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Topologically nontrivial counterexamples to Sard's theorem

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We prove the following dichotomy: if $n = 2, 3$ and $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ is not homotopic to a constant map, then there is an open set $\Omega \subset \mathbb{S}^{n+1}$ such that $\operatorname{rank} df = n$ on Ω and $f(\Omega)$ is dense in $\mathbb{S}^n$, while for any $n \geq 4$, there is a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ that is not homotopic to a constant map and such that $\operatorname{rank} df < n$ everywhere. The result in the case $n \geq 4$ answers a question of Larry Guth.

1 Introduction

In 1942, Sard [16] proved that if $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$ for $k > \max\{m - n, 0\}$, then the set of critical values of f has measure zero; see also [17]. In particular, if $\mathcal{M}^m$ and $\mathcal{N}^n$ are closed manifolds of dimensions $m \geq n$, respectively, $k > m - n$, and $f \in C^k(\mathcal{M}^m, \mathcal{N}^n)$ is a surjective map, then there is an open set $\Omega \subset \mathcal{M}^m$ such that $\operatorname{rank} df = n$ everywhere in Ω and $f(\Omega)$ is dense in $\mathcal{N}^n$.

Sard's theorem is no longer true for $k \leq \max\{m - n, 0\}$. In fact, Kaufman showed in [14] that, for each $n \geq 2$, there exists a surjective map $f \in C^1([0, 1]^{n+1}, [0, 1]^n)$ with $\operatorname{rank} df \leq 1$ *everywhere*. However, Kaufman's mapping is a limit of a uniformly convergent sequence of mappings into finite, one dimensional, piecewise linear trees, so it is topologically trivial in the sense that mimicking Kaufman's construction in the case of C^1 mappings between spheres $\mathbb{S}^{n+1}$ and $\mathbb{S}^n$ would result in a mapping $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ with $\operatorname{rank} f \leq 1$ everywhere that is homotopic to a constant map. Indeed, mappings into trees are contractible and so is their limit.

Since the homotopy groups $\pi_{n+1}(\mathbb{S}^n) \neq 0$ are non-trivial for $n \geq 2$ (see e.g. [12]), one may ask whether it is possible to construct a Kaufman type example that is not homotopic to a constant map.

A mapping $f \in C^2(\mathbb{S}^{n+1}, \mathbb{S}^n)$ that is not homotopic to a constant mapping is surjective and hence, according to Sard's theorem, there is an open set $\Omega \subset \mathbb{S}^{n+1}$ having the property that

$$\operatorname{rank} df = n \text{ everywhere in } \Omega \text{ and } f(\Omega) \text{ is dense in } \mathbb{S}^n. \tag{1.1}$$

In particular, there is no mapping $f \in C^2(\mathbb{S}^{n+1}, \mathbb{S}^n)$, satisfying $\operatorname{rank} df < n$ everywhere, that is not homotopic to a constant map. Note that the condition $\operatorname{rank} df < n$ is much weaker than Kaufman's $\operatorname{rank} df \leq 1$. This leads to two natural questions.

QUESTION 1. *Is it possible to construct a mapping $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$, $n \geq 2$, such that $\operatorname{rank} df < n$ everywhere and f is not homotopic to a constant one?*

QUESTION 2. *Let $n \geq 2$ and $1 \leq m < n$ be given. Is it possible to construct a mapping $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ such that $\operatorname{rank} df \leq m$ everywhere and f is not homotopic to a constant one?*

We note in passing that, in the more general context of C^1-mappings between closed manifolds, it is easy to give examples of mappings and manifolds answering both questions. Consider, for example, the smooth map $f : \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, $(x, y, z) \mapsto (x, y_o)$, where $y_o \in \mathbb{S}^1$, which is not homotopic to a constant map but satisfies $\operatorname{rank} df = 1$ everywhere.

The questions stated above are essentially due to Larry Guth [10, p. 1889], who asked: *We don't know any homotopically non-trivial C^1 maps from $\mathbb{S}^m$ to $\mathbb{S}^n$ with* $\operatorname{rank} < n$. *Does one exist?* Guth [10, Main Theorem]

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obtained a partial answer to Question 2 by showing a lower bound for the rank of the derivative of homotopically non-trivial maps.

Theorem 1.1 (Guth)**.** If $n \geq 2$ and $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ satisfies $\operatorname{rank} df < \left[\frac{n+2}{2}\right]$, then f is homotopic to a constant map. □

Here $[x]$ stands for the integer part of x. In particular, the following maps are necessarily homotopic to constant maps:

$$
\begin{aligned}
&f \in C^1(\mathbb{S}^3, \mathbb{S}^2), \quad \text{or} \quad f \in C^1(\mathbb{S}^4, \mathbb{S}^3), \quad \text{with } \operatorname{rank} df < 2, \\
&f \in C^1(\mathbb{S}^5, \mathbb{S}^4), \quad \text{or} \quad f \in C^1(\mathbb{S}^6, \mathbb{S}^5), \quad \text{with } \operatorname{rank} df < 3, \\
&f \in C^1(\mathbb{S}^7, \mathbb{S}^6), \quad \text{or} \quad f \in C^1(\mathbb{S}^8, \mathbb{S}^7), \quad \text{with } \operatorname{rank} df < 4.
\end{aligned}
\tag{1.2}
$$

On the other hand, in the case of mappings $f \in C^1(\mathbb{S}^4, \mathbb{S}^3)$, Guth proved a stronger result than that in (1.2). Namely he proved in [10, Proposition 13.4] that if $f \in C^1(\mathbb{S}^4, \mathbb{S}^3)$ and $\operatorname{rank} df < 3$, then f is homotopic to a constant map. This led him to the following conjecture:

Conjecture 1. (Guth) *Let $n \geq 5$ be odd. If $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ and $\operatorname{rank} df < \left[\frac{n+3}{2}\right]$, then f is homotopic to a constant map.*

Note that if $n \geq 2$ is even, then the above claim is true by Theorem 1.1, and when $n = 3$, it is true by [10, Proposition 13.4], but it is an open problem when $n \geq 5$ is odd. Guth also conjectured that the above estimate for the rank is sharp:

Conjecture 2. (Guth) *If $n \geq 4$, then there is a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ with $\operatorname{rank} df \leq \left[\frac{n+3}{2}\right]$ that is not homotopic to a constant map.*

Note that the if $n = 2, 3$, then the above claim is obvious, because $\left[\frac{n+3}{2}\right] = n$, so any map satisfies the rank condition.

The above conjectures were communicated to Piotr Hajłasz by Larry Guth.

The aim of this paper is to prove the following result.

Theorem 1.2. For $n = 2, 3$ and each map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ not homotopic to a constant map, there is an open set $\Omega \subset \mathbb{S}^{n+1}$ such that $\operatorname{rank} df = n$ on Ω and $f(\Omega)$ is dense in $\mathbb{S}^n$. In contrast, for each $n \geq 4$, there is a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ that is not homotopic to a constant map and such that $\operatorname{rank} df < n$ everywhere. □

The case $n = 2$ is relatively easy, it follows from Theorem 1.1, but, in fact, it has been known before, see comments to Theorem 1.4. The known proofs are based on estimates of the Hopf invariant. For the sake of completeness we provide a variant of such a proof. The case $n = 3$ was proved in [10, Proposition 13.4] with a difficult argument based on the Steenrod squares. We provide a very different, and a more elementary proof based on a generalized Hopf invariant introduced in [11]. However, the case $n \geq 4$ is new. It answers Question 1 and Conjecture 2 for $n = 4$ in the affirmative.

Modifying our proof slightly, we could show that if $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$, $n \geq 3$, and $\operatorname{rank} df < 3$, then f is homotopic to a constant map. This is consistent with the estimates obtained by Guth when $n \leq 5$. However, in higher dimensions Theorem 1.1 gives a better estimate.

Note that when $n = 2$ or $n = 3$ and $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ is not homotopic to a constant map, then the conclusion (1.1) of Sard's theorem is still true, despite the fact that the mapping f has less regularity than required in Sard's theorem.

We find it somewhat surprising that the situation changes in the dimension $n = 4$. For example, $\pi_4(\mathbb{S}^3) = \pi_5(\mathbb{S}^4) = \mathbb{Z}_2$, so the homotopy groups of the spheres are the same when $n = 3$ and $n = 4$, but the claim of Theorem 1.2 is different in these dimensions.

The map constructed in the proof of Theorem 1.2 in the case $n \geq 4$ has $\operatorname{rank} df = n - 1$ on a set of positive measure. We do not know whether there exists a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ which is not homotopic to a constant map and satisfies $\operatorname{rank} df \leq n - 2$ everywhere. Looking for such a map would be a first step towards answering Conjecture 2.

The first part of Theorem 1.2 is a consequence of a slightly stronger result:

Theorem 1.3. If $n = 2, 3$ and $f : \mathbb{S}^{n+1} \to \mathbb{S}^n$ is Lipschitz continuous and not homotopic to a constant map, then there is a set $A \subset \mathbb{S}^{n+1}$ of positive measure such that f is differentiable at every point of A, $\operatorname{rank} df = n$ on A and $f(A)$ is dense in $\mathbb{S}^n$. □

If now $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ and $\operatorname{rank} df = n$ on A, then $\operatorname{rank} df = n$ on an open set that contains A, so the first part of Theorem 1.2 follows.

For $n = 2$, Theorem 1.3 follows from the following more general result related to the Hopf invariant. This result is known and it follows from the so called Hopf invariant inequality, see [8, Section 3.6], [9, pp. 358-359], [10, p. 1805 and p. 1818], [15].

