TOPOLOGICAL OBSTRUCTIONS TO CONTINUITY OF ORLICZ-SOBOLEV MAPPINGS OF FINITE DISTORTION

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ABSTRACT. In the paper we investigate continuity of Orlicz-Sobolev mappings $W^{1,P}(M,N)$ of finite distortion between smooth Riemannian n-manifolds, $n \geq 2$, under the assumption that the Young function P satisfies the so called divergence condition $\int_1^\infty P(t)/t^{n+1}\, dt = \infty$. We prove that if the manifolds are oriented, N is compact, and the universal cover of N is *not* a rational homology sphere, then such mappings are continuous. That includes mappings with $Df \in L^n$ and, more generally, mappings with $Df \in L^n \log^{-1} L$. On the other hand, if the space $W^{1,P}$ is larger than $W^{1,n}$ (for example if $Df \in L^n \log^{-1} L$), and the universal cover of N is homeomorphic to $\mathbb{S}^n$, $n \neq 4$, or is diffeomorphic to $\mathbb{S}^n$, $n = 4$, then we construct an example of a mapping in $W^{1,P}(M,N)$ that has finite distortion and is discontinuous. This demonstrates a new *global-to-local phenomenon*: both finite distortion and continuity are local properties, but a seemingly local fact that finite distortion implies continuity is a consequence of a global topological property of the target manifold N.

1. INTRODUCTION

Throughout the paper we assume that $n \geq 2$.

Vodop'janov and Gol'dšteĭn [33] proved that if a mapping $f : \Omega \to \mathbb{R}^n$ of class $W^{1,n}$, defined on a domain $\Omega \subset \mathbb{R}^n$, has positive Jacobian, $J_f > 0$, almost everywhere, then f is continuous (i.e. f has a continuous representative). In fact, Vodop'janov and Gol'dšteĭn proved a slightly stronger result, that $W^{1,n}$ mappings of finite distortion are continuous.

We say that a mapping $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, has *finite distortion* if there is a function $K : \Omega \to [0, \infty)$ such that $|Df(x)|^n \leq K(x)J_f(x)$ a.e. Taking $K(x) = |Df(x)|^n/J_f(x)$ we see that $W^{1,n}$ mappings with almost everywhere positive Jacobian have finite distortion, so continuity of mappings with finite distortion implies continuity of mappings with positive Jacobian. The notion of finite distortion generalizes with no change to definition to mappings between oriented, smooth manifolds of the same dimension.

Recently, the result of Vodop'janov and Gol'dšteĭn has been extended to the case of mappings between manifolds [9].

Theorem 1. *Let M and N be smooth, oriented, n-dimensional Riemannian manifolds without boundary and assume additionally that N is compact. If $f \in W^{1,n}(M,N)$ has*

finite distortion, then f is continuous. In particular, if $f \in W^{1,n}(M, N)$ has positive Jacobian almost everywhere, then f is continuous.

In the Euclidean setting the result of Vodop'janov and Gol'dšteǐn has been extended to Orlicz-Sobolev spaces that are larger than the Sobolev space $W^{1,n}$, see [16], [17, Theorem 7.5.2].

Theorem 2. *Let P be a Young function satisfying the doubling condition* (5), *the growth condition* (7) *and the divergence condition* (8). *If $f \in W^{1,P}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, has finite distortion, then f is continuous.*

For more information about Orlicz-Sobolev spaces, see Section 4. If $P(t) = t^n$, then $W^{1,P} = W^{1,n}$, so the result of Vodop'janov and Gol'dšteǐn follows from Theorem 2. Perhaps the most important Young function, other than $P(t) = t^n$, satisfying the assumptions of Theorem 2 is $P(t) = \frac{t^n}{\log(e+t)}$. The corresponding Orlicz-Sobolev space $W^{1,P}$ is larger than $W^{1,n}$. If Ω is bounded and has smooth boundary, then the space $W^{1,P}(\Omega)$ consists of functions such that

$$|Df| \in L^n \log^{-1} L, \quad \text{that is} \quad \int_{\Omega} \frac{|Df|^n}{\log(e + |Df|)} < \infty.$$

This is a much weaker condition than L^n integrability of $|Df|$.

Orlicz-Sobolev spaces with Young functions satisfying the assumptions of Theorem 2 ($L^n \log^{-1} L$ in particular) turn out to be critical in the study of regularity of distributional Jacobians, degree theory, properties of monotone mappings and the theory of Hardy-Orlicz spaces [8, 10, 12, 16, 17, 18].

In this paper we address the problem of extending Theorem 2 to the case of Orlicz-Sobolev mappings between manifolds. Clearly, we want the Young function to satisfy the conditions described in Theorem 2, but since we want the Orlicz-Sobolev space to be larger than the Sobolev space $W^{1,n}$, we impose one more growth condition (6a). It turns out that in the case of manifolds, the answer to the question whether $W^{1,P}(M, N)$ mappings of finite distortion (or even with almost everywhere positive Jacobian) are continuous depends on delicate topological properties of the target manifold N. The main result of the paper reads as follows.

Theorem 3. *Let M and N be smooth, oriented, n-dimensional Riemannian manifolds without boundary and assume additionally that N is compact. Assume that a Young function P satisfies conditions* (5), (6a), (7) *and* (8).

- *If the universal cover of N is not a rational homology sphere, then $W^{1,P}(M, N)$ mappings of finite distortion (in particular mappings with almost everywhere positive Jacobian) are continuous.*
- *If the universal cover of N is homeomorphic (when $n \neq 4$) or diffeomorphic (when $n = 4$) to $\mathbb{S}^n$, then there are mappings in $W^{1,P}(M, N)$ of finite distortion that are discontinuous. If in addition $M = \mathbb{B}^n$ is a Euclidean ball, one can construct a discontinuous mapping in $W^{1,P}(\mathbb{B}^n, N)$ that has almost everywhere positive Jacobian.*

Remark 4. In fact in the first part (continuity) of Theorem 3 we do not need the growth condition (6a), but a weaker condition (6b), ensuring that $W^{1,n}(M, N) \subset W^{1,P}(M, N)$ (see Theorems 5 and 6). The growth condition (6a) is only needed for the construction of a counterexample – it guarantees that the space $W^{1,P}$ is strictly larger than $W^{1,n}$ and for $W^{1,n}$ mappings continuity is always guaranteed by Theorem 1.

We say that a compact n-manifold without boundary is a *rational homology sphere*, if it has the same deRham cohomology as the standard sphere $\mathbb{S}^n$. Rational homology spheres were investigated in [8, 10, 12] in the context of the degree theory of Orlicz-Sobolev mappings. Quasiregular mappings and mappings of finite distortion with values into rational homology spheres have also been studied in [5, 29]. For more information about rational homology spheres, see Section 2.

It follows from the Poincaré conjecture (when $n = 3, 4$) that in dimensions $n = 2, 3, 4$ simply connected rational homology spheres are homeomorphic (but for $n = 4$ not necessarily diffeomorphic) to $\mathbb{S}^n$, so Theorem 3 completely solves the problem in dimensions 2 and 3. In dimension 4 the situation is complicated by the possible existence of *exotic spheres*, see the discussion in Section 5. However, in higher dimensions there is a gap, because there are many examples of simply connected rational homology spheres that are not spheres, see Section 2.

In geometry, the *local-to-global principle* means that: local properties of mappings imply their global properties. However, Theorem 3 shows a new, dual *global-to-local phenomenon*: both having finite distortion (or positive Jacobian) and continuity are local properties, but a seemingly local fact that finite distortion implies continuity is a consequence of a global topological property of the target manifold N.

If the Young function P satisfies the assumptions of Theorem 3, the derivative of a mapping $f \in W^{1,P}(M, N)$ does not necessarily belong to L^n and hence there is no apparent reason why the Jacobian J_f should be integrable. In fact, the discontinuous mappings discussed in the second part of Theorem 3 do not have integrable Jacobian. However, if we know that the Jacobian J_f is integrable, then continuity of f follows without any topological assumptions about N. This is the second main result of the paper.

Theorem 5. *Let M and N be smooth, oriented, n-dimensional Riemannian manifolds without boundary and assume additionally that N is compact. Assume that a Young function P satisfies conditions* (5), (6b), (7) *and* (8). *If $f \in W^{1,P}(M, N)$ has finite distortion and $J_f \in L^1_{\mathrm{loc}}(M)$, then f is continuous.*

Now the first part (continuity) of Theorem 3 follows from Theorem 5 and the last main result.

