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Sensitivity Analysis of Continuous-Time Systems based on Power Spectral Density

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***Abstract*— Bode integrals of sensitivity and sensitivity-like functions along with complementary sensitivity and complementary sensitivity-like functions are conventionally used for describing performance limitations of a feedback control system. In this paper, we show that in the case when the disturbance is a wide sense stationary process the (complementary) sensitivity Bode integral and the (complementary) sensitivity-like Bode integral are identical. A lower bound of the continuous-time complementary sensitivity-like Bode integral is also derived and examined with the linearized flight-path angle tracking control problem of an F-16 aircraft.**

I. Introduction

The last two decades have witnessed a tremendous progress in communication technologies and their use in feedback control systems. A great deal of attention has been given to understanding the fundamental limitations of closed-loop systems in the presence of communication channels [1]–[6]. The main contribution of these papers was to derive performance limitations of stochastic nonlinear systems in the presence of limited information. While [1]–[3], [5], [6] looked into discrete-time systems and investigated the Bode-like integrals using Kolmogorov's entropy-rate equality [7], the results in [4] provided an extension to continuous-time systems by resorting to mutual information rates. In these papers, the notion of the sensitivity-like function was introduced to derive Bode-like integrals and corresponding lower bounds, which can be considered as a generalization of the classical result of Bode integrals for linear time-invariant (LTI) deterministic systems [8]. The classical result in [8] states that for open-loop stable transfer functions the Bode integral equals zero, while for unstable open-loop transfer functions it is lower bounded by the sum of unstable poles of the open-loop transfer function [9], [10]. Similar to the sensitivity function in a LTI system, the complementary sensitivity function is also used for robustness and performance analysis of closed-loop systems [11]. We notice that the result on the complementary sensitivity Bode integral was once hindered by the unboundedness of the integrand in high frequencies [12]. This issue was later overcome in [11] by adopting a weighted Bode integral of the complementary sensitivity function, proven to be lower bounded by the sum of the reciprocals of non-minimum phase zeros. Seminal results on this topic were reported also in [13], [14].

Performance limitations of stochastic systems in the presence of limited information were analyzed through sensitivity-like function $S(\omega)$ in [1]–[4] and the complementary sensitivity-like function $T(\omega)$ in [5], [6]. Taking an information-theoretic approach was the key to get Bode integrals extended to stochastic nonlinear systems. Unlike the frequency-domain approach, which explicitly depends on the input-output relationship of the feedback systems (transfer function), the focus of the information-theoretic approach is on the signals. The lower bound for sensitivity-like Bode integral for continuous-time systems was first put forward in [4]:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log|S(\omega)|d\omega \geq \sum_{\lambda\in\mathcal{UP}} p_i. \tag{1}$$

This result can be applied to systems with nonlinear controllers, which is an improvement upon the prior results based on the frequency-domain approach [8]–[15]. However, to the best of authors' knowledge, a lower bound for the complementary sensitivity-like Bode integral for continuous-time systems has not been derived yet. The unboundedness of the integrand in high frequencies as stated in [12] and the challenge in representing the weighted Bode-like integral with information-theoretic tools similar to [11] have been the main obstacles on this path.

In this paper, we provide a partial answer to the question: *What is the relationship between Bode integrals of the (complementary) sensitivity function and the (complementary) sensitivity-like function?* We answer this question for the continuous-time linear feedback system with a wide sense stationary input, while some partial answers on discrete-time systems can be found in [2], [6]. We notice that while Kolmogorov's entropy-rate equality has been used for discrete-time systems in [1]–[3], [5], [6] to obtain a lower bound for the sensitivity Bode-like integral, a seminal result on mutual information rates from [16, p. 181] was used in [4] to obtain a similar bound for continuous-time systems. In this paper, we resort to power spectral density (PSD) to analyze the sensitivity and the complementary sensitivity of continuous-time systems. With the convenience brought by this new tool, we first time find a lower bound and an information-theoretic representation for the complementary sensitivity Bode-like integral. The sensitivity properties of an F-16 aircraft in the flight-path angle tracking problem are analyzed.