Theorem 1.4. If $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ is Lipschitz and $\mathcal{H}f \neq 0$, then there is a set $A \subset \mathbb{S}^{4n-1}$ of positive measure such that f is differentiable at every point of A, $\operatorname{rank} df = 2n$ on A and $f(A)$ is a dense subset of $\mathbb{S}^{2n}$. □

Our proof is similar to the other known proofs. We decided to include the details as they play an important role in the proof of Corollary 3.6.

Since a map $f : \mathbb{S}^3 \to \mathbb{S}^2$ is not homotopic to a constant map if and only if the Hopf invariant $\mathcal{H}f \neq 0$ is non-trivial (see Remark 3.2), we readily obtain that Theorem 1.4 yields Theorem 1.3 in this case. We note, again in passing, that for each $n \geq 2$ there are mappings $\mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ which have a trivial Hopf invariant but which are not homotopic to a constant map.

The proof of Theorem 1.4 (and hence the proofs of Theorems 1.2 and 1.3 when $n = 2$) is based on a generalized Hopf invariant defined and studied in [11]. This is a non-standard generalization that requires the use of the L^p-Hodge decomposition. The proof of Theorem 1.3 in the case $n = 3$ (and hence the proof of Theorem 1.2 when $n = 3$) is based on a mixture of methods from geometric measure theory (Eilenberg's inequality), ideas behind the proof of the Freudenthal suspension theorem [12] and the generalized Hopf invariant from [11]. As explained above, Theorem 1.3 proves the first part of Theorem 1.2.

The second part of Theorem 1.2, i.e. the case of dimensions $n \geq 4$, is also a consequence of a more general result; recall that $\pi_n(\mathbb{S}^{n-1}) = \mathbb{Z}_2$ for $n \geq 4$ (see e.g. [12]).

Theorem 1.5. If $k + 1 \leq m < 2k - 1$ and $\pi_m(\mathbb{S}^k) \neq 0$, then there is a mapping $f \in C^1(\mathbb{S}^{m+1}, \mathbb{S}^{k+1})$ that is not homotopic to a constant map and such that $\operatorname{rank} df \leq k$ everywhere. □

The proof of Theorem 1.5 is based on a beautiful and surprising construction of Wenger and Young [18, Theorem 2], who proved that *if* $k + 1 \leq m < 2k - 1$ *and* $g : \mathbb{S}^m \to \mathbb{S}^k$ *is Lipschitz continuous, then there is a Lipschitz extension* $G : \overline{\mathbb{B}}^{m+1} \to \mathbb{R}^{k+1}$ *such that* $\operatorname{rank} dG \leq k$ *almost everywhere.* Since we are interested in C^1 mappings rather than Lipschitz ones, we have to modify their construction to make sure that we can find a C^1 extension G when g is C^1. Our construction is explicit, while the arguments in [18] are based on homotopy theory.

The article is organized in the following way. In Section 2 we recall well known facts related to suspension and the Freudenthal suspension theorem. This material will be needed in the proofs of Theorem 1.3 (for $n = 3$) and of Theorem 1.5. In Section 3 we discuss the generalized Hopf invariant introduced in [11] and we end the section with the proof of Theorem 1.4, which easily follows from the properties of the generalized Hopf invariant. The generalized Hopf invariant will also be used in the proof of Theorem 1.3 for $n = 3$. Recall that Theorem 1.4 implies Theorems 1.2 and 1.3 for $n = 2$. In Section 4 we prove Theorem 1.3 for $n = 3$. This completes the proofs of Theorems 1.2 and 1.3 for $n = 2, 3$. In the final Section 5 we prove Theorem 1.5, which implies Theorem 1.2 for $n \geq 4$.

Notation used in the article is pretty standard. By $\mathbb{B}^\ell$ will always denote *open* balls, while the symbol B can be used to denote open or closed balls. The hemispheres $\mathbb{S}^n_\pm$ will always be closed. By a smooth mapping we will always mean a C^∞ smooth one. By a smooth diffeomorphism defined on a closed domain $\overline{\Omega}$ we mean a diffeomorphism that smoothly extends to a diffeomorphism in a larger domain that contains $\overline{\Omega}$.

2 The Freudenthal suspension theorem

With a continuous map $f : \mathbb{S}^n \to \mathbb{S}^k$ we can associate the *suspension map* $Sf : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$, which maps n-spheres parallel to the equator to the corresponding k-spheres parallel to the equator. On each of such spheres the map Sf is a scaled copy of f.

Some basic and easy to verify properties of the suspension map are listed in the next three lemmata.

Lemma 2.1. If the maps $f, g : \mathbb{S}^n \to \mathbb{S}^k$ are homotopic, then their suspensions $Sf, Sg : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$ are homotopic as well. □

The homotopy between Sf and Sg is simply the suspension of the homotopy between f and g.

Lemma 2.2. If a map $f : \mathbb{S}^n \to \mathbb{S}^k$ is homotopic to a constant map, then its suspension $Sf : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$ is homotopic to a constant map. □

Indeed, since f is homotopic to a constant map f_o, Sf is homotopic to Sf_o (Lemma 2.1), but the image of Sf_o is a single meridian in $\mathbb{S}^{k+1}$, which is contractible, so Sf_o (and hence Sf) is homotopic to a constant map.

Lemma 2.3. If $F : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$ maps the equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$ to the equator $\mathbb{S}^k \subset \mathbb{S}^{k+1}$, the upper hemisphere $\mathbb{S}^{n+1}_+$ to the upper hemisphere $\mathbb{S}^{k+1}_+$ and the lower hemisphere $\mathbb{S}^{n+1}_-$ to the lower hemisphere $\mathbb{S}^{k+1}_-$, then F is homotopic to the suspension Sf of the mapping $f = F|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^k$ between the equators. □

The homotopy is defined as the continuous family of mappings F_t, $0 \leq t \leq 1$, such that on each n-sphere parallel to the equator whose vertical distance to the equator is no larger than t, the mapping F_t coincides with Sf and on polar caps consisting of points with the vertical distance to the equator at least t, the mapping F_t is a scaled version of the mapping F on the hemispheres. Then $F_0 = F$ and $F_1 = Sf$.

The next result is the celebrated Freudenthal suspension theorem [12, Corollary 4.24].

Lemma 2.4. If $n \leq 2k-1$, then every map $F : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$ is homotopic to the suspension Sf of a map $f : \mathbb{S}^n \to \mathbb{S}^k$. If in addition $n < 2k-1$, then $Sf : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$ is homotopic to a constant map if an only if $f : \mathbb{S}^n \to \mathbb{S}^k$ is homotopic to a constant map. □

It follows from Lemma 2.2 that if f is homotopic to a constant map, then Sf is homotopic to a constant map. However, if $n = 2k-1$, it may happen that Sf is homotopic to a constant map, even though f is not.

Some ideas from the proof of the Freudenthal theorem are also used in Section 4. Actually, the ideas from Section 4 have been used in [6] to find a somewhat new proof of the Freudenthal theorem (Lemma 2.4) that uses only elementary methods from differential topology.

The usual statement of the Freudenthal theorem is that the reduced suspension homomorphism

$$\Sigma : \pi_n(\mathbb{S}^k) \to \pi_{n+1}(\mathbb{S}^{k+1})$$

is an epimorphism for $n \leq 2k-1$ and an isomorphism for $n < 2k-1$. However, we do not need to use the reduced suspension in the article, merely the version stated in Lemma 2.4.

It is important to note that even if f is smooth, the suspension map Sf is not smooth at the north and south poles. For example, as observed above, the suspension Sf_o of a constant map $f_o : \mathbb{S}^n \to \mathbb{S}^k$ maps $\mathbb{S}^{n+1}$ into one meridian in $\mathbb{S}^{k+1}$. If we go along a great circle in $\mathbb{S}^{n+1}$ that passes through poles at a constant speed, then in the image of Sf_o we will go back and forth along one meridian, suddenly changing the direction of the constant speed at the poles, showing that the derivative of Sf_o is discontinuous at the poles. The discontinuity of the derivative of the suspension will cause some technical problems in the proof of Theorem 1.5. However, we can easily correct the suspension to a smooth mapping. If we parameterize the hemispheres $\mathbb{S}^{n+1}_\pm$ and $\mathbb{S}^{k+1}_\pm$ as graphs over the balls $\bar{\mathbb{B}}^{n+1}$ and $\bar{\mathbb{B}}^{k+1}$, then in these coordinate systems the suspension $Sf : \mathbb{S}^{n+1}_\pm \to \mathbb{S}^{k+1}_\pm$ restricted to the hemispheres becomes

$$\Phi(x) = |x| f\left(\frac{x}{|x|}\right).$$

The mapping Φ has discontinuous derivative at the origin, which corresponds to the discontinuity of the derivative of Sf at the poles. If $\lambda_\varepsilon : [0,1] \to [0,1]$ is a smooth and non-decreasing function such that $\lambda_\varepsilon(t) = 0$ on $[0,\varepsilon]$ and $\lambda_\varepsilon(t) = t$ on $[1-\varepsilon, 1]$, then the mapping

$$\Phi_\varepsilon(x) = \lambda_\varepsilon(|x|) f\left(\frac{x}{|x|}\right)$$

is smooth and it coincides with Φ near the boundary of $\bar{\mathbb{B}}^{n+1}$. The mapping Φ_ε induces a smooth mapping $S_\varepsilon f : \mathbb{S}^{n+1} \to \mathbb{S}^{k+1}$ that is homotopic to Sf and coincides with Sf in a neighborhood of the equator.