Theorem 6. *Let M and N be smooth, oriented, n-dimensional Riemannian manifolds without boundary and assume that N is compact and the universal cover of N is not a rational homology sphere. If a Young function P satisfies conditions* (5), (6b), (7), (8) *and $f \in W^{1,P}(M, N)$ has non-negative Jacobian (in particular if f has finite distortion), then the Jacobian J_f is locally integrable, $J_f \in L^1_{\mathrm{loc}}(M)$.*

Remark 7. Note that in Theorems 5 and 6 the growth condition (6a) is not needed.

Remark 8. Theorem 6 is related to [12, Theorem 6.6], but the proof is very different and we do not know if the technique used in [12] can be adapted to prove Theorem 6.

The article is organized in the following way. In Section 2 we discuss rational homology spheres. In Section 3 we recall basic results from the classical theory of Sobolev spaces. This section is followed by Section 4 devoted to definitions and facts from the theory of Orlicz-Sobolev spaces needed in the paper. As a byproduct of methods developed in that section, we provide a new proof of density of smooth mappings in the class of Orlicz-Sobolev mappings between manifolds, Corollary 21. In Section 5 we construct discontinuous maps in $W^{1,P}(M, N)$ that have finite distortion. This proves the second part of Theorem 3. In Sections 6 and 7 we prove Theorems 6 and 5, respectively. These results, along with results of Section 5, complete the proof of Theorem 3.

In the final Section 6 we prove the continuity part of Theorem 3 which, along with the results of Section 5, completes the proof of Theorem 3.

1.1. **Notation.** The notation used in the article is pretty standard. By C we will denote a generic constant whose value may change even within a single string of estimates. By writing, for example, $C = C(n, \alpha)$ we will mean that the constant C depends on n and α only. The Lebesgue measure of a set A (both in $\mathbb{R}^n$ and on a manifold) will be denoted by $|A|$. The volume of the unit Euclidean ball in $\mathbb{R}^n$ will be denoted by ω_n, so the volume of the unit sphere $\mathbb{S}^{n-1}$ is $n\omega_n$. The barred integral will denote the integral average

$$\fint_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu.$$

The measure on a hypersurface in $\mathbb{R}^n$ will be denoted by $d\sigma$. The characteristic function of a set E will be denoted by χ_E. The closure of E is denoted by $\overline{E}$. By $\|\cdot\|_p$ we denote the L^p-norm with respect to the Lebesgue measure. Whenever we write about smooth functions or mappings, we mean C^∞-smooth.

Acknowledgements. The authors would like to thank Armin Schikorra for a helpful discussion about fractional Sobolev spaces.

2. Rational homology spheres

Let us recall that a compact n-dimensional Riemannian manifold N without boundary is a *rational homology sphere* if its deRham cohomology groups are the same as these of an n-dimensional sphere, i.e. $H^i_{dR}(N) = \mathbb{R}$ for $i = 0$ and $i = n$ and $H^i_{dR}(N) = 0$ for all other values of i. The importance of this condition comes from the following lemma ([8, Theorem 2.1], see also [35]).

Lemma 9. *Let N be a smooth, compact, connected, oriented n-dimensional manifold without boundary. Then there is a smooth mapping $f : \mathbb{S}^n \to N$ of non-zero degree if and only if the universal cover of N is a rational homology sphere.*

Thus if the universal cover of N *is not* a rational homology sphere, then every smooth mapping from an n-dimensional sphere to N is of degree zero. The same holds for Lipschitz mappings from $\mathbb{S}^n$ to N, since every Lipschitz mapping is homotopic to a smooth one and the degree is a homotopy invariant.

Rational homology spheres include spheres themselves, integral homology spheres like the celebrated Poincaré sphere and more general Brieskorn manifolds, and many others. The book [36] provides numerous 3-dimensional examples, we refer the reader also to [8, Section 2]. On the other hand, the following well known proposition holds:

Proposition 10. *If N is an n-dimensional rational homology sphere and*

    a) *$n = 2$, then N is diffemorphic to $\mathbb{S}^2$,*
    b) *$n = 3$ and N is simply connected, then N is diffeomorphic to $\mathbb{S}^3$,*
    c) *$n = 4$, then N is homeomorphic to $\mathbb{S}^4$.*

Proof. Case a) follows from classification of closed surfaces (see e.g. [14, Section 9.3]): an orientable surface without boundary is uniquely (up to a diffeomorphism) determined by its genus, which, for a rational homology sphere, must be 0.

For a proof of c) see e.g. [8, Proposition 2.6].

The case $n = 3$ is settled similarly as $n = 4$, and we present a sketch of arguments here. The missing details and references can be found in [8, Proposition 2.6].

If N is connected, orientable and compact, then $H_0(N, \mathbb{Z}) = \mathbb{Z}$ and $H_3(N, \mathbb{Z}) = \mathbb{Z}$. Since N is also simply connected, $H_1(N, \mathbb{Z}) = 0$, and the Universal Coefficients Theorem gives $H^1(N, \mathbb{Z}) = 0$. Also, Poincaré duality shows that $H_2(N, \mathbb{Z}) = H^1(N, \mathbb{Z}) = 0$. Thus N is an integral homology sphere; homology Whitehead's theorem yields that N is a homotopy sphere. Finally, Perelman's theorem on Poincaré's Conjecture proves that N is indeed diffeomorphic to a 3-dimensional sphere. $\qquad\square$

Corollary 11. *Assume N is a smooth, compact, connected, oriented n-dimensional manifold without boundary such that its universal cover $\tilde{N}$ is a rational homology sphere. Then*

    • *if $n = 2$ then N is diffeomorphic to $\mathbb{S}^2$,*
    • *if $n = 3$, then $\tilde{N}$ is diffeomorphic to $\mathbb{S}^3$,*
    • *if $n = 4$, then N is homeomorphic to $\mathbb{S}^4$.*

Whether N is diffeomorphic to $\mathbb{S}^4$ when $n = 4$, remains a long standing open problem.

However, in dimension 5 and higher, there are simply connected rational homology spheres that are not spheres, e.g. the Wu manifold $SU(3)/SO(3)$, [26, Theorem 6.7] and [3, Remark, p. 374]. See [35] and [3, Lemma 1.1] for more examples.

3. SOBOLEV SPACES

In this section we collect technical results from the theory of Sobolev spaces that are needed in the paper. All the results discussed here are well known except perhaps for Lemma 14, which is also known, but very difficult to find in the literature.

The Sobolev space $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, consists of weakly differentiable functions such that $\|f\|_{1,p} = \|f\|_p + \|Df\|_p < \infty$. Similarly we define the Sobolev space $W^{1,p}(M)$, where M is a Riemannian manifold.

Assume M and N are smooth, Riemannian manifolds without boundary, with N compact. Assume also that N is isometrically embedded in $\mathbb{R}^k$ for some $k \in \mathbb{N}$. Then the class of Sobolev mappings $W^{1,p}(M, N)$ is defined as

$$W^{1,p}(M,N) = \left\{ f \in W^{1,p}_{\mathrm{loc}}(M, \mathbb{R}^k) : |Df| \in L^p(M) \text{ and } f(x) \in N \text{ for a.e. } x \in M \right\}. \quad (1)$$

If M is compact, then $f \in L^p(M, \mathbb{R}^k)$. However, we cannot require integrability of the mapping f in the non-compact case (especially when the measure of M is infinite): if the measure of M is infinite and if N is embedded into $\mathbb{R}^k$ in a way that the origin is at a positive distance to N, then no mapping $f : M \to N \subset \mathbb{R}^k$ is integrable with exponent $1 \leq p < \infty$. Since our results are of local nature in M (continuity and finite distortion are defined locally), integrability of f is not important to us.

3.1. Pointwise inequalities.
For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the Hardy-Littlewood maximal function is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{\mathbb{B}^n(x,r)} |f(y)| \, dy.$$

It is well known, [31], that the maximal function is bounded in L^p when $1 < p < \infty$. This and Chebyshev's inequality imply that if $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then

$$t^p |\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}| \to 0 \quad \text{as } t \to \infty. \quad (2)$$

If $f \in W^{1,1}_{\mathrm{loc}}$, then the following pointwise inequality is true, see for example [1, 11, 24].

$$|f(x) - f(y)| \leq C(n)|x - y|(\mathcal{M}|Df|(x) + \mathcal{M}|Df|(y)) \quad \text{for almost all } x, y \in \mathbb{R}^n. \quad (3)$$

A simple argument (see [11, p. 97]) shows that if we choose the representative of f defined by

$$f(x) := \limsup_{r \to 0} \fint_{\mathbb{B}^n(x,r)} f(y) \, dy \quad \text{for every } x \in \mathbb{R}^n,$$

then the inequality (3) is true for *all* $x, y \in \mathbb{R}^n$.

