# DESIGN OF UNIMODULAR SEQUENCE SETS WITH GOOD CORRELATION AND COMPLEMENTARY CORRELATION PROPERTIES

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## ABSTRACT

The design of sequences with good correlation properties has been of long-standing interest in signal processing owing to their applications in diverse fields of technology. The computational construction of such sequences typically relies on the minimization of Integrated Sidelobe Level (ISL)-related metrics that effectively lower the correlation magnitude for out-of-phase lags. In this paper, we address the design of sets of unimodular sequences with good cross-correlation and complementary cross-correlation properties over a zero-correlation-zone. The sequence sets are designed by employing a complete second-order characterization, which has been proven to be beneficial for identification and sensing systems that employ widely linear (WL) signal processing. Numerical results show the potential of the generated sequences in identifying both strictly linear and widely linear systems.

***Index Terms*—** Unimodular sequences, correlation, complementary correlation, weighted integrated sidelobe level

## 1. INTRODUCTION

In many areas of applied science and technology, it is common practice to employ probing sequences that possess specific characteristics suited to extract desired information from the systems under study [1–3]. One of the descriptors usually considered for a discrete sequence $x(n)$ is its auto-correlation function. The auto-correlation is a measure of similarity of the sequence $x(n)$ with its own time-shifted versions. Considerable effort has been devoted to analyze discrete-time sequences that possess *good* correlation properties, especially in wireless communication and sensing (see [4] and references therein). A discrete sequence is said to have good correlation properties if the magnitudes of the out-of-phase correlation lags are low, ideally zero. In communications, this characteristic is imposed to guarantee resilience to multi-path interference and facilitate the channel estimation task at the receiver. With the evolution of communication systems, sets of sequences $\{x_m(n)\}_{n=0}^{N-1}$ for $m = 1, 2, \ldots, M$, were considered to handle multiple users, imposing restrictions on their cross-correlation properties. For a pair of sequences $x_m(n)$ and $x_{m'}(n)$, their cross-correlation function is defined as,

$$r_{m,m'}(l) = \sum_{n=l}^{N-1} x_m^*(n) x_{m'}(n-l), \tag{1}$$

for $-(N-1) \leq l \leq (N-1)$. In order to mitigate interference due to multiple users, it is desirable to generate sequences which are uncorrelated to each other. The literature on this topic is quite extensive (e.g. see [5, 7, 8]). However, it can be noted that designing such sequences for length $N \sim 10^4$ or larger is still deemed to be impractical using the current standard computational tools.

Recently, the use of numerical techniques to generate sequences with good cross-correlation properties have gained increased attention [5–12]. These methods seek to minimize a function that incorporates a weighted sum of the energy in the sidelobe levels of the sequences, referred to as the weighted integrated sidelobe level (WISL). The optimization is achieved by using alternating projections via the Fast Fourier Transform [5], quasi-Newton methods [3, 6, 16], and the majorization-minimization method [8–11] among others. Nonetheless, there has been less attention to developing methods for generating sequences that take into account a complete second order description [17, 19, 20] in order to consider especially WL systems [14], which essentially incorporates the complementary correlation (or relation) function [13, 15] in the design method. The complementary correlation of a pair of sequences $x_m(n)$ and $x_{m'}(n)$, can be given as,

$$\gamma_{m,m'}(l) = \sum_{n=l}^{N-1} x_m(n) x_{m'}(n-l), \tag{2}$$

for $-(N-1) \leq l \leq (N-1)$. As has been documented, the correlation and the complementary correlation functions can be used to completely characterize the second order statistics of complex signals. In wireless communications, WL systems

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have been used to analyze baseband models that consider radio frequency impairments such as in-phase and quadrature-phase (I/Q) imbalances [15].

In this paper, we address the design of sequence sets with good correlation and complementary correlation properties through optimizing an objective function which minimizes the second order descriptors of the sequences over a subset of lags. It is shown via numerical examples, that the statistical properties of the designed sequences can be fully exploited to identify strictly linear (SL) systems and WL systems.