3 The generalized Hopf invariant

For a smooth map $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$, the classical *Hopf invariant* is defined as follows (see [4]). Let α_o be the volume form on $\mathbb{S}^{2n}$ with $\int_{\mathbb{S}^{2n}} \alpha_o = 1$. Then $df^*\alpha_o = f^* d\alpha_o = 0$. Since the de Rham cohomology $H^{2n}(\mathbb{S}^{4n-1})$ is trivial, $H^{2n}(\mathbb{S}^{4n-1}) = 0$, there is a smooth $(2n-1)$-form ω on $\mathbb{S}^{4n-1}$ such that $f^*\alpha_o = d\omega$ and *the Hopf invariant of* f is defined by

$$\mathcal{H}f = \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega.$$

The Hopf invariant is invariant under homotopies ([4, Proposition 17.22]), so it can be defined for any continuous map $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$. However, it is no longer given by the above formula if f is not sufficiently smooth.

Lemma 3.1. The Hopf invariant is a non-zero group homomorphism $\mathcal{H} : \pi_{4n-1}(\mathbb{S}^{2n}) \to \mathbb{Z}$ and it is an isomorphism when $n = 1$. □

Remark 3.2. For the proof that $\mathcal{H}$ is a group homomorphism, see [12, Proposition 4B.1]. Hopf [13, Satz II, Satz II'] proved that for any n, there is a map $h : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ with $\mathcal{H}h \neq 0$ and hence the homomorphism $\mathcal{H} : \pi_{4n-1}(\mathbb{S}^{2n}) \to \mathbb{Z}$ is non-zero. Since the Hopf invariant of the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ equals 1 ([4, Example 17.23]), $\mathcal{H} : \pi_3(\mathbb{S}^2) \to \mathbb{Z}$ is an isomorphism. However, for $n \geq 2$ the Hopf invariant is never an isomorphism. Indeed, Adams [1] proved that mappings with Hopf invariant equal 1 exist only when $n = 1, 2$ and 4, so these are the only cases when one may suspect $\mathcal{H}$ to be an isomorphism, but $\pi_7(\mathbb{S}^4) = \mathbb{Z} \times \mathbb{Z}_{12}$ and $\pi_{15}(\mathbb{S}^8) = \mathbb{Z} \times \mathbb{Z}_{120}$, so $\mathcal{H}$ cannot be an isomorphism. $\square$

Let $f : \mathbb{S}^{4n-1} \to \mathbb{R}^m$, $m \geq 2n+1$, be a Lipschitz map such that $\operatorname{rank} df \leq 2n$ almost everywhere. Let α be any C^∞-smooth $2n$-form on $\mathbb{R}^m$. Following [11] we define a generalized Hopf invariant $\mathcal{H}_\alpha f$ as described below.

According to Lemma 5.4 in [11], the form $f^*\alpha \in L^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$ is weakly closed. Since the L^2-de Rham cohomology of $\mathbb{S}^{4n-1}$ in dimension $2n$ is zero ([11, Proposition 4.5]), there is a Sobolev form $\omega \in W^{1,2}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$ such that $d\omega = f^*\alpha$, and we define

$$\mathcal{H}_\alpha f = \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega.$$

The main properties of $\mathcal{H}_\alpha$ are described in the following results (see Propositions 5.5 and 5.8 in [11]).

Lemma 3.3. If $\omega_1, \omega_2 \in W^{1,2}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$ and $d\omega_1 = d\omega_2$ a.e., then the forms $\omega_i \wedge d\omega_i$, $i = 1, 2$, are integrable and

$$\int_{\mathbb{S}^{4n-1}} \omega_1 \wedge d\omega_1 = \int_{\mathbb{S}^{4n-1}} \omega_2 \wedge d\omega_2.$$

In particular, the definition of $\mathcal{H}_\alpha f$ does not depend on the choice of the form ω. $\square$

Lemma 3.4. Let $f, g : \mathbb{S}^{4n-1} \to \mathbb{R}^m$, $m \geq 2n+1$, be Lipschitz mappings such that $\operatorname{rank} df \leq 2n$ and $\operatorname{rank} dg \leq 2n$ almost everywhere, and let α be a smooth $2n$-form on $\mathbb{R}^m$. If $H : [0,1] \times \mathbb{S}^{4n-1} \to \mathbb{R}^m$ is a Lipschitz homotopy from f to g that satisfies $\operatorname{rank} dH \leq 2n$ almost everywhere, then $\mathcal{H}_\alpha f = \mathcal{H}_\alpha g$. $\square$

This means the generalized Hopf invariant $\mathcal{H}_\alpha f$ is invariant under homotopies whose rank of the derivative does not exceed $2n$.

If α is a smooth $2n$-form on $\mathbb{R}^{2n+1}$ whose restriction to $\mathbb{S}^{2n}$ coincides with the fixed volume form α_o and $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ is a Lipschitz map, then $\operatorname{rank} df \leq 2n$ almost everywhere and hence the generalized Hopf invariant $\mathcal{H}_\alpha f$ is well defined.

Corollary 3.5. If $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ is Lipschitz continuous, then $\mathcal{H}_\alpha f = \mathcal{H} f$. $\square$

Proof. If $g : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ is smooth, then $\mathcal{H}_\alpha g = \mathcal{H} g$, because in that case the definition of $\mathcal{H}_\alpha g$ is identical with the classical definition of the Hopf invariant. If g is homotopic to f, then there is also a Lipschitz homotopy $H : [0,1] \times \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ between f and g (by a standard approximation argument). Since H takes values in $\mathbb{S}^{2n}$, $\operatorname{rank} dH \leq 2n$ a.e. and hence Lemma 3.4 yields

$$\mathcal{H}_\alpha f = \mathcal{H}_\alpha g = \mathcal{H} g = \mathcal{H} f,$$

where the last equality follows from the homotopy invariance of the classical Hopf invariant. $\blacksquare$

The next result is essentially contained in [11, Theorem 1.7].

Corollary 3.6. If $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ is Lipschitz continuous with $\mathcal{H} f \neq 0$ and $F : \overline{\mathbb{B}}^{4n} \to \mathbb{R}^{2n+1}$ is a Lipschitz extension of f, then $\operatorname{rank} dF = 2n+1$ on a set of positive $4n$-dimensional measure. $\square$

Proof. Let f and F be as in the statement. Suppose to the contrary that $\operatorname{rank} dF \leq 2n$ almost everywhere. Then the Lipschitz homotopy

$$H(t,\theta) : [0,1] \times \mathbb{S}^{4n-1} \to \mathbb{R}^{2n+1}, \qquad H(t,\theta) = F(t\theta)$$

from the constant map $g(\theta) = F(0)$ to f satisfies $\operatorname{rank} dH \leq 2n$ almost everywhere. Since the mappings f, g, H satisfy assumptions of Lemma 3.4, we have that Lemma 3.4 together with Corollary 3.5 yield

$$\mathcal{H} f = \mathcal{H}_\alpha f = \mathcal{H}_\alpha g = 0,$$

which is a contradiction. $\blacksquare$

Theorem 1.4 is now a straightforward consequence of Corollary 3.5.

Proof of Theorem 1.4. First we will prove that $\operatorname{rank} df = 2n$ on a set of positive measure. Suppose to the contrary that $\operatorname{rank} df < 2n$ almost everywhere. Let α be as in Corollary 3.5. Then $f^*\alpha = 0$, so $\mathcal{H}_\alpha f = 0$ and hence $\mathcal{H} f = \mathcal{H}_\alpha f = 0$, which is a contradiction. This proves that the set A where df exists and satisfies $\operatorname{rank} df = 2n$ has positive measure. It remains to prove that the set $f(A)$ is dense in $\mathbb{S}^{2n}$. Suppose to the contrary that $f(A) \cap \mathbb{B}(y_o, \varepsilon) = \varnothing$ for some ball $\mathbb{B}(y_o, \varepsilon) \subset \mathbb{S}^{2n}$. Then, stretching along meridians with y_o regarded as the north pole, we can find a Lipschitz homotopy between f and a mapping f_1 which maps A to the south pole. Hence $\operatorname{rank} df_1 < 2n$ almost everywhere, so by the first part of the proof $\mathcal{H} f_1 = 0$ and therefore $0 \neq \mathcal{H} f = \mathcal{H} f_1 = 0$, which is a contradiction. $\blacksquare$

4 Proof of Theorem 1.3

Recall that Theorem 1.4 completes the proof of Theorem 1.3 (and hence that of Theorem 1.2) when $n = 2$. Thus we can assume that $n = 3$.

Let $f : \mathbb{S}^4 \to \mathbb{S}^3$ be a Lipschitz map that is not homotopic to a constant map. Let A be the set of points where f is differentiable and $\operatorname{rank} df = 3$. We need to prove that A has positive measure and that its image $f(A)$ is dense in $\mathbb{S}^3$. Once we prove that the set A has positive measure, the fact that the set $f(A)$ is dense in $\mathbb{S}^3$ will follow from the same argument as the one in the last step in the proof of Theorem 1.4. Thus it remains to prove that the measure of A is positive. Assume, to the contrary, that $\operatorname{rank} df \leq 2$ almost everywhere.