The paper is organized as follows: Section II introduces

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the preliminaries on Bode integrals and information theory. Section III investigates the relationship between the sensitivity and the sensitivity-like Bode integrals. Section IV investigates the complementary sensitivity and the complementary sensitivity-like Bode integrals and proposes a lower bound for the latter. Section V presents a numerical example. Section VI draws the conclusion.

II. Preliminaries

Consider a continuous-time feedback configuration $\mathcal{P}$ depicted in Figure 1,

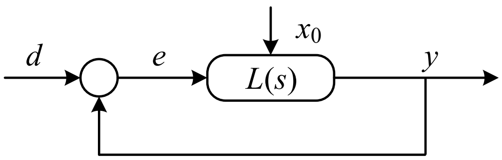

Fig. 1. Continuous-time feedback control system.

where $d(t) \in \mathbb{R}$ is the disturbance input, $y(t) \in \mathbb{R}$ is the output, $e(t) = d(t) - y(t)$ is the error signal, $x(t) \in \mathbb{R}^n$ is the state, and $L(s)$ denotes the open-loop transfer function from $e(t)$ to $y(t)$

$$L(s) = L(j\omega) = \int_0^\infty l(t) \cdot \mathrm{e}^{-j\omega t} dt, \tag{2}$$

with $l(t)$ being the impulse response of the system. In a deterministic setting, the initial condition x_0 in the configuration of Figure 1 is assumed zero. In a stochastic setting, one assumes that the differential entropy of the initial condition is finite [1]–[4]. Further discussion on these two different types of initial conditions is available in [6]. Let the open-loop transfer function $L(s)$ in Figure 1 be

$$L(s) = \frac{Y(s)}{E(s)} = c \cdot \frac{\prod_{j=1}^m (s - z_j)}{\prod_{i=1}^n (s - p_i)}, \tag{3}$$

where $m \leq n$, and $c > 0$. Inspired by [11], consider the following frequency transformation

$$\tilde{s} = j\tilde{\omega} = (j\omega)^{-1} = s^{-1}, \tag{4}$$

where $\tilde{\omega} = -\omega^{-1}$. Applying (4) to transfer function (3), the system with following transfer function $\tilde{L}(\tilde{s})$ is defined as the auxiliary system:

$$\tilde{L}(\tilde{s}) = \frac{\tilde{Y}(\tilde{s})}{\tilde{E}(\tilde{s})} = c \cdot \frac{\tilde{s}^n \cdot \prod_{j=1}^m (1 - \tilde{s} \cdot z_j)}{\tilde{s}^m \cdot \prod_{i=1}^n (1 - \tilde{s} \cdot p_i)} = L(s), \tag{5}$$

which can be depicted by the diagram in Figure 2.

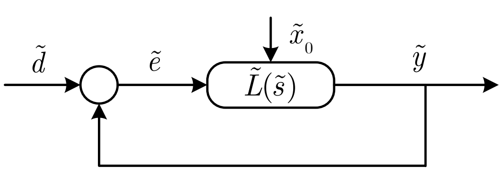

Fig. 2. Auxiliary system.

The Laplace transforms of the signals in the auxiliary system and the signals in the original system satisfy

$$\tilde{D}(\tilde{s}) = \tilde{D}(s^{-1}) = D(\tilde{s}^{-1}) = D(s), \tag{6}$$

which will also hold if d is replaced by e or y. It is worth noting that although the auxiliary system $\tilde{L}(\tilde{s})$ may not be proper, no intermediate result will be derived from this auxiliary system. The inverse system $\tilde{L}^{-1}(\tilde{s})$ is defined by swapping the input $\tilde{e}$ and the output $\tilde{y}$ of the auxiliary system. The transfer function of this inverse system then becomes:

$$\tilde{L}^{-1}(\tilde{s}) = \frac{\tilde{E}(\tilde{s})}{\tilde{Y}(\tilde{s})} = \frac{1}{c} \cdot \frac{1}{\tilde{s}^{n-m}} \cdot \frac{\prod_{i=1}^n (1 - \tilde{s} \cdot p_i)}{\prod_{j=1}^m (1 - \tilde{s} \cdot z_j)}, \tag{7}$$

which is illustrated in Figure 3.