*Notation*. Throughout the paper, we denote matrices with bold-face uppercase letters. The $N$ dimensional discrete Fourier transform matrix is denoted by $\mathbf{F}_N$. For a function of $n$ variables, collected in the vector $\boldsymbol{\Phi} = [\phi_0, \phi_1, \ldots, \phi_n]^T$, its partial derivatives with respect to $\{\phi_i\}_{i=0}^{n}$ are represented in a compact form by $\frac{\partial f(\boldsymbol{\Phi})}{\partial \boldsymbol{\Phi}} = [\frac{\partial f}{\partial \phi_0}, \ldots, \frac{\partial f}{\partial \phi_n}]^T$.

## 2. PROBLEM FORMULATION

The problem addressed herein is the design of unimodular sequences that possess good correlation and complementary correlation properties for a given set of lags. Let $\{x_m(n)\}_{n=0,m=1}^{N-1,M}$ denote a set of $M$ sequences with length $N$. We are interested in determining the sequences that minimize the following criterion [20],

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 \tag{3}$$

where

$$\mathcal{C}_1 = \sum_{m=1}^{M} \sum_{\substack{l=-(N-1),\\ l\neq 0}}^{N-1} \left(\alpha_{m,m}^{(l)}\right)^2 |r_{m,m}(l)|^2 , \tag{4}$$

$$\mathcal{C}_2 = \sum_{m=1}^{M} \sum_{l=-(N-1)}^{N-1} \left(\beta_{m,m}^{(l)}\right)^2 |\gamma_{m,m}(l)|^2 , \tag{5}$$

$$\mathcal{C}_3 = \sum_{m=1}^{M} \sum_{\substack{m'=1,\\ m'\neq m}}^{M} \sum_{l=-(N-1)}^{N-1} \left(\alpha_{m,m'}^{(l)}\right)^2 |r_{m,m'}(l)|^2 , \tag{6}$$

$$\mathcal{C}_4 = \sum_{m=1}^{M} \sum_{\substack{m'=1,\\ m'\neq m}}^{M} \sum_{l=-(N-1)}^{N-1} \left(\beta_{m,m'}^{(l)}\right)^2 |\gamma_{m,m'}(l)|^2 , \tag{7}$$

subject to the restriction $|x_m(n)|^2 = 1$ for $n = 0, 1, \ldots, N-1$ and $m = 1, 2, \ldots, M$. The scalars $\{\alpha_{m,m}^{(l)}\}$ and $\{\beta_{m,m'}^{(l)}\}$ are positive real numbers used for the purpose of weighing the correlation and complementary correlation functions of the sequences in the set to be designed.

**Remark:** Note that, in above formulations $\mathcal{C}_1$ and $\mathcal{C}_3$ are responsible for minimizing the auto-correlation and the cross-correlation functions of the sequences in the said set and the corresponding weights $\{\alpha_{m,m'}^{(l)}\}$ define the zero-correlation-zone (ZCZ) region. Similarly, $\mathcal{C}_2$ and $\mathcal{C}_4$ impose low complementary correlation values according to the weights $\{\beta_{m,m'}^{(l)}\}$. Moreover, (3) can be considered to define a Generalized Weighted Integrated Sidelobe Level (GWISL) [18,20], where a complete second order characterization (that encompasses the correlation and the relation functions) is considered to construct the set of interest. In fact, if we set all $\{\beta_{m,m'}^{(l)}\}$ to 0, then (3) reduces to a WISL criterion as reported in [5] and [11].

## 3. NUMERICAL DESIGN APPROACH

In this section, we take advantage of the inherent symmetries in each of the functions given in (4-7), to write (3) in vector notation. This is done with the objective of deriving a closed form expression for the gradient that enables us to use a quasi-Newton based method to solve the stated optimization problem. Note that, due to the unimodularity restriction, i.e. $x_m(n) = e^{i\phi_n^{(m)}}$ $\forall\, n, m$, the problem reduces to find the values of $MN$ phases that minimize (3).

