Active Anomaly Detection in Heterogeneous Processes

Boshuang Huang, Kobi Cohen, and Qing Zhao, *Fellow, IEEE*

***Abstract*— An active inference problem of detecting anomalies among heterogeneous processes is considered. At each time, a subset of processes can be probed. The objective is to design a sequential probing strategy that dynamically determines which processes to observe at each time and when to terminate the search so that the expected detection time is minimized under a constraint on the probability of misclassifying any process. This problem falls into the general setting of sequential design of experiments pioneered by Chernoff in 1959, in which a randomized strategy, referred to as the Chernoff test, was proposed and shown to be asymptotically optimal as the error probability approaches zero. For the problem considered in this paper, a low-complexity deterministic test is shown to enjoy the same asymptotic optimality while offering significantly better performance in the finite regime and faster convergence to the optimal rate function, especially when the number of processes is large. Furthermore, the proposed test offers considerable reduction in computation complexity.**

***Index Terms*— Active hypothesis testing, sequential design of experiments, anomaly detection, dynamic search, target whereabout.**

I. Introduction

WE CONSIDER the problem of detecting an anomalous process among M heterogeneous processes. Borrowing terminologies from target search, we refer to these processes as cells and the anomalous process as the target which can locate in any of the M cells. At each time, K ($1 \leq K < M$) cells can be probed simultaneously to search for the target. Each search of cell i generates a noisy observation drawn i.i.d. over time from two different distributions f_i and g_i, depending on whether the target is absent or present. The objective is to design a sequential search strategy that dynamically determines which cells to probe at each time and when to terminate the search so that the expected detection time is minimized under a constraint on the probability of declaring a wrong location of the target.

The above problem is prototypical of searching for rare events in a large number of data streams or a large system. The rare events could be opportunities (e.g., financial trading opportunities or transmission opportunities in dynamic spectrum access [1]), unusual activities in surveillance feedings, frauds in financial transactions, attacks and intrusions in communication and computer networks, anomalies in infrastructures (such as bridges, buildings, and the power grid) that may indicate catastrophes. Depending on the application, a cell may refer to an autonomous data stream with a continuous data flow or a system component that only generates data when probed.

A. Main Results

The anomaly detection problem considered in this paper is a special case of active hypothesis testing originated from Chernoff's seminal work on sequential design of experiments in 1959 [2]. Compared with the classic passive sequential hypothesis testing pioneered by Wald [3], where the observation model under each hypothesis is predetermined, active hypothesis testing has a control aspect that allows the decision maker to choose the experiment to be conducted at each time. Different experiments generate observations from different distributions under each hypothesis. Intuitively, as more observations are gathered, the decision maker becomes more certain about the true hypothesis, which in turn leads to better choices of experiments.

In [2], Chernoff proposed a *randomized* strategy, referred to as the Chernoff test, and established its asymptotic (as the error probability diminishes) optimality.[1] This randomized test chooses, at each time, a probability distribution that governs the selection of the experiment to be carried out at this time. This distribution is obtained by solving a minimax problem so that the next observation generated under the random action can best differentiate the current maximum likelihood estimate of the true hypothesis (using all past observations) from its closest alternative, where the closeness is measured by the Kullback-Liebler (KL) divergence. Due to the complexity in solving this minimax problem at each time, the Chernoff test

[1]The asymptotic optimality of the Chernoff test was shown under the assumption that the hypotheses are distinguishable under every experiment.

Manuscript received June 27, 2017; revised August 3, 2018; accepted August 10, 2018. Date of publication August 21, 2018; date of current version March 15, 2019. This work was supported in part by the National Science Foundation under Grant CCF-1815559 and in part by the Army Research Office under Grant W911NF-17-1-0464. K. Cohen was supported in part by the Cyber Security Research Center at Ben-Gurion University of the Negev and in part by the U.S.-Israel Binational Science Foundation under Grant 2017723. This paper was presented at the 2018 IEEE International Conference on Acoustics, Speech, and Signal Processing, April.

B. Huang and Q. Zhao are with the School of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14853 USA (e-mail: bh467@cornell.edu; qz16@cornell.edu).

K. Cohen is with the Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel (e-mail: yakovsec@bgu.ac.il).

Communicated by G. V. Moustakides, Associate Editor for Sequential Methods.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2018.2866257

can be expensive to compute and cumbersome to implement, especially when the number of hypotheses or the number of experiments is large.

