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Optimal Control of Heat Transfer in Unsteady Stokes Flows

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Abstract— **This paper is concerned with establishing a rigorous mathematical framework to address optimal control designs for heat transfer in unsteady Stokes flows. In particular, we focus on the problem of enhancing convection-cooling between two fluids via controlling the velocity of the cold fluid flow, where both internal and boundary controls will be investigated. This essentially leads to a bilinear control problem. We present a rigorous proof of the existence of an optimal control and derive the first-order necessary conditions for optimality by using a variational inequality. Finally, we show that the uniqueness of the optimal controller can be obtained if the control weight is sufficiently large.**

I. INTRODUCTION

We consider an optimal control problem of convection-cooling between two fluids via an active control of the cold fluid flow velocity, which is governed by the unsteady Stoke equations. This is motivated by the control design of a tubular counter-current heat exchanger; see, e.g., [1], [2], [3], [9], [10], [11], [21], [22]. Effective circulation plays an important role in keeping the cold fluid with suitably low temperatures evenly. Very often liquid-cooled systems use a circulation pump to avoid the occurrence of hot spots. The current work aims at formulating an optimal strategy for enhancing circulation of the cold fluid flow in order to achieve optimal heat transfer.

Consider a simple convection-cooling model demonstrated in Fig. 1, where $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are open bounded with smooth boundary $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Assume that the fluid in Ω_1 is heated by the walls, while cooled down by the cold fluid in Ω_2. The convection-cooling between these two fluids happens at the interface $\Gamma_e = \partial\Omega_1 \cap \partial\Omega_2$. The precise mathematical model is governed by the following diffusion-convection equations.

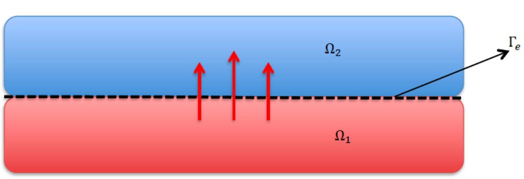

Fig. 1. Convection-cooling between two fluids

$$\frac{\partial T_1}{\partial t} = \kappa_1 \Delta T_1 - v_1 \cdot \nabla T_1, \quad x \in \Omega_1, \tag{1}$$

$$\kappa_1 \frac{\partial T_1}{\partial n_1}|_{\Gamma_e} = a(T_2 - T_1), \quad T_1|_{\partial\Omega_1 \setminus \Gamma_e} = \theta, \tag{2}$$

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where T_1 represents a (scaled) temperature of a hot fluid in $\Omega_1 \in \mathbb{R}^2$ convected by an incompressible fluid flow with velocity v_1, n_1 is the outward normal unit vector with respect to Ω_1, and θ is a given heating function acting on the boundary $\Omega_1 \setminus \Gamma_e$. Here T_2 is the (scaled) temperature of a cold fluid in Ω_2 satisfying

$$\frac{\partial T_2}{\partial t} = \kappa_2 \Delta T_2 - v_2 \cdot \nabla T_2, \quad x \in \Omega_2, \tag{3}$$

$$\kappa_2 \frac{\partial T_2}{\partial n_2}|_{\Gamma_e} = a(T_1 - T_2), \quad T_2|_{\partial\Omega_2 \setminus \Gamma_e} = 0. \tag{4}$$

The initial conditions are given by

$$T_1(x,0) = T_{10}(x) \quad \text{and} \quad T_2(x,0) = T_{20}(x). \tag{5}$$

In our model equations, $\kappa_i, i = 1,2$ stand for the diffusivity and a is the heat transfer coefficient.

We assume that the hot fluid flow velocity v_1 is prescribed and driven by some time-dependent external body force. Moreover, v_1 is divergence free with no-penetration boundary condition, i.e.,

$$\nabla \cdot v_1 = 0, \quad x \in \Omega_1 \quad \text{and} \quad v_1 \cdot n_1|_{\partial\Omega_1} = 0. \tag{6}$$

In our previous work [17] we considered that the cold fluid flow is governed by the steady Stokes equations. No dynamics was incorporated for the flow velocity during the cooling process. In the present paper we consider that the cold fluid flow velocity v_2 is governed by the unsteady Stokes equations, which are steered by internal (distributed) or boundary controls. We formulate different types of control designs in the next section.

II. OPTIMAL CONTROL DESIGNS

A. Internal Control

Internal control is employed to steer the convection by providing energy to the system in the interior of the flow domain. For example, stirring a fluid back and forth can generate fluctuating velocities with respect to the flow barriers, therefore engenders transport across them for achieving better heat transfer [24], [25]. To formulate an internal control problem, we consider that the control acts on a subdomain $\omega \subset \Omega$ with a smooth boundary $\partial\omega_2$. The controlled Stokes equations become

$$\frac{\partial v_2}{\partial t} = \Delta v_2 - \nabla p_2 + m_\omega u_\omega, \quad \nabla \cdot v_2 = 0, \quad x \in \Omega_2. \tag{7}$$

with no-slip boundary conditions imposed on Γ

$$v_2|_{\partial\Omega_2} = 0, \tag{8}$$

where $m_\omega(x)$ is a sufficiently smooth function with compact support at ω and u is the control input. The initial condition is given by $v_2(0) = v_{20}$.