It easily follows that if $f \in W^{1,1}(\mathbb{B}^n)$, where $\mathbb{B}^n \subset \mathbb{R}^n$ is a ball of any radius, then the inequality (3) is still true for all $x, y \in \mathbb{B}^n$, where we put $|Df| = 0$ outside $\mathbb{B}^n$, see [13, Lemma 4 and (7)].

If M is a compact Riemannian manifold with or without without boundary, we can define the maximal function on M with averages over geodesic balls. Then the maximal function is bounded in $L^p(M)$, $1 < p < \infty$, (2) is true (with the same proof) and $f \in W^{1,1}(M)$ satisfies the pointwise inequality

$$|f(x) - f(y)| \leq C(M)|x - y|(\mathcal{M}|Df|(x) + \mathcal{M}|Df|(y)) \quad \text{for *all* } x, y \in M, \quad (4)$$

with a suitable choice of a representative of f. This inequality immediately gives the following well known

Lemma 12. *Assume $f \in W^{1,1}(M)$, where M is a compact Riemannian manifold with or without boundary. Then f, restricted to the set $\{\mathcal{M}|Df| \leq t\}$, is Ct-Lipschitz for some constant C depending on M only.*

3.2. **Morrey's inequality.** The next result is a version of the classical Morrey's lemma in the case of Sobolev functions defined on a sphere.

Lemma 13. *If $\mathbb{S}^{n-1}(r)$ is an $(n-1)$-dimensional Euclidean sphere of radius r and $f \in W^{1,\alpha}(\mathbb{S}^{n-1}(r))$ for some $\alpha > n-1$, then f has a $C^{0,1-\frac{n-1}{\alpha}}$-Hölder continuous representative which satisfies*

$$\underset{\mathbb{S}^{n-1}(r)}{\operatorname{osc}} f = \sup_{x,y \in \mathbb{S}^{n-1}(r)} |f(x) - f(y)| \leq C(n,\alpha) r \left(\fint_{\mathbb{S}^{n-1}(r)} |Df|^\alpha \, d\sigma \right)^{1/\alpha}.$$

3.3. **Traces.** If $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $1 < p < \infty$, we say that u belongs to the fractional Sobolev space $W^{1-\frac{1}{p},p}(\partial\Omega)$ if $\|u\|_{1-\frac{1}{p},p} = \|u\|_{L^p(\partial\Omega)} + S_p(u) < \infty$, where

$$S_p(u) = \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p-2}} \, d\sigma(x) \, d\sigma(y) \right)^{1/p}.$$

Gagliardo [7] (see also [23, Chapter 15]) proved that the trace operator $\operatorname{Tr} : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ is bounded and there is an extension operator $\operatorname{Ext} : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$ such that $\operatorname{Tr} \circ \operatorname{Ext} = \operatorname{id}$ on $W^{1-\frac{1}{p},p}(\partial\Omega)$. In other words, the fractional Sobolev space $W^{1-\frac{1}{p},p}(\partial\Omega)$ provides a complete description of traces of $W^{1,p}(\Omega)$ functions.

In the article we will need the following known fact that was proven in [4]. A self contained and elementary (but difficult) proof can be found in [23] (see Theorem 14.32, Remark 14.35 and Proposition 14.40). This result also follows from a sequence of results (as indicated below) in [32].

Lemma 14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. If $n > 2$, then $W^{1,n-1}(\partial\Omega) \subset W^{1-\frac{1}{n},n}(\partial\Omega)$. That is, there is a bounded extension operator $\operatorname{Ext} : W^{1,n-1}(\partial\Omega) \to W^{1,n}(\Omega)$. If $n = 2$, then for any $\alpha > 1$ we have $W^{1,\alpha}(\partial\Omega) \subset W^{\frac{1}{2},2}(\partial\Omega)$. That is, there is a bounded extension operator $\operatorname{Ext} : W^{1,\alpha}(\partial\Omega) \to W^{1,2}(\Omega)$.*

Proof. If $n > 2$, then using the following results from [32]: Theorem 2.5.6, Theorem 2.7.1, Proposition 2.3.2.2(8), Theorem 2.5.7 and 2.5.7(9) (in that order) we obtain the following relations for function spaces on $\mathbb{R}^{n-1}$:

$$W^{1,n-1}(\mathbb{R}^{n-1}) = H^1_{n-1} = F^1_{n-1,2} \subset F^{1-\frac{1}{n}}_{n,n} = B^{1-\frac{1}{n}}_{n,n} = \Lambda^{1-\frac{1}{n}}_{n,n} = W^{1-\frac{1}{n},n}(\mathbb{R}^{n-1}).$$

The above identification of Sobolev spaces and Triebel-Lizorkin spaces fails for $n = 2$, and in that case we need to argue in a slightly different way. Note that we can assume that $1 < \alpha < 2$. Then using the results from [32]: Theorem 2.5.6, Proposition 2.3.2.2(8), Theorem 2.7.1, Proposition 2.3.2.2(9), Theorem 2.5.7(5) and 2.5.7(9) (in that order) we obtain

$$W^{1,\alpha}(\mathbb{R}) = H^1_\alpha = F^1_{\alpha,2} \subset F^{\frac{1}{\alpha}}_{\alpha,2} \subset F^{\frac{1}{2}}_{2,2} = B^{\frac{1}{2}}_{2,2} = \Lambda^{\frac{1}{2}}_{2,2} = W^{\frac{1}{2},2}(\mathbb{R}).$$

□

Remark 15. In fact, if $n = 2$, the space $W^{1,1}(\partial\Omega)$ does not embed into $W^{\frac{1}{2},2}(\partial\Omega)$, see [4], [23, Exercise 14.36] and [30, Proposition 4].

Remark 16. The reasoning in the case $n = 2$ is slightly different than in the case $n > 2$ and it has to be different as is explained below. If we apply the argument from the case $n > 2$ to $W^{1,\alpha}(\mathbb{R})$, $1 < \alpha < 2$, then we obtain

$$W^{1,\alpha} = H^1_\alpha = F^1_{\alpha,2} \subset F^{\frac{1}{2}}_{\frac{2\alpha}{2-\alpha},\frac{2\alpha}{2-\alpha}} = B^{\frac{1}{2}}_{\frac{2\alpha}{2-\alpha},\frac{2\alpha}{2-\alpha}} = \Lambda^{\frac{1}{2}}_{\frac{2\alpha}{2-\alpha},\frac{2\alpha}{2-\alpha}} = W^{\frac{1}{2},\frac{2\alpha}{2-\alpha}}(\mathbb{R}).$$

Since $\frac{2\alpha}{2-\alpha} > 2$ it seems that we get what we wanted

$$W^{1,\alpha}(\partial\Omega) \subset W^{\frac{1}{2},\frac{2\alpha}{2-\alpha}}(\partial\Omega) \subset W^{\frac{1}{2},2}(\partial\Omega).$$

However, very surprisingly, the last inclusion is **false** as was shown in [27].

Remark 17. The extension operator Ext : $W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$ can be defined by an explicit integral formula [23, Theorem 15.21] from which it follows that the extension is smooth in Ω and is continuous up to the boundary if the function that we extend is continuous on $\partial\Omega$.

4. ORLICZ-SOBOLEV SPACES

In this section we briefly describe basic properties of Orlicz-Sobolev spaces that are used in the paper. Since we do not need any delicate results from the theory of Orlicz-Sobolev spaces, we will try to keep the definitions as simple as possible. For more information about Orlicz and Orlicz-Sobolev spaces, see e.g. [2, 17, 22, 34]. Our approach is closely related to that in [12, Section 4].

We assume that $P : [0,\infty) \to [0,\infty)$ is convex, strictly increasing, with $P(0) = 0$. A function satisfying these conditions is called a *Young function*. We will always assume that P satisfies the *doubling condition*:

- **doubling condition** or Δ_2-**condition**

    there exists $K > 0$ such that $P(2t) \leq K\, P(t)$ for all $t \geq 0$. (5)

We also consider other conditions:

- **growth conditions**

$$\frac{P(t)}{t^n} \to 0 \quad \text{as } t \to \infty, \tag{6a}$$

$$P(t) \leq Ct^n \text{ for some } C > 0 \text{ and all } t \geq 1. \tag{6b}$$

    the function $t^{-\alpha}P(t)$ is non-decreasing for some $n > \alpha > n - 1$, (7)

- **divergence condition**

$$\int\limits_1^\infty \frac{P(t)}{t^{n+1}}\, dt = \infty. \tag{8}$$

Let (X, μ) be a measure space. If a Young function P satisfies the doubling condition (5), then the *Orlicz space* $L^P(X)$ is defined as the class of all measurable functions f such that $\int_X P(|f|)\, d\mu < \infty$. Clearly, $L^P \subset L^1_{loc}$.