It is not difficult to see that the problem at hand is a special case of the general active hypothesis testing problem. Specifically, the available experiments are in the form of different subsets of K cells to probe, and the number of experiments is $\binom{M}{K}$. Under each hypothesis that cell m ($m = 1, \ldots, M$) is the target, the distribution of the next observation (a vector of dimension K) depends on which K cells are chosen. The Chernoff test thus directly applies. Unfortunately, with the large number of hypotheses and the large number of experiments, it can be computationally prohibitive to obtain the Chernoff test.

In this paper, we show that the anomaly detection problem considered here exhibits sufficient structures to admit a low-complexity *deterministic* policy with strong performance. In particular, we develop a deterministic test that *explicitly* specifies which K cells to search at each given time and show that this test enjoys the same asymptotic optimality as the Chernoff test.[2] Furthermore, extensive simulation examples have demonstrated significant performance gain over the Chernoff test in the finite regime and faster convergence to the optimal rate function, especially when M is large. In contrast to the Chernoff test, the proposed test requires little offline or online computation. The test can also be extended to cases with multiple targets as discussed in Section V. Its asymptotic optimality is preserved for $K = 1$.

Often, when a solution is simpler, establishing its optimality becomes harder. This is indeed the case here. In Chernoff test, since the distribution of the random action depends only on the current maximum likelihood estimate of the underlying hypothesis which becomes time-invariant after an initial phase with a bounded duration, the stochastic behaviors of the test statistics, namely, the log-likelihood ratios (LLRs), are independent over time. In contrast, the deterministic actions under the proposed policy result in strong time and spacial (across processes) dependencies in the dynamic evolutions of the LLRs. Establishing the asymptotic optimality becomes much more involved.

B. Related Work

Chernoff's pioneering work on sequential design of experiments focuses on binary composite hypothesis testing [2]. Variations and extensions have been studied in [4]–[9], where the problem was referred to as controlled sensing for hypothesis testing in [5]–[7] and active hypothesis testing in [8] and [9]. As variants of the Chernoff test, the tests developed in [4]–[9] are all randomized tests.

There is an extensive literature on dynamic search and target whereabout problems under various scenarios. We discuss here existing studies within the sequential inference setting, which is the most relevant to this work. Two models on prior information about the targets have been considered in the literature: the exclusive model which assumes a fixed number of targets and the independent model which assumes each cell may contain a target with a given prior probability independent of other cells. These two models were juxtaposed in [10] and [11] under different objective functions. The studies in [12]–[16] focus on the exclusive model. In particular, homogeneous Poisson point processes with unknown rates was investigated and an asymptotically optimal randomized test was developed in [12]. In [13], the problem of tracking a target that moves as a Markov Chain in a finite discrete environment is studied and a search strategy that provides the most confident estimate is developed. The studies in [17]–[20] focus on the independent model. The problem of searching among Gaussian signals with rare mean and variance values was studied and an adaptive group sampling strategy was developed in [17]. In [18], the problem of quickly detecting anomalous components under the objective of minimizing system-wide cost incurred by all anomalous components was studied. In [19], an important case of multichannel sequential change detection is studied and an asymptotic framework in which the number of sensors tends to infinity was proposed.

Asymptotically optimal search policies over homogeneous processes were established in [21] under a non-parametric setting with finite discrete distributions and in [22] under a parametric composite hypothesis setting with continuous distributions. The objective of minimizing operational cost as opposed to detection delay led to a different problem from the one considered in this paper. Other related work on quickest search over multiple processes under various models and formulations includes [10], [14], [20], [23] and references therein. Sequential spectrum sensing within both the passive and active hypothesis testing frameworks has also received extensive attention in the application domain of cognitive radio networks (see, for example, [24]–[27] and references therein). The readers are also referred to [28] for a comprehensive survey on the problem of detecting outlying sequences.