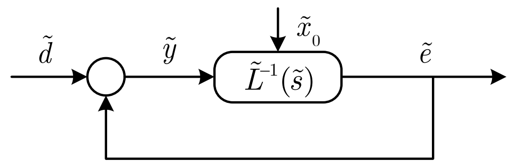

Fig. 3. Inverse of auxiliary system.

One can easily verify that if all the closed-loop poles of the original system is stable, the closed-loop poles of the inverse system will also be stable. To generalize the results of this paper to MIMO systems, interested readers can refer to [6], [17]. Before we continue to formulate the (complementary) sensitivity analysis problem, some basic definitions are given below following [4], [7].

Definition 1 (Wide Sense Stationary) A second order random process $\{x\}$ is called wide sense stationary, if

$$\begin{aligned} \mathbb{E}[x(t)] &= \mathbb{E}[x(t+v)], \\ \mathrm{Cov}[x(t), x(t+\tau)] &= \mathrm{Cov}[x(v), x(v+\tau)], \end{aligned} \tag{8}$$

where $\mathbb{E}$ denotes expectation.

Definition 2 (Mutual Information & Mutual Information Rate) The mutual information between two continuous-time stochastic processes x and y is defined as

$$I(x;y) = \int_Y \int_X f(x,y) \log \frac{f(x,y)}{f(x)f(y)} dxdy, \tag{9}$$

where $f(x,y)$ is the joint probability distribution function, and $f(x)$ and $f(y)$ are the marginal probability distribution functions. The mutual information rate is defined as

$$I_\infty(x;y) = \lim_{t\to\infty} \frac{I(x^t; y^t)}{t}. \tag{10}$$

Definition 3 (Class $\mathbb{F}$ Function; See [4] or [16, p. 182]) We define class $\mathbb{F}$ function in the following way:

$$\mathbb{F} = \{l : l(\omega) = p(\omega)(1 - \varphi(\omega)), l(\omega) \in \mathbb{C}, \omega \in \mathbb{R}\}, \tag{11}$$

where $p(\cdot)$ is rational and $\varphi(\cdot)$ is a measurable function, such that $0 \leq \phi \leq 1$ for all $\omega \in \mathbb{R}$ and $\int_{\mathbb{R}} |\log(1 - \varphi(\omega))| d\omega < \infty$.

The sensitivity function $S(j\omega)$ of the feedback system in Figure 1 is defined as the closed-loop transfer function from the disturbance input d to the tracking error e:

$$S(j\omega) = \frac{E(j\omega)}{D(j\omega)} = \frac{1}{1 + L(j\omega)}. \tag{12}$$

The complementary sensitivity function $T(j\omega)$ is defined as the closed-loop transfer function from the disturbance input d to the measurement output y:

$$T(j\omega)=\frac{Y(j\omega)}{D(j\omega)}=\frac{L(j\omega)}{1+L(j\omega)}. \tag{13}$$

The integrals of $S(j\omega)$ and $T(j\omega)$ over the whole frequency domain are referred to as Bode integrals and satisfy the following equalities [9], [10], [13]:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left|\frac{S(j\omega)}{S(j\infty)}\right|d\omega=\lim_{s\to\infty}\frac{s[S(s)-S(\infty)]}{2\cdot S(\infty)}+\sum_{p_i\in\mathcal{UP}}p_i, \tag{14}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left|\frac{T(j\omega)}{T(0)}\right|\frac{d\omega}{\omega^2}=\frac{1}{2T(0)}\lim_{s\to 0}\frac{dT(s)}{ds}+\sum_{z_i\in\mathcal{UZ}}\frac{1}{z_i}, \tag{15}$$

where $\mathcal{UP}$ and $\mathcal{UZ}$ respectively denote the set of unstable poles and the set of non-minimum phase zeros of the plant $\mathcal{P}$. Since (14) and (15) are derived in frequency domain using transfer functions, they cannot be applied to nonlinear systems.