The doubling condition implies that L^P is a linear space and actually it is a Banach space with respect to the *Luxemburg norm*

$$\|f\|_P = \inf \left\{ k > 0 : \int_X P(|f|/k)\, d\mu \leq 1 \right\}.$$

We say that a sequence (f_k) of functions in $L^P(X)$ converges to f *in mean*, if

$$\lim_{k \to \infty} \int_X P(|f_k - f|)\, d\mu = 0.$$

It is well known and easy to show that under the doubling condition convergence in mean is equivalent to convergence in the Luxemburg norm.

Given a Young function P satisfying (5), we define the *Orlicz-Sobolev space* $W^{1,P}(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ as the class of functions in $W^{1,1}_{loc}(\Omega)$ such that

$$\|f\|_{1,P} = \|f\|_P + \|Df\|_P < \infty. \tag{9}$$

$W^{1,P}(\Omega)$ is a Banach space and smooth functions, $C^\infty(\Omega)$, are dense in it. Similarly as in the case of L^P, it is not difficult to show that $f_k \to f$ in $W^{1,P}$ if and only if we have convergence in mean, i.e.

$$\int_\Omega P(|f - f_k|) + P(|Df - Df_k|)\, dx \to 0 \quad \text{as } k \to \infty.$$

The definition of the Orlicz-Sobolev space can be easily extended to the case of functions on a Riemannian manifold, $W^{1,P}(M)$. Then Orlicz-Sobolev mappings between manifolds $W^{1,P}(M, N)$ are defined by analogy to (1).

The next lemma was proved in [12]: part a) is obvious. For part b) see [12, Proposition 4.7]. Part c) is an easy consequence of part b) and for part d) see [12, Proposition 4.1] and Lemma 13. For the sake of completeness we will include a short and self-contained proof.

Lemma 18. *Assume* $f \in W^{1,P}(M)$, *where* M *is a smooth, compact, Riemannian n-manifold with or without boundary. Assume moreover that P satisfies conditions* (5), (7) *and* (8). *Then*

a) $f \in W^{1,\alpha}(M)$,

b) $\displaystyle \liminf_{t \to \infty} t^{n-\alpha} \int_{\{|Df| > t\}} |Df|^\alpha = 0$,

c) $\displaystyle \liminf_{t \to \infty} t^n\, |\{x \in M \,:\, \mathcal{M}|Df| > t\}| = 0$,

d) *if* $x \in M \setminus \partial M$ *and* $R > 0$, *then for any* $\varepsilon > 0$, *the set of radii* $r \in (0, R)$ *such that* $\mathrm{osc}_{\mathbb{S}^{n-1}(x,r)} f = \sup_{y,z \in \mathbb{S}^{n-1}(x,r)} |f(y) - f(z)| < \varepsilon$ *has positive linear measure.*

Remark 19. Since for almost all $r \in (0, R)$ we have $f \in W^{1,\alpha}(\mathbb{S}^{n-1}(x, r))$, where $\alpha > n-1$, the function f restricted to $\mathbb{S}^{n-1}(x, r)$ has a Hölder continuous representative and the oscillation in d) is defined for that representative.

Proof. Note that (7) implies that $P(t) \geq P(1)t^\alpha$ for $t \geq 1$, which immediately gives a).

Let α be as in (7) and set $\Psi(t) = t^{\alpha-n}(t^{-\alpha}P(t))'$. Then, integrating by parts, we get that

$$\int_1^k \Psi(t)\, dt = \int_1^k t^{\alpha-n}(t^{-\alpha}P(t))'\, dt = t^{-n}P(t)\Big|_1^k + (n-\alpha)\int_1^k \frac{P(t)}{t^{n+1}}\, dt$$

and letting $k \to \infty$ yields

$$\int_1^\infty \Psi(t)\, dt = +\infty. \tag{10}$$

On the other hand, for all $T > 1$,

$$\left(\int_1^T \Psi(t)\, dt\right) \inf_{t>1}\left(t^{n-\alpha}\int_{\{|Df|>t\}} |Df|^\alpha\, dx\right) \leq \int_1^T \Psi(t)t^{n-\alpha}\int_{\{|Df|>t\}} |Df|^\alpha\, dx\, dt$$

$$\overset{*}{\leq} \int_{\{|Df|>1\}} |Df|^\alpha \left(\int_1^{|Df|} \Psi(t)t^{n-\alpha}\, dt\right) dx = \int_{\{|Df|>1\}} |Df|^\alpha \left(\int_1^{|Df|} (t^{-\alpha}P(t))'\, dt\right) dx$$

$$= \int_{\{|Df|>1\}} |Df|^\alpha \left(|Df|^{-\alpha}P(|Df|) - P(1)\right) dx \leq \int_M P(|Df|)\, dx < \infty,$$

where the inequality $\overset{*}{\leq}$ follows from the obvious implication

$$(1 \leq t \leq T, |Df| > t) \Rightarrow (|Df| > 1, 1 \leq t < |Df|).$$

Letting $T \to \infty$ and using (10) we see that $\inf_{t>1}\left(t^{n-\alpha}\int_{\{|Df|>t\}} |Df|^\alpha\right) = 0$, which, in turn, implies b).

Let $h = |Df|\chi_{\{|Df|>t/2\}}$. Then $|Df| \leq h + t/2$, so $\{\mathcal{M}|Df| > t\} \subset \{\mathcal{M}h > t/2\}$ and Chebyshev's inequality along with boundedness of the maximal function on $L^\alpha(M)$ yield

$$t^n|\{\mathcal{M}|Df| > t\}| \leq t^n|\{\mathcal{M}h > t/2\}| \leq t^n \left(\frac{2}{t}\right)^\alpha \|\mathcal{M}h\|_\alpha^\alpha$$

$$\leq C(M, \alpha)t^{n-\alpha}\|h\|_\alpha^\alpha = Ct^{n-\alpha}\int_{\{|Df|>t/2\}} |Df|^\alpha.$$

Combining this estimate with b) immediately yields c).

It remains to prove d).

There is $0 < R' \leq R$ such that the exponential map $\exp_x : T_xM \to M$ maps Euclidean balls $\mathbb{B}^n(0, r) \subset T_xM$ onto Riemannian balls $\mathbb{B}^n(x, r) \subset M$, in a diffeomorphic way, for all

$0 < r \leq R'$ (see [19]). In particular, it maps Euclidean spheres centered at 0 onto spheres in M centered at x. Thus we can assume that $\mathbb{B}^n(x, R') = \mathbb{B}^n(0, R')$ is the Euclidean ball.

According to Lemma 13 it suffices to show that the set of $r \in (0, R')$ such that

$$r \left(\fint_{\mathbb{S}^{n-1}(0,r)} |Df|^\alpha \, d\sigma \right)^{1/\alpha} < \varepsilon$$

has positive linear measure. Suppose to the contrary, that there is $\varepsilon > 0$ such that

$$r \left(\fint_{\mathbb{S}^{n-1}(0,r)} |Df|^\alpha \, d\sigma \right)^{1/\alpha} \geq \varepsilon \quad \text{for almost all } r \in (0, R').$$

Integration in spherical coordinates gives that for every $0 < \rho < R'$

$$\int_{\mathbb{B}^n(0,\rho)} |Df|^\alpha \, dx \geq \frac{n\omega_n \varepsilon^\alpha \rho^{n-\alpha}}{n - \alpha}.$$

For $t > 0$, $|Df|^\alpha \leq |Df|^\alpha \chi_{\{|Df|>t\}} + t^\alpha$, so

$$\frac{n\omega_n \varepsilon^\alpha \rho^{n-\alpha}}{n - \alpha} \leq \int_{\mathbb{B}^n(0,\rho)} |Df|^\alpha \, dx \leq \int_{\{|Df|>t\}} |Df|^\alpha + \omega_n \rho^n t^\alpha.$$

Let $t > 0$ be such that

$$\frac{n\omega_n \varepsilon^\alpha \rho^{n-\alpha}}{n - \alpha} = 2\omega_n \rho^n t^\alpha, \quad \text{that is} \quad t = \left(\frac{n}{2(n-\alpha)} \right)^{1/\alpha} \frac{\varepsilon}{\rho}.$$