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B. Boundary Control

For boundary control, we consider that the control inputs act tangentially along the wall through Dirichlet or Navier slip boundary conditions. This is motived by the observation that wall rotation or movement enhances heat transport in the whole fluid domain [12], [13], [14], [15], [23]. Based on these results, we investigate the control inputs that tangentially act along the wall of the cold fluid to increase circulation for enhancing heat transfer.

In this paper, we establish a rigorous theoretical framework to study this phenomenon by employing two types of controls for the cold fluid flow velocity field through non-penetration boundary conditions and characterize the velocity field for achieving optimal transport process. The boundary conditions are formulated as follows.

$$v_2 \cdot n_2|_{\partial\Omega_2} = 0 \quad \text{and} \quad \gamma_k v_2|_{\partial\Omega_2} = u, \quad k = 1,2, \tag{9}$$

where n_2 is the outward normal unit vector with respect to Ω_2 and $n_2 = -n_1$ on Γ_e. The boundary input u depends on time and spatial position along the wall of Ω_2, which is the manipulated function that generates the fluid flow in Ω_2. γ_1 and γ_2 are the trace operators defined through the Dirichlet and the Navier slip boundary conditions, respectively. To be more specific,

$$\gamma_1 v_2|_{\partial\Omega_2} = u \tag{10}$$

and

$$\gamma_2 v_2|_{\partial\Omega_2} = (\mathbb{T}(v_2)\cdot n_2)_{\tau_2} + \alpha v_2|_{\partial\Omega_2} = u, \tag{11}$$

where $\mathbb{T}(v_2) = 2\nu\mathbb{D}(v_2)$ stands for the stress tensor with $\mathbb{D}(v_2) = (1/2)(\nabla v_2 + (\nabla v_2)^T)$, τ_2 stands for the outward tangental unit vector with respect to Ω_2, and $(\mathbb{T}(v_2)\cdot n_2)_{\tau_2}$ is the tangential component of $(\mathbb{T}(v_2)\cdot n_2)$. The friction between the fluid and the boundary $\partial\Omega_2$ is proportional to $-v_2$ with the positive coefficient of proportionality α.

C. Formulation of the Cost Functional

The objective of this paper is to minimize the average temperature T_1 in Ω_1 over the time interval $[0,T]$, with an optimal input u

$$J(u) = \frac{1}{2}\int_0^T |\int_{\Omega_1} T_1(x,t)\,dx|^2\,dt + \frac{\gamma}{2}\int_0^T \|u\|^2_{U_{ad}}\,dt. \tag{P}$$

where $\gamma > 0$ is the control weight parameter and U_{ad} is the set of admissible controls. To set up the abstract formulation for the velocity and the temperature, we define for $s \geq 0$,

$$V_0^s(\Omega_2) = \{v \in H^s(\Omega_2)\colon \operatorname{div} v = 0,\ v|_{\partial\Omega_2} = 0\},$$
$$V_n^s(\Omega_i) = \{v \in H^s(\Omega_i)\colon \operatorname{div} v = 0,\ v\cdot n_i|_{\partial\Omega_i} = 0\},\ i = 1,2,$$
$$V_n^0(\partial\Omega_2) = \{u \in L^2(\partial\Omega_2)\colon u\cdot n_2|_{\partial\Omega_2} = 0\},$$
$$H^1_{\partial\Omega\backslash\Gamma_e}(\Omega_2) = \{T \in H^1(\Omega_2) : T|_{\partial\Omega_2\backslash\Gamma_e} = 0\}.$$

Let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ stand for the L^2-inner products in the interior of the domain and on the boundary, respectively. In the sequel, the symbol c denotes a generic positive constant.

Due to one-way coupling between the temperature and the velocity, investigating the optimal control design for optimal heat transfer is tied to understanding the control problem of the Stokes flows. Since the velocity and the temperature are coupled nonlinearly via the advective term $v_2\cdot\nabla T_2$ in the temperature equation (3), linear control of the Stokes flow essentially leads to a bilinear control of the temperature. Therefore, problem (P) is a nonconvex optimization problem. For convenience of our discussion, we rewrite the controlled velocity field in a more compact form

$$\frac{\partial v_2}{\partial t} = Av_2 + B_k u, \quad k = 0,1,2, \tag{12}$$
$$v_2(0) = v_{20}, \tag{13}$$

where $A = \mathbb{P}\Delta$ is the Stokes operator associated with the homogenous Dirichlet or Navier slip boundary conditions given by (9), with domain

$$D(A) = \{v \in H^2(\Omega)\colon \operatorname{div} v = 0,\ v_2\cdot n_2|_{\partial\Omega_2} = 0 \text{ and } \gamma_k v|_{\partial\Omega_2} = 0\}, \quad k = 1,2,$$

