Automatica 103 (2019) 346–353

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

$\mathcal{L}_1$ adaptive sampled-data control for uncertain multi-input multi-output systems[☆]

Hamidreza Jafarnejadsani[a,*], Hanmin Lee[b], Naira Hovakimyan[a]

[a] Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
[b] Department of Aerospace Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

ARTICLE INFO

Article history:
Received 6 April 2017
Received in revised form 4 September 2018
Accepted 5 January 2019
Available online 25 February 2019

Keywords:
Multi-input multi-output system
$\mathcal{L}_1$ adaptive sampled-data control
Output-feedback
Disturbance compensation
Digital controller
Non-minimum-phase system

ABSTRACT

This paper develops a sampled-data (SD) controller for uncertain multi-input multi-output (MIMO) systems, possibly with non-minimum phase zeros, using the $\mathcal{L}_1$ adaptive control architecture. The proposed controller compensates for disturbances and uncertainties within the bandwidth of a low-pass filter. Sufficient conditions for robust stability are obtained for the closed-loop system with SD controller, where the input/output signals are held constant over a sampling period. It is shown that the hybrid closed-loop system can recover the performance of a continuous-time reference system as the sampling time of the SD controller tends to zero. Simulation examples are provided to validate the theoretical findings.

1. Introduction

Most controllers in modern cyber–physical systems (CPS) are implemented on digital computers using sample and hold mechanisms, where the control systems can be dealt with in the sampled-data (SD) framework. The SD nature of controller implementation plays significant role in the analysis of infrastructures such as power grids, transportation, and financial systems (Chen & Francis, 1995; Naghnaeian, Hirzallah, & Voulgaris, 2015; Nesic & Teel, 2004). The controller design in the SD framework can potentially address uncertainties in CPS, which involve digital controllers interacting with physical systems.

SD control systems are extensively analyzed in the literature (Dullerud & Glover, 1993; Fridman, 2010; Sivashankar & Khargonekar, 1994). The SD control designs are mainly based on the controller emulation methods, where an SD controller is developed in two stages; first, a continuous-time controller which satisfies certain performance/robustness requirements is designed; next, a discrete-time controller is obtained for digital implementation using an approximation technique (Chu, Qian, Yang, Xu, & Liu, 2015; Lin & Wei, 2016; Nesic, Teel, & Carnevale, 2009). The main issue in this approach is the selection of the sampling period that guarantees stability of the SD system with the emulated controller. In practice, the sampling period cannot be chosen arbitrarily small due to hardware limitations, such as the limits in central processing unit (CPU) and communication links. On the other hand, a larger sampling period reduces the performance and robustness of digital controllers. The conditions under which the SD controllers recover the properties of the underlying continuous-time design are investigated in Nesic et al. (2009) and Teel, Nesic, and Kokotovic (1998).

In Ahmed Ali, Langlois, and Guermouche (2014), Khalil (2004) and Ahrens, Tan, and Khalil (2009), the problem of SD output-feedback control is addressed by introducing high-gain observers to estimate the unmeasured states. Output-feedback stabilization of nonlinear systems with SD control has been studied in Lam (2012) and Shim and Teel (2003). Refs. Chu et al. (2015), Lin and Wei (2016, 2017), Liu, Ma, and Jia (2016), Qian and Du (2012) and Zhang and Yang (2016) have addressed the problem of SD output-feedback control for systems with uncertainties and disturbances for a class of single-input single-output (SISO) nonlinear systems under a lower-triangular linear growth condition. In Lin and Wei (2016, 2017), non-minimum phase nonlinear systems are considered. Nonlinear sampled-data systems with full state-feedback are addressed in Guillaume, Bastin, and Campion (1994), Wu and Ding (2007) and Laila, Navarro-López, and Astolfi (2011).

This paper develops an SD output-feedback control approach for nonlinear uncertain MIMO systems, using the $\mathcal{L}_1$ adaptive

☆ Research supported by National Science Foundation, USA. The material in this paper was partially presented at the 2017 American Control Conference, May 24–26, 2017, Seattle, WA, USA. This paper was recommended for publication in revised form by Associate Editor Brett Ninness under the direction of Editor Torsten Söderström.

* Corresponding author.
E-mail addresses: jafarne2@illinois.edu (H. Jafarnejadsani), hlee170@illinois.edu (H. Lee), nhovakim@illinois.edu (N. Hovakimyan).

https://doi.org/10.1016/j.automatica.2019.01.007
0005-1098/

control structure. The $\mathcal{L}_1$ adaptive control theory is extended to the SD framework, while maintaining the key benefits of a continuous-time $\mathcal{L}_1$ controller implementation (Cao & Hovakimyan, 2008, 2009, 2010; Hovakimyan & Cao, 2010). Compared to continuous-time design, the SD approach of this paper provides a more accurate model for CPS, with hybrid discrete/continuous nature. Conditions are derived, under which the SD controller uniformly recovers the performance of the underlying continuous-time control design. The unknown nonlinearities are assumed to be locally Lipschitz. In addition, the system under consideration can have non-minimum phase dynamics in this paper. The controller compensates for disturbances within the bandwidth of a lowpass filter, and similar to other $\mathcal{L}_1$ controllers, achieves uniform transient and steady-state performance. In this paper, using the method of controller emulation, a discrete-time $\mathcal{L}_1$ adaptive controller is derived from a continuous-time reference system. Uniform bounds between the response of the closed-loop SD system and the reference system are derived, which can be made arbitrarily small as the sampling time tends to zero. We notice that the performance of $\mathcal{L}_1$ adaptive controller has been verified on manned and unmanned aerial vehicles, as well as several high-fidelity simulation models (Ackerman et al., 2016; Sun, Choe, Xargay, & Hovakimyan, 2016; Xargay, Hovakimyan, Dobrokhodov, Kaminer, Cao, & Gregory, 2012).

The rest of the paper is organized as follows. Section 2 presents the problem formulation. In Section 3, the structure of the digital controller is presented. The closed-loop SD system is analyzed in Section 4. Section 5 presents a simulation example. Finally, Section 6 concludes the paper.

2. Problem formulation

Throughout this paper, $\|x_\tau\|_{\mathcal{L}_\infty}$ denotes the $\mathcal{L}_\infty$ norm of the truncated signal $x_\tau(t)$ for original $x(t) \in \mathbb{R}^n$, given as

$$x_\tau(t) = x(t), \qquad \forall t \leq \tau,$$
$$x_\tau(t) = 0_{n\times 1}, \quad \text{otherwise.}$$

The notation $\|\cdot\|_p$ represents vector or matrix p-norms with $1 \leq p \leq \infty$. The right pseudo-inverse of a full row-rank matrix $A \in \mathbb{R}^{q\times n}$ is denoted by $A^\dagger$, and can be computed as $A^\dagger = A^\top \left(AA^\top\right)^{-1}$ such that $AA^\dagger = \mathbb{I}_q$. Finally, s is used for the Laplace transform.

Consider the following MIMO system

$$\begin{aligned}\dot{x}(t) &= A_\mathrm{p}x(t) + B_\mathrm{p}\left(u(t) + f(t, x(t))\right), \quad x(0) = x_0,\\ y(t) &= C_\mathrm{p}x(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^q$ is the input signal, and $y(t) \in \mathbb{R}^q$ is the system output vector. Also, $\{A_\mathrm{p} \in \mathbb{R}^{n\times n}, B_\mathrm{p} \in \mathbb{R}^{n\times q}, C_\mathrm{p} \in \mathbb{R}^{q\times n}\}$ is a known observable-controllable triple. The unknown initial condition $x_0 \in \mathbb{R}^n$ is assumed to be inside an arbitrarily large set, so that $\|x_0\|_\infty \leq \rho_0 < \infty$ for some known $\rho_0 > 0$. Let $f(t, x) \in \mathbb{R}^q$ represent the time-varying uncertainties, physical failures, and disturbances subject to the following assumption.

Assumption 1. There exist K_δ for arbitrary $\delta > 0$, and constant $L_0 > 0$ such that

$$\|f(t, x_2) - f(t, x_1)\|_\infty \leq K_\delta \|x_2 - x_1\|_\infty,$$
$$\|f(t, 0)\|_\infty \leq L_0,$$

hold for all $\|x_i\|_\infty \leq \delta, \ i \in \{1, 2\}$, uniformly in $t \geq 0$.