Starting with [1], [2], information theoretic tools were leveraged to derive performance limitations and Bode-like results for nonlinear systems. Instead of considering the sensitivity function $S(j\omega)$, in [2], [4] sensitivity-like function $S(\omega)$ was introduced based on the properties of signals:

$$S(\omega)=\sqrt{\frac{\phi_e(\omega)}{\phi_d(\omega)}}, \tag{16}$$

where $\phi_x(\omega)$ denotes the PSD of a stationary signal x:

$$\phi_x(\omega)=\int_{-\infty}^{\infty}r_x(\tau)\cdot \mathrm{e}^{-j\omega\tau}d\tau, \tag{17}$$

and $r_x(\tau)=r_{xx}(t+\tau,t)$ denotes the auto-covariance of the signal x with

$$r_{xy}(v,t)=\mathrm{Cov}[x(v),y(t)].$$

The complementary sensitivity-like function was defined for discrete-time systems in [5]. Following the same philosophy, the following definition of the complementary sensitivity-like function is adopted in this paper:

$$T(\omega)=\sqrt{\frac{\phi_y(\omega)}{\phi_d(\omega)}}. \tag{18}$$

As we mentioned previously, the lower bound for Bode integral of $T(\omega)$ in continuous-time systems has not been studied yet. In the following sections, we first discuss the relationship between the (complementary) sensitivity and the (complementary) sensitivity-like Bode integrals and then propose a lower bound for the Bode integral of $T(\omega)$. Some lemmas and assumptions that we adopt in this paper are listed next.

Lemma 1 (See [4] or [16, p. 181]) Suppose that two one-dimensional continuous-time processes x and y form a stationary Gaussian process (x,y). Then

$$I_\infty(x,y)\geq-\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{|\phi_{xy}(\omega)|^2}{\phi_x(\omega)\phi_y(\omega)}\right)d\omega. \tag{19}$$

The equality holds, if ϕ_x and ϕ_y belong to the class $\mathbb{F}$.

Assumption 1 The disturbance input $d(t)$ is a zero-mean wide sense stationary process.

Remark 1 Compared with [1]–[3], [5], which assumed that d is an asymptotically stationary process, Assumption 1 is relatively stringent. However, this assumption is commonly adopted among the results on continuous-time systems in terms of signals, [4], [18].

Assumption 2 For the transfer function $L(s)$ the amount of zeros at $s=0$ does not exceed the amount of poles at $s=0$.

Remark 2 We only adopt this assumption when establishing a lower bound for the complementary sensitivity-like Bode integral. This assumption ensures that the inverse system $\tilde{L}^{-1}(\tilde{s})$ is proper, *e.g.* for a double integrator vehicle with first order actuator dynamics $L(s)=1/[s^2\cdot(0.1s+1)]$ from [19], we have $\tilde{L}^{-1}=(\tilde{s}+0.1)/\tilde{s}^3$. Similar assumption was adopted in [20], when investigating the string instability (sensitivity) via a frequency-domain approach.

III. Sensitivity and Sensitivity-Like Functions

We first investigate the relationship between Bode integrals of sensitivity function $S(j\omega)$ and sensitivity-like function $S(\omega)$ of the closed-loop configuration in Figure 1. The following theorem states this relationship.

Theorem 1 *When the disturbance input $d(t)$ is wide sense stationary, Bode integrals of the sensitivity and the sensitivity-like functions satisfy*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log S(\omega)d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\log|S(j\omega)|d\omega. \tag{20}$$

Proof. The proof is given in Appendix A.

IV. Complementary Sensitivity and Sensitivity-Like Functions

The relationship between Bode integrals of complementary sensitivity function $T(j\omega)$ and the complementary sensitivity-like function $T(\omega)$ in Figure 1 is summarized in the following corollary.

Corollary 2 *When the disturbance input $d(t)$ is wide sense stationary, Bode integrals of complementary sensitivity function $T(j\omega)$ and complementary sensitivity-like function $T(\omega)$ satisfy*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log T(\omega)\frac{d\omega}{\omega^2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\log|T(j\omega)|\frac{d\omega}{\omega^2}. \tag{21}$$

Proof. The proof is given in Appendix B.