### 3.1. Correlation and complementary correlation of extended sequences

In order to simplify (3), the following vectors are considered $\boldsymbol{\Phi}_m = [\phi_0^{(m)}, \phi_1^{(m)}, \ldots, \phi_{N-1}^{(m)}]^T$, for $m = 1, 2, \ldots, M$ and $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1^T\ \boldsymbol{\Phi}_2^T \cdots \boldsymbol{\Phi}_N^T]_{MN\times 1}^T$. Additionally, we define $\bar{\mathbf{x}}_m = [e^{i\phi_0^{(m)}}, e^{i\phi_1^{(m)}}, \ldots, e^{i\phi_{N-1}^{(m)}}, 0, 0, \ldots, 0]_{2N\times 1}^T$ by extending sequences $\{x_m(n)\}_{n=0}^{N-1}$ with $N$ zeros padded in the end. These extended sequences are used to compute the correlation and complementary correlation between $x_m(n)$ and $x_{m'}(n)$, for the lags $l = 0, 1, \ldots, N-1$, via a Discrete Fourier Transform (DFT) given as,

$$\bar{\mathbf{r}}_{m,m'} = \frac{1}{2N}\mathbf{F}_{2N}\left((\mathbf{F}_{2N}\bar{\mathbf{x}}_m)^* \circ (\mathbf{F}_{2N}\bar{\mathbf{x}}_{m'})\right) \tag{8}$$

and

$$\bar{\boldsymbol{\gamma}}_{m,m'} = \frac{1}{2N}\mathbf{F}_{2N}\left((\mathbf{R}_{2N}\mathbf{F}_{2N}\bar{\mathbf{x}}_m) \circ (\mathbf{F}_{2N}\bar{\mathbf{x}}_{m'})\right), \tag{9}$$

respectively. The matrix $\mathbf{R}_{2N}$ in (9) is given by $\mathbf{R}_{2N} = [\mathbf{e}_0, \mathbf{e}_{2N-1}, \ldots, \mathbf{e}_1]_{2N\times 2N}$ where $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2N-1}\}$ denotes the standard basis for $\mathbb{C}^{2N\times 1}$.

### 3.2. On the optimization with respect to (w.r.t.) $\mathcal{C}_1$

The function $\mathcal{C}_1$ can be simplified, if we impose the restrictions $\alpha_{m,m}^{(-l)} = \alpha_{m,m}^{(l)}$ for $l = 1, 2, \ldots, N-1$ (symmetric weights for positive and negative correlation lags). Furthermore, by making use of the Hermitian symmetry of the auto-correlation function, $r_{m,m}(-l) = r_{m,m}^*(l)$, we can write $\mathcal{C}_1$

as,

$$\mathcal{C}_1(\mathbf{\Phi}) = \sum_{m=1}^{M} \bar{\mathbf{r}}_{m,m}^H \mathbf{U}_{m,m} \bar{\mathbf{r}}_{m,m} \quad (10)$$

where $\mathbf{U}_{m,m} = \text{diag}[0, \alpha_{m,m}^{(1)}, \alpha_{m,m}^{(2)}, \ldots, \alpha_{m,m}^{(N-1)}, 0, 0, \ldots, 0]$. From (10), it is possible to compute the partial derivatives of $\mathcal{C}_1$ with respect to $\phi_0^{(m)}, \ldots, \phi_{N-1}^{(m)}$, by

$$\frac{\partial \mathcal{C}_1(\mathbf{\Phi})}{\partial \mathbf{\Phi}_m} = 2\text{Re}\left\{ \left( \frac{\partial \bar{\mathbf{r}}_{m,m}(\mathbf{\Phi}_m)}{\partial \mathbf{\Phi}_m} \right)^T \mathbf{U}_{m,m}^T \bar{\mathbf{r}}_{m,m}^* \right\} \quad (11)$$