The control input, which is implemented via a zero-order hold mechanism with the time period of $T_\mathrm{s} > 0$, is given by

$$u(t) = u_\mathrm{d}[i], \quad t \in [iT_\mathrm{s}, (i+1)T_\mathrm{s}), \quad i \in \mathbb{Z}_{\geq 0}, \tag{2}$$

where $u_\mathrm{d}[i]$ is a discrete-time control input signal. The output $y(t)$ is sampled with the sampling time of T_s, such that the discrete-time output measurement $y_\mathrm{d}[i]$ is given by

$$y_\mathrm{d}[i] = y\left(iT_\mathrm{s}\right). \tag{3}$$

Assumption 2. The desired dynamics are defined by

$$M(s) \triangleq C_\mathrm{m}\left(s\mathbb{I}_{n_\mathrm{m}} - A_\mathrm{m}\right)^{-1} B_\mathrm{m}, \tag{4}$$

where the triple $\{A_\mathrm{m} \in \mathbb{R}^{n_\mathrm{m}\times n_\mathrm{m}}, B_\mathrm{m} \in \mathbb{R}^{n_\mathrm{m}\times q}, C_\mathrm{m} \in \mathbb{R}^{q\times n_\mathrm{m}}\}$ represents a minimal state–space realization. The desired system $M(s)$ should satisfy one of the following conditions:

- the triple $(A_\mathrm{m}, B_\mathrm{m}, C_\mathrm{m})$ is selected such that $C_\mathrm{m}B_\mathrm{m}$ is nonsingular, A_m is Hurwitz, and $M(s)$ does not have a non-minimum-phase transmission zero,
- or, if the system defined by $(A_\mathrm{p}, B_\mathrm{p}, C_\mathrm{p})$ does not have a non-minimum-phase transmission zero, one can select

$$A_\mathrm{m} = A_\mathrm{p} - B_\mathrm{p}F, \quad B_\mathrm{m} = B_\mathrm{p}, \quad C_\mathrm{m} = C_\mathrm{p}, \tag{5}$$

where $F \in \mathbb{R}^{q\times n}$ is selected such that $A_\mathrm{p} - B_\mathrm{p}F$ is Hurwitz. In this case $C_\mathrm{m}B_\mathrm{m}$ can be rank deficient.

The desired response $y_\mathrm{m}(t)$ is given by the Laplace transform $y_\mathrm{m}(s) = M(s)K_\mathrm{g}r(s)$, where

$$K_\mathrm{g} \triangleq -\left(C_\mathrm{m}A_\mathrm{m}^{-1}B_\mathrm{m}\right)^{-1},$$

and $r(s)$ is the Laplace transform of $r(t)$ given by

$$r(t) = r_\mathrm{d}[i], \quad t \in [iT_\mathrm{s}, (i+1)T_\mathrm{s}), \quad i \in \mathbb{Z}_{\geq 0}, \tag{6}$$

where $r_\mathrm{d}[i]$ is a given discrete-time reference command. The command signal is assumed to be bounded, such that $\|r_\mathrm{d}[i]\|_\infty \leq M_\mathrm{r}, \ i \in \mathbb{Z}_{\geq 0}$, where M_r is a known positive constant.

In the following, a sampled-data $\mathcal{L}_1$ adaptive controller is formulated for cyber–physical systems to compensate for physical failures, uncertainties, and disturbances, such that the output $y(t)$ of the system in (1) tracks the desired response $y_\mathrm{m}(t)$.

3. The proposed adaptive sampled-data controller

In this section, the proposed adaptive SD controller is presented. The conditions for selection of the control parameters and the detailed analysis of the closed-loop system are provided in Section 4. Elements of the output-feedback $\mathcal{L}_1$ adaptive SD controller are given next.

Let $T_s > 0$ be the sampling time of the digital controller. Consider a strictly proper stable transfer function $C(s)$ such that $C(0) = \mathbb{I}_q$. In the $\mathcal{L}_1$ adaptive control structure, $C(s)$ represents the low-pass filter at the control input (Hovakimyan & Cao, 2010). Also, define $O(s) \triangleq C(s)M^{-1}(s)C_\mathrm{m}\left(s\mathbb{I}_{n_\mathrm{m}} - A_\mathrm{m}\right)^{-1}$, and let $\left\{A_\mathrm{o} \in \mathbb{R}^{v\times v}, \ B_\mathrm{o} \in \mathbb{R}^{v\times q}, \ C_\mathrm{o} \in \mathbb{R}^{q\times v}\right\}$ be a minimal state-space realization such that

$$C_\mathrm{o}(s\mathbb{I}_v - A_\mathrm{o})^{-1}B_\mathrm{o} = O(s). \tag{7}$$

The control law is given by

$$\begin{aligned}x_\mathrm{u}[i+1] &= e^{A_\mathrm{o}T_\mathrm{s}}x_\mathrm{u}[i] + A_\mathrm{o}^{-1}\left(e^{A_\mathrm{o}T_\mathrm{s}} - \mathbb{I}_v\right)B_\mathrm{o}e^{-A_\mathrm{m}T_\mathrm{s}}\hat{\sigma}_\mathrm{d}[i],\\ u_\mathrm{d}[i] &= K_\mathrm{g}r_\mathrm{d}[i] - C_\mathrm{o}x_\mathrm{u}[i], \quad x_\mathrm{u}[0] = 0, \quad i \in \mathbb{Z}_{\geq 0},\end{aligned} \tag{8}$$

where $\hat{\sigma}_\mathrm{d}[\cdot] \in \mathbb{R}^n$ is given by the adaptation law in (13), and $r_\mathrm{d}[\cdot]$ is a given discrete-time reference command.

The output predictor is given by

$$\begin{aligned}\hat{x}_\mathrm{d}[i+1] &= e^{A_\mathrm{m}T_\mathrm{s}}\hat{x}_\mathrm{d}[i] + A_\mathrm{m}^{-1}(e^{A_\mathrm{m}T_\mathrm{s}} - \mathbb{I}_{n_\mathrm{m}})\left(B_\mathrm{m}u_\mathrm{d}[i] + \hat{\sigma}_\mathrm{d}[i]\right),\\ \hat{y}_\mathrm{d}[i] &= C_\mathrm{m}\hat{x}_\mathrm{d}[i], \quad \hat{x}_\mathrm{d}[0] = C_\mathrm{m}^\dagger y_0,\end{aligned} \tag{9}$$

where $u_{\mathrm{d}}(t)$ is provided by (8), and $y_0 \triangleq C_{\mathrm{p}}x_0$ is the known initial output.

Given $A_{\mathrm{m}} \in \mathbb{R}^{n_{\mathrm{m}}\times n_{\mathrm{m}}}$ is Hurwitz, there exists a positive definite matrix $P \in \mathbb{R}^{n_{\mathrm{m}}\times n_{\mathrm{m}}}$ solving $A_{\mathrm{m}}^{\top}P + PA_{\mathrm{m}} = -Q$ for a given positive definite matrix $Q \in \mathbb{R}^{n_{\mathrm{m}}\times n_{\mathrm{m}}}$. Define

$$\Lambda \triangleq \begin{bmatrix} C_{\mathrm{m}} \\ D\sqrt{P} \end{bmatrix}, \tag{10}$$

where $\sqrt{P}$ satisfies $P = \sqrt{P}^{\top}\sqrt{P}$, and $D \in \mathbb{R}^{(n_{\mathrm{m}}-q)\times n_{\mathrm{m}}}$ is a matrix that is in the null space of $C_{\mathrm{m}}\left(\sqrt{P}\right)^{-1}$, *i.e.*

$$D\left(C_{\mathrm{m}}\left(\sqrt{P}\right)^{-1}\right)^{\top} = 0\,. \tag{11}$$

Further, let $\Phi(\cdot)$ be the $n_{\mathrm{m}} \times n_{\mathrm{m}}$ matrix

$$\Phi\left(T_{\mathrm{s}}\right) \triangleq \int_0^{T_{\mathrm{s}}} e^{\Lambda A_{\mathrm{m}}\Lambda^{-1}(T_{\mathrm{s}}-\tau)}\Lambda d\tau\,. \tag{12}$$