From Corollary 2, we know that Bode integrals of $T(j\omega)$ and $T(\omega)$ are equivalent, when the disturbance input is wide sense stationary. The following theorem gives a lower bound for the Bode integral of $T(\omega)$ in continuous-time setting.

Theorem 3 *When the original system in Figure 1 is mean-square stable and the inverse frequency noise $\tilde{d}$ is wide sense* *stationary, one has:*

$$I_\infty(\tilde{y};\tilde{e}) - I_\infty(\tilde{d};\tilde{e}) \geq \sum_{z_i\in\mathcal{UZ}} \frac{1}{z_i}, \tag{22}$$

where $\mathcal{UZ}$ is the set of unstable zeros of the plant $\mathcal{P}$, and $\tilde{e}$ and $\tilde{y}$ are the signals defined in the (inverse) auxiliary system. Moreover, when the disturbance input $\tilde{d}$ is Gaussian stationary, the complementary sensitivity-like Bode integral satisfies

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \log T(\omega)\frac{d\omega}{\omega^2} \geq \sum_{z_i\in\mathcal{UZ}} \frac{1}{z_i}. \tag{23}$$

Proof. By the frequency transform (4), we can rewrite the complementary sensitivity-like Bode integral defined in (21) as follows

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \log T(\omega)\frac{d\omega}{\omega^2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \log T(-\tilde{\omega}^{-1})d\tilde{\omega} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \log \tilde{T}(\tilde{\omega})d\tilde{\omega},\end{aligned} \tag{24}$$

where by Corollary 2 the complementary sensitivity-like function of auxiliary system $\tilde{T}(\tilde{\omega})$ satisfies

$$\tilde{T}(\tilde{\omega}) = \sqrt{\frac{\phi_{\tilde{y}}(\tilde{\omega})}{\phi_{\tilde{d}}(\tilde{\omega})}} = \sqrt{\frac{\phi_y(-\tilde{\omega}^{-1})}{\phi_d(-\tilde{\omega}^{-1})}} = T(\omega). \tag{25}$$

Meanwhile, since the complementary sensitivity-like function of the auxiliary system is identical to the sensitivity-like function of the inverse system, our task becomes to seek a lower bound for the sensitivity Bode-like integral for the inverse system shown in Figure 3. Since the inverse frequency noise $\tilde{d}$ is a wide sense stationary process, applying Theorem 4.8 in [4] to the inverse system, we have

$$I_\infty(\tilde{y};\tilde{e}) - I_\infty(\tilde{d};\tilde{e}) \geq \sum_{z_i\in\mathcal{UZ}} \frac{1}{z_i}. \tag{26}$$

When the disturbance $\tilde{d}$ is stationary Gaussian, according to (25) and Theorem 4.8 in [4], we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \log T(\omega)\frac{d\omega}{\omega^2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \log \tilde{T}(\tilde{\omega})d\tilde{\omega} \\ &= I_\infty(\tilde{y};\tilde{e}) - I_\infty(\tilde{d};\tilde{e}) \geq \sum_{z_i\in\mathcal{UZ}} \frac{1}{z_i}.\end{aligned} \tag{27}$$

This completes the proof. ∎

Remark 3 Since $\log T(\omega) = \log|T(j\omega)|$ tends to infinity as $\omega\to\infty$, similar to (15), we define the Bode-like integral of $T(\omega)$ with a weighting factor $1/\omega^2$ in (23). We note that this weighting factor induces some restrictions when analyzing the complementary sensitivity via information-theoretic approach, such as the requirement of stationary Gaussian condition on the inverse frequency signal.

Remark 4 When the disturbance $\tilde{d}$ is Gaussian stationary and the initial condition $\tilde{x}_0$ is Gaussian, by Lemma 1 we can express the mutual information rate $I_\infty(\tilde{y},\tilde{e})$ in terms of the density functions of e and y:

$$\begin{aligned}I_\infty(\tilde{y},\tilde{e}) &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{|\phi_{\tilde{y}\tilde{e}}(\tilde{\omega})|^2}{\phi_{\tilde{y}}(\tilde{\omega})\phi_{\tilde{e}}(\tilde{\omega})}\right)d\tilde{\omega} \\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{|\phi_{ye}(\omega)|^2}{\phi_{y}(\omega)\phi_{e}(\omega)}\right)\frac{d\omega}{\omega^2}.\end{aligned} \tag{28}$$

The expression of $I_\infty(\tilde{d};\tilde{e})$ can be readily implied.