$\mathbb{P}\colon L^2(\Omega_2) \to V_n^0(\Omega_2)$ is the Leray projector and B_k is the control input operator defined by the way how the control is introduced to the system. Note that A is strictly negative and self-adjoint. For the internal control,

$$B_0 = \mathbb{P}m_\omega,$$

whereas for the boundary control,

$$B_k = -AR_k, \quad k = 1,2,$$

where R_k is the boundary lifting operator that maps the boundary data to an interior function. To be more precise, if $v_2 = R_k u$ for $u \in L^2(\partial\Omega_2)$, then v_2 satisfies the boundary value problem associated with different types of boundary conditions

$$\nu\Delta v_2 - \nabla p = 0, \quad \nabla\cdot v_2 = 0, \tag{14}$$
$$v_2\cdot n_2|_\Gamma = 0 \quad \text{and} \quad \gamma_k v_2|_\Gamma = u, \quad k = 1,2. \tag{15}$$

With the Dirichlet trace γ_1, B_1 is so called the Dirichlet boundary control input operator. The tangential Dirichlet boundary control is thoroughly studied in [4], [5]. If let B_1^* be the L^2-adjoint operator of B_1, then

$$B_1^*\phi = -R_1^*A = -\frac{\partial\phi}{\partial n}|_{\partial\Omega_2}, \quad \forall\phi \in D(A). \tag{16}$$

In contrast, with the Navier slip trace γ_2, B_2 is called the Navier slip boundary control input operator. A general existence and regularity theory of the Stokes problem associated with nonhomogenous Navier slip boundary conditions can be found in (cf. [8]). In particular, if $u \in H^{-1/2}(\partial\Omega_2)$, then there exists a unique weak solution $(v_2,p_2) \in V^1(\Omega_2)\times L^2(\Omega_2)$ to (14)–(15). Moreover, if $u \in H^{1/2}(\partial\Omega_2)$, then there exists a unique strong solution $(v_2,p_2) \in V^2(\Omega_2)\times H^1(\Omega_2)$. If let B_2^* be the L^2-adjoint operator of B_2, then

$$B_2^*\phi = -R_2^*A\phi = \phi|_{\partial\Omega_2}, \quad \forall\phi \in D(A). \tag{17}$$

Moreover, the velocity can be solved by using the variation of parameters formula

$$v(t) = e^{At}v_0 + (L_k u)(t), \quad k = 0,1,2, \tag{18}$$

where e^{At} is an analytic semigroup generated by A on $V_n^0(\Omega)$, L_k is given by

$$(L_k u)(t) = \int_0^t e^{A(t-\tau)} B_k u(\tau)\, d\tau, \quad k = 0,1,2, \tag{19}$$

and its $L^2(0,T;\cdot)$-adjoint L_i^* is given by

$$(L_k^* \phi)(t) = \int_t^T B_k^* e^{-A(\tau - t)} \phi(\tau)\, d\tau. \tag{20}$$

To simplify the notation, we replace L_k by L without ambiguity for a generic approach in the rest of our paper.

Recall that problem (P) is well-posed if for any initial condition (T_{10}, T_{20}, v_{20}), there exit a control $u \in U_{ad}$ and the corresponding solution (T_1, T_2, v_2) of the governing system (1)–(6) and (12)–(13), such that the cost functional $J(u)$ is finite. However, the derivation of U_{ad} requires a differentiability analysis of the control-state map. In this work, the first-order optimality conditions associated with problem (P) will be derived by using a variational inequality [20], that is, if u is an optimal solution to problem (P), then

$$J'(u) \cdot (g - u) \geq 0, \quad g \in U_{ad}, \tag{21}$$

where $J'(u) \cdot g$ stands for the Gâteaux derivative of J with respect to u in every direction $g \in U_{ad}$. Therefore, U_{ad} will be chosen such that $J(u)$ is Gâteaux differentiable. In fact, for the current model we chose

$$U_{ad} = L^2(0,T;L^2(\omega)) \tag{22}$$

for the internal control, and

$$U_{ad} = L^2(0,T;V_n^0(\partial\Omega_2)) \tag{23}$$

for both Dirichlet and Navier slip boundary controls.

III. Well-posedness of the Model and Existence of an Optimal Solution

The well-posedness of system (1)–(5) with steady Stokes flows and the existence of an optimal solution to problem (P) are addressed in [17]. In this work we assume that $T_{10} \in L^\infty(\Omega_1)$ and $T_{20} \in L^\infty(\Omega_2)$. Slightly modifying the proof of [17, Theorem 1], we have the following result for the case with unsteady Stokes flows. Some components of the proof can be also found in [6, Theorem 1].