for $m = 1, 2, \ldots, M$, where $\left[\frac{\partial \bar{\mathbf{r}}_{m,m}}{\partial \mathbf{\Phi}_m}\right]_{2N \times N}$ denotes the Jacobian of $\bar{\mathbf{r}}_{m,m}$. Furthermore, the columns of $\frac{\partial \bar{\mathbf{r}}_{m,m}}{\partial \mathbf{\Phi}_m}$ are computed from,

$$\frac{\partial \bar{\mathbf{r}}_{m,m}}{\partial \phi_n^{(m)}} = \frac{\mathbf{F}_{2N}}{N} \text{Re}\left\{ (\mathbf{F}_{2N} \bar{\mathbf{x}}_m)^* \circ \left( \mathbf{F}_{2N} \frac{\partial \bar{\mathbf{x}}_m}{\partial \phi_n^{(m)}} \right) \right\} \quad (12)$$

for $n = 0, 1, \ldots, N-1$. Now, from the structure of each of the vectors containing the elements of the extended sequences, we have that $\frac{\partial \bar{\mathbf{x}}_m}{\partial \phi_n^{(m)}} = i e^{i\phi_n^{(m)}} \mathbf{e}_n$. Hence, the gradient of $\mathcal{C}_1$ can be computed from (11) as, $\frac{\partial \mathcal{C}_1(\mathbf{\Phi})}{\partial \mathbf{\Phi}} = [(\frac{\partial \mathcal{C}_1}{\partial \mathbf{\Phi}_1})^T, \ldots, (\frac{\partial \mathcal{C}_1}{\partial \mathbf{\Phi}_M})^T]^T$.

### 3.3. On the optimization w.r.t. $\mathcal{C}_2$

If symmetric weights are used for positive and negative lags, and the even symmetry of the complementary correlation function is taken into account, this is $\gamma_{m,m}(-l) = \gamma_{m,m}(l)$, then (5) can be cast in vector notation as,

$$\mathcal{C}_2(\mathbf{\Phi}) = \sum_{m=1}^{M} \bar{\boldsymbol{\gamma}}_{m,m}^H \mathbf{V}_{m,m} \bar{\boldsymbol{\gamma}}_{m,m} \quad (13)$$

where, $\mathbf{V}_{m,m} = \text{diag}[\beta_{m,m}^{(0)}, \beta_{m,m}^{(1)}, \ldots, \beta_{m,m}^{(N-1)}, 0, \ldots, 0]_{2N \times 2N}$ for $m = 1, 2, \ldots, M$. Furthermore, $\frac{\partial \mathcal{C}_2}{\partial \mathbf{\Phi}_m}$, is evaluated through the expression,

$$\frac{\partial \mathcal{C}_2(\mathbf{\Phi})}{\partial \mathbf{\Phi}_m} = 2\text{Re}\left\{ \left( \frac{\partial \bar{\boldsymbol{\gamma}}_{m,m}}{\partial \mathbf{\Phi}_m} \right)^T \mathbf{V}_{m,m} \bar{\boldsymbol{\gamma}}_{m,m}^* \right\}. \quad (14)$$

The columns of the Jacobian matrix $\left[\frac{\partial \bar{\boldsymbol{\gamma}}_{m,m}}{\partial \mathbf{\Phi}_m}\right]_{2N \times N}$, are computed in turn by,

$$\begin{aligned} \frac{\partial \bar{\boldsymbol{\gamma}}_{m,m}}{\partial \phi_n^{(m)}} &= \frac{\mathbf{F}_{2N}}{2N} \left( (\mathbf{R}_{2N} \mathbf{F}_{2N} \bar{\mathbf{x}}_m) \circ \left( \mathbf{F}_{2N} \frac{\partial \bar{\mathbf{x}}_m}{\partial \phi_n^{(m)}} \right) + \right. \\ &+ \left. \left( \mathbf{R}_{2N} \mathbf{F}_{2N} \frac{\partial \bar{\mathbf{x}}_m}{\partial \phi_n^{(m)}} \right) \circ (\mathbf{F}_{2N} \bar{\mathbf{x}}_m) \right). \quad (15) \end{aligned}$$

Similarly, the gradient of $\mathcal{C}_2$ is obtained from the $M$ vectors $\frac{\partial \mathcal{C}_2}{\partial \mathbf{\Phi}_m}$, arranged as $\frac{\partial \mathcal{C}_2}{\partial \mathbf{\Phi}} = [(\frac{\partial \mathcal{C}_2}{\partial \mathbf{\Phi}_1})^T, \ldots, (\frac{\partial \mathcal{C}_2}{\partial \mathbf{\Phi}_M})^T]^T$.