V. An Illustrative Example

With the lower bound of the complementary sensitivity Bode-like integral given in Theorem 3, we now investigate the control trade-offs in an aircraft flight-path angle tracking problem. Considering an F-16 aircraft with Mach $= 0.7$ and altitude $h = 10,000$ ft, the linearized longitudinal dynamics can be described by the following state-space model [21].

$$A = \begin{bmatrix}-11.707 & 0 & -75.666\\ 0 & 11.141 & -79.908\\ 0.723 & 0.907 & -1.844\end{bmatrix},\quad B = \begin{bmatrix}0\\0\\0.117\end{bmatrix},$$
$$C = \begin{bmatrix}0, & 0, & 1\end{bmatrix},$$

where the input is elevator deflection $\delta_e(t)$, and the output is flight-path angle $\gamma(t)$. With zero initial condition, the longitudinal dynamics in state-space form can be equivalently described by the following transfer function

$$G(s) = \frac{0.117\cdot(s+11.71)(s-11.14)}{(s+2.979)(s-1.051)(s+0.4826)},$$

which contains a non-minimum phase zero at $s = 11.14$ and an unstable pole at $s = 1.051$. Consider the following two PID controllers with different sets of parameters:

$$C_1(s) = -0.4 - 0.06\cdot\frac{1}{s} - 1\cdot\frac{100}{1+100\cdot 1/s},$$
$$C_2(s) = 2\cdot C_1(s),$$

where $100/(1+100/s)$ is an approximation of the derivative term in PID controller, and the open-loop transfer functions $L_1(s) = G(s)C_1(s)$ and $L_2(s) = G(s)C_2(s)$.

With the plant transfer function $G(s)$ and control mapping $C_1(s)$, we first verify the lower bounds of Bode-like integrals. By Lemma 1, we can compute the Bode-like integral in (23) with the complementary sensitivity function defined by $L_1(s)$, which gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log T(\omega)\ \frac{d\omega}{\omega^2} = 0.915 \geq 8.977\times 10^{-2} = \sum_{z_i\in\mathcal{UZ}}\frac{1}{z_i}.$$

The sensitivity Bode-like integral can also be computed as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log S(\omega)d\omega = 6.925 \geq 1.051 = \sum_{p_i\in\mathcal{UP}} p_i.$$

Remark 5 Although both the sensitivity and complementary sensitivity Bode-like integrals are bounded in this example, for arbitrary causal transfer functions $L(s)$ that are closed-loop stable, these two Bode-like integrals are not guaranteed to be bounded. A comprehensive discussion on the boundedness of sensitivity Bode integral subject to the different conditions of the open-loop transfer functions $L(s)$ is available in [22].

With the linearized longitudinal dynamics $G(s)$ and controller mappings $C_1(s)$ and $C_2(s)$, by Theorem 1 and Corollary 2, the magnitudes of complementary sensitivity-like functions and sensitivity-like functions are given in Figure 4, in which the solid lines denote the data with $C_1(s)$ and the dashed lines represent the data with $C_2(s)$. Subject to disturbance $d(t)$, the complementary sensitivity-like and sensitivity-like functions shown in Figure 4 tell that control mapping $C_1(s)$ performs better in disturbance mitigation in higher frequencies ($\omega > 5$ rad $\cdot$ s^{-1}), while control mapping $C_2(s)$ performs better when attenuating the disturbance of lower frequencies ($\omega < 5$ rad $\cdot$ s^{-1}), which can be explained by inequalities (1) and (23), since the area below the solid line should equal to the area below the dashed line when the control mappings do not contain any unstable pole and non-minimum phase zero. This phenomenon is also known as the water-bed effect [23].