Then

$$\int_{\{|Df|>t\}} |Df|^\alpha \, dx \geq \omega_n \rho^n t^\alpha = \omega_n \varepsilon^n \left(\frac{n}{2(n-\alpha)} \right)^{n/\alpha} t^{\alpha-n},$$

$$\text{thus} \qquad C(n, \alpha)\varepsilon^n \leq t^{n-\alpha} \int_{\{|Df|>t\}} |Df|^\alpha \, dx.$$

When $\rho \to 0$, we have $t \to \infty$, and hence

$$\liminf_{t \to \infty} t^{n-\alpha} \int_{\{|Df|>t\}} |Df|^\alpha \, dx \geq C(n, \alpha)\varepsilon^n,$$

which contradicts b). $\qquad\square$

Lemma 20. *Assume $f \in W^{1,P}(M, N)$, where M and N are compact Riemannian manifolds and $\dim M = n$. M may have boundary, but N has no boundary. Assume moreover that P satisfies conditions (5), (7) and (8). Then there exists a constant C depending on M and N only, a sequence $t_i \to \infty$, and continuous maps $f_i \in C(M, N)$ such that*

    a) *f_i is Ct_i-Lipschitz,*
    b) *f_i coincides with f on the set $\{\mathcal{M}|Df| \leq t_i\}$,*
    c) *$\lim_{i \to \infty} t_i^n |\{x \in M \; : \; \mathcal{M}|Df| > t_i\}| = 0$.*

Proof. Recall that, by Lemma 18 c), $\liminf_{t \to \infty} t^n |\{x \in M \ : \ \mathcal{M}|Df| > t\}| = 0$. Thus, let (t_i) denote such an increasing sequence that $\lim_{i \to \infty} t_i = +\infty$ and

$$\lim_{i \to \infty} t_i^n |\{x \in M \ : \ \mathcal{M}|Df| > t_i\}| = 0. \tag{11}$$

By Lemma 12,

$$f\big|_{\{\mathcal{M}|Df| \leq t_i\}} \colon \{\mathcal{M}|Df| \leq t_i\} \to N \subset \mathbb{R}^k \qquad \text{is } Ct_i\text{-Lipschitz.}$$

Let $F_i : M \to \mathbb{R}^k$ denote its Ct_i-Lipschitz extension to the whole M. Then F_i coincides with f on $\{\mathcal{M}|Df| \leq t_i\}$ and is Ct_i-Lipschitz; however, it does not necessarily map M to N.

Let d_i be the radius of the largest Riemannian ball contained in the open set $\{\mathcal{M}|Df| > t_i\}$. Then, for any $x \in \{\mathcal{M}|Df| > t_i\}$ we can find $y \in \{\mathcal{M}(|Df|) \leq t_i\}$ such that the Riemannian distance $d(x, y)$ is not more than d_i. Then,

$$\operatorname{dist}(F_i(x), N) \leq |F_i(x) - f(y)| = |F_i(x) - F_i(y)| \leq Ct_i|x - y| \leq Ct_i d_i.$$

Note also that, by (11), we have $\lim_{i \to \infty} t_i^n d_i^n = 0$, thus $\lim_{i \to \infty} t_i d_i = 0$, which implies that $\operatorname{dist}(F_i(x), N) \leq Ct_i d_i \xrightarrow{i \to \infty} 0$. This convergence is uniform in x, thus the image of the mapping F_i lies in a tubular neighborhood of N. As N is compact, for a sufficiently small tubular neighborhood of N there is a well defined, smooth and Lipschitz nearest point projection π onto N; taking $f_i = \pi \circ F_i$ gives the desired sequence. Claims a) and b) are then obvious. $\qquad \square$

As a corollary we obtain a new and much shorter proof of Theorem 5.2 in [12]. For some generalizations of this result, see [6].

Corollary 21. *Assume $f \in W^{1,P}(M, N)$, where M and N are compact Riemannian manifolds and $\dim M = n$. M may have boundary, but N has no boundary. Assume moreover that P satisfies conditions (5), (6b), (7) and (8). Then $C^\infty(M, N)$ is dense in $W^{1,P}(M, N)$.*

Proof. Assume $f \in W^{1,P}(M, N)$. By Lemma 20 there is a sequence $t_i \to \infty$ and Ct_i-Lipschitz functions $f_i \in C(M, N)$, $i = 1, 2, \ldots$, that coincide with f on the sets $\{\mathcal{M}|Df| \leq t_i\}$, respectively.

We shall prove that f_i converge to f in $W^{1,P}(M, N)$ *in the mean.*

Let $A_N = \max\{|x| \ : \ x \in N\}$ be the maximum distance of a point in N to the origin. Naturally, $\|f_i\|_\infty \leq A_N$. Also, since f_i is Ct_i-Lipschitz, $|Df_i| \leq Ct_i$ a.e.

Convexity of P and the doubling condition (5) imply immediately that for $a, b \geq 0$ we have $P(a + b) \leq \frac{K}{2}(P(a) + P(b))$. Also, by (6b), we have $P(t) \leq Ct^n$ for large t.

$$\int\limits_M (P(|f - f_i|) + P(|Df - Df_i|)) = \int\limits_{\{\mathcal{M}|Df|>t_i\}} (P(|f - f_i|) + P(|Df - Df_i|))$$

$$\leq \frac{K}{2} \int\limits_{\{\mathcal{M}|Df|>t_i\}} (P(|f|) + P(A_N) + P(|Df|) + P(Ct_i)) \tag{12}$$

$$\leq \frac{K}{2} \int\limits_{\{\mathcal{M}|Df|>t_i\}} (P(|f|) + P(|Df|)) + \frac{K}{2}|\{\mathcal{M}|Df| > t_i\}|(P(A_N) + Ct_i^n)$$

$$= I_i + J_i.$$

The term I_i tends to 0 as $i \to \infty$, because the function $P(|f|) + P(|Df|)$ is integrable, and the measure of $\{\mathcal{M}|Df| > t_i\}$ goes to zero. Likewise, $J_i \to 0$ as $i \to \infty$, by Lemma 20 c).

This proves the density of Lipschitz maps in $W^{1,P}(M, N)$. To pass to smooth maps, it suffices to show that we can approximate a Lipschitz map $g \in C(M, N)$ by a smooth one *in the mean.*

Since N is isometrically embedded in $\mathbb{R}^k$, there is a sequence $\tilde{g}_i \in C^\infty(M, \mathbb{R}^k)$ of smooth maps uniformly approximating g, obtained by a standard convolution with a mollifier. One immediately checks that $\tilde{g}_i$ have derivatives uniformly bounded by the Lipschitz constant L of g. Then we compose the $\tilde{g}_i$ with the nearest point projection π onto N (by uniform convergence of $\tilde{g}_i$, for large i the images of $\tilde{g}_i$ lie arbitrarily close to N), getting $g_i = \pi \circ \tilde{g}_i \in C^\infty(M, N)$. Then $|Dg_i|$ are uniformly bounded by CL for some C, g_i converge uniformly to g and Dg_i converge to Dg a.e. in M. The Dominated Convergence Theorem yields that g_i converge to g in $W^{1,P}(M, N)$ in the mean. $\qquad\square$

5. DISCONTINUOUS MAPPINGS IN $W^{1,P}(M, N)$

Proposition 22. *Assume M, N are smooth, oriented n-dimensional Riemannian manifolds without boundary, N is compact and the universal cover $\tilde{N}$ of N is either diffeomorphic (if $n = 4$) or homeomorphic (if $n \neq 4$) to $\mathbb{S}^n$. Assume moreover that P satisfies conditions (5), (6a), (7) and (8). Then there exists a discontinuous mapping $G \in W^{1,P}(M, N)$ of finite distortion. If in addition $M = \mathbb{B}^n$ is a Euclidean ball, then we can construct a discontinuous mapping $G \in W^{1,P}(\mathbb{B}^n, N)$ that has almost everywhere positive Jacobian.*

