Q_p$ and F is holomorphic at $\tilde{p}$, then $F(\tilde{p}) \in Q'_{F(p)}$. In particular, if $\tilde{p} \in Q_p \cap M$, then $F(\tilde{p}) \in Q'_{F(p)} \cap \mathbb{S}^{2N-1} = \{F(p)\}$. That is, $F(\tilde{p}) = F(p)$.

Notice that $Q_p \cap M$ is a compact set and contains infinitely many points. Let $\hat{p}$ be an accumulation point of $Q_p \cap M$. Clearly, by what we argued above, F is not one-to-one in any neighborhood of $\hat{p}$. This shows that F is constant. Indeed, suppose F is not a constant map. We then conclude that F is a holomorphic embedding near $\hat{p}$ by a standard Hopf lemma type argument for both M_ϵ and $\mathbb{S}^{2N-1}$ are strongly pseudo-convex.

Now, let ϵ, ε_0 be sufficient small such that all these lemmas hold. Let F be a holomorphic map defined in a small neighborhood U of $p \in M_\epsilon$ that sends an open piece of M_ϵ into $\mathbb{S}^{2N-1}$. Then F is a rational map. Pick $p_0 = (0, 1) \in M_\epsilon$. Notice

that the associated Segre variety $Q_{p_0} = \{(z, 1) : z \in \mathbb{C}\}$ is an irreducible complex variety in $\mathbb{C}^2$. Then it follows that F is a constant. We have thus sketched the proof of the main Theorem.

4 Non-embeddable Examples in the Positive Signature Case

Let n, ℓ be two integers with $1 < \ell \leq n/2$. For any ϵ , define

$$M_\epsilon := \{[z_0, \dots, z_{n+1}] \in \mathbb{P}^{n+1} : \\ \|z\|^2 \left(-\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 \right) + \epsilon (|z_1|^4 - |z_{n+1}|^4) = 0\}.$$

Here $\|z\|^2 = \sum_{j=0}^{n+1} |z_j|^2$ as usual. For $\epsilon = 0$, M_ϵ reduces to the generalized sphere with signature ℓ , which is the boundary of the generalized ball.

For $0 < \epsilon \ll 1$, M_ϵ is a compact smooth real-algebraic hypersurface with Levi form non-degenerate of the same signature ℓ .

Theorem (Huang-Zaitsev [13]) *There is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds: M_ϵ is a smooth real-algebraic hypersurface in $\mathbb{P}^{n+1}$ with non-degenerate Levi form of signature ℓ at every point. There does not exist any holomorphic embedding from any open piece of M_ℓ into $\mathbb{H}_\ell^{N+1}$.*

When $0 < \epsilon \ll 1$, since M_ϵ is a small algebraic deformation, we see that M_ϵ must also be a compact real-algebraic Levi non-degenerate hypersurface in $\mathbb{P}^{n+1}$ with signature ℓ diffeomorphic to the generalized ball $\mathbb{B}_\ell^{n+1} \subset \mathbb{P}^{n+1}$.

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