### 3.4. On the optimization w.r.t. $\mathcal{C}_3$

The function $\mathcal{C}_3$ can be expressed as,

$$\begin{aligned} \mathcal{C}_3(\mathbf{\Phi}) &= \sum_{m=1}^{M-1} \sum_{m'=m+1}^{M} \bar{\mathbf{r}}_{m,m'}^H \mathbf{U}_{m,m'} \bar{\mathbf{r}}_{m,m'} + \\ &+ \sum_{m=1}^{M-1} \sum_{m'=m+1}^{M} \bar{\mathbf{r}}_{m',m}^H \mathbf{U}_{m',m} \bar{\mathbf{r}}_{m',m} \quad (16) \end{aligned}$$

if the restrictions $\alpha_{m,m'}^{(-l)} = \alpha_{m',m}^{(l)}$ are imposed. Their matrices $\mathbf{U}_{m,m'}$ and $\mathbf{U}_{m',m}$ contain the weights $\alpha_{m,m'}^{(l)}$ and $\alpha_{m',m}^{(l)}$, respectively in their diagonals. Furthermore, $\frac{\partial \mathcal{C}_3}{\partial \mathbf{\Phi}_m}$ is computed from,

$$\begin{aligned} \frac{\partial \mathcal{C}_3(\mathbf{\Phi})}{\partial \mathbf{\Phi}_m} &= 2 \sum_{l=1}^{m-1} \text{Re}\left\{ \left( \frac{\partial \bar{\mathbf{r}}_{l,m}}{\partial \mathbf{\Phi}_m} \right) \mathbf{U}_{l,m}^T \bar{\mathbf{r}}_{l,m}^* \right\} + \\ &+ 2 \sum_{l=1}^{m-1} \text{Re}\left\{ \left( \frac{\partial \bar{\mathbf{r}}_{m,l}}{\partial \mathbf{\Phi}_m} \right) \mathbf{U}_{m,l}^T \bar{\mathbf{r}}_{m,l}^* \right\} + \\ &+ 2 \sum_{l=m+1}^{M} \text{Re}\left\{ \left( \frac{\bar{\mathbf{r}}_{l,m}}{\partial \mathbf{\Phi}_m} \right)^T \mathbf{U}_{l,m}^T \bar{\mathbf{r}}_{l,m}^* \right\} + \\ &+ 2 \sum_{l=m+1}^{M} \text{Re}\left\{ \left( \frac{\partial \bar{\mathbf{r}}_{m,l}}{\partial \mathbf{\Phi}_m} \right)^T \mathbf{U}_{m,l}^T \bar{\mathbf{r}}_{m,l}^* \right\}. (17) \end{aligned}$$

The columns of the matrices $\left[\frac{\bar{\mathbf{r}}_{m',m}}{\partial \mathbf{\Phi}_m}\right]_{2N \times N}$ and $\left[\frac{\bar{\mathbf{r}}_{m,m'}}{\partial \mathbf{\Phi}_m}\right]_{2N \times N}$ are obtained in a similar way by evaluating $\frac{\bar{\mathbf{r}}_{m,m'}}{\partial \phi_n^{(m)}}$ and $\frac{\bar{\mathbf{r}}_{m',m}}{\partial \phi_n^{(m)}}$ respectively for $n = 0, 1, \ldots, N-1$. The gradient of $\mathcal{C}_3$ is obtained from its $M$ sub-blocks given in (17).