The adaptation law is given by

$$\hat{\sigma}_{\mathrm{d}}[i] = -\Phi^{-1}\left(T_{\mathrm{s}}\right)e^{\Lambda A_{\mathrm{m}}\Lambda^{-1}T_{\mathrm{s}}}\mathbf{1}_{n_{\mathrm{m}}q}\tilde{y}_{\mathrm{d}}[i], \tag{13}$$

where $\tilde{y}_{\mathrm{d}}[i] = \hat{y}_{\mathrm{d}}[i] - y_{\mathrm{d}}[i]$, and $\mathbf{1}_{n_{\mathrm{m}}q}$ is given by

$$\mathbf{1}_{n_{\mathrm{m}}q} \triangleq \begin{bmatrix} \mathbb{I}_q \\ 0_{(n_{\mathrm{m}}-q)\times q} \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{m}}\times q}. \tag{14}$$

4. Analysis of the closed-loop sampled-data system

This section provides the analysis of stability and performance of the closed-loop SD system with the proposed controller. Also, the conditions for selection of the control parameters T_{s} and $C(s)$ are provided. First, we define a few variables of interest and design constraints. Let

$$\begin{aligned}
P(s) &\triangleq C_{\mathrm{p}}\left(s\mathbb{I}_n - A_{\mathrm{p}} + B_{\mathrm{p}}F\right)^{-1}B_{\mathrm{p}},\\
H_0(s) &\triangleq \left(s\mathbb{I}_n - A_{\mathrm{p}} + B_{\mathrm{p}}F\right)^{-1}B_{\mathrm{p}},\\
H_1(s) &\triangleq \left(\mathbb{I}_q + \left(M^{-1}(s)P(s) - \mathbb{I}_q\right)C(s)\right)^{-1},\\
H_2(s) &\triangleq H_0(s) - H_0(s)C(s)H_1(s)\left(M^{-1}(s)P(s) - \mathbb{I}_q\right),\\
H_3(s) &\triangleq H_1(s)M^{-1}(s)P(s),\\
H_4(s) &\triangleq H_1(s)\left(M^{-1}(s)P(s) - \mathbb{I}_q\right),\\
H_5(s) &\triangleq H_0(s)C(s)H_1(s)M^{-1}(s),\\
G(s) &\triangleq H_0(s) - H_5(s)P(s),
\end{aligned} \tag{15}$$

where $F \in \mathbb{R}^{q\times n}$ is selected such that $A_{\mathrm{p}} - B_{\mathrm{p}}F$ is Hurwitz, as mentioned in Assumption 2. We define an auxiliary system with the same input–output mapping as the system (1), using the state–space matrices $(A_{\mathrm{m}}, B_{\mathrm{m}}, C_{\mathrm{m}})$ of the desired dynamics. The uncertainties are lumped into a variable denoted by $\sigma(t)$ in the auxiliary system. The control input $u(t)$ compensates for the matched uncertainty $\sigma(t)$ to recover the desired output tracking response (introduced in Assumption 2). Let the auxiliary system be

$$\begin{aligned}
\dot{x}_{\mathrm{a}}(t) &= A_{\mathrm{m}}x_{\mathrm{a}}(t) + B_{\mathrm{m}}\left(u(t) + \sigma(t)\right), \quad x_{\mathrm{a}}(0) = C_{\mathrm{m}}^{\dagger}y_0,\\
y(t) &= C_{\mathrm{m}}x_{\mathrm{a}}(t),
\end{aligned} \tag{16}$$

where $x_{\mathrm{a}}(t) \in \mathbb{R}^{n_{\mathrm{m}}}$ is the state vector, the Laplace transform of $\sigma(t)$ is given by

$$\sigma(s) = M^{-1}(s)\left(\left(P(s) - M(s)\right)u(s) + P(s)w(s) + H_{\mathrm{in}}(s)x_0\right),$$

with

$$H_{\mathrm{in}}(s) \triangleq C_{\mathrm{p}}\left(s\mathbb{I}_n - A_{\mathrm{p}} + B_{\mathrm{p}}F\right)^{-1} - C_{\mathrm{m}}\left(s\mathbb{I}_{n_{\mathrm{m}}} - A_{\mathrm{m}}\right)^{-1}C_{\mathrm{m}}^{\dagger}C_{\mathrm{p}},$$

and $w(s)$ is the Laplace transform of $w(t)$ given by

$$w(t) \triangleq Fx(t) + f\left(t, x(t)\right). \tag{17}$$

Since the full state measurement is not available, $Fx(t)$ is unknown. Therefore, $Fx(t)$ is added to the uncertainty term $f(t,\ x(t))$, and the addition of the two unknown signals is denoted by $w(t)$.

Remark 1. Given that $M(s)$ does not have an unstable transmission zero, $M^{-1}(s)P(s)$ is proper and stable. In addition, Assumption 2 implies that $sM^{-1}(s)H_{\mathrm{in}}(s)$ is proper and stable. Therefore, $\sigma(t)$, defined in (16), is a casual signal.

Further, for every $\delta > 0$, let

$$L_\delta \triangleq \frac{\bar{\gamma}_1 + \delta}{\delta}\left(K_{(\bar{\gamma}_1+\delta)} + \|F\|_\infty\right), \tag{18}$$

where K_δ is introduced in Assumption 1, and $\bar{\gamma}_1$ is an arbitrarily small positive constant. It can be shown that the following bound on $w(t)$ holds

$$\|w_t\|_{\mathcal{L}_\infty} \le L_\delta\,\|x_t\|_{\mathcal{L}_\infty} + L_0. \tag{19}$$

The design of the controller proceeds by considering a strictly proper stable transfer function $C(s)$ such that $C(0) = \mathbb{I}_q$. The selection of $C(s)$ must ensure that

$$H_1(s) \quad \text{is stable}, \tag{20}$$

where $H_1(s)$ is defined in (15), and

$$C(s)M^{-1}(s) \quad \text{is proper}. \tag{21}$$

Also, for a given ρ_0, there should exist $\rho_{\mathrm{r}} > \rho_0$ such that the following $\mathcal{L}_1$-norm condition holds

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_{\mathrm{r}} - \rho_1 - \rho_2}{L_{\rho_{\mathrm{r}}}\rho_{\mathrm{r}} + L_0}, \tag{22}$$

where

$$\begin{aligned}
\rho_1 &\triangleq \left\|s(s\mathbb{I}_n - A_{\mathrm{p}} + B_{\mathrm{p}}F)^{-1} - sH_5(s)H_{\mathrm{in}}(s)\right\|_{\mathcal{L}_1}\rho_0,\\
\rho_2 &\triangleq \left\|H_2(s)K_{\mathrm{g}}\right\|_{\mathcal{L}_1}M_{\mathrm{r}}.
\end{aligned} \tag{23}$$

Remark 2. If the system with state-space matrices $(A_{\mathrm{p}},\ B_{\mathrm{p}},\ C_{\mathrm{p}})$ does not have a non-minimum-phase transmission zero, one can select the desired system as $M(s) = P(s)$ (as stated in Assumption 2), where $P(s)$ is introduced in (15). Then, we have $H_1(s) = \mathbb{I}_q$. Also, $G(s)$ can be rewritten as

$$G(s) = H_0(s)\left(\mathbb{I}_q - C(s)\right). \tag{24}$$

Therefore, a filter with sufficiently high bandwidth (*i.e.*, $C(s) \approx \mathbb{I}_q$), and high relative degree such that $C^{-1}(s)M(s)$ is proper, always satisfies the conditions in (20)–(22). In the case $(A_{\mathrm{p}},\ B_{\mathrm{p}},\ C_{\mathrm{p}})$ defines a non-minimum phase system, the selection of $C(s)$ and $M(s)$ that would verify (20)–(22) is not trivial as reported in Kharisov and Hovakimyan (2011).

Remark 3. Selection of the filter $C(s)$ provides a trade-off between performance in terms of disturbance compensation and robustness in terms of input-delay margin. A mixed-norm optimization of the filter for $\mathcal{L}_1$ adaptive control structure can be found in Jafarnejadsani, Sun, Lee, and Hovakimyan (2017).