The idea is to show that f can be deformed to a Lipschitz map with the rank of the derivative less than or equal to 2, that maps the equator $\mathbb{S}^3$ of $\mathbb{S}^4$ to the equator $\mathbb{S}^2$ of $\mathbb{S}^3$ and hemispheres to hemispheres. Since the map f is not homotopic to the constant map, it easily follows from Lemmata 2.2 and 2.3 that the map between the equators is not homotopic to a constant map either. Hence its Hopf invariant is not zero (Lemma 3.1). However, it follows now from Corollary 3.6 that the rank of the derivative of its extension to the upper (or lower) hemisphere has to be equal 3 on a set of positive measure, which is a contradiction.

The idea of deformation of the map to a map that sends the equator to the equator and hemispheres to the hemispheres is closely related to the proof of the Freudenthal suspension theorem (i.e., Lemma 2.4). Indeed, this and Lemma 2.3 imply that $f : \mathbb{S}^4 \to \mathbb{S}^3$ is homotopic to the suspension of a map between equators, which is a part of the statement of Freudenthal's theorem. However, we cannot use the Freudenthal theorem in the proof of Theorem 1.3 directly, because the homotopy between f and the suspension map may possibly increase the rank of the derivative, i.e. the homotopic suspension map may have the rank of the derivative equal 3 on a set of positive measure and we will not obtain any contradiction.

In the first step of the construction of the deformation we need to find two points $y_1, y_2 \in \mathbb{S}^3$ with 'small' pre-images $f^{-1}(y_1)$, $f^{-1}(y_2)$. To do this we use the next lemma.

Lemma 4.1. If $f : \mathcal{M}^m \to \mathcal{N}^n$ is a Lipschitz map between closed Riemannian manifolds of dimensions m and n respectively, then $\mathscr{H}^{m-n}(f^{-1}(y)) < \infty$ for a.e. $y \in \mathcal{N}^n$. □

Here $\mathscr{H}^{m-n}$ stands for the Hausdorff measure. This lemma is a direct consequence of Eilenberg's inequality [5, Theorem 13.3.1]. In particular, for almost all $y \in \mathbb{S}^3$, $\mathscr{H}^1(f^{-1}(y)) < \infty$ and we would like to conclude that for almost all $y_1, y_2 \in \mathbb{S}^3$, $\mathscr{H}^2(f^{-1}(y_1) \times f^{-1}(y_2)) < \infty$. However, it cannot be directly concluded from the estimate for the Hausdorff measure of the factors. In fact, the Hausdorff dimension of the Cartesian product of compact sets A, B can be larger than the sum of Hausdorff dimensions of the sets A and B: Theorem 5.11 in [7] provides an example of compact sets $A, B \subset \mathbb{R}$, each of Hausdorff dimension zero, and such that $\mathscr{H}^1(A \times B) > 0$. Fortunately, a small trick allows us to show that $\mathscr{H}^2(f^{-1}(y_1) \times f^{-1}(y_2)) < \infty$ for almost all $y_1, y_2 \in \mathbb{S}^3$ as a direct consequence of Lemma 4.1: Since the map

$$F : \mathbb{S}^4 \times \mathbb{S}^4 \to \mathbb{S}^3 \times \mathbb{S}^3, \qquad F(x_1, x_2) = (f(x_1), f(x_2))$$

is Lipschitz continuous, it follows from Lemma 4.1 that

$$\mathscr{H}^2(f^{-1}(y_1) \times f^{-1}(y_2)) = \mathscr{H}^2(F^{-1}(y_1, y_2)) < \infty$$

for almost all $y_1, y_2 \in \mathbb{S}^3$.

Choose such points $y_1, y_2 \in \mathbb{S}^3$, $y_1 \neq y_2$. The sets $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are compact and disjoint. We want to show that there is a diffeomorphism of $\mathbb{S}^4$ that moves one of the sets to a small neighborhood of a north pole and the other one to a small neighborhood of a south pole. To construct such a diffeomorphism it will be easier to work in $\mathbb{R}^4$ rather than with $\mathbb{S}^4$, but that can be easily achieved. Let $z \in \mathbb{S}^4 \setminus (f^{-1}(y_1) \cup f^{-1}(y_2))$ and consider the stereographic projection from $\mathbb{S}^4$ onto $\mathbb{R}^4$ with z as a north pole. With a slight abuse of notation we will denote by $f^{-1}(y_1)$ and $f^{-1}(y_2)$ the corresponding (compact and disjoint) images in $\mathbb{R}^4$.

Consider the map

$$\pi : f^{-1}(y_1) \times f^{-1}(y_2) \to \mathbb{RP}^3 \tag{4.1}$$

which assigns to any pair of points $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ the line passing through x_1 and x_2. It is easy to see that π is Lipschitz, because the distance between the sets $f^{-1}(y_1)$ and $f^{-1}(y_2)$ is positive. Since $\mathscr{H}^2(f^{-1}(y_1) \times f^{-1}(y_2)) < \infty$, the set

$$\pi(f^{-1}(y_1) \times f^{-1}(y_2)) \subset \mathbb{RP}^3$$

is compact and has finite two-dimensional Hausdorff measure. Since the space $\mathbb{RP}^3$ is three dimensional, the mapping (4.1) is not surjective and we can find

$$v \in \mathbb{RP}^3 \setminus \pi(f^{-1}(y_1) \times f^{-1}(y_2)).$$

This simply means that the lines parallel to v and passing though the points of $f^{-1}(y_1)$ do not intersect $f^{-1}(y_2)$. Denote the union of all lines parallel to v and passing through $f^{-1}(y_1)$ by V. Note that V is closed, so there is an open set U containing V, whose closure is disjoint from the compact set $f^{-1}(y_2)$. The direction v defines a vector field on V which can be extended to a bounded and smooth vector field with support contained in U. The flow of this vector field defines a one-parameter group of diffeomorphisms of $\mathbb{R}^4$ which moves $f^{-1}(y_1)$ arbitrarily far away and does not move $f^{-1}(y_2)$. Making a small adjustment so that the vector field vanishes near the infinity we may transform it back to $\mathbb{S}^4$ through the stereographic projection. As a consequence we find a diffeomorphism of $\mathbb{S}^4$ which does not move $f^{-1}(y_2)$ but moves $f^{-1}(y_1)$ arbitrarily close to the north pole.

Now, we can find a one-parameter group of diffeomorphisms that moves points along meridians towards the south pole but does not move the image of $f^{-1}(y_1)$ that is already near the north pole. This family of diffeomorphisms moves $f^{-1}(y_2)$ to the interior of the closed lower hemisphere $\mathbb{S}^4_-$ of $\mathbb{S}^4$. This simply means that there is a diffeomorphism $\Phi : \mathbb{S}^4 \to \mathbb{S}^4$ such that $\Phi(f^{-1}(y_1)) \subset \operatorname{int} \mathbb{S}^4_+$ and $\Phi(f^{-1}(y_2)) \subset \operatorname{int} \mathbb{S}^4_-$. Observe that Φ is homotopic to the identity, because Φ is constructed through two one-parameter groups of diffeomorphisms connecting Φ to the identity (or simply because any orientation preserving diffeomorphism of $\mathbb{S}^4$ is homotopic to the identity, as a mapping of degree 1). Hence $f_1 = f \circ \Phi^{-1}$ is homotopic to f. Let $\mathbb{S} = \mathbb{S}^4_+ \cap \mathbb{S}^4_-$ be the equator of $\mathbb{S}^4$.

Note that $y_1 \notin f_1(\mathbb{S}^4_-)$ because $f_1^{-1}(y_1) = \Phi(f^{-1}(y_1)) \subset \operatorname{int} \mathbb{S}^4_+$. Similarly $y_2 \notin f_1(\mathbb{S}^4_+)$. Hence, for a small $\varepsilon > 0$,

$$\mathbb{B}(y_1, \varepsilon) \cap f_1(\mathbb{S}^4_-) = \mathbb{B}(y_2, \varepsilon) \cap f_1(\mathbb{S}^4_+) = \varnothing.$$

Let $\Psi_t : \mathbb{S}^3 \to \mathbb{S}^3$ be a continuous family of smooth mappings such that $\Psi_0 = \mathrm{id}$ and Ψ_1 retracts $\mathbb{S}^3 \setminus (\mathbb{B}(y_1, \varepsilon) \cup \mathbb{B}(y_2, \varepsilon))$ onto an equator $\tilde{\mathbb{S}}$ of $\mathbb{S}^3$ that separates the sets $\mathbb{B}(y_1, \varepsilon)$ and $\mathbb{B}(y_2, \varepsilon)$. Namely, Ψ_1 stretches the balls $\mathbb{B}(y_i, \varepsilon) \cap \mathbb{S}^3$, for $i = 1, 2$, in $\mathbb{S}^3$ onto the hemispheres $\mathbb{S}^3_\pm$ of $\mathbb{S}^3$, retracting everything what is between these ε-balls onto the equator.

Clearly, f_1 (and hence f) is homotopic to

$$f_2 = \Psi_1 \circ f_1 = \Psi_1 \circ f \circ \Phi^{-1}.$$

Note that

$$f_2(\mathbb{S}) \subset \tilde{\mathbb{S}}, \quad f_2(\mathbb{S}^4_+) \subset \mathbb{S}^3_+, \quad \text{and} \quad f_2(\mathbb{S}^4_-) \subset \mathbb{S}^3_-. \tag{4.2}$$

Indeed,

$$f_1(\mathbb{S}^4_+) \subset \mathbb{S}^3 \setminus \mathbb{B}(y_2, \varepsilon), \quad \text{so} \quad f_2(\mathbb{S}^4_+) \subset \Psi_1(\mathbb{S}^3 \setminus \mathbb{B}(y_2, \varepsilon)) = \mathbb{S}^3_+.$$

Similarly, $f_2(\mathbb{S}^4_-) \subset \mathbb{S}^3_-$ and hence

$$f_2(\mathbb{S}) = f_2(\mathbb{S}^4_+ \cap \mathbb{S}^4_-) \subset \mathbb{S}^3_+ \cap \mathbb{S}^3_- = \tilde{\mathbb{S}}.$$

Note that $\operatorname{rank} df_2 = \operatorname{rank} d(\Psi_1 \circ f \circ \Phi^{-1}) \leq 2$ a.e. by the chain rule.