VI. Conclusions

We discussed the relationship between Bode integrals of (complementary) sensitivity functions and (complementary) sensitivity-like functions. A lower bound for the continuous-time complementary sensitivity Bode-like integral was derived based on the power spectral densities of signals. The lower bound was later examined with the linearized flight-path angle tracking control problem of an F-16 aircraft. Future discussions may include relaxing distribution condition on the disturbance signal and generalizing these results to nonlinear systems.

Appendix

A. Proof of Theorem 1

Since $d(t) = e(t)+y(t)$, the density function $\phi_d(\omega)$ in (16) satisfies

$$\begin{aligned}\phi_d(\omega) &= \int_{-\infty}^{\infty} r_d(\tau)\cdot \mathrm{e}^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} [r_e(\tau) + r_{ey}(\tau) + r_{ye}(-\tau) + r_y(\tau)]\, \mathrm{e}^{-j\omega\tau} d\tau \\ &= \phi_e(\omega) + \phi_{ey}(\omega) + \phi_{ye}(\omega) + \phi_y(\omega) \qquad (29)\end{aligned}$$

Letting $\tau = v - t$, and noticing that $y(t) = \int_0^\infty l(v')e(t - v')dv'$, subject to Assumption 1, the covariances r_e, r_{ey}, r_{ye} and r_y, in (29) satisfy

$$\begin{aligned}r_e(v,t) &= \mathrm{Cov}[e(t+v-t), e(t)] \\ &= \mathrm{Cov}[e(t+\tau), e(t)] = r_e(\tau) \qquad (30a)\\ r_{ey}(v,t) &= \mathrm{Cov}[e(v), \int_0^\infty l(v')e(t-v')dv'] \\ &= \int_0^\infty l(v')r_e(v'+\tau)dv' \\ &= r_{ey}(\tau) \\ r_{ye}(v,t) &= \mathrm{Cov}[\int_0^\infty l(v')e(v-v')dv', e(t)] \\ &= \int_0^\infty l(v')r_e(-v'+\tau)dv' \qquad (30b)\\ &= r_{ye}(\tau) \\ r_y(v,t) &= \mathrm{Cov}[\int_0^\infty l(v')e(v-v')dv', \int_0^\infty l(t')e(t-t')dt'] \\ &= \int_0^\infty\int_0^\infty l(v')l(t')\cdot r_e(\tau - v' + t')dv'dt' \\ &= r_y(\tau) \qquad (30c)\end{aligned}$$

Hence the spectral density functions ϕ_{ey}, ϕ_{ye}, and ϕ_y, in (29) satisfy

$$\begin{aligned}\phi_{ey}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} r_{ey}(\tau)\cdot \mathrm{e}^{-j\omega\tau}d\tau \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{j\omega v'} l(v') \int_{-\infty}^{\infty} \mathrm{e}^{-j\omega(\tau+v')} r_e(\tau+v')d\tau dv' \\ &= L(-j\omega)\phi_e(\omega) \qquad (31a)\\ \phi_{ye}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} r_{ye}(\tau)\cdot \mathrm{e}^{-j\omega\tau}d\tau \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-j\omega v'} l(v') \int_{-\infty}^{\infty} \mathrm{e}^{-j\omega(\tau-v')} r_e(\tau-v')d\tau dv' \\ &= L(j\omega)\phi_e(\omega) \qquad (31b)\\ \phi_y(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} r_y(\tau)\cdot \mathrm{e}^{-j\omega\tau}d\tau \\ &= \int_0^\infty \mathrm{e}^{j\omega t'} l(t') \int_0^\infty \mathrm{e}^{-j\omega s'} l(s')\cdot \qquad (31c)\\ &\quad \int_{-\infty}^{\infty} \mathrm{e}^{-j\omega(\tau-s'+t')} r_e(\tau - s' + t')d\tau ds' dt' \\ &= L(-j\omega)L(j\omega)\phi_e(\omega)\end{aligned}$$