Let $P_1 \in \mathbb{R}^{q\times q}$ and $P_2 \in \mathbb{R}^{(n_{\mathrm{m}}-q)\times(n_{\mathrm{m}}-q)}$ be positive definite matrices:

$$P_1 \triangleq \left(C_{\mathrm{m}}\sqrt{P}^{-1}\sqrt{P}^{-\top}C_{\mathrm{m}}^{\top}\right)^{-1}, \quad P_2 \triangleq (DD^{\top})^{-1}. \tag{25}$$

Define

$$\begin{bmatrix}\eta_1^{\top}(t) & \eta_2^{\top}(t)\end{bmatrix} \triangleq \mathbf{1}_{n_{\mathrm{m}}q}^{\top}e^{\Lambda A_{\mathrm{m}}\Lambda^{-1}t}, \tag{26}$$

where $\eta_1(t) \in \mathbb{R}^{q\times q}$ and $\eta_2(t) \in \mathbb{R}^{(n_m-q)\times q}$, and

$$\kappa(T_s) \triangleq \int_0^{T_s} \left\| \mathbf{1}_{n_m q}^\top e^{\Lambda A_m \Lambda^{-1}(T_s-\tau)} \Lambda B_m \right\|_2 d\tau. \tag{27}$$

Define the function

$$\Gamma(T_s) \triangleq \alpha_1(T_s) \left\| (s\mathbb{I}_v - A_o)^{-1} B_o \right\|_{\mathcal{L}_1} + \alpha_2(T_s), \tag{28}$$

where

$$\alpha_1(T_s) \triangleq \max_{t\in[0,\,T_s]} \left\| C_o \left(e^{A_o t} - \mathbb{I}_v \right) \right\|_\infty,$$

$$\alpha_2(T_s) \triangleq \max_{t\in[0,\,T_s]} \int_0^t \left\| C_o e^{A_o(t-\tau)} B_o \right\|_\infty d\tau.$$

Let

$$\begin{aligned}
\Upsilon(T_s) &= \left\| e^{-A_m T_s} \Phi^{-1}(T_s)\, e^{\Lambda A_m \Lambda^{-1} T_s} \mathbf{1}_{n_m q} \right\|_\infty, \\
\Psi(T_s) &= \left\| H_5(s) C_m \left(s\mathbb{I}_{n_m} - A_m \right)^{-1} \left(e^{A_m T_s} - \mathbb{I}_{n_m} \right) \right\|_{\mathcal{L}_1}, \\
\Omega_1(T_s) &= \left\| H_2(s) C(s) M^{-1}(s) \right\|_{\mathcal{L}_1} \left(1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r} \right)^{-1} \\
&+ \|H_2(s)\|_{\mathcal{L}_1} \left(\Gamma(T_s) + \Psi(T_s) \right) \Upsilon(T_s) \left(1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r} \right)^{-1}, \\
\Theta(T_s) &= \|H_3(s)\|_{\mathcal{L}_1} L_{\rho_r} \Omega_1(T_s) + \left\| H_4(s) C(s) M^{-1}(s) \right\|_{\mathcal{L}_1} \\
&\quad + \|H_4(s)\|_{\mathcal{L}_1} \left(\Gamma(T_s) + \Psi(T_s) \right) \Upsilon(T_s), \\
\rho_\Delta &= \|H_3(s)\|_{\mathcal{L}_1} \left(L_{\rho_r} \rho_r + L_0 \right) + \left\| H_4(s) K_g \right\|_{\mathcal{L}_1} M_r \\
&\quad + \left\| s H_1(s) M^{-1}(s) H_{in}(s) \right\|_{\mathcal{L}_1} \rho_0, \\
\Omega_2(T_s) &= \left\| C(s) M^{-1}(s) \right\|_{\mathcal{L}_1} + \|C(s)\|_{\mathcal{L}_1} L_{\rho_r} \Omega_1(T_s) \\
&\quad + \left(\Gamma(T_s) + \Psi(T_s) \right) \Upsilon(T_s), \\
\rho_{ur} &= \|C(s) H_3(s)\|_{\mathcal{L}_1} \left(L_{\rho_r} \rho_r + L_0 \right) \\
&\quad + \left\| s C(s) H_1(s) M^{-1}(s) H_{in}(s) \right\|_{\mathcal{L}_1} \rho_0 \\
&\quad + \left\| \left(\mathbb{I}_q - C(s) H_4(s) \right) K_g \right\|_{\mathcal{L}_1} M_r,
\end{aligned} \tag{29}$$

where $H_i(\cdot)$'s are defined in (15). Next, we introduce the functions

$$\beta_1(T_s) \triangleq \max_{t\in[0,\,T_s]} \|\eta_1(t)\|_2, \quad \beta_2(T_s) \triangleq \max_{t\in[0,\,T_s]} \|\eta_2(t)\|_2, \tag{30}$$

where $\eta_1(t)$ and $\eta_2(t)$ are given in (26). Also

$$\beta_3(T_s) \triangleq \max_{t\in[0,\,T_s]} \eta_3(t, T_s), \quad \beta_4(T_s) \triangleq \max_{t\in[0,\,T_s]} \eta_4(t), \tag{31}$$

where

$$\begin{aligned}
&\eta_3(t, T_s) \triangleq \\
&\int_0^t \left\| \mathbf{1}_{n_m q}^\top e^{\Lambda A_m \Lambda^{-1}(t-\tau)} \Lambda \Phi^{-1}(T_s)\, e^{\Lambda A_m \Lambda^{-1} T_s} \mathbf{1}_{n_m q} \right\|_2 d\tau, \\
&\eta_4(t) \triangleq \int_0^t \left\| \mathbf{1}_{n_m q}^\top e^{\Lambda A_m \Lambda^{-1}(t-\tau)} \Lambda B_m \right\|_2 d\tau.
\end{aligned} \tag{32}$$

For $\bar{\gamma}_0 > 0$, let

$$\begin{aligned}
\Delta_1(\bar{\gamma}_0) &\triangleq \rho_\Delta + \Theta(T_s)\bar{\gamma}_0, \\
\Delta_2(\bar{\gamma}_0) &\triangleq \lambda_{\max}\left(\Lambda^{-\top} P \Lambda^{-1} \right) \left(\frac{2\sqrt{q}\,\Delta_1(\bar{\gamma}_0) \left\| \Lambda^{-\top} P B_m \right\|_2}{\lambda_{\min}\left(\Lambda^{-\top} Q \Lambda^{-1} \right)} \right)^2,
\end{aligned} \tag{33}$$

where, ρ_Δ and $\Theta(\cdot)$ are defined in (29). Also, let

$$\varsigma(\bar{\gamma}_0, T_s) \triangleq \|\eta_2(T_s)\|_2 \sqrt{\frac{\Delta_2(\bar{\gamma}_0)}{\lambda_{\max}(P_2)}} + \sqrt{q}\,\kappa(T_s)\Delta_1(\bar{\gamma}_0), \tag{34}$$

where $\eta_2(\cdot)$ is defined in (26) and $\kappa(\cdot)$ is given in (27).

Finally, define

$$\begin{aligned}
\gamma_0(\bar{\gamma}_0, T_s) \triangleq\ & \beta_1(T_s)\varsigma(\bar{\gamma}_0, T_s) + \beta_2(T_s)\sqrt{\frac{\Delta_2(\bar{\gamma}_0)}{\lambda_{\max}(P_2)}} \\
&+ \beta_3(T_s)\varsigma(\bar{\gamma}_0, T_s) + \sqrt{q}\,\beta_4(T_s)\Delta_1(\bar{\gamma}_0).
\end{aligned} \tag{35}$$

Lemma 4. *For all $\bar{\gamma}_0 > 0$, the following relationships hold:*

$$\lim_{T_s\to 0} \gamma_0(\bar{\gamma}_0, T_s) = 0, \tag{36}$$

where $\gamma_0(\cdot,\ \cdot)$ is given in (35).

Proof. It is similar to the proof of Lemma 3.3.1 in Hovakimyan and Cao (2010) and hence omitted here. ∎

Lemma 5. *There exist $T_s > 0$ and arbitrarily small positive constant $\bar{\gamma}_0$, such that*

$$\gamma_0(\bar{\gamma}_0, T_s) < \bar{\gamma}_0, \quad \Omega_1(T_s)\bar{\gamma}_0 < \bar{\gamma}_1, \tag{37}$$

where $\bar{\gamma}_1$ is introduced in (18) and $\gamma_0(\cdot,\cdot)$ is defined in (35), while $\Omega_1(\cdot)$ is given in (29).