Since $f_2 : \mathbb{S}^4 \to \mathbb{S}^3$ is homotopic to f, it is not homotopic to a constant map. Now Lemma 2.3 and (4.2) yield that the mapping f_2 is homotopic to the suspension Sh of the map $h = f_2|_{\mathbb{S}} : \mathbb{S} \to \tilde{\mathbb{S}}$. Since f_2 is not homotopic to a constant map, Sh is not homotopic to a constant map either. This and Lemma 2.2 imply that $h : \mathbb{S} \to \tilde{\mathbb{S}}$ is not homotopic to a constant map.

As h is a mapping from a 3-sphere to a 2-sphere that is not homotopic to a constant map, its Hopf invariant is non-zero (Lemma 3.1). Also, h is Lipschitz and the Lipschitz extension f_2 of h maps $\mathbb{S}^4_+$ to $\mathbb{S}^3_+$, thus it follows from Corollary 3.6 that df_2 has rank 3 on a subset of $\mathbb{S}^4_+$ of positive measure, which is a contradiction. This completes the proof of Theorem 1.3 and hence that of Theorem 1.2 when $n = 2, 3$. □

Now it remains to prove Theorem 1.5.

5 Proof of Theorem 1.5

Let $\lambda_{s,r} : \mathbb{R} \to \mathbb{R}$, for $0 < s < r < 1$, be a smooth, odd, and non-decreasing function such that $\lambda_{s,r}(t) = 1$ when $|t| > r$ and $\lambda(t) = t$ for $|t| < s$.

The smooth mapping $\Lambda : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$

$$(x_1, x_2, \ldots, x_{k+1}) \xmapsto{\Lambda} (\lambda_{s,r}(x_1), \ldots, \lambda_{s,r}(x_{k+1})) \tag{5.1}$$

maps $\mathbb{R}^{k+1}$ onto the cube $[-1, 1]^{k+1}$ in such a way that the interior neighborhood of the boundary $\partial[-1, 1]^{k+1}$ is mapped onto the boundary, and the complement $\mathbb{R}^{k+1} \setminus [-1, 1]^{k+1}$ is also smoothly mapped onto $\partial[-1, 1]^{k+1}$. Note, however, that $\Lambda|_{\partial[-1,1]^{k+1}} \neq \mathrm{id}$, and hence Λ is not a retraction.

Obviously, $x \mapsto \frac{1}{2}\Lambda(2x)$ has the same properties, with $[-1/2, 1/2]^{k+1}$ in place of $[-1,1]^{k+1}$, and similarly we can rescale and shift this mapping to be used on any other cube in $\mathbb{R}^{k+1}$.
Since $\pi_m(\mathbb{S}^k) \neq 0$, there is a $\phi \in C^\infty(\mathbb{S}^m, \partial[-\frac{1}{2},\frac{1}{2}]^{k+1})$ that represents a non-trivial element in $\pi_m(\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}) = \pi_m(\mathbb{S}^k)$. Clearly, we can choose ϕ to be continuous, but smoothing ϕ and then composing it with a projection onto the boundary of the cube as described above gives a mapping onto $\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$ that is C^∞ smooth as a mapping into $\mathbb{R}^{k+1}$.

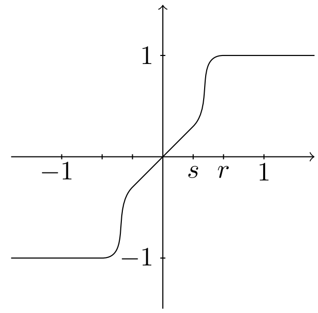

Fig. 1: Function $\lambda_{s,r}$.

We reduce the proof of Theorem 1.5 to the following lemma.

Lemma 5.1. There is a mapping $F \in C^1(\overline{\mathbb{B}}^{m+1}, [-\frac{1}{2},\frac{1}{2}]^{k+1})$ satisfying $\operatorname{rank} dF \leq k$ everywhere such that F maps the boundary $\partial\mathbb{B}^{m+1} = \mathbb{S}^m$ to $\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$ and $F|_{\partial\mathbb{B}^{m+1}} = \phi$. □

Before we prove the lemma, we show how Theorem 1.5 follows from it. Once we have a mapping F as above, we glue two copies of this mapping along the common boundary $\partial\mathbb{B}^{m+1} = \mathbb{S}^m$. We obtain a mapping into two copies of $[-\frac{1}{2},\frac{1}{2}]^{k+1}$ glued along the common boundary $\partial[-\frac{1}{2},\frac{1}{2}]^{k+1} \approx \mathbb{S}^k$, so we essentially obtain a mapping into $\mathbb{S}^{k+1}$.

To do it in a smooth way and to have $\mathbb{S}^{k+1}$ as the target, let $\xi : [0,\infty) \to [0,\infty)$ be a smooth function satisfying $\xi(t) \leq 1/t$ for all $t > 0$, $\xi(t) = 1$ for $t \in [0, 1/8]$ and $\xi(t) = 1/t$ for $t \geq 1/4$. Then $\Xi^\ell : \mathbb{R}^\ell \to \mathbb{R}^\ell$, $\Xi^\ell(x) = \xi(|x|)x$, is smooth and maps $\mathbb{R}^\ell$ to $\overline{\mathbb{B}}^\ell$ and maps $\partial[-\frac{1}{2},\frac{1}{2}]^\ell$ onto $\partial\mathbb{B}^\ell$.

Since the composition does not increase the rank of the derivative, the derivative of the function $\tilde{f} = \Xi^{k+1} \circ F \circ \Xi^{m+1} : \overline{\mathbb{B}}^{m+1} \to \overline{\mathbb{B}}^{k+1}$ has rank at most k everywhere and it is constant along radii near $\partial\mathbb{B}^{m+1}$, which is guaranteed by Ξ^{m+1}. Thus the radial derivative of $\tilde{f}$ vanishes at $\partial\mathbb{B}^{m+1}$, and $\tilde{f}$ maps $\partial\mathbb{B}^{m+1} = \mathbb{S}^m$ onto $\partial\mathbb{B}^{k+1} = \mathbb{S}^k$. Let $\Phi_\pm : \overline{\mathbb{B}}^{k+1} \to \mathbb{S}^{k+1}_\pm$ be diffeomorphisms of $\overline{\mathbb{B}}^{k+1}$ onto the closed upper and lower hemispheres that are smooth up to the boundary and equal to the identity on $\partial\mathbb{B}^{k+1}$. Then we smoothly glue two copies of $\tilde{f}$, defining $f : \mathbb{S}^{m+1} \to \mathbb{S}^{k+1}$ by the formula

$$f(x_1,\dots,x_{m+1},x_{m+2}) = \begin{cases} \Phi_+ \circ \tilde{f}(x_1,\dots,x_{m+1}) & \text{if } x_{m+2} \geq 0, \\ \Phi_- \circ \tilde{f}(x_1,\dots,x_{m+1}) & \text{if } x_{m+2} \leq 0. \end{cases}$$

The mapping $f|_{\mathbb{S}^m} : \mathbb{S}^m \to \mathbb{S}^k$, where $\mathbb{S}^m \subset \mathbb{S}^{m+1}$ and $\mathbb{S}^k \subset \mathbb{S}^{k+1}$ are equators, is not homotopic to a constant map, because the mapping ϕ is not homotopic to a constant map. The mapping $f : \mathbb{S}^{m+1} \to \mathbb{S}^{k+1}$ is homotopic to the suspension of $f|_{\mathbb{S}^m}$ (Lemma 2.3) and since $m < 2k-1$, it is not homotopic to the constant map (Lemma 2.4). This completes the proof of Theorem 1.5 and it remains to prove Lemma 5.1.

Proof of Lemma 5.1. In what follows, we denote by $\mathbb{B}^\ell$ the unit ball in $\mathbb{R}^\ell$ centered at the origin. We also denote by σB a ball concentric with B and with radius $\sigma > 0$ times that of B.

We shall repeatedly use the following geometric facts:

Lemma 5.2. Let $B_1,\dots,B_j \subset \mathbb{B}^\ell$ and $\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_j \subset \mathbb{B}^\ell$ be two families of pairwise disjoint, closed balls. Then there exists a smooth diffeomorphism $\Psi : \overline{\mathbb{B}}^\ell \to \overline{\mathbb{B}}^\ell$, $\Psi|_{\partial\mathbb{B}^\ell} = \mathrm{id}$, which maps B_i to $\tilde{B}_i$ for $i = 1,2,\dots,j$ in such a way that $\Psi|_{B_i}$ is a translation and scaling. □

Consider the cubical $(k+1)$-dimensional complex obtained by partitioning the unit cube $[-\frac{1}{2},\frac{1}{2}]^{k+1}$ into n^{k+1} equal cubes of edge-length $1/n$; denote these cubes by J_i, $i = 1,2,\dots,N = n^{k+1}$ and by $S = \bigcup_{i=1}^N \partial J_i$ the k-skeleton of the complex.