Substituting (29) and (31) into the sensitivity-like function $S(\omega)$ defined in (16), we can rewrite the sensitivity-like function as follows

$$S(\omega) = \sqrt{\frac{\phi_e(\omega)}{[1+L(-j\omega)]\cdot[1+L(j\omega)]\cdot\phi_e(\omega)}}. \qquad (32)$$

When $\phi_e(\omega) \not\equiv 0$, we have

$$S(\omega) = \sqrt{S(-j\omega)\cdot S(j\omega)}. \qquad (33)$$

Since $S(-j\omega) = \bar{S}(j\omega)$, where $\bar{S}(j\omega)$ is the complex conjugate of $S(j\omega)$, the equality (20) in Theorem 1 can be retrieved from

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \log S(\omega)\; d\omega &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \log\left[S(-j\omega)\cdot S(j\omega)\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \log|S(j\omega)| d\omega. \qquad (34)\end{aligned}$$

This completes the proof. ■

B. Proof of Corollary 2

Substituting (29) and (31) into the complementary sensitivity-like function $T(\omega)$ defined in (18), we can then rewrite $T(\omega)$ as follows

$$T(\omega) = \sqrt{\frac{L(-j\omega)L(j\omega)\cdot\phi_e}{[1+L(-j\omega)]\cdot[1+L(j\omega)]\cdot\phi_e}} \qquad (35)$$

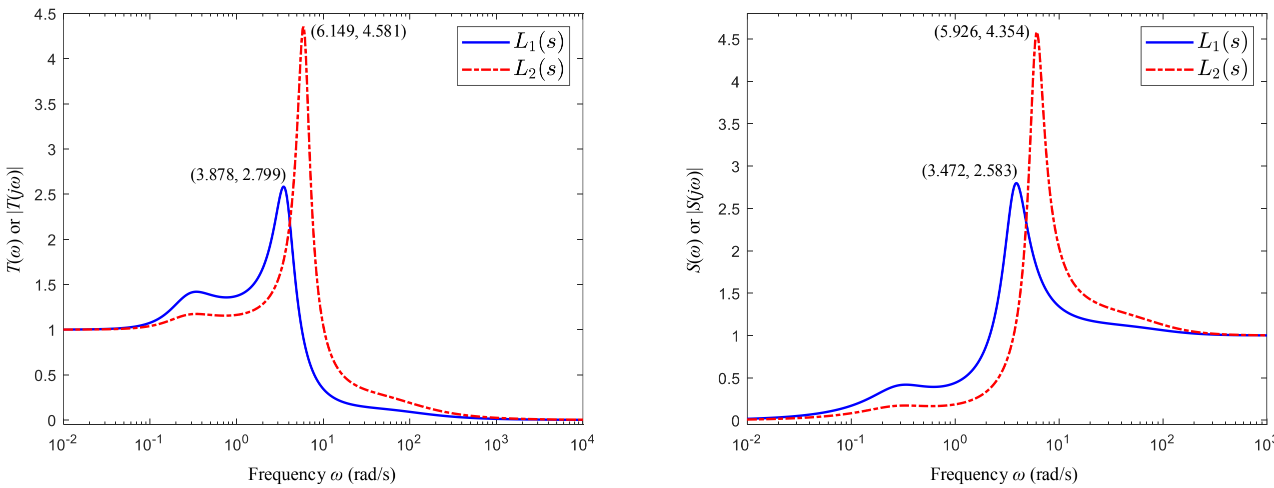

Fig. 4. Complementary sensitivity-like functions (left) and sensitivity-like integrals (right).

When $\phi_e(\omega) \not\equiv 0$, it follows that

$$T(\omega) = \sqrt{T(-j\omega) \cdot T(j\omega)} \tag{36}$$

Since $T(-j\omega) = \bar{T}(j\omega)$, where $\bar{T}(j\omega)$ is the complex conjugate of $T(j\omega)$, the equality (21) in Corollary 2 can be retrieved from

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\log T(\omega)\frac{d\omega}{\omega^2} &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\log[T(-j\omega)\cdot T(j\omega)]\frac{d\omega}{\omega^2} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\log|T(j\omega)|\frac{d\omega}{\omega^2}\end{aligned} \tag{37}$$

This completes the proof. ■

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