Proof. It is straightforward to verify that $\Omega_1(T_s)$ is a bounded function as T_s tends to zero. In addition, Lemma 4 shows that $\gamma_0(\bar{\gamma}_0, T_s)$ approaches arbitrarily closely to zero for all $\bar{\gamma}_0$ with sufficiently small T_s. Therefore, there always exist constants T_s and $\bar{\gamma}_0$ that satisfy the inequalities in (37).

Lemma 6. *For arbitrary $\xi = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^{n_m}$, where $y \in \mathbb{R}^q$ and $z \in \mathbb{R}^{(n_m-q)}$, there exist positive definite $P_1 \in \mathbb{R}^{q\times q}$ and $P_2 \in \mathbb{R}^{(n_m-q)\times(n_m-q)}$ such that*

$$\xi^\top \left(\Lambda^{-1} \right)^\top P \Lambda^{-1} \xi = y^\top P_1 y + z^\top P_2 z, \tag{38}$$

where Λ is given in (10), and P_1 and P_2 are defined in (25).

Proof. The proof of Lemma 6 is found in Hovakimyan and Cao (2010).

Consider the following closed-loop reference system

$$\begin{aligned}
\dot{x}_{ref}(t) &= A_p x_{ref}(t) + B_p \left(u_{ref}(t) + f(t, x_{ref}(t)) \right), \\
u_{ref}(s) &= K_g r(s) - C(s)\sigma_{ref}(s), \\
y_{ref}(t) &= C_p x_{ref}(t), \quad x_{ref}(0) = x_0,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
&\sigma_{ref}(s) = \\
&[(P(s) - M(s))\, C(s) + M(s)]^{-1} (P(s) - M(s))\, K_g r(s) \\
&+ [(P(s) - M(s))\, C(s) + M(s)]^{-1} (P(s) w_{ref}(s) + H_{in}(s) x_0),
\end{aligned} \tag{40}$$

and $w_{ref}(s)$ is the Laplace transform of $w_{ref}(t)$ given by

$$w_{ref}(t) = F x_{ref}(t) + f(t, x_{ref}(t)). \tag{41}$$

The reference system can be rewritten as

$$\begin{aligned}
y_{ref}(s) &= M(s) K_g r(s) + M(s) \left(\mathbb{I}_q - C(s) \right) \sigma_{ref}(s) \\
&\quad + C_m \left(s\mathbb{I}_q - A_m \right)^{-1} C_m^\dagger y_0.
\end{aligned} \tag{42}$$

From (42), we notice that the unknown uncertainty $\sigma_{ref}(t)$, given by the Laplace transform in (40), is mitigated within the bandwidth of $C(s)$, and the desired response (in Assumption 2) is recovered. The reference system in (39) defines the *achievable performance* by the closed-loop sampled-data system given in (1), (8)–(13), as the sampling time T_s of the digital controller tends to zero. In the following, we first prove that $\sigma_{ref}(t)$ is bounded, and the reference system in (39) is stable. Then, we establish uniform bounds between the closed-loop system defined by (1), (8)–(13), and the reference system.

Lemma 7. *For the closed-loop reference system in (39), subject to the conditions in (20)–(22), if $\|x_0\|_\infty \le \rho_0$, then*

$$\|x_{\mathrm{ref}}\|_{\mathcal{L}_\infty} < \rho_{\mathrm{r}}, \tag{43}$$

$$\|u_{\mathrm{ref}}\|_{\mathcal{L}_\infty} < \rho_{\mathrm{ur}}, \tag{44}$$

where ρ_{r} is introduced in (22), and ρ_{ur} is given in (29).

Proof. The proof can be found in Jafarnejadsani (2018). ■

Remark 8. We can rewrite $\sigma_{\mathrm{ref}}(s)$ in (40) as

$$\sigma_{\mathrm{ref}}(s) = H_1(s)\left(M^{-1}(s)P(s) - \mathbb{I}_q\right)K_{\mathrm{g}}r(s) + H_1(s)\left(M^{-1}(s)P(s)w_{\mathrm{ref}}(s) + M^{-1}(s)H_{\mathrm{in}}(s)x_0\right).$$

Then, Remark 1 implies that $\sigma_{\mathrm{ref}}(s)$ is casual. In addition, the stability of $H_1(s)$ in (20) together with the results of Lemma 7 implies that $\sigma_{\mathrm{ref}}(s)$ is bounded:

$$\|\sigma_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \rho_\Delta, \tag{45}$$

where ρ_Δ is defined in (29).

In the proposed SD control structure, discrete-time output predictor dynamics are introduced in (9), where the unknown uncertainty $\sigma(t)$ (formulated in (16)) is replaced with an adaptation variable $\hat{\sigma}_{\mathrm{d}}[i]$. We consider a continuous-time equivalent state-space model of the predictor dynamics in (9) given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A_{\mathrm{m}}\hat{x}(t) + B_{\mathrm{m}}u(t) + \hat{\sigma}(t), \quad \hat{x}(0) = C_{\mathrm{m}}^{\dagger}y_0, \\ \hat{y}(t) &= C_{\mathrm{m}}\hat{x}(t), \end{aligned} \tag{46}$$

where

$$\hat{\sigma}(t) = \hat{\sigma}_{\mathrm{d}}[i], \quad t \in [iT_{\mathrm{s}}, (i+1)T_{\mathrm{s}}), \quad i \in \mathbb{Z}_{\ge 0}, \tag{47}$$

and $u(t)$ is given in (2) and (8). Since $\hat{\sigma}(t)$ and $u(t)$ in (46) are held constant over sampling intervals, we notice that (9) is a step-invariant discrete-time approximation of (46) such that

$$\hat{y}(iT_{\mathrm{s}}) = \hat{y}_{\mathrm{d}}[i]. \tag{48}$$

Let $\tilde{x}(t) = \hat{x}(t) - x_{\mathrm{a}}(t)$, where $x_{\mathrm{a}}(t)$ is defined in (16). Then, the prediction error dynamics between (16) and (46) are given by

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_{\mathrm{m}}\tilde{x}(t) + \hat{\sigma}(t) - B_{\mathrm{m}}\sigma(t), \quad \tilde{x}(0) = 0_{n_{\mathrm{m}}\times 1}, \\ \tilde{y}(t) &= C_{\mathrm{m}}\tilde{x}(t), \end{aligned} \tag{49}$$

where $\hat{\sigma}(t)$ is defined in (47).

Lemma 9. *Consider the closed-loop system defined by (1), (8)–(13), and the closed-loop reference system in (39). The following upper bound holds*

$$\|(x_{\mathrm{ref}} - x)_t\|_{\mathcal{L}_\infty} \le \Omega_1(T_{\mathrm{s}})\|\tilde{y}_t\|_{\mathcal{L}_\infty},$$

where $\Omega_1(\cdot)$ is given in (29), and $\tilde{y}(t)$ is the prediction error defined in (49).

Proof. The proof can be found in Jafarnejadsani (2018). ■

Theorem 10. *Consider the system in (1) and the controller in (8)–(13) subject to the conditions in (20)–(22). Assume that T_{s} is selected sufficiently small such that the inequalities in (37) hold. If $\|x_0\|_\infty \le \rho_0$, then*

$$\|\tilde{y}\|_{\mathcal{L}_\infty} < \bar{\gamma}_0, \tag{50}$$

$$\|x_{\mathrm{ref}} - x\|_{\mathcal{L}_\infty} < \Omega_1(T_{\mathrm{s}})\bar{\gamma}_0, \quad \|u_{\mathrm{ref}} - u\|_{\mathcal{L}_\infty} < \Omega_2(T_{\mathrm{s}})\bar{\gamma}_0, \tag{51}$$

where $\tilde{y}(t)$ is the prediction error defined in (49), and $\bar{\gamma}_0 > 0$ is a given arbitrarily small constant that satisfies (37). Also, $\Omega_1(T_{\mathrm{s}})$ and $\Omega_2(T_{\mathrm{s}})$ are defined in (29).