### 3.5. On the optimization w.r.t. $\mathcal{C}_4$

The function $\mathcal{C}_4$ is obtained from (6) by replacing $\mathbf{r}_{m,m'}$ with $\gamma_{m,m'}$. Similarly, the matrices $\mathbf{U}_{m,m'}$ and $\mathbf{U}_{m',m}$ are substituted with $\mathbf{V}_{m,m'}$ and $\mathbf{V}_{m',m}$, which are constructed by considering the weights $\beta_{m,m'}^{(l)}$ for $l = 0, 1, \ldots, N-1$. The vector $\frac{\partial \mathcal{C}_4}{\partial \mathbf{\Phi}_m}$ is obtained from (17) as well by replacing $\bar{\mathbf{r}}_{m,m'}$ and $\bar{\mathbf{r}}_{m',m}$ with $\bar{\boldsymbol{\gamma}}_{m,m'}$ and $\bar{\boldsymbol{\gamma}}_{m',m}$, respectively.

### 3.6. The optimization method

A gradient based approach, the L-BFGS method as implemented in the SciPy toolbox [21], is hereby used to solve the optimization problem given by (3). This technique has been considered in the design of set of sequences with good correlation [6] and good complementary correlation properties [17, 18]. The algorithm requires as inputs, the cost function, its gradient and a suitable starting point to initialize the optimization algorithm which has been detailed in the following section.

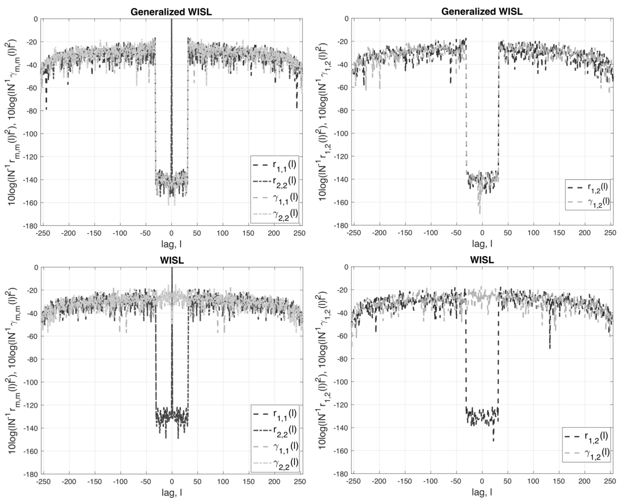


**Fig. 1**. Correlation and complementary correlation of the designed set $\{x_m(n)\}_{n=0,m=1}^{255,2}$ using GWISL (top) and WISL (bottom) criterion.

## 4. NUMERICAL SIMULATIONS

### 4.1. Design of a set of sequences with $N = 256$, $M = 2$

We consider the design of a set of $M = 2$ sequences with length $N = 256$. The weights that define the auto- and complementary auto-correlation windows were selected as $\alpha_{m,m}^{(l)} = 1$ for $l \in [-31, -1] \cup [1, 31]$ and $\beta_{m,m}^{(l)} = 1$ for $l = [-31, 31]$, and zero otherwise. Concerning the cross- and complementary cross-correlation weights, $\alpha_{m,m'}^{(l)} = \beta_{m,m'}^{(l)} = 1$ for $l \in [-31, 31]$ and $\alpha_{m,m}^{(l)} = \beta_{m,m}^{(l)} = 0$ were used for $m, m' = 1, 2$. The normalized correlation and complementary cross-correlation of the resultant set are depicted in Fig. 1 and are labeled as Generalized WISL (GWISL). For comparison, a set of sequences with good correlation properties for $l \in [-31, 31]$ is also considered. The set was generated with the algorithm in [10] and the results obtained are labeled as WISL. For both cases, a common initialization point was used to start the algorithms, halting the iterations once $\frac{f^{(n-1)} - f^{(n)}}{\max\{f^{(n-1)}, f^{(n)}, 1\}} < 2.22 \times 10^{-9}$ was reached.

### 4.2. Application to MIMO system identification

As the second numerical example, we exhibit an identification of Multiple-Input-Multiple-Output (MIMO) systems with $P = 2$ inputs and $Q = 2$ outputs for strictly linear (SL) and widely linear (WL) systems.