A prior study by Cohen and Zhao considered the problem for homogeneous processes (i.e., $f_i \equiv f$ and $g_i \equiv g$) [15]. This work builds upon this prior work and addresses the problem in heterogeneous systems where the absence distribution f_i and the presence distribution g_i are different across processes. Allowing heterogeneity significantly complicates the design of the test and the analysis of asymptotic optimality. Since each process has different observation distributions, the rate at which the state of a cell can be inferred is different across processes. To achieve asymptotic optimality, the decision maker must carefully balance the search time among the observed processes, which makes both the algorithm design and the performance analysis much more involved under the heterogeneous case. Specifically, in terms of algorithm design, when dealing with homogeneous processes, the search strategy is often static in nature [10], [12], [15], [21]. In contrast, the asymptotically optimal search strategy developed here for heterogeneous processes dynamically changes based on the current belief about the location of the target. In terms of performance analysis, when dealing with homogeneous processes, the resulting rate function (which is inversely proportional to the search time) always obeys a certain averaging over the KL divergences between normal and abnormal distributions of

[2]The asymptotic optimality of the proposed test holds for all but at most three singular values of K (see Theorem 3).

all processes. This observation follows from the fact that the decision maker completes gathering the required information from all the processes at approximately the same time due to the homogeneity. In contrast, when searching over heterogeneous processes, the overall rate function does not always obey a simple averaging across the KL divergences of all processes. In Section IV, we show that the search time can be analyzed by considering two separate scenarios, referred to as the balanced and the unbalanced cases. The balanced case holds when a judicious allocation of probing resources can ensure the information gathering from all the processes be completed at approximately the same time, in which case the rate function is a weighted average among the heterogeneous processes. The unbalanced case occurs when there is a process with a sufficiently small KL divergence that it dominates the overall rate function of the search. This case is unique to the heterogeneous processes considered here and needs to be addressed with new analytical techniques.

Besides the active inference approach to anomaly detection considered in this paper, there is a growing body of literature on various approaches to the general problem of anomaly detection. We refer the readers to [29] and [30] for comprehensive surveys on this topic.

II. Problem Formulation

We consider the problem of detecting a single target located in one of M cells. If the target is in cell m, we say that hypothesis H_m is true. The *a priori* probability that H_m is true is denoted by π_m, where $\sum_{m=1}^{M}\pi_m = 1$. To avoid trivial solutions, it is assumed that $0 < \pi_m < 1$ for all m.

When cell m is observed at time n, an observation $y_m(n)$ is drawn, independent of previous observations. If cell m contains a target, $y_m(n)$ follows distribution $g_m(y)$. Otherwise, $y_m(n)$ follows distribution $f_m(y)$. Let $\mathbf{P}_m$ be the probability measure under hypothesis H_m and $\mathbf{E}_m$ the operator of expectation with respect to the measure $\mathbf{P}_m$.

An active search strategy Γ consists of a stopping rule τ governing when to terminate the search, a decision rule δ for determining the location of the target at the time of stopping, and a sequence of selection rules $\{\phi(n)\}_{n\geq 1}$ governing which K cells to probed at each time n. Let $\mathbf{y}(n)$ be the set of all cell selections and observations up to time n. A deterministic selection rule $\phi(n)$ at time n is a mapping from $\mathbf{y}(n-1)$ to $\{1, 2, \ldots, M\}^K$. A randomized selection rule $\phi(n)$ is a mapping from $\mathbf{y}(n-1)$ to probability mass functions over $\{1, 2, \ldots, M\}^K$.

We adopt a Bayesian approach as in Chernoff's original study [2] by assigning a cost of c for each observation and a loss of 1 for a wrong declaration. Note that c represents the ratio of the sampling cost to the cost of wrong detections. The Bayes risk under strategy Γ when hypothesis H_m is true is given by:

$$R_m(\Gamma) \triangleq \alpha_m(\Gamma) + c\mathbf{E}_m(\tau|\Gamma), \tag{1}$$

where $\alpha_m(\Gamma) = \mathbf{P}_m(\delta \neq m|\Gamma)$ is the probability of declaring $\delta \neq m$ under H_m and $\mathbf{E}_m(\tau|\Gamma)$ is the detection delay under H_m. The average Bayes risk is given by:

$$R(\Gamma) = \sum_{m=1}^{M} \pi_m R_m(\Gamma) = P_e(\Gamma) + c\mathbf{E}(\tau|\Gamma), \tag{2}$$

where $P_e(\Gamma)$ and $\mathbf{E}(\tau|\Gamma)$ are the error probability and detection delay averaged under the given prior $\{\pi_m\}$. The objective is to find a strategy Γ that minimizes the Bayes risk $R(\Gamma)$:

$$\inf_{\Gamma} \quad R(\Gamma). \tag{3}$$

A strategy Γ^* is *asymptotically optimal* if

$$\lim_{c\to 0} \frac{R(\Gamma^*)}{\inf_\Gamma R(\Gamma)} = 1, \tag{4}$$

which is denoted as

$$R(\Gamma^*) \sim \inf_{\Gamma} R(\Gamma). \tag{5}$$

III. The Deterministic DGFi Policy

In this section we propose a deterministic policy, referred to as the DGFi policy, indicating the key quantities $\{D(g_i||f_i),\ D(f_i||g_i)\}_{i=1}^{M}$ that govern the selection rule of the proposed policy.