Theorem 1: Assume that $v_1 \in L^2(0,T;V_n^0(\Omega_1))$ and $v_2 \in L^2(0,T;V_n^0(\Omega_2))$. For $T_{i0} \in L^\infty(\Omega_i), i = 1,2$, and $\theta \in L^\infty(0,T;H^1(\partial\Omega_1 \setminus \Gamma_e))$, there exists a unique solution (T_1, T_2) to the linear system of equations (1)–(5) satisfying

$$\begin{aligned} &\|T_i\|_{L^\infty(0,T;L^\infty(\Omega_i))} + \|T_i\|_{L^2(0,T;H^1(\Omega_i))} + \|\frac{dT_i}{dt}\|_{L^2(0,T;(H^1(\Omega_i))')} \\ &\quad + \|v_i \cdot \nabla T_i\|_{L^2(0,T;(H^1(\Omega_i))')} \leq C(T_{10}, T_{20}, \theta), \end{aligned} \tag{24}$$

for $i = 1,2$.

Note that by Agmon's inequality, $\|T_1\|_{L^\infty} \leq c\|T_1\|_{H^{1+\delta}}$, $0 < \delta < 1/2$, for dimension $d = 2$. It suffices to have $\theta \in L^\infty(0,T;H^1(\partial\Omega_1 \setminus \Gamma_e))$ based on the regularity of the Dirichlet trace.

Next we establish the well-posedness of the Gâteaux derivatives of the state variables with respect to the control input u. Let $z_i = T_i'(u) \cdot h, i = 1,2$, and $w = v_2'(u) \cdot h$ denote the Gâteaux derivatives of T_i and v_2 with respect to u in every direction h in U_{ad}, respectively. Then $w = Lh$ by (18) and $z_i, i = 1,2$, satisfy the following coupled linear equations

$$\frac{\partial z_1}{\partial t} = \kappa_1 \Delta z_1 - v_1 \cdot \nabla z_1, \tag{25}$$

$$\frac{\partial z_2}{\partial t} = \kappa_2 \Delta z_2 - (w \cdot \nabla T_2 + v_2 \cdot \nabla z_2) \tag{26}$$

with boundary conditions

$$\kappa_1 \frac{\partial z_1}{\partial n_1}|_{\Gamma_e} = a(z_2 - z_1), \quad z_1|_{\partial\Omega_1 \setminus \Gamma_e} = 0, \tag{27}$$

$$\kappa_2 \frac{\partial z_2}{\partial n_2}|_{\Gamma_e} = a(z_1 - z_2), \quad z_2|_{\partial\Omega_1 \setminus \Gamma_e} = 0, \tag{28}$$

and initial conditions

$$z_1(x,0) = z_2(x,0) = 0. \tag{29}$$

Comparing (25)–(29) with (1)–(5), the essential difference lies in the term $w \cdot \nabla T_2$ in (26). It is easy to show that (25)–(29) is well-posed if $T_2 \in L^\infty(0,T;L^\infty(\Omega_2))$ and $w \in L^2(0,T;V_n^0(\Omega_2))$. Let $v_{20} \in V_n^0(\Omega_2)$. Then latter holds immediately for $u \in U_{ad}$ given by (22) or (23), due to the regularity of L associated with different types of controls (cf. [4], [5], [7], [16], [18]).

The following theorem provides the existence of an optimal solution to problem (P). The proof follows the similar procedures as constructed in [17, Theorem 2].

Theorem 2: Assume that $v_1 \in L^2(0,T;V_n^0(\Omega_1))$. For $T_{i0} \in L^\infty(\Omega_i), i = 1,2$, $\theta \in L^\infty(0,T;H^1(\partial\Omega_1 \setminus \Gamma_e))$, and $v_{20} \in V_n^0(\Omega_2)$, there exists an optimal solution $u^* \in U_{\mathrm{ad}}$ to problem (P) subject to (1)–(5) and (12)–(13).

IV. Conditions of Optimality and Uniqueness of the Optimal Controller

A. First-order Necessary Conditions for Optimality

In this section, we derive the first-order conditions of optimality for problem (P) by using a variational inequality (21). Define the operator $D\colon L^2(\Omega_1) \to \mathbb{R}$ by

$$DT_1(x,t) = \frac{1}{|\Omega_1|} \int_{\Omega_1} T_1(x,t) dx.$$

Then problem (P) can be rewritten as

$$J(u) = \frac{1}{2} \int_0^T (D^* DT_1, T_1)_{\Omega_1} dt + \frac{\gamma}{2} \int_0^T (u,u)_{U_{ad}} dt. \tag{30}$$

If u is an optimal solution to (30), then (21) implies

$$J'(u) \cdot h = \int_0^T (D^* DT_1, z_1)_{\Omega_1} dt + \gamma \int_0^T (u,h)_{U_{ad}} dt \geq 0, \tag{31}$$

for every $h \in U_{ad}$.