First, let us explain the difference between the dimension 4 and all other dimensions (recall that, throughout the paper, $n \geq 2$). In dimensions $n = 2, 3, 5, 6$, if a smooth n-manifold $\tilde{N}$ is homeomorphic to $\mathbb{S}^n$, then it is diffeomorphic to $\mathbb{S}^n$. When $n = 2$ it follows from classification of surfaces. When $n = 3$, it follows from Hauptvermutung for 3-manifolds [28, Theorems 3 and 4] and from the fact that a combinatorial 3-manifold has a unique smoothing [15, Theorem 4.2]. For the cases $n = 5, 6$, see [20, Remark on p. 505]. In dimensions $n \geq 7$ there exist *exotic spheres*: smooth manifolds $\tilde{N}$ homeomorphic, but not diffeomorphic to $\mathbb{S}^n$, see [20, Remark on p. 505]. The first example of an exotic sphere was discovered by Milnor [25] in dimension $n = 7$.
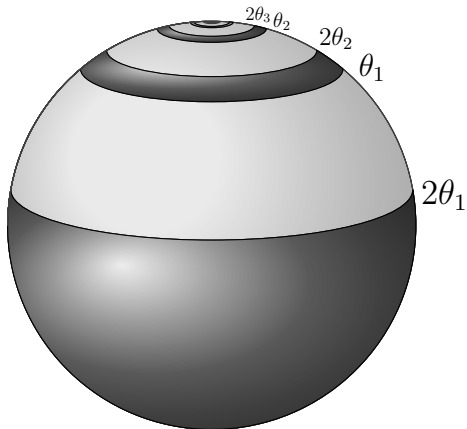
FIGURE 1. Example 23: Dark shaded bands are preserved by F, while light shaded ones are stretched onto the whole sphere so that they cover the whole sphere twice, in an orientation preserving way.

Assume $\tilde{N}$ is an exotic sphere and $n = \dim \tilde{N} \geq 7$. Even though there is no diffeomorphism between $\mathbb{S}^n$ and $\tilde{N}$, one can find a bi-Lipschitz homeomorphism between these spaces, see [37, Theorem 2] and [38, Corollary 4.5]. We can even assume it is a diffeomorphism outside a single point, see [21, Corollary 1.15].

The situation in dimension 4 is more complicated: it is not known whether there exist any exotic sphere of that dimension, or, if there exists one, whether it is bi-Lipschitz-homeomorphic to the standard $\mathbb{S}^4$. Therefore we need the assumption that if $n = 4$, $\tilde{N}$ is diffeomorphic to $\mathbb{S}^4$, as a safeguard against the possible existence of exotic spheres in dimension 4. We could weaken it, asking that $\tilde{N}$ is bi-Lipschitz equivalent to $\mathbb{S}^4$.

The main ingredient of the proof of Proposition 22 is an example of a discontinuous mapping $F \in W^{1,P}(\mathbb{S}^n, \mathbb{S}^n)$ of positive Jacobian a.e. This example has appeared first in [12, Section 3.4], and the details of the construction are carefully presented there. In [9, Section 4] a slightly simplified construction in the case $P(t) = t^n / \log(e + t)$ is given. We refer the reader to these two sources, giving here only a sketch of the construction.

Example 23. Let $S_\alpha^\beta = \{(z \sin \theta, \cos \theta) \ : \ z \in \mathbb{S}^{n-1}, \alpha \leq \theta \leq \beta\} \subset \mathbb{S}^n$ be the spherical slice bounded by latitude spheres $\theta = \alpha$ and $\theta = \beta$ (with $\theta = 0$ denoting the north and $\theta = \pi$ the south pole of $\mathbb{S}^n$).

We choose a sequence of latitude angles θ_i, $i = 1, 2, \dots$ such that

$$\theta_0 = \pi > \frac{\pi}{2} \geq 2\theta_1 > \theta_1 \geq 2\theta_2 > \theta_2 \geq 2\theta_3 > \cdots > 0.$$

The mapping $F : \mathbb{S}^n \to \mathbb{S}^n$ is identity on the spherical slices $S_{2\theta_{k+1}}^{\theta_k}$ and on the southern cap $S_{2\theta_1}^\pi$. The slice $S_{\theta_k}^{2\theta_k}$, for $k = 1, 2, \dots$, is stretched (linearly in the latitude angle) in such a way that it covers the whole sphere twice, keeping the latitude circle $\theta = \theta_k$ fixed, mapping $\theta = 3\pi\theta_k/(2\pi + \theta_k)$ to the south pole, $\theta = 4\pi\theta_k/(2\pi + \theta_k)$ to the north pole and finally the latitude circle $\theta = 2\theta_k$ back to itself (Figure 23). We can do it in an orientation-preserving way, so the Jacobian determinant of F is positive inside the slice.

Obviously, F is discontinuous in the north pole $\mathfrak{n}$ and, since the image of each slice $S_{\theta_k}^{2\theta_k}$ has measure $2|\mathbb{S}^n|$, the Jacobian of F is not integrable.

As shown in [12], if the Young function P satisfies the conditions (5), (6a), (7) and (8), one can choose the sequence θ_i in such a way that the mapping F belongs to the Orlicz-Sobolev class $W^{1,P}(\mathbb{S}^n, \mathbb{S}^n)$.

Proof of Proposition 22. Let $p : \mathbb{S}^n \to N$ denote a covering map. This is an orientation preserving (and thus of positive Jacobian) local diffeomorphism, if the universal cover $\tilde{N}$ is diffeomorphic to $\mathbb{S}^n$, and a bi-Lipschitz, orientation preserving local homeomorphism, if $\tilde{N}$ is an exotic sphere.

Next, let $\mathbb{B}$ be a Riemannian ball in M, diffeomorphic to a Euclidean ball. There is a Lipschitz map $q : M \to \mathbb{S}^n$ which maps $M \setminus \mathbb{B}$ to the south pole $\mathfrak{s}$ of $\mathbb{S}^n$ and which is an orientation preserving diffeomorphism between $\mathbb{B}$ and $\mathbb{S}^n \setminus \{\mathfrak{s}\}$; thus q has finite distortion. If $M = \mathbb{B}$, then q is smooth and has everywhere positive Jacobian. The mapping G is given as the composition

$$M \xtwoheadrightarrow[]{} \overbrace{\xrightarrow{\quad q \quad} \mathbb{S}^n \xrightarrow{\quad F \quad} \mathbb{S}^n}^{G} \xrightarrow{\quad p \quad} N.$$

Then, if $\mathfrak{n}$ denotes the north pole of $\mathbb{S}^n$, G is discontinuous at $x = q^{-1}(\mathfrak{n})$: any neighborhood of x is mapped by G onto the whole N. At the same time, p and q are Lipschitz and F is in $W^{1,P}(\mathbb{S}^n, \mathbb{S}^n)$, thus the composition lies in $W^{1,P}(M, N)$. $\square$

6. Proof of Theorem 6

Let us choose a Riemannian ball $\mathbb{B}^n \subset M$, diffeomorphic to an n-dimensional Euclidean ball and such that $f|_{\partial \mathbb{B}^n} \in W^{1,\alpha}(\partial \mathbb{B}^n)$ and that the image $f(\partial \mathbb{B}^n)$ has small diameter in N. Recall that for any $x \in M$ there exist arbitrarily small balls centered at x with the above property (this follows immediately from Remark 19 and Lemma 18 d)).

By Lemma 14 there is an extension $\tilde{g} \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^k)$ of $f|_{\partial \mathbb{B}^n}$. The extension $\tilde{g}$ can be defined by an explicit integral fomula [23, Theorem 15.21], so that at any point of $\mathbb{B}^n$, the mapping $\tilde{g}$ is an integral average of values of $f|_{\partial \mathbb{B}^n}$. Hence the image of $\tilde{g}$ is contained in the convex hull of $f(\partial \mathbb{B}^n)$. Since we can assume that the diameter of $f(\partial \mathbb{B}^n)$ is sufficiently small, we can assume that values of $\tilde{g}$ belong to a tubular neighborhood U of N, from which there is a smooth Lipschitz retraction $\pi : U \to N$. Taking $g = \pi \circ \tilde{g} \in W^{1,n}(\mathbb{B}^n, N)$ we obtain an extension of $f|_{\partial \mathbb{B}^n}$ with values in N. Clearly, the Jacobian J_g is integrable in $\mathbb{B}^n$.

Let us thus consider a mapping F from the n-dimensional sphere $\mathbb{S}^n$ to N, which on the upper hemisphere $\mathbb{S}^n_+$, identified with $\mathbb{B}^n$, equals f, and on the lower hemisphere $\mathbb{S}^n_-$, again identified with $\mathbb{B}^n$, equals g, see Figure 2. As $f \in W^{1,P}(\mathbb{B}^n, N)$ and $g \in W^{1,n}(\mathbb{B}^n, N) \subset W^{1,P}(\mathbb{B}^n, N)$, we have that $F \in W^{1,P}(\mathbb{S}^n, N)$.