Proof. The proof can be found in Jafarnejadsani (2018). ■

Remark 11. Lemmas 4 and 5 indicate that an arbitrarily small bound $\bar{\gamma}_0$ on the prediction error can be achieved as T_{s} goes to zero. In addition, it can be shown that $\Omega_1(T_{\mathrm{s}})$ and $\Omega_2(T_{\mathrm{s}})$ are bounded as T_{s} tends to zero. Therefore, the uniform bounds in (51) can be made arbitrarily small. This implies that the closed-loop sampled-data system recovers the performance of the continuous-time reference system in (39) as the sampling time goes to zero.

Corollary 12. *The system in (1) with the controller in (8)–(13) subject to the conditions in (20)–(22), and (37), is semi-globally practically input to state stable (SPISS) (Lee, 2017; Nesic & Laila, 2002), if the system defined by the triple $(A_{\mathrm{p}}, B_{\mathrm{p}}, C_{\mathrm{p}})$ does not have a non-minimum-phase transmission zero.*

Proof. The proof can be found in Jafarnejadsani (2018). ■

5. Simulation examples

Two flight control examples are provided to validate the theoretical claims, and to verify the effectiveness of the proposed SD controller. The first example is the simulation of the lateral dynamics of F-16 aircraft with two inputs and two outputs. In the second example, a controller for the F-16 flight-path angle tracking is developed, where the dynamics from the control input (elevator deflection) to the flight-path angle is non-minimum-phase and unstable.

5.1. Aircraft lateral dynamics

A model for the lateral dynamics of F-16 aircraft at the airspeed of $V = 502$ ft/s and the angle of attack $\alpha = 2.11°$, found in Young, Cao, Hovakimyan, and Lavretsky (2006), is given by

$$A_{\mathrm{p}} = \begin{bmatrix} -0.3320 & 0.064 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0393 \\ -30.6490 & 0 & -3.6784 & 0.6646 \\ 8.5395 & 0 & -0.0254 & -0.4764 \end{bmatrix},$$

$$B_{\mathrm{p}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.7331 & 0.1315 \\ -0.0319 & -0.0620 \end{bmatrix}, \quad C_{\mathrm{p}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The state vector of the lateral dynamics model is $x(t) = [\beta(t), \phi(t), p_{\mathrm{s}}(t), r_{\mathrm{s}}(t)]^\top$, where the variables β, ϕ, p_{s} and r_{s} represent the angle of sideslip, the roll angle, the stability axis roll and yaw rates, respectively. The system dynamics are stable, however the eigenvalues are slow. The objective is to design a control input $u_{\mathrm{d}}[i] = [\delta_{\mathrm{a}}[i], \delta_{\mathrm{r}}[i]]^\top$, where δ_{a} and δ_{r} are the aileron and the rudder deflections, such that the output vector $y(t) = [\beta(t), \phi(t)]^\top$ tracks the reference command $r(t)$ given by (6), where $r_{\mathrm{d}}[i]$ is

$$r_{\mathrm{d}}[i] = \begin{bmatrix} 0.2\left(-\frac{0.5}{1+e^{iT_{\mathrm{s}}-8}} + \frac{1}{1+e^{iT_{\mathrm{s}}-30}} - 0.5\right) \\ 0.2\left(-\frac{0.5}{1+e^{iT_{\mathrm{s}}-8}} + \frac{1}{1+e^{iT_{\mathrm{s}}-30}} - 0.2\right) \end{bmatrix}, \quad i \in \mathbb{Z}_{\ge 0}, \tag{52}$$

and T_s is the sampling time. The desired tracking dynamics $M(s)$ are given by the state-space matrices

$$A_m = \begin{bmatrix} -4.0538 & -4.5045 & -0.8386 & -2.0633 \\ 1.2602 & 1.1254 & -2.0913 & 1.0746 \\ 2.7591 & 4.2500 & -1.4731 & 1.3436 \\ 3.1833 & -1.6250 & 6.5772 & -3.3832 \end{bmatrix},$$

$$B_m = \begin{bmatrix} -0.0021 & 0.1053 \\ -0.0402 & 0.0347 \\ 0.1562 & -0.0134 \\ -0.1722 & -0.0174 \end{bmatrix},$$

$$C_m = \begin{bmatrix} 0.0234 & 0.0894 & 0.0908 & 0.0597 \\ -0.2073 & 0.6566 & -0.2254 & -0.1419 \end{bmatrix}.$$

In this simulation, input uncertainties of the form

$$f_{\delta_a}(t, x(t)) = 0.01\,(|\beta(t)| + |\phi(t)| + p_s(t)\cos(4t)) + 0.02 r_s(t)\sin(t) + 0.25\cos(0.8t),$$
$$f_{\delta_r}(t, x(t)) = 0.01\,(|p_s(t)| + |r_s(t)| + \phi(t)\cos(0.7t)) + 0.02 r_s(t)\sin(4t) + 0.25\sin(1.1t)$$

are considered. The non-zero initial condition is $x_0 = [0 \text{ rad}, 0.06 \text{ rad}, 0.02 \text{ rad/s}, -0.02 \text{ rad/s}]^\top$, leading to $y_0 = [0 \text{ rad}, 0.06 \text{ rad}]^\top$. Next, we select the design parameters for the sampled-data $\mathcal{L}_1$ controller. Let $\rho_0 = M_r = 0.25$, $K_\delta = 0.05$, $\bar{\gamma}_0 = 0.1$, $\bar{\gamma}_1 = 9 \times 10^3$, and $F = 0_{2\times 4}$. With $\rho_r = 6.7 \times 10^3$, the uncertainty bounds $L_{\rho_r} = 0.1172$ and $L_0 = 0.25$ (which satisfy (18)), and the filter

$$C(s) = \begin{bmatrix} \frac{10}{s+10} & 0 \\ 0 & \frac{40}{s+40} \end{bmatrix},$$

the stability conditions in (20) and (22) are met. For the selected parameters, we can calculate $\rho_1 = 20.549$ and $\rho_2 = 9.633$. Then, the right hand side of (22) is equal to 8.492, which is larger than $\|G(s)\|_{\mathcal{L}_1} = 0.256$, and thus the inequity in (22) is verified. Finally, by selecting the sampling time $T_s = 10^{-7}$ sec, we have $\gamma_0(\bar{\gamma}_0, T_s) = 0.0956$ and $\Omega_1(T_s) = 8.996 \times 10^4$. Therefore, we can verify that the inequalities in (37) hold. In Fig. 1, the response of the closed-loop SD system is shown. The output tracks the desired response in the presence of the disturbances, as illustrated in Fig. 1(a). The control input is shown in Fig. 1(b). Fig. 2 shows the response of the closed-loop SD system for the step reference commands $r(t) = [0.05 \text{ rad}, 0.2 \text{ rad}]^\top$, $r(t) = [0.075 \text{ rad}, 0.3 \text{ rad}]^\top$, and $r(t) = [0.1 \text{ rad}, 0.4 \text{ rad}]^\top$, in the presence of uncertainties and time delay of 0.01 sec at the control input. In this simulation, the sampling time of the SD controller is $T_s = 0.01$ sec. We notice that the controller leads to scaled control inputs and outputs for scaled reference commands.