Define the adjoint system of (1)–(5) associated with the cost functional (30) by

$$-\frac{\partial q_1}{\partial t} = \kappa_1 \Delta q_1 + v_1 \cdot \nabla q_1 + D^* DT_1, \tag{32}$$

$$-\frac{\partial q_2}{\partial t} = \kappa_2 \Delta q_2 + w_k \cdot \nabla q_2, \tag{33}$$

with boundary conditions

$$\kappa_1 \frac{\partial q_1}{\partial n_1}|_{\Gamma_e} = a(q_2 - q_1), \quad q_1|_{\partial\Omega_1 \setminus \Gamma_e} = 0 \tag{34}$$

$$\kappa_2 \frac{\partial q_2}{\partial n_2}|_{\Gamma_e} = a(q_1 - q_2), \quad q_1|_{\partial\Omega_1 \setminus \Gamma_e} = 0, \tag{35}$$

where $w = Lu$. The final time conditions are given by

$$q_1(x,T) = q_2(x,T) = 0. \tag{36}$$

Then

$$q_1 \in C([0,T];L^2(\Omega_1)) \cap L^2(0,T;H^1(\Omega_1)), \quad T > 0,$$

and

$$q_2 \in C([0,T];L^2(\Omega_2)) \cap L^2(0,T;H^1(\Omega_2)), \quad T > 0. \tag{37}$$

The first-order necessary conditions for optimality are stated in the following theorem. The result can be obtained by slightly modifying the proof of [17, Theorem 3] and utilizing the linearity of operator L. We provide the complete proof for the convenience of the readers.

Theorem 3: Let $v_1 \in L^2(0,T;V_n^0(\Omega_1))$, $T_{i0} \in L^2(\Omega_i), i = 1,2$, and $\theta \in L^2(0,T;H^1(\partial\Omega_1 \setminus \Gamma_e))$. Assume that $u^{\text{opt}} \in U_{ad}$ is an optimal solution to problem (P). If (T_1, T_2, v_2) is the corresponding solution to the system governed by (1)–(5) and (12)–(13) and (q_1, q_2) is the solution to the adjoint system (32)–(36), then

$$u^{opt} = -\frac{1}{\gamma} L^* \mathbb{P}(q_2 \nabla T_2), \tag{38}$$

where L^* is the $L^2(0,T;\cdot)$-adjoint operator of L give by (20), which is defined through different types of controls.

Proof: By (31) and (32), we have

$$J'(u) \cdot h = \int_0^T (-\frac{\partial q_1}{\partial t} - \kappa_1 \Delta q_1 - v_1 \cdot \nabla q_1, z_1)_{\Omega_1} dt + \gamma \int_0^T (u,h)_{U_{ad}} dt, \tag{39}$$

where

$$\begin{aligned} &\int_0^T (-\frac{\partial q_1}{\partial t} - \kappa_1 \Delta q_1 - v_1 \cdot \nabla q_1, z_1)_{\Omega_1} dt \\ &= \int_0^T (q_1, \frac{\partial z_1}{\partial t} - \kappa_1 \Delta z_1 + v_1 \cdot \nabla z_1)_{\Omega_1} dt \\ &\quad + \int_0^T \langle a(z_2 - z_1), q_1 \rangle_{\Gamma_e} - \langle a(q_2 - q_1), z_1 \rangle_{\Gamma_e} dt \\ &= \int_0^T \langle a z_2, q_1 \rangle_{\Gamma_e} - \langle a q_2, z_1 \rangle_{\Gamma_e} dt. \end{aligned} \tag{40}$$

Moreover, with the help of (26) and (33) we have

$$\begin{aligned} 0 &= \int_0^T (-\frac{\partial q_2}{\partial t} - \kappa_2 \Delta q_2 - (Lu) \cdot \nabla q_2, z_2)_{\Omega_2} dt \\ &= \int_0^T (q_2, \frac{\partial z_2}{\partial t} - \kappa_2 \Delta z_2 + (Lu) \cdot \nabla z_2)_{\Omega_2} dt \\ &\quad + \int_0^T \langle a(z_1 - z_2), q_2 \rangle_{\Gamma_e} - \langle a(q_1 - q_2), z_2 \rangle_{\Gamma_e} dt \\ &= -\int_0^T (q_2, (Lh) \cdot \nabla T_2)_{\Omega_2} dt + \int_0^T \langle a z_2, q_1 \rangle_{\Gamma_e} - \langle a q_2, z_1 \rangle_{\Gamma_e} dt. \end{aligned} \tag{41}$$

From (39)–(41) we get

$$\begin{aligned} &\int_0^T (-\frac{\partial q_1}{\partial t} - \kappa_1 \Delta q_1 - v_1 \cdot \nabla q_1, z_1)_{\Omega_1} dt \\ &= \int_0^T (q_2, (Lh) \cdot \nabla T_2)_{\Omega_2} dt = \int_0^T (L^* \mathbb{P}(q_2 \nabla T_2), h)_{U_{ad}} dt. \end{aligned} \tag{42}$$