#### *4.2.1. MIMO-SL system*

We consider that the output of the systems is given by $y_q(n) = \sum_{p=0}^{1} (h_{p,q} \star x_p)(n) + \nu(n)$ for $q = 1, 2$, where $\{h_{p,q}(n)\}_{n=0}^{3}$ is the response of the filter that relates the

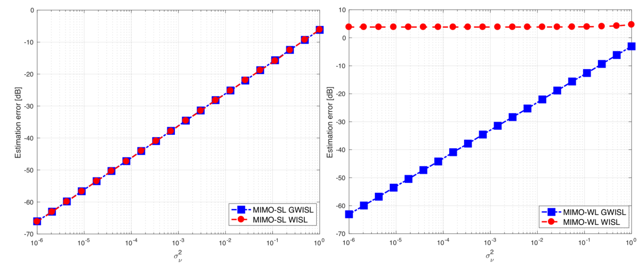


**Fig. 2**. Estimation error of a $2 \times 2$ MIMO-SL (left) and MIMO-WL (right) system using WISL and GWISL criterion for different noise variances.

$p$th input ($p = 1, 2$) with $y_q(n)$. The taps of the filters are complex numbers with their real and imaginary part drawn from a standard normal distribution. Each of the output branches is affected with additive white Gaussian noise, with $E\{|\nu(n)|^2\} = \sigma_\nu^2$. The MIMO-SL system is sounded with two sequences $x_p(n)$ of length $N = 64$, that posses good correlation and complementary correlation in a window of length 16. The identification is done with a matched filter at each output, $\hat{h}_{p,q}(l) = N^{-1}(\mathbf{x}_p^{(l)})^H \mathbf{y}_q$ , $\mathbf{y}_q = [y_q(0), y_q(1), \ldots, y_q(N + L - 2)]^T$ and $\mathbf{x}_p^{(l)} = [\mathbf{0}_{l\times 1}^T \; \mathbf{x}_p^T \; \mathbf{0}_{L-l-1\times 1}^T]^T$ with $\mathbf{x}_p = [x_p(0), x_p(1), \ldots, x_p(N - 1)]^T$. The results of the estimation process are depicted in Fig. 2, where the estimation error is plotted as a function of the average power of the noise $\sigma_\nu^2$ (trace labeled as MIMO-SL GWISL). For comparison, the estimation process was done also with a set of sequences of length $N = 64$ and good correlation properties in a window of length 16, generated with the algorithm in [10] (trace labeled as MIMO-SL WISL).

#### *4.2.2. MIMO-WL system*

In the WL case, the output of the system is given by $y_q(n) = \sum_{p=0}^{1} (h_{p,q}^{(1)} \star x_p)(n) + \sum_{p=0}^{1} (h_{p,q}^{(2)} \star x_p^*)(n) + \nu(n)$, where $\{h_{p,q}^{(1)}(l)\}_{l=0}^{3}$ and $\{h_{p,q}^{(2)}(l)\}_{l=0}^{3}$ are the filter responses that relate the $p$th input with $y_q(n)$. The filter coefficients were generated similarly, and estimated with $\hat{h}_{p,q}^{(1)} = N^{-1}(\mathbf{x}_p^{(l)})^H \mathbf{y}_q$ and $\hat{h}_{p,q}^{(2)} = N^{-1}(\mathbf{x}_p^{(l)})^T \mathbf{y}_q$. The estimation error, using the previous set of sequences, is depicted in Fig. 2. It can be verified that sequences which minimize the GWISL criterion, are able to identify MIMO-SL and MIMO-WL systems, by making use of the second order statistics of the output signals. However, for sequences that minimize only the WISL, there is interference that degrades the estimates for the WL scenario.

## 5. CONCLUSIONS

We have proposed the design of unimodular sequences by considering a complete second order characterization which encompasses its correlation and complementary correlation functions. It is shown via numerical examples, that due to the properties of the proposed sequences, they are well suited to estimate both SL and WL systems.

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