5.2. Aircraft flight-path angle

We consider the problem of flight-path angle, γ, tracking, using the elevator deflection, δ_e, for a longitudinal model of an F-16 aircraft. The state-space model, from Shkolnikov and Shtessel (2001), with $\gamma(t)$ as the output and $\delta_e(t)$ as the input, is non-minimum-phase and unstable, and is given by the matrices

$$A_p = \begin{bmatrix} -11.707 & 0 & -75.666 \\ 0 & 11.141 & -79.908 \\ 0.723 & 0.907 & -1.844 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 0.117 \end{bmatrix},$$
$$C_p = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

for Mach $= 0.7$ and altitude of $h = 10{,}000$ ft. This system has an unstable pole at $s = 1.051$, and a non-minimum-phase zero at $s = 11.141$. The state vector is $x(t) = [x_1(t),\ x_2(t),\ x_3(t)]^\top$, and the

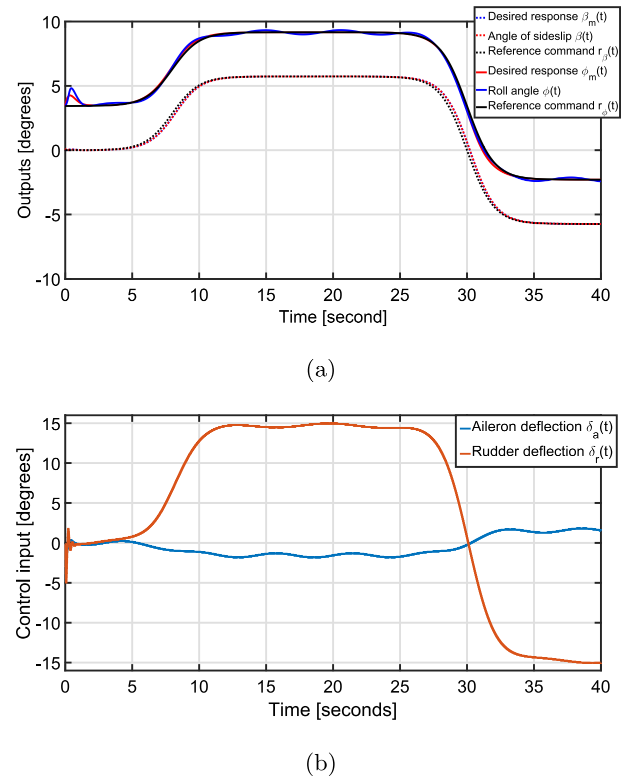


Fig. 1. The outputs of the closed-loop lateral dynamics, $\beta(t)$ and $\phi(t)$, track the desired responses $\beta_m(t)$ and $\phi_m(t)$ for the given reference command in (52). (a) The outputs, the reference commands, and the desired responses. (b) The control inputs $\delta_a(t)$ and $\delta_r(t)$.

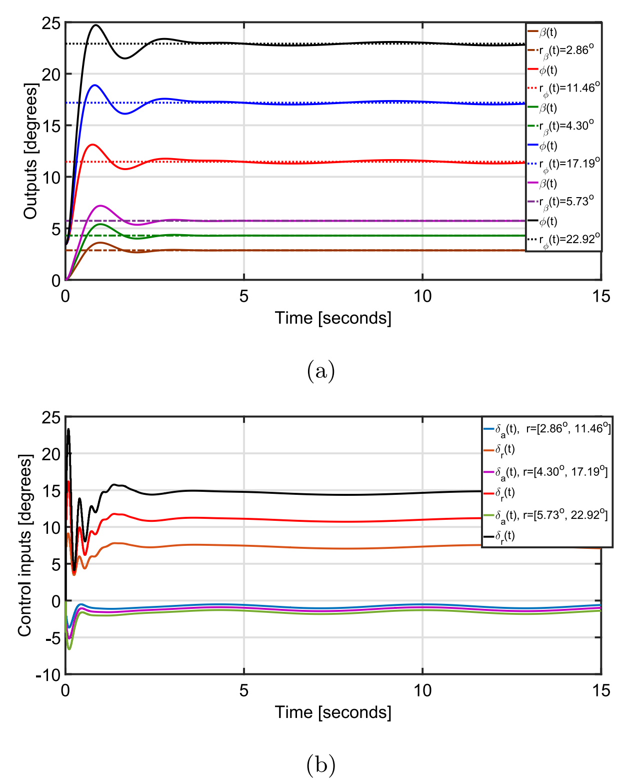
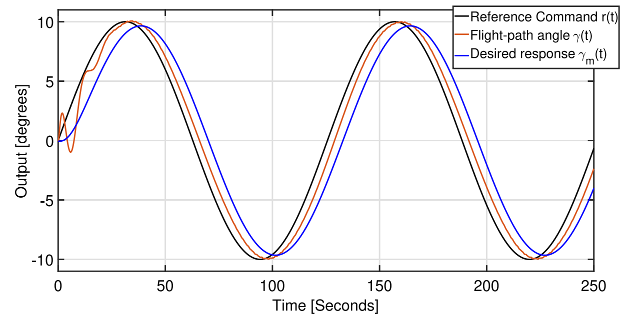
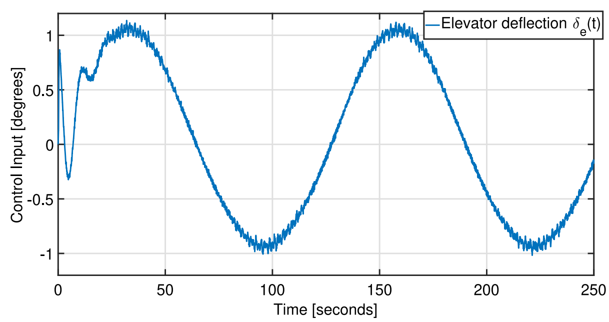


Fig. 2. Scaled responses of the closed-loop lateral dynamics to scaled reference inputs. (a) Scaled reference commands and system outputs. (b) Scaled control inputs.



(a)



(b)

Fig. 3. The flight-path angle, $\gamma(t)$, tracks the desired $\gamma_m(t)$ for a sinusoidal reference command. (a) The output, the reference command, and the desired response. (b) The control input $\delta_e(t)$.

output is $\gamma(t) = x_3(t)$. We choose the desired dynamics $M(s)$ and the filter $C(s)$ as

$$M(s) = -\frac{469.6\, s^2 + 1.384 \times 10^4\, s + 9.76 \times 10^4}{2174\, s^3 + 7868\, s^2 + 4348\, s + 579.8},$$

$$C(s) = \frac{17^4}{(s+17)^4}.$$

This choice of $M(s)$ and $C(s)$ satisfies the condition (20). The sampling time of the SD controller is $T_s = 0.01$ sec. The initial condition of the simulation is $x_0 = [0.001,\ 0,\ -0.001]^\top$, and the nonlinear input disturbance is given by

$$f(t, x(t)) = 0.001x_1(t)x_2(t)\cos(5t) + 0.001\sin(x_1(t)x_2(t)) + 0.003x_2(t)x_3(t)\sin(3t).$$

In addition, a delay of 0.03 s is considered at the control input. A white noise with the power spectral density of 10^{-10} and the sample time of 0.01 sec is considered at the measured output. The simulation results (Fig. 3) indicate that the digital controller is robust to measurement noise, input delay, and nonlinear disturbances. The closed-loop system with the SD controller is stable and tracks the desired flight-path angle in the presence of the uncertainties as illustrated in Fig. 3(a). The control input is shown in Fig. 3(b). While many output feedback approaches based on high-gain observer amplify the noise at the control input, the filter in the SD $\mathcal{L}_1$ controller limits the noise amplification at the input channel.

6. Conclusions

This paper develops an adaptive sampled-data controller for a class of uncertain MIMO systems, possibly with non-minimum-phase zeros. Sufficient conditions for robust stability are obtained. It is shown that the closed-loop SD system recovers the performance and robustness of a continuous-time reference system, as the sampling time of the digital controller tends to zero. This paper provides a robust approach for digital implementation of output-feedback controllers. The simulation examples validate the theoretical claims. Future directions include relaxing the conservatism of sufficient conditions in general and a constructive approach for filter selection for non-minimum phase systems.

References

Ackerman, Kasey A., Xargay, Enric, Choe, Ronald, Hovakimyan, Naira, Cotting, M. Christopher, Jeffrey, Robert B., et al. (2016). $\mathcal{L}_1$ stability augmentation system for Calspan's variable-stability learjet. In *AIAA guidance, navigation and control conference*.

Ahmed Ali, Sofiane, Langlois, Nicolas, & Guermouche, Mohamed (2014). Sampled-data disturbance observer for a class of nonlinear systems. In *19th world congress, vol. 19* (pp. 3346–3351).

Ahrens, Jeffrey H., Tan, Xiaobo, & Khalil, Hassan K. (2009). Multirate sampled-data output feedback control with application to smart material actuated systems. *IEEE Transactions on Automatic Control*, *54*(11), 2518–2529.

Cao, Chengyu, & Hovakimyan, Naira (2008). Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture with guaranteed transient performance. *IEEE Transactions on Automatic Control*, *53*(2), 586–591.