Thus if u^{opt} is an optimal solution to (30), then

$$J'(u^{opt}) \cdot h = \int_0^T (L^* \mathbb{P}(q_2 \nabla T_2), h)_{U_{ad}} dt + \gamma \int_0^T (u^{opt}, h)_{U_{ad}} dt \quad \geq 0,$$

for every $h \in U_{ad}$. This yields the desired result (38). ■

Remark 4: As mentioned in [7, Remark 6], since the Leray projector $\mathbb{P}\colon L^2(\Omega_2) \to V_n^0(\Omega_2)$ can be extended from $H^s(\Omega_2), s > 0$, to $V_n^s(\Omega_2)$, its adjoint $\mathbb{P}^*\colon V_n^0(\Omega_2) \to L^2(\Omega_2)$ can be extended as a bounded operator from $(V_n^s(\Omega_2))'$ to $(H^s(\Omega_2))'$ by

$$(\mathbb{P}^* \psi, \varphi)_{(H^s(\Omega_2))', H^s(\Omega_2)} = (\psi, \mathbb{P}\varphi)_{(V_n^s(\Omega_2))', V_n^s(\Omega_2)}, \tag{43}$$

for $\psi \in (V_n^s(\Omega))'$, $\varphi \in V_n^s(\Omega)$. Therefore, if $q_2 \nabla T_2 \in L^2(0,T;H^s(\Omega_2))$, where $s < 0$, then we replace $\mathbb{P}$ by $\mathbb{P}^*$ in the last equality of (42). In this case,

$$u^{opt} = -\frac{1}{\gamma} L^*(\mathbb{P}^*(q_2 \nabla T_2)). \tag{44}$$

B. Uniqueness of the Optimal Controller

In this section, we discuss the uniqueness of the optimal controller to problem (P). Due to the limited space, we focus on the discussion for the case with Navier slip boundary control. The proof can be easily applied to the internal control, whereas the Dirichlet boundary control case will be addressed in our future paper. The main result is given by the following theorem.

Theorem 5: Let $v_1 \in L^2(0,T;V_n^0(\Omega_1))$, $T_{i0} \in L^2(\Omega_i), i = 1,2$, and $\theta \in L^2(0,T;H^1(\partial\Omega_1 \setminus \Gamma_e))$. For γ sufficiently large, there exists at most one optimal controller $u^{opt} \in U_{ad}$ to problem (P) with Navier slip boundary control, which can be solved from (38).

Proof: Assume that there are two pairs of optimal solutions to problem (P), denoted by $(u^1, T_1^1, T_2^1, v_2^1)$ and $(u^2, T_1^2, T_2^2, v_2^2)$. The corresponding solutions to the adjoint problem (32)–(36) are denoted by (q_1^1, q_2^1) and (q_1^2, q_2^2). Then $U = u^1 - u^2$, $\Theta^i = T_i^1 - T_i^2, i = 1,2$, $W = v_2^1 - v_2^2 = LU$, and $Q^i = q_i^1 - q_i^2, i = 1,2$, satisfy the following system

$$\frac{\partial \Theta^1}{\partial t} = \kappa_1 \Delta \Theta^1 - v_1 \cdot \nabla \Theta^1, \tag{45}$$

$$\frac{\partial \Theta^2}{\partial t} = \kappa_2 \Delta \Theta^2 - (LU) \cdot \nabla T_2^1 - v_2^2 \cdot \nabla \Theta^2, \tag{46}$$

with boundary conditions

$$\kappa_1 \frac{\partial \Theta^1}{\partial n_1}|_{\Gamma_e} = a(\Theta^2 - \Theta^1), \quad \Theta^1|_{\partial\Omega_1\setminus\Gamma_e} = 0, \tag{47}$$

$$\kappa_2 \frac{\partial \Theta^2}{\partial n_2}|_{\Gamma_e} = a(\Theta^1 - \Theta^2), \quad \Theta^2|_{\partial\Omega_2\setminus\Gamma_e} = 0, \tag{48}$$

and

$$-\frac{\partial Q^1}{\partial t} = \kappa_1 \Delta Q^1 + v_1 \cdot \nabla Q^1 + D^* D Q^1, \tag{49}$$

$$-\frac{\partial Q^2}{\partial t} = \kappa_2 \Delta Q^2 + (LU)\cdot \nabla q_2^1 + (Lu^2)\cdot \nabla Q^2, \tag{50}$$

with boundary conditions

$$\kappa_1 \frac{\partial Q^1}{\partial n_1}|_{\Gamma_e} = a(Q^2 - Q^1), \quad Q_1|_{\partial\Omega_1\setminus\Gamma_e} = 0, \tag{51}$$

$$\kappa_2 \frac{\partial Q^2}{\partial n_2}|_{\Gamma_e} = a(Q^1 - Q^2), \quad Q_2|_{\partial\Omega_2\setminus\Gamma_e} = 0, \tag{52}$$

where

$$U = -\frac{1}{\gamma} L^*(\mathbb{P}(Q^2 \nabla T_2^1 + q_2^2 \nabla \Theta^2)). \tag{53}$$

The initial conditions for Θ^1 and Θ^2 are given by $\Theta^1(0) = \Theta^2(0) = 0$. The final time conditions for Q^1 and Q^2 are given by $Q^1(T) = Q^2(T) = 0$.