Lemma 5.3. There exists a smooth mapping $R : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ with the following properties:

- R maps a neighborhood of S to S:
  - for each cube J_i, if B_i is the $(k+1)$-dimensional ball inscribed into J_i, then $J_i \setminus \frac{1}{2}B_i$ is mapped onto ∂J_i,
  - R projects $\mathbb{R}^{k+1} \setminus [-\frac{1}{2},\frac{1}{2}]^{k+1}$ onto $\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$,
- R is the same, up to translation, in each of the cubes J_i,
- R is homotopic to identity on ∂J_i for each i, and on $\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$.

□

Proof. Observe that the unit ball $\mathbb{B}^{k+1}$ is inscribed in the cube $[-1,1]^{k+1}$. The mapping Λ defined in (5.1) maps the cube $[-1,1]^{k+1}$ onto itself in such a way that the interior neighborhood of the boundary $\partial[-1,1]^{k+1}$ is mapped onto the boundary. If we choose $s=\frac{1}{4\sqrt{k+1}}$, $r=2s$, then everything in $[-1,1]^{k+1}$ lying outside the cube $[-\frac{1}{2\sqrt{k+1}},\frac{1}{2\sqrt{k+1}}]^{k+1}$, in particular $[-1,1]^{k+1}\setminus\frac{1}{2}\mathbb{B}^{k+1}$, is mapped onto $\partial[-1,1]^{k+1}$.

The mapping is obviously smooth and homotopic to identity on $\partial[-1,1]^{k+1}$.

We can use Λ (rescaled and translated) to project an interior neighborhood of the boundary of each J_i onto that boundary of J_i. The resulting mapping is of class C^∞ on the whole cube $[-\frac{1}{2},\frac{1}{2}]^{k+1}$. Even the corners of the cubes J_i do not cause any problems, because the entire neighborhood of each of the corners is mapped into the corner and the mapping is C^∞ there as it is a constant one. This way R is defined already on $[-\frac{1}{2},\frac{1}{2}]^{k+1}$. Finally, for x outside $[-\frac{1}{2},\frac{1}{2}]^{k+1}$ we have well defined nearest point projection $\pi:\mathbb{R}^{k+1}\setminus[-\frac{1}{2},\frac{1}{2}]^{k+1}\to\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$; we take $R(x)=R(\pi(x))$. Although, π is not smooth, it is easy to see that the mapping R is smooth, because in the normal directions to faces of any dimension, the mapping $R\circ\pi$ is constant in a neighborhood of the point on the edge where we take the normal direction. ∎

We concentrate now on the construction of F. We begin by choosing $N=n^{k+1}$ disjoint closed balls $\mathbb{B}_i$, of radius $\frac{2}{n}$, all inside $\frac{1}{2}\mathbb{B}^{m+1}$ (see Figure 2). This is possible if we choose n large enough. Indeed, the ball $\frac{1}{2}\overline{\mathbb{B}}^{m+1}$ contains a cube of edge length $\frac{1}{\sqrt{m+1}}$, and in it one can fit at least $\left[\frac{n}{5\sqrt{m+1}}\right]^{m+1}$ cubes of edge length $\frac{5}{n}$ with pairwise disjoint interiors, where $[t]$ is the integer part of t.

For $n>(10\sqrt{m+1})^{m+1}$, we have

$$\left[\frac{n}{5\sqrt{m+1}}\right]^{m+1}>\left(\frac{n}{10\sqrt{m+1}}\right)^{m+1}=n^m\frac{n}{(10\sqrt{m+1})^{m+1}}>n^{k+1}.$$

Finally, in the interior of each of these cubes one can find a closed ball $\overline{\mathbb{B}}_i$, of radius $\frac{2}{n}$, concentric with the cube. Since the balls do not touch the boundaries of the cubes, they are pairwise disjoint.

Let $\overline{\Omega}_o=\overline{\mathbb{B}}^{m+1}\setminus\bigcup_{i=1}^N\mathbb{B}_i$. This will be the domain of the initial step of our construction, which will be then iterated inside each of the balls $\mathbb{B}_i$. Note that $\overline{\Omega}_0$ is an $(m+1)$-manifold with $N+1$ boundary components, all of which are m-spheres.

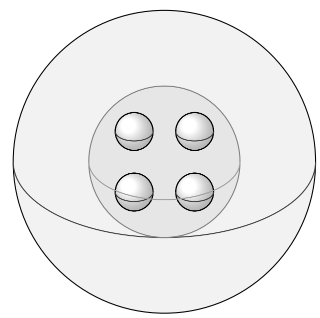

Fig. 2: Disjoint balls $\overline{\mathbb{B}}_i$ of radius $\frac{2}{n}$ lie inside $\frac{1}{2}\mathbb{B}^{m+1}$.

Using Lemma 5.2, we find a diffeomorphism $G_1:\overline{\mathbb{B}}^{m+1}\to\overline{\mathbb{B}}^{m+1}$ such that

- G_1 is identity on $\partial\mathbb{B}^{m+1}$,
- it maps the balls $\overline{\mathbb{B}}_i$ into N identical closed balls $\hat{K}_i$, arranged along the x_{m+1} axis, mapping these balls by a translation and scaling, see Figure 3.

Since n is large, the balls $\hat{K}_i$ might have to be smaller than the balls $\overline{\mathbb{B}}_i$.

For the next step in the construction we will need the following lemma.

Lemma 5.4. Let $\phi\in C^\infty(\mathbb{S}^m,\partial[-\frac{1}{2},\frac{1}{2}]^{k+1})$ be as in Lemma 5.1. Then there is a smooth map $h\in C^\infty(\mathbb{S}^{m-1},\mathbb{S}^{k-1})$, not homotopic to a constant map, such that for any map $P\colon\mathbb{S}^k\to\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$ that is homotopic to the radial projection $\pi\colon\mathbb{S}^k\to\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$, the map $P\circ Sh:\mathbb{S}^m\to\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$ is homotopic to ϕ. □

Remark 5.5. Recall that the operation of smooth suspension S_ε has been defined at the end of Section 2 and it follows from Lemma 5.4 that $P\circ S_\varepsilon h$ is homotopic to ϕ. □

Proof of Lemma 5.4. Since π is a homeomorphism, the map $\tilde{\phi}=\pi^{-1}\circ\phi:\mathbb{S}^m\to\mathbb{S}^k$ is well defined and not homotopic to a constant map. Next, $m\leq 2k-2$, thus it follows from Lemma 2.4 that $\tilde{\phi}$ is homotopic to the suspension Sh of a map $h\in C^\infty(\mathbb{S}^{m-1},\mathbb{S}^{k-1})$. Hence $\pi\circ Sh$ is homotopic to ϕ and thus $P\circ Sh$ is homotopic to ϕ for any map P as in the statement of the lemma. The map h is not homotopic to a constant map by Lemma 2.2. ∎

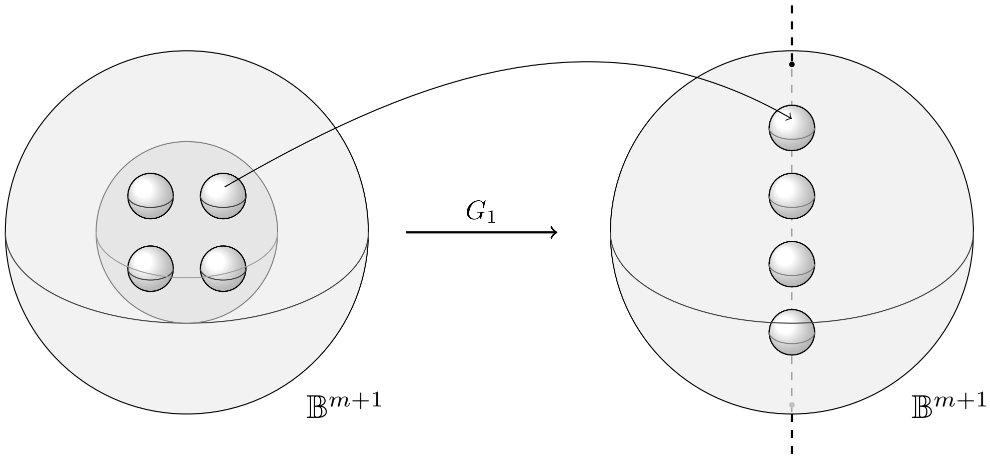

Fig. 3: The diffeomorphism G_1 rearranges the balls $\overline{\mathbb{B}}_i$, possibly shrinking them, so that their centers lie on the x_{m+1} axis. It maps $\overline{\mathbb{B}}_i$ to $\hat{K}_i$ by a similarity (scaling+translation) transformation.

We extend h radially to the mapping

$$\mathbb{R}^m \ni x \xmapsto{H_1} |x|\, h\left(\frac{x}{|x|}\right) \in \mathbb{R}^k,$$

so each sphere (centered at the origin) of radius r is mapped to the sphere of radius r by a scaled version of the mapping h. Then we extend H_1 to the mapping

$$\mathbb{R}^{m+1} \ni (x,t) \xmapsto{H_2} \left(|x|\, h\left(\frac{x}{|x|}\right), t\right) \in \mathbb{R}^{k+1}.$$

The mapping H_2 maps the $(m+1)$-balls of radius r centered at the t-axis (i.e. x_{m+1} axis) to $(k+1)$-balls of radius r centered at the t-axis (i.e. x_{k+1} axis). Thus it maps $\overline{\mathbb{B}}^{m+1}$ onto $\overline{\mathbb{B}}^{k+1}$ and each of the balls $\hat{K}_i$ to a corresponding ball K_i in $\mathbb{R}^{k+1}$.