Next, we apply Lemma 20 to the mapping F, obtaining a sequence $t_i \to \infty$ and Ct_i-Lipschitz continuous mappings $F_i : \mathbb{S}^n \to N$, $i = 1, 2, \ldots$ such that

$$t_i^n |\{\mathcal{M}|DF| > t_i\}| \to 0 \quad \text{as } i \to \infty, \tag{13}$$

and F_i coincides with F on the set $\{\mathcal{M}|DF| \le t_i\}$.
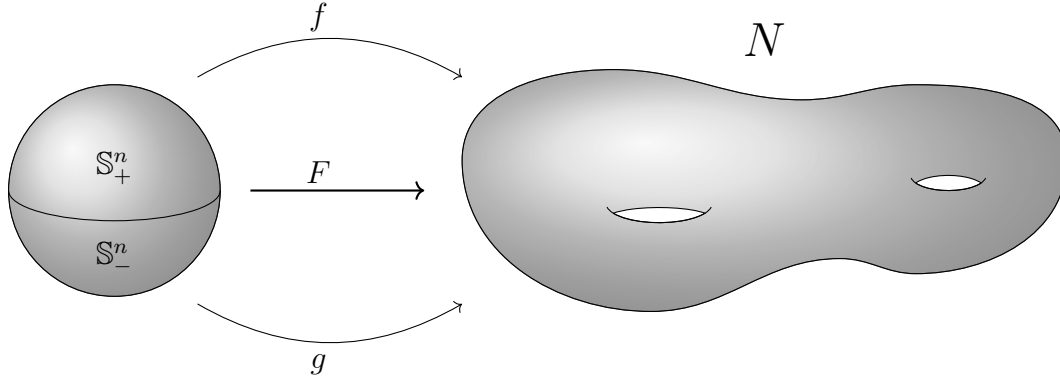
FIGURE 2. Construction of the mapping F.

Since, by assumption, the universal cover of the target manifold N is not a rational homology sphere, by Lemma 9 for $i = 1, 2 \ldots$ the mapping F_i has degree zero, thus

$$\int_{\mathbb{S}^n} J_{F_i} = 0.$$

Therefore

$$\int_{\mathbb{S}^n_+} J_{F_i} = -\int_{\mathbb{S}^n_-} J_{F_i} = -\int_{\mathbb{S}^n_- \cap \{\mathcal{M}|DF| \leq t_i\}} J_g - \int_{\mathbb{S}^n_- \cap \{\mathcal{M}|DF| > t_i\}} J_{F_i}$$

$$= -\int_{\mathbb{S}^n_-} J_g + \int_{\mathbb{S}^n_- \cap \{\mathcal{M}|DF| > t_i\}} J_g - \int_{\mathbb{S}^n_- \cap \{\mathcal{M}|DF| > t_i\}} J_{F_i}$$

$$= \left(-\int_{\mathbb{S}^n_-} J_g \right) + I_i - K_i.$$

It follows from (13) that the measure of the set of integration in I_i tends to 0 when $i \to \infty$. This, together with the fact that J_g is an integrable function, shows that $I_i \to 0$ with $i \to \infty$.

Condition (13) allows us to prove the same for K_i. Namely, since F_i is Lipschitz with the Lipschitz constant Ct_i, we have $|J_{F_i}| \leq Ct_i^n$ and

$$|K_i| \leq Ct_i^n |\mathbb{S}^n_- \cap \{\mathcal{M}|DF| > t_i\}| \leq Ct_i^n |\{\mathcal{M}|DF| > t_i\}| \xrightarrow{i \to \infty} 0.$$

Hence

$$\int_{\mathbb{S}^n_+} J_{F_i} \xrightarrow{i \to \infty} -\int_{\mathbb{S}^n_-} J_g < \infty. \tag{14}$$

However,

$$\int_{\mathbb{S}^n_+} J_{F_i} = \int_{\mathbb{S}^n_+ \cap \{\mathcal{M}|DF| \leq t_i\}} J_f + \int_{\mathbb{S}^n_+ \cap \{\mathcal{M}|DF_i| > t_i\}} J_{F_i} = I'_i + K'_i \to \int_{\mathbb{S}^n_+} J_f. \tag{15}$$

Indeed, $J_f \geq 0$, and the measure of the complement of the set of integration in I_i' tends to zero, $|\mathbb{S}_+^n \cap \{\mathcal{M}|DF_i| > t_i\}| \xrightarrow{i \to \infty} 0$, so the Monotone Convergence Theorem yields $I_i' \to \int_{\mathbb{S}_+^n} J_f$ as $i \to \infty$. Also, by the same argument as when estimating K_i,

$$|K_i'| \leq Ct_i^n|\{\mathcal{M}|DF| > t_i\}| \xrightarrow{i \to \infty} 0.$$

Comparing (14) and (15) we obtain

$$\int_{\mathbb{B}^n} J_f = -\int_{\mathbb{S}_-^n} J_g < \infty$$

which proves integrability of J_f in $\mathbb{B}^n$. $\qquad\qquad\square$

7. Proof of Theorem 5

The key step of the proof is the following lemma, which is essentially [9, Proposition 7], under weaker assumptions.

Lemma 24. *Assume that M and N are smooth, oriented, n-dimensional Riemannian manifolds without boundary and assume additionally that N is compact. If P satisfies conditions (5), (6b), (7), (8) and $f \in W^{1,P}(M, N)$ has finite distortion and locally integrable Jacobian, $J_f \in L^1_{\mathrm{loc}}$, then for every point $x \in N$ and every $\varepsilon > 0$ there is $r > 0$ such that* $\mathrm{diam}\, f(\mathbb{B}^n(x,r)) < \varepsilon$.

Remark 25. To be more precise, we will show that almost all points of the ball $\mathbb{B}^n(x,r)$ are mapped on a set of diameter less than ε, but then we can find a representative of f such that $\mathrm{diam}\, f(\mathbb{B}^n(x,r)) < \varepsilon$.

By Lemma 24, we can find $r > 0$ such that both $\mathbb{B}^n(x,r)$ and $f(\mathbb{B}^n(x,r))$ lie then in local charts, so the problem is reduced to the Euclidean one and the result follows from Theorem 2.

In the proof of Lemma 24 we need a simple geometric lemma.

Lemma 26. *Assume N is a connected Riemannian manifold. Let D, D' be Riemannian balls in N with disjoint closures and such that balls that are concentric with D and D' and with twice the radius are diffeomorphic to n-dimensional Euclidean balls. Then there is a Lipschitz retraction $\pi : N \setminus \overline{D}' \to \overline{D}$.*

For a detailed construction of such a retraction see [9, Proof of Proposition 7].

Proof of Lemma 24. Let $\varepsilon > 0$ be so small that

$$\delta = \inf_{\substack{D \subset N \\ \mathrm{diam}\, D < \varepsilon}} |N \setminus D| > 0,$$

where the infimum is taken over all balls $D \subset N$ of diameter less than ε.

By Lemma 18 d), for every $x \in M$ there is a sufficiently small $r > 0$ such that $\mathbb{B}^n = \mathbb{B}^n(x,r)$ satisfies

- $f|_{\partial \mathbb{B}^n} \in W^{1,\alpha}(\partial \mathbb{B}^n)$,
- $f(\partial \mathbb{B}^n)$ is contained in a ball $D \subset N$ of diameter less than ε,
- $\int_{\mathbb{B}^n} J_f < \delta$.

To complete the proof it suffices to show that almost all points of $\mathbb{B}^n$ are mapped into $\overline{D}$, i.e.

$$|A| = 0, \quad \text{where} \quad A = f^{-1}(N \setminus \overline{D}) \cap \mathbb{B}^n.$$

Suppose to the contrary that $|A| > 0$. We claim that

$$\int_A J_f > 0. \tag{16}$$

Consider the function $h : \mathbb{B}^n \to \mathbb{R}$ defined by $h(x) = \text{dist}(f(x), \overline{D})$. Since the distance function $\text{dist}(\cdot, \overline{D})$ is 1-Lipschitz, $h \in W^{1,P}(\mathbb{B}^n) \subset W^{1,1}(\mathbb{B}^n)$ and hence $h \in W_0^{1,1}(\mathbb{B}^n)$, because the trace of h on the boundary $\partial \mathbb{B}^n$ equals zero. Clearly $Dh = 0$ on $\mathbb{B}^n \setminus A$, because $h = 0$ on that set. On the other hand h is not constant since $h > 0$ on the set A of positive measure so the derivative Dh cannot be equal zero a.e. in $\mathbb{B}^n$ (as otherwise we would have $h = 0$ in $\mathbb{B}^n$) so Dh must be non-zero on a subset of A of positive measure. Since the mapping f has finite distortion, (16) follows.