Cao, Chengyu, & Hovakimyan, Naira (2009). $\mathcal{L}_1$ adaptive output-feedback controller for non-strictly-positive-real reference systems: missile longitudinal autopilot design. *Journal of Guidance, Control and Dynamics*, *32*(3), 717–726.

Cao, Chengyu, & Hovakimyan, Naira (2010). Stability margins of $\mathcal{L}_1$ adaptive control architecture. *IEEE Transactions on Automatic Control*, *55*, 480–487.

Chen, Tongwen, & Francis, Bruce (1995). *Optimal sampled-data control systems*.

Chu, Hongyan, Qian, Chunjiang, Yang, Jiquan, Xu, Shengyuan, & Liu, Yijian (2015). Almost disturbance decoupling for a class of nonlinear systems via sampled-data output feedback control. *International Journal of Robust and Nonlinear Control*.

Dullerud, Geir, & Glover, Keith (1993). Robust stabilization of sampled-data systems to structured LTI perturbations. *IEEE Transactions on Automatic Control*, *38*(10), 1497–1508.

Fridman, Emilia (2010). A refined input delay approach to sampled-data control. *Automatica*, *46*(2), 421–427.

Guillaume, A.-M., Bastin, Georges, & Campion, Guy (1994). Sampled-data adaptive control of a class of continuous nonlinear systems. *International Journal of Control*, *60*(4), 569–594.

Hovakimyan, Naira, & Cao, Chengyu (2010). *$\mathcal{L}_1$ adaptive control theory: guaranteed robustness with fast adaptation*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

Jafarnejadsani, Hamidreza (2018). Robust adaptive sampled-data control design for MIMO systems: applications in cyber-physical security. Ph.D. thesis, University of Illinois at Urbana-Champaign.

Jafarnejadsani, Hamidreza, Sun, Donglei, Lee, Hanmin, & Hovakimyan, Naira (2017). Optimized $\mathcal{L}_1$ adaptive controller for trajectory tracking of an indoor quadrotor. *Journal of Guidance, Control and Dynamics*.

Khalil, Hassan K. (2004). Performance recovery under output feedback sampled-data stabilization of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, *49*(12), 2173–2184.

Kharisov, Evgeny, & Hovakimyan, Naira (2011). L1 adaptive output feedback controller for minimum phase systems. In *American control conference (ACC), 2011* (pp. 1182–1187). IEEE.

Laila, Dina Shona, Navarro-López, Eva M., & Astolfi, Alessandro (2011). Sampled-data adaptive control for a class of nonlinear systems with parametric uncertainties. *IFAC Proceedings Volumes*, *44*(1), 1261–1266.

Lam, H. K. (2012). Stabilization of nonlinear systems using sampled-data output-feedback fuzzy controller based on polynomial-fuzzy-model-based control approach. *IEEE Transactions on Systems, Man and Cybernetics, Part B (Cybernetics)*, *42*(1), 258–267.

Lee, Hanmin (2017). L1 adaptive control for nonlinear and non-square multivariable systems. Ph.D. thesis, University of Illinois at Urbana-Champaign.

Lin, Wei, & Wei, Wei (2016). Robust stabilization of nonminimum-phase systems with uncertainty by sampled-data output feedback. In *55th IEEE conference on decision and control* (pp. 4078–4083).

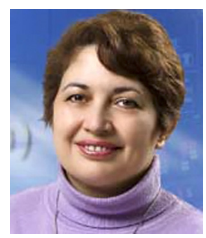
Lin, Wei, & Wei, Wei (2017). Robust control of a family of uncertain nonminimum-phase systems via continuous-time and sampled-data output feedback. *International Journal of Robust and Nonlinear Control*.

Liu, Wenhui, Ma, Qian, & Jia, Xiaglei (2016). Sampled-data output feedback control for a class of nonlinear systems with mismatched disturbances. In *35th Chinese control conference* (pp. 380–385).

Naghnaeian, Mohammad, Hirzallah, Nabil, & Voulgaris, Petros G. (2015). Dual rate control for security in cyber-physical systems. In *2015 54th IEEE conference on decision and control* (pp. 1415–1420).
Nesic, Dragan, & Laila, Dina Shona (2002). A note on input-to-state stabilization for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control*, *47*(7), 1153–1158.
Nesic, Dragan, & Teel, Andrew R. (2004). Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, *49*(10), 1650–1667.
Nesic, Dragan, Teel, Andrew R., & Carnevale, Daniele (2009). Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control*, *54*(3), 619–624.
Qian, Chunjiang, & Du, Haibo (2012). Global output feedback stabilization of a class of nonlinear systems via linear sampled-data control. *IEEE Transactions on Automatic Control*, *57*(11), 2934–2939.
Shim, Hyungbo, & Teel, Andrew R. (2003). Asymptotic controllability and observability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback. *Automatica*, *39*(3), 441–454.
Shkolnikov, Ilya A., & Shtessel, Yuri B. (2001). Aircraft nonminimum phase control in dynamic sliding manifolds. *Journal of Guidance, Control and Dynamics*, *24*(3), 566–572.
Sivashankar, N., & Khargonekar, Pramod P. (1994). Characterization of the L_2-induced norm for linear systems with jumps with applications to sampled-data systems. *SIAM Journal on Control and Optimization*, *32*(4), 1128–1150.
Sun, Donglei, Choe, Ronald, Xargay, Enric, & Hovakimyan, Naira (2016). An $\mathcal{L}_1$ adaptive backup flight control law for transport aircraft with vertical-tail damage. In *AIAA guidance, navigation, and control conference*.
Teel, Andrew R., Nesic, Dragan, & Kokotovic, Petar V. (1998). A note on input-to-state stability of sampled-data nonlinear systems. In *Proceedings of the 37th IEEE conference on decision and control, vol. 3* (pp. 2473–2478).
Wu, Buzhou, & Ding, Zhengtao (2007). A sampled-data scheme for adaptive control of nonlinear systems. In *2007 American control conference* (pp. 2893–2898).
Xargay, Enric, Hovakimyan, Naira, Dobrokhodov, Vladimir, Kaminer, Isaac, Cao, Chengyu, & Gregory, Irene M. (2012). $\mathcal{L}_1$ adaptive control in flight. *Intelligent Systems: AIAA Progress in Aeronautics and Astronautics*.
Young, Amanda, Cao, Chengyu, Hovakimyan, Naira, & Lavretsky, Eugene (2006). An adaptive approach to nonaffine control design for aircraft applications. In *AIAA guidance, navigation, and control conference and exhibit* (p. 6343).
Zhang, Chuanlin, & Yang, Jun (2016). Semi-global sampled-data output feedback disturbance rejection control for a class of uncertain nonlinear systems. *International Journal of Systems Science*, 1–12.

Hamidreza Jafarnejadsani received his B.S. and M.S. degrees both in Mechanical Engineering from the University of Tehran in 2011 and the University of Calgary in 2013, respectively. He got his Ph.D. degree in Mechanical Engineering from the University of Illinois at Urbana-Champaign (UIUC) in 2018. He is currently a postdoctoral research associate in the Department of Computer Science at UIUC. His research interests include robust and adaptive control with applications in the safety and security of cyber–physical and sampled-data systems.

Hanmin Lee received his B.S. and M.S. degrees in Mechanical and Aerospace Engineering from Seoul National University, Seoul, South Korea, in 2003 and 2005, respectively, and the Master's degree from the Department of Mathematics, University of Illinois at Urbana-Champaign (UIUC), Champaign, IL, USA, in 2016. He received his Ph.D. degree in aerospace engineering at UIUC in 2017.



Naira Hovakimyan received her M.S. degree in Theoretical Mechanics and Applied Mathematics in 1988 from Yerevan State University in Armenia. She got her Ph.D. in Physics and Mathematics in 1992 from the Institute of Applied Mathematics of Russian Academy of Sciences in Moscow, majoring in optimal control and differential games. She is currently a W. Grafton and Lillian B. Wilkins Professor of Mechanical Science and Engineering at the University of Illinois at Urbana-Champaign. Her research interests are in control and optimization, autonomous systems, neural networks, game theory and their applications in aerospace, robotics, mechanical, agricultural, electrical, petroleum, biomedical engineering and elderly care.