Moreover, since H_1 is a scaled version of h on each of the spheres centered at the origin, it follows that the restriction of H_2 to the boundaries of the balls

$$H_2 : \partial\mathbb{B}^{m+1} \to \partial\mathbb{B}^{k+1} \quad \text{and} \quad H_2 : \partial\hat{K}_i \to \partial K_i \quad \text{for } i = 1, \ldots, N, \tag{5.2}$$

is the same mapping $Sh : \mathbb{S}^m \to \mathbb{S}^k$ (homotopic to $\tilde{\phi}$), up to a similarity in source and target.

In particular, we have

$$H_2 : \overline{\mathbb{B}}^{m+1} \setminus \bigcup_{i=1}^{N} \operatorname{int} \hat{K}_i \to \overline{\mathbb{B}}^{k+1} \setminus \bigcup_{i=1}^{N} \operatorname{int} K_i$$

and we will consider the mapping H_2 restricted to that set only.

As explained at the end of Section 2, the mapping H_1 is not smooth at the origin and hence H_2 is not smooth along the x_{m+1}-axis. In particular, the restrictions of H_2 in (5.2) are not smooth at the poles of the spheres. However, the mappings (5.2) are homotopic to the smooth suspension $S_\varepsilon h$ discussed in Section 2. Therefore we may modify H_2 to obtain a mapping that coincides with a scaled version of $S_\varepsilon h$ on each of the spheres $\partial\mathbb{B}^{m+1}$ and $\partial\hat{K}_i$ and is smooth in a neighborhood of each of the spheres. The resulting mapping is still not smooth on a compact subset of the x_{m+1}-axis that is in the interior of the set $\overline{\mathbb{B}}^{m+1} \setminus \bigcup_{i=1}^{N} \hat{K}_i$ and hence it does not touch the spheres. A standard mollification argument allows us to smooth it out and finally we obtain a smooth map

$$H : \overline{\mathbb{B}}^{m+1} \setminus \bigcup_{i=1}^{N} \operatorname{int} \hat{K}_i \to \overline{\mathbb{B}}^{k+1} \setminus \bigcup_{i=1}^{N} \operatorname{int} K_i$$

that coincides with a scaled version of $S_\varepsilon h$ on each of the spheres $\partial\mathbb{B}^{m+1}$ and $\partial\hat{K}_i$ (see Figure 4).

Let the unit cube $Q = [-\frac{1}{2}, \frac{1}{2}]^{k+1} \subset \mathbb{R}^{k+1}$ be divided into an even grid of $N = n^{k+1}$ cubes J_i, of edge length $1/n$.

Note that $Q \subset \frac{1}{2}\sqrt{k+1}\,\overline{\mathbb{B}}^{k+1}$.

In the next step we again use Lemma 5.2 to find a diffeomorphism G_2 that maps $\overline{\mathbb{B}}^{k+1}$ to $\frac{1}{2}\sqrt{k+1}\overline{\mathbb{B}}^{k+1}$ in such a way that

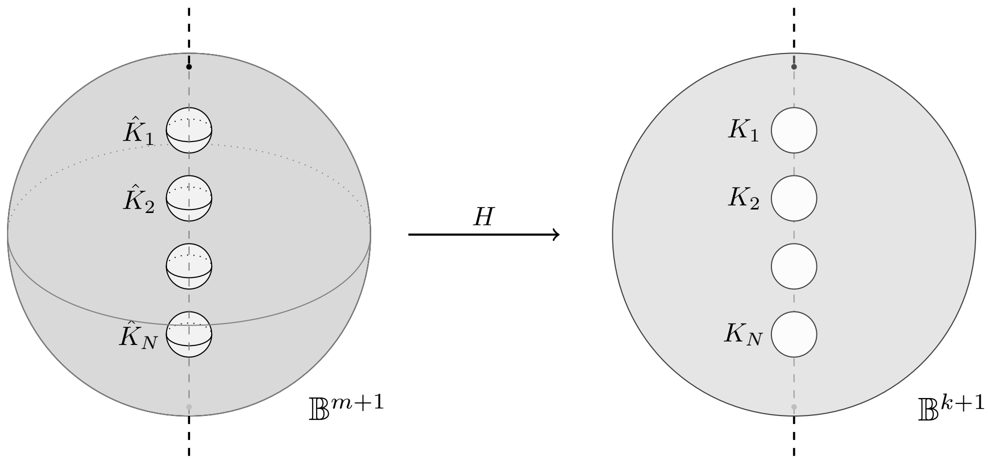

Fig. 4: The smooth mapping H maps the spheres $\partial\mathbb{B}^{m+1}$ and $\partial\hat{K}_i$ onto $\partial\mathbb{B}^{k+1}$ and ∂K_i by a scaled copy of $S_\varepsilon h$.

- G_2 maps $\partial\mathbb{B}^{k+1}$ to $\partial(\frac{1}{2}\sqrt{k+1}\mathbb{B}^{k+1})$ by similarity (in fact, scaling),
- G_2 maps each of the balls K_i into a ball L_i such that the ball $\frac{11}{10}L_i$ is inscribed into the cube J_i, and $G_2|_{K_i}$ is a similarity (translation+scaling).

The diffeomorphism G_2 is depicted in Figure 5.

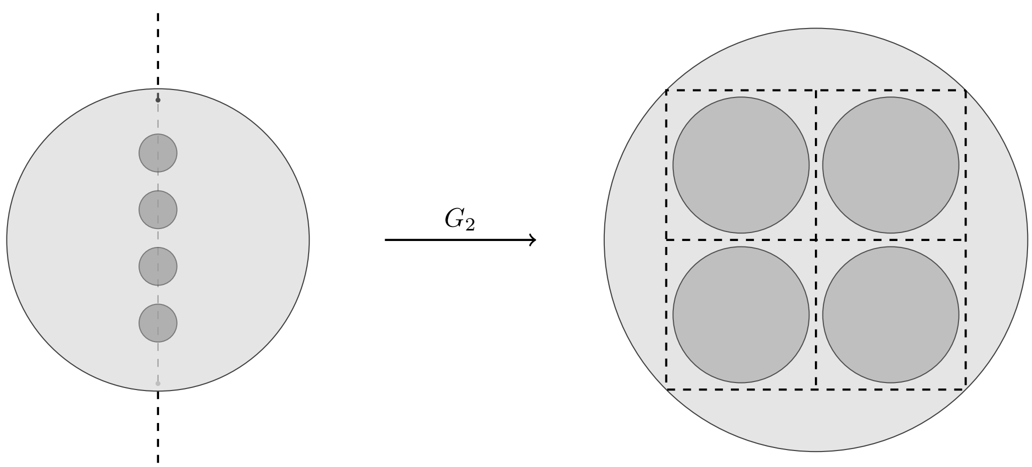

Fig. 5: The diffeomorphism G_2 rearranges the balls K_i in $\mathbb{B}^{k+1}$, mapping $\partial\mathbb{B}^{k+1}$ by scaling to a ball of radius $\frac{1}{2}\sqrt{k+1}$ and balls K_i to balls L_i, almost inscribed into a grid obtained by partitioning the unit cube $[-\frac{1}{2},\frac{1}{2}]^{k+1}$ into $N=n^{k+1}$ cubes of edge length $\frac{1}{n}$.

Finally, we use the mapping R defined in Lemma 5.3 to project $\frac{1}{2}\sqrt{k+1}\overline{\mathbb{B}}^{k+1}\setminus\bigcup_{i=1}^N L_i$ onto the k dimensional complex $S=\bigcup_i \partial J_i$, see Figure 6.

Let $\hat{F}=R\circ G_2\circ H\circ G_1:\overline{\Omega}_o=\overline{\mathbb{B}}^{m+1}\setminus\bigcup_{i=1}^N \mathbb{B}_i\to S$.

On the boundary of each ball $\overline{\mathbb{B}}_i$ the mapping $\hat{F}|_{\partial\mathbb{B}_i}\to\partial J_i$ is, up to a similarity in source and image, identical with some fixed mapping $g:\mathbb{S}^m\to\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$, that is homotopic to ϕ by Lemma 5.4 and Remark 5.5, see Figure 7. Similarly, $\hat{F}|_{\partial\mathbb{B}^{m+1}}:\partial\mathbb{B}^{m+1}\to\partial[-\frac{1}{2},\frac{1}{2}]^{k+1}$ is homotopic to ϕ (but not necessarily equal, up to scaling, to g).

Since we want to iterate the construction, by gluing into each of $\overline{\mathbb{B}}_i$ a rescaled copy of the mapping $\hat{F}$, we want to ensure that the maps indeed glue in a C^∞ manner (although C^1 would be enough). To this end, we want to have $\hat{F}|_{\partial\mathbb{B}^{m+1}}$ and $\hat{F}|_{\partial\mathbb{B}_i}$ equal, up to scaling, to the mapping ϕ (given in the statement of Lemma 5.1). Moreover, for the maps to glue in a C^∞ manner we want the map $\hat{F}$, in a neighborhood of the boundary of $\overline{\Omega}_o=\overline{\mathbb{B}}^{m+1}\setminus\bigcup_i\mathbb{B}_i$, to be constant in the normal directions to the boundary of Ω_o.

At the moment, $\hat{F}|_{\partial\mathbb{B}^{m+1}}$ and $\hat{F}|_{\partial\mathbb{B}_i}$ are only homotopic to ϕ. Let us thus correct $\hat{F}$ in three steps in the following way.