By $\mathbb{B}_\sigma^n$, for $\sigma > 0$, we shall denote a ball concentric with $\mathbb{B}^n$, but of radius σ times that of $\mathbb{B}^n$.

Using Lemma 14 we extend the Hölder continuous (by Lemma 13) function $f|_{\partial \mathbb{B}^n}$ to a $W^{1,n}$ function $\tilde{f}$ on an annulus $\mathbb{B}_{1+2\delta}^n \setminus \mathbb{B}^n$, for some small $\delta > 0$. We can choose the extension $\tilde{f}$ to be smooth in the annulus and continuous up to the boundary (see Remark 17). Hence if δ is sufficiently small, values of $\tilde{f}$ on the annulus $\mathbb{B}_{1+2\delta}^n \setminus \mathbb{B}^n$ belong to a tubular neighborhood of N and composition with the nearest point projection shows that we can assume that the values of the extension $\tilde{f}$ belong to N. Then, if δ is small enough, continuity of $\tilde{f}$ shows that $\tilde{f}(\mathbb{B}_{1+2\delta}^n \setminus \mathbb{B}^n) \subset D$.

Let

$$g = \begin{cases} \tilde{f} & \text{on } \mathbb{B}_{1+2\delta}^n \setminus \mathbb{B}^n, \\ f & \text{on } \mathbb{B}^n. \end{cases}$$

Recall that

$$\int_{\mathbb{B}^n} J_f < \delta \leq |N \setminus D|.$$

Since the extension $\tilde{f}$ belongs to $W^{1,n}$ and $W^{1,n}$ mappings have integrable Jacobians, we can take $\delta > 0$ so small that

$$\int_{\mathbb{B}_{1+2\delta}^n \setminus \mathbb{B}^n} |J_g| < \int_A J_f \quad \text{and} \quad \int_{\mathbb{B}_{1+2\delta}^n} |J_g| < |N \setminus D|. \tag{17}$$

Note that $g \in W^{1,P}(\mathbb{B}_{1+2\delta}^n, N)$. Applying Lemma 20 to g defined on the manifold (with boundary) $\mathbb{B}_{1+2\delta}^n$ we obtain a sequence $t_i \to \infty$ and Ct_i-Lipschitz mappings g_i such that g_i coincides with g on $\mathbb{B}_{1+2\delta}^n \cap \{\mathcal{M}|Dg| \leq t_i\}$.

The function $x \mapsto \mathcal{M}|Dg|(x)$ is bounded on $\partial\mathbb{B}_{1+\delta}^n$: since g is smooth in the annulus $\mathbb{B}_{1+2\delta}^n \setminus \mathbb{B}^n$, averages of $|Dg|$ on small balls centered at $\partial\mathbb{B}_{1+\delta}^n$ are uniformly bounded while averages on larger balls are bounded by $C \int_{\mathbb{B}_{1+2\delta}^n} |Dg|$. Thus we can choose i large enough to have $\partial\mathbb{B}_{1+\delta}^n \subset \{\mathcal{M}|Dg| \le t_i\}$. Then $g_i = g$ on $\partial\mathbb{B}_{1+\delta}^n$, in particular $g_i(\partial\mathbb{B}_{1+\delta}^n) = \tilde{f}(\partial\mathbb{B}_{1+\delta}^n) \subset D$.

Next, we see that, by Lemma 20,

$$|g_i(\overline{\mathbb{B}}_{1+\delta}^n)| \le \int_{\mathbb{B}_{1+\delta}^n} |J_{g_i}| = \int_{\mathbb{B}_{1+\delta}^n} |J_g| + \int_{\mathbb{B}_{1+\delta}^n \cap \{\mathcal{M}|Dg|>t_i\}} |J_{g_i}| - |J_g| \xrightarrow{i\to\infty} \int_{\mathbb{B}_{1+\delta}^n} |J_g| < |N \setminus D|,$$

since $|J_g|$ is integrable and

$$\left| \int_{\mathbb{B}_{1+\delta}^n \cap \{\mathcal{M}|Dg|>t_i\}} |J_{g_i}| - |J_g| \right| \le Ct_i^n |\{\mathcal{M}|Dg| > t_i\}| + \int_{\mathbb{B}_{1+\delta}^n \cap \{\mathcal{M}|Dg|>t_i\}} |J_g| \xrightarrow{i\to\infty} 0.$$

Therefore, for i sufficiently large we have $|g_i(\overline{\mathbb{B}}_{1+\delta}^n)| < |N \setminus D|$, and since the set $g_i(\overline{\mathbb{B}}_{1+\delta}^n)$ is closed in N, there exists a Riemannian ball $D' \subset N \setminus \overline{D}$ such that $D' \cap g_i(\overline{\mathbb{B}}_{1+\delta}^n) = \varnothing$.

Let $\pi : N \setminus \overline{D}' \to \overline{D}$ be the Lipschitz retraction given by Lemma 26. Then $g_i|_{\partial\mathbb{B}_{1+\delta}^n} = \pi \circ g_i|_{\partial\mathbb{B}_{1+\delta}^n}$, because $g_i(\partial\mathbb{B}_{1+\delta}^n) \subset D$. Now, we repeat the construction as in Figure 2 and in the proof of Theorem 6: we construct a Lipschitz mapping G_i from a sphere into N, that equals $g_i|_{\overline{\mathbb{B}}_{1+\delta}^n}$ on the northern hemisphere and $\pi \circ g_i|_{\overline{\mathbb{B}}_{1+\delta}^n}$ on the southern hemisphere. Since the mapping $G_i : \mathbb{S}^n \to N$ is not surjective (it omits the ball D' in N), its degree equals zero, thus

$$\int_{\mathbb{B}_{1+\delta}^n} J_{g_i} = \int_{\mathbb{B}_{1+\delta}^n} J_{\pi \circ g_i}. \tag{18}$$

On the other hand, since $\pi \circ g_i$ maps $\mathbb{B}_{1+\delta}^n \cap g_i^{-1}(N \setminus \overline{D})$ onto the boundary of $\overline{D}$ that has dimension $n-1$, the Jacobian $J_{\pi \circ g_i}$ equals zero on that set, so

$$J_{\pi \circ g_i} = \begin{cases} J_{g_i} & \text{in } \mathbb{B}_{1+\delta}^n \cap g_i^{-1}(\overline{D}), \\ 0 & \text{in } \mathbb{B}_{1+\delta}^n \cap g_i^{-1}(N \setminus \overline{D}). \end{cases}$$

Thus

$$\int_{\mathbb{B}_{1+\delta}^n \cap g_i^{-1}(N \setminus \overline{D})} J_{g_i} = 0. \tag{19}$$

However,

$$\int_{\mathbb{B}_{1+\delta}^n \cap g_i^{-1}(N \setminus \overline{D})} J_{g_i} = \int_{\mathbb{B}_{1+\delta}^n \cap g_i^{-1}(N \setminus \overline{D}) \cap \{\mathcal{M}|Dg|>t_i\}} J_{g_i} + \int_{\mathbb{B}_{1+\delta}^n \cap g^{-1}(N \setminus \overline{D}) \cap \{\mathcal{M}|Dg|\le t_i\}} J_g$$

$$= S_i + T_i.$$

We can estimate $|S_i|$ by $Ct_i^n|\{\mathcal{M}|Dg| > t_i\}|$, thus $S_i \to 0$ as $i \to \infty$. The estimate for T_i is more difficult. We have

$$T_i = \int_{\mathbb{B}^n \cap f^{-1}(N\setminus\overline{D}) \cap \{\mathcal{M}|Dg| \le t_i\}} J_f + \int_{(\mathbb{B}^n_{1+\delta} \setminus \mathbb{B}^n) \cap g^{-1}(N\setminus\overline{D}) \cap \{\mathcal{M}|Dg| \le t_i\}} J_g$$

$$\ge \int_{A \cap \{\mathcal{M}|Dg| \le t_i\}} J_f - \int_{\mathbb{B}^n_{1+\delta} \setminus \mathbb{B}^n} |J_g| \xrightarrow{i \to \infty} \int_A J_f - \int_{\mathbb{B}^n_{1+\delta} \setminus \mathbb{B}^n} |J_g| \overset{(16),(17)}{>} 0$$

which contradicts (19).

This concludes the proof of Lemma 24 and hence that of Theorem 5. $\qquad\square$

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