Applying the L^2-estimate to (45)–(46), respectively, yields

$$\frac{1}{2}\frac{dt\|\Theta^1\|_{L^2}}{dt} + \kappa_1 \|\nabla\Theta^1\|_{L^2} = \int_{\Gamma_e} a(\Theta^2 - \Theta^1)\Theta^1\, dx - \int_\Omega (v_1\cdot\nabla\Theta^1)\Theta^1\, dx, \tag{54}$$

$$\frac{1}{2}\frac{dt\|\Theta^2\|_{L^2}}{dt} + \kappa_2 \|\nabla\Theta^2\|_{L^2} = \int_{\Gamma_e} a(\Theta^1 - \Theta^2)\Theta^2\, dx - \int_\Omega ((LU)\cdot\nabla T_2^1)\Theta^2\, dx - \int_\Omega (v_2^2\cdot\nabla\Theta^2)\Theta^2\, dx, \tag{55}$$

Recall that the velocity is divergent free with no-penetration conditions. We get

$$\int_\Omega (v_1\cdot\nabla\Theta^1)\Theta^1\, dx = \frac{1}{2}\int_\Omega v_1\cdot\nabla(\Theta^1)^2\, dx = \frac{1}{2}\Big(\int_{\partial\Omega_1} v_1\cdot n_1(\Theta^1)^2\, dx - \int_\Omega \nabla\cdot v_1(\Theta^1)^2\, dx\Big) = 0.$$

Similarly, $\int_\Omega (v_2^2\cdot\nabla\Theta^2)\Theta^2\, dx = 0$. Adding (54)–(55) gives

$$\begin{aligned}
&\frac{1}{2}\frac{dt\|\Theta^1\|_{L^2}^2}{dt} + \frac{1}{2}\frac{dt\|\Theta^2\|_{L^2}^2}{dt} + \kappa_1\|\nabla\Theta^1\|_{L^2}^2 + \kappa_2\|\nabla\Theta^2\|_{L^2}^2 \\
&\quad + a(\|\Theta^1\|_{L^2(\Gamma_e)}^2 + \|\Theta^2\|_{L^2(\Gamma_e)}^2) \\
&\le 2a\|\Theta^1\|_{L^2(\Gamma_e)}\|\Theta^2\|_{L^2(\Gamma_e)} + \int_\Omega ((LU)\cdot\nabla T_2^1)\Theta^2\, dx \\
&\le a(\|\Theta^1\|_{L^2(\Gamma_e)} + \|\Theta^2\|_{L^2(\Gamma_e)}^2) - \int_\Omega (LU)T_2^1\cdot\nabla\Theta^2\, dx \\
&\le a(\|\Theta^1\|_{L^2(\Gamma_e)} + \|\Theta^2\|_{L^2(\Gamma_e)}^2) + \|LU\|_{L^2}\|T_2^1\|_{L^\infty}\|\nabla\Theta^2\|_{L^2} \\
&\le a(\|\Theta^1\|_{L^2(\Gamma_e)} + \|\Theta^2\|_{L^2(\Gamma_e)}^2) + c\|LU\|_{L^2}^2\|T_2^1\|_{L^\infty}^2 \\
&\quad + \frac{\kappa_2}{2}\|\nabla\Theta^2\|_{L^2}^2.
\end{aligned} \tag{56}$$

After simplification, we integrate (56) on both sides from 0 to T. Then making use of (24) yields

$$\begin{aligned}
&\|\Theta^1(T)\|_{L^2} + \|\Theta^2(T)\|_{L^2} + 2\kappa_1\int_0^T \|\nabla\Theta^1\|_{L^2}^2\, dt \\
&\quad + \kappa_2\int_0^T \|\nabla\Theta^2\|_{L^2}^2\, dt \le c\int_0^T \|LU\|_{L^2}^2\, dt \sup_{t\in[0,T]}\|T_2^1\|_{L^\infty}^2 \\
&\le C(T_{10}, T_{20}, \theta)\int_0^T \|LU\|_{L^2}^2\, dt.
\end{aligned} \tag{57}$$

To proceed, we first recall the regularity of L and L^* in the case of Navier slip boundary conditions (c.f [7], [16], [18], [19]). We have that

$$L \in \mathscr{L}\big(L^2(0,T;(V_n^{3/2}(\partial\Omega_2))'), L^2(0,T;V_n^0(\Omega_2))\big), \tag{58}$$

and

$$L^* \in \mathscr{L}\big(L^2(0,T;(V_n^{3/2}(\Omega_2))'), L^2(0,T;V_n^0(\partial\Omega_2))\big). \tag{59}$$