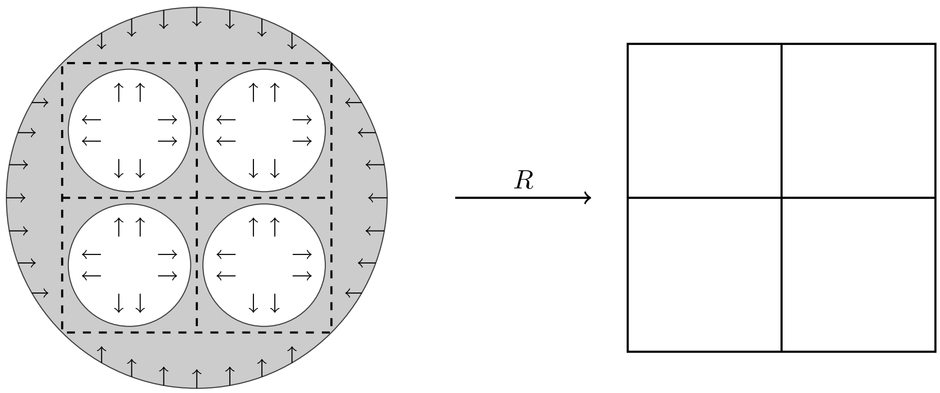

Fig. 6: In the final step we use the mapping R to project the image of $\overline{\Omega}_o$, here in darker shade, onto the k-complex S.

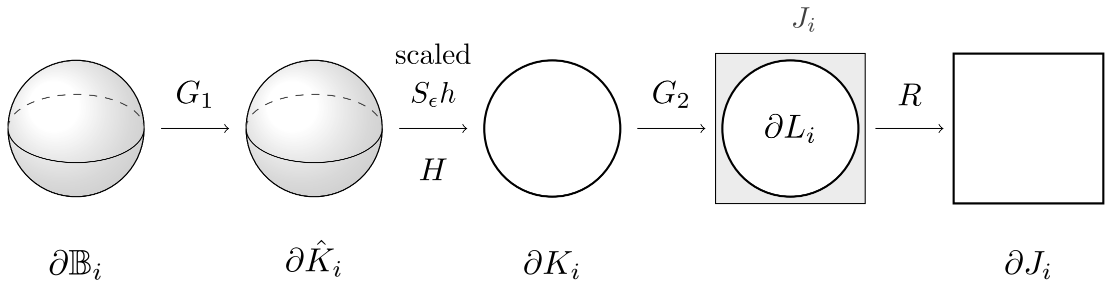

Fig. 7: The fate of $\partial\mathbb{B}_i$ throughout the construction.

- First, we modify $\hat{F}$ to a smooth map $\hat{F}_1$ which coincides with $\hat{F}$ on $\frac{2}{3}\overline{\mathbb{B}}^{m+1} \setminus \bigcup_{i=1}^N \mathbb{B}_i$, and $\hat{F}_1|_{\partial\mathbb{B}^{m+1}}$ equals ϕ. This is possible, because $\hat{F}_1|_{\partial(\frac{2}{3}\mathbb{B}^{m+1})}$ and ϕ are smoothly homotopic (up to the scaling that identifies $\partial(\frac{2}{3}\mathbb{B}^{m+1})$ with $\partial\mathbb{B}^{m+1}$) as mappings into $\partial[-\frac{1}{2}, \frac{1}{2}]^{k+1}$.
- Next, we want to ensure that the mapping is constant along radii on $\overline{\mathbb{B}}^{m+1} \setminus \frac{3}{4}\mathbb{B}^{m+1}$ (which is a smaller annulus than $\overline{\mathbb{B}}^{m+1} \setminus \frac{2}{3}\mathbb{B}^{m+1}$). We can do it by pre-composing $\hat{F}_1$ with the mapping

$$\overline{\mathbb{B}}^{m+1} \ni x \overset{\Phi_1}{\longmapsto} \lambda_{\frac{2}{3},\frac{3}{4}}(|x|)\frac{x}{|x|} \in \overline{\mathbb{B}}^{m+1},$$

  where the function $\lambda_{s,t}$ is as defined at the beginning of the proof. The mapping $\hat{F}_2 = \hat{F}_1 \circ \Phi_1$ is constant along the radii on $\mathbb{B}^{m+1} \setminus \frac{3}{4}\mathbb{B}^{m+1}$ and $\hat{F}_2$ equals ϕ on $\partial\mathbb{B}^{m+1}$.
- Using the same argument as above we can modify $\hat{F}_2$ in a small neighborhood of each of the boundaries $\partial\mathbb{B}_i$ so that the resulting mapping F_o equals ϕ (up to scaling) on each of the spheres $\partial\mathbb{B}_i$, is constant in normal directions near $\partial\mathbb{B}_i$ and maps $\partial\mathbb{B}_i$ onto ∂J_i.

The mapping $F_o : \overline{\Omega}_o \to S \subset [-\frac{1}{2}, \frac{1}{2}]^{k+1}$ is our initial step of the construction.

To inductively fill the map F_o into the holes $\mathbb{B}_i$ (up to a Cantor set), we associate to each $\overline{\mathbb{B}}_i$ its center x_i and the similarity map

$$\sigma_i \colon \overline{\mathbb{B}}^{m+1} \to \overline{\mathbb{B}}_i, \qquad \sigma_i(x) = \frac{2}{n}x + x_i;$$

recall that the radius of $\mathbb{B}_i$ equals $\frac{2}{n}$.

Each mapping σ_i maps $\overline{\Omega}_o$ into the ball $\overline{\mathbb{B}}_i$, so that if

$$\overline{\Omega}_1 = \bigcup_{i=1}^N \sigma_i(\overline{\Omega}_o) \quad \text{and} \quad D_1 = \overline{\Omega}_o \cup \overline{\Omega}_1,$$

then the set D_1 is obtained by adding to $\overline{\Omega}_o$ scaled copies of $\overline{\Omega}_o$ inside each of the holes $\mathbb{B}_i$. The set $\overline{\Omega}_o$ has n^{k+1} holes, each of radius $\frac{2}{n}$ while D_1 has $(n^{k+1})^2$ holes, each of radius $(\frac{2}{n})^2$.

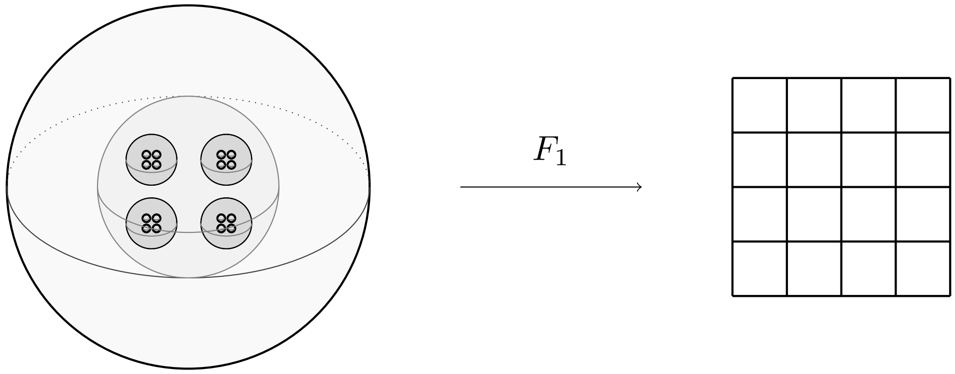

Fig. 8: The second iteration F_1.

We define now inductively

$$\overline{\Omega}_\ell = \bigcup_{i=1}^{N} \sigma_i(\overline{\Omega}_{\ell-1}), \qquad D_\ell = \bigcup_{j=0}^{\ell} \overline{\Omega}_j$$

so D_ℓ has $(n^{k+1})^{\ell+1}$ holes, each of radius $(\frac{2}{n})^{\ell+1}$.

Let $D = \bigcup_{\ell=0}^{\infty} D_\ell$ so $C = \overline{\mathbb{B}}^{m+1} \setminus D$ is a Cantor set. The mapping F_o has already been defined and we define inductively

$$F_\ell : D_\ell \to \left[-\frac{1}{2}, \frac{1}{2}\right]^{k+1}$$

for each $\ell \geq 1$ by

$$F_\ell|_{D_{\ell-1}} = F_{\ell-1}, \quad F_\ell|_{\sigma_i(\Omega_{\ell-1})} = \tau_i \circ F_{\ell-1} \circ \sigma_i^{-1}, \quad i = 1, 2, \ldots, N,$$

where

$$\tau_i : \left[-\frac{1}{2}, \frac{1}{2}\right]^{k+1} \to J_i \overset{\text{isometric}}{\approx} \left[-\frac{1}{2n}, \frac{1}{2n}\right]^{k+1}$$

is the translation and scaling transformation.

Now $F : D \to [-\frac{1}{2}, \frac{1}{2}]^{k+1}$ is given by $F|_{D_\ell} = F_\ell$ for each $\ell = 0, 1, 2, \ldots$ This map is smooth in D and it continuously extends to the Cantor set C. Moreover

$$\|dF_\ell|_{\Omega_\ell}\|_\infty = \frac{1}{n}\|dF_{\ell-1}|_{\Omega_{\ell-1}}\|_\infty \frac{n}{2} = \ldots = \frac{1}{2^\ell}\|dF_o|_{\Omega_o}\|_\infty \to 0.$$

Thus F is continuously differentiable also on the Cantor set C, with $DF|_C \equiv 0$. This completes the proof (and the article). ■

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