With the help of the optimality condition (53), (58)–(59), and Remark 4 we have

$$\begin{aligned}
&\int_0^T \|LU\|_{L^2}^2\, dt \le \frac{1}{\gamma^2}\int_0^T \|LL^*(\mathbb{P}(Q^2\nabla T_2^1 + q_2^2\nabla\Theta^2))\|_{L^2}^2\, dt \\
&\le \frac{2}{\gamma^2}\int_0^T (\|\mathbb{P}^*(Q^2\nabla T_2^1)\|_{H^{-1-\delta}}^2 + \|\mathbb{P}^*(q_2^2\nabla\Theta^2)\|_{H^{-1-\delta}}^2)\, dt,
\end{aligned} \tag{60}$$

where $\delta > 0$ is sufficiently small. Moreover,

$$\|\mathbb{P}^*(Q^2\nabla T_2^1)\|_{(H^{1+\delta}(\Omega_2))'} = \sup_{\varphi\in V_n^{1+\delta}(\Omega_2)} \frac{|\int_\Omega (Q^2\nabla T_2^1)\cdot(\mathbb{P}\varphi)\, dx|}{\|\varphi\|_{H^{1+\delta}}}$$

$$\le \sup_{\varphi\in H^{1+\delta}(\Omega_2)} \frac{c\|Q^2\|_{L^2}\|\nabla T_2^1\|_{L^2}\|\varphi\|_{L^\infty}}{\|\varphi\|_{H^{1+\delta}}} \tag{61}$$

$$\le \sup_{\varphi\in H^{1+\delta}(\Omega_2)} \frac{c\|Q^2\|_{L^2}\|\nabla T_2^1\|_{L^2}\|\varphi\|_{H^{1+\delta}}}{\|\varphi\|_{H^{1+\delta}}} \tag{62}$$

$$\le c\|Q^2\|_{L^2}\|\nabla T_2^1\|_{L^2}.$$

From (61) to (62), we used Agmon's inequality for dimension $d = 2$.

Similarly, for the second term on the right hand side of (60) we have

$$\|\mathbb{P}^*(q_2^2\nabla\Theta^2)\|_{(H^{1+\delta}(\Omega_2))'} \le c\|q_2^2\|_{L^2}\|\nabla\Theta^2\|_{L^2}.$$

Thus (60) becomes

$$\begin{aligned}
\int_0^T \|LU\|_{L^2}^2\, dt &\le \frac{1}{\gamma^2}\int_0^T (c\|Q^2\|_{L^2}^2\|\nabla T_2^1\|_{L^2}^2 \\
&\quad + c\|q_2^2\|_{L^2}^2\|\nabla\Theta^2\|_{L^2}^2)\, dt \\
&\le \frac{c}{\gamma^2}\Big(\sup_{t\in[0,T]}\|Q^2\|_{L^2}^2\int_0^T \|\nabla T_2^1\|_{L^2}^2\, dt \\
&\quad + \sup_{t\in[0,T]}\|q_2^2\|_{L^2}^2\int_0^T \|\nabla\Theta^2\|_{L^2}^2\, dt\Big).
\end{aligned}$$

With the help of the regularity results for T_2^1, q_2^2, and Θ^2 given by (24), (37), and (57), we further obtain that

$$\int_0^T \|LU\|_{L^2}^2\,dt \le \frac{1}{\gamma^2} C(T_{10}, T_{20}, \theta)\Big(\sup_{t\in[0,T]} \|Q^2\|_{L^2}^2 + \int_0^T \|LU\|_{L^2}^2\,dt\Big). \tag{63}$$

Moreover, applying the L^2-estimate for Q^2 in (50) gives

$$\sup_{t\in[0,T]} \|Q^2\|_{L^2}^2 \le c \int_0^T \|LU\|_{L^2}^2\,dt. \tag{64}$$

Combining (63) with (64) yields

$$\int_0^T \|LU\|_{L^2}^2 \le \frac{1}{\gamma^2} C(T_{10}, T_{20}, \theta) \int_0^T \|LU\|_{L^2}^2.$$

If let γ be sufficiently large such that

$$C(T_{10}, T_{20}, \theta)\frac{1}{\gamma^2} < 1 \quad \text{or} \quad \gamma > (C(T_{10}, T_{20}, \theta))^{1/2},$$

then $\int_0^T \|LU\|_{L^2}^2\,dt = 0$. Finally, by the linearity of L we derive that $U = 0$. Uniqueness of the optimal solution is established for γ sufficiently large. ∎

V. Conclusion

In this paper we present a rigorous mathematical framework for optimal control of cooling via convection in unsteady Stokes fluid flows. Both internal and boundary controls are addressed. We prove the existence of an optimal control and establish the first-order necessary conditions for optimality for solving the optimal control by using a variational inequality. Moreover, we derive the uniqueness of the optimal Navier slip boundary control when γ is sufficiently large. The gradient based iterative schemes will be employed to implement control designs in our future work. It will also be interesting to investigate the cooling rate with respect to the control actuation.

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