A. DGFi Under Single-Cell Probing

We first consider the case of $K = 1$. Let $\mathbf{1}_m(n)$ be the indicator function, where $\mathbf{1}_m(n) = 1$ if cell m is observed at time n, and $\mathbf{1}_m(n) = 0$ otherwise. This indicator function clearly depends on the selection rule, which we omit in the notation for simplicity. Let

$$\ell_m(n) \triangleq \log \frac{g_m(y_m(n))}{f_m(y_m(n))}, \tag{6}$$

and

$$S_m(n) \triangleq \sum_{t=1}^{n} \ell_m(t)\mathbf{1}_m(t) \tag{7}$$

be the LLR and the observed sum LLRs of cell m at time n, respectively. Let $D(g||f)$ denote the KL divergence between two distributions g and f which is given by[3]

$$D(g||f) \triangleq \int_{-\infty}^{\infty} \log \frac{g(x)}{f(x)} g(x)\, dx. \tag{8}$$

Illustrated in Fig. 1 are typical sample paths of the sum LLRs of $M = 4$ cells, where, without loss of generality, we assume that cell 1 is the target. Note that the sum LLR of cell 1 is a random walk with a positive expected increment $D(g_1||f_1)$, whereas the sum LLR of cell m ($m = 2, 3, 4$) is a random walk with a negative expected increment $-D(f_m||g_m)$. Thus, when the gap between the largest sum LLR and the second largest sum LLR is sufficiently large, we can declare with a sufficient accuracy that the cell with the largest sum LLR is the target. This is the intuition behind the stopping rule and the decision rule under DGFi.

[3] We assume that g_i is absolutely continuous with respect to f_i ($i = 1, \ldots, M$) and vise versa, which ensures that all KL divergences are finite.

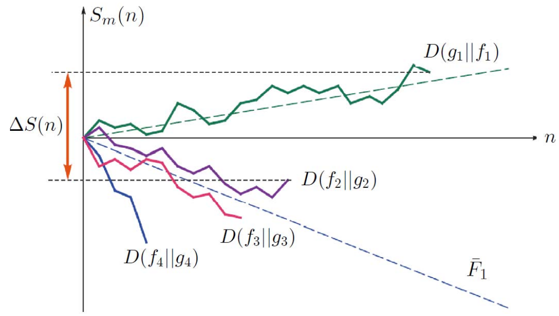

Fig. 1. Typical sample paths of sum LLRs.

Specifically, we define $m^{(i)}(n)$ as the index of the cell with the i^{th} largest observed sum LLRs at time n. Let

$$\Delta S(n) \triangleq S_{m^{(1)}(n)}(n) - S_{m^{(2)}(n)}(n) \tag{9}$$

denote the difference between the largest and the second largest observed sum LLRs at time n. The stopping rule and the decision rule under the DGFi policy are given by:

$$\tau = \inf\{n \ : \ \Delta S(n) \geq -\log c\}, \tag{10}$$

and

$$\delta = m^{(1)}(\tau). \tag{11}$$

We now specify the selection rule of the DGFi policy. The intuition behind the selection rule is to select a cell from which the observation can increase $\Delta S(n)$ at the fastest rate. The selection rule is thus given by comparing the rate at which $S_{m^{(1)}(n)}(n)$ increases with the rate at which $S_{m^{(2)}(n)}(n)$ decreases. If $S_{m^{(1)}(n)}(n)$ is expected to increase faster than $S_{m^{(2)}(n)}(n)$ decreases, cell $m^{(1)}(n)$ is chosen. Otherwise, cell $m^{(2)}(n)$ is chosen. This leads to the following selection rule:

$$\phi(n) = \begin{cases} m^{(1)}(n), & \text{if } D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)}) \geq \bar{F}_{m^{(1)}(n)} \\ m^{(2)}(n), & \text{otherwise,} \end{cases} \tag{12}$$

where

$$\bar{F}_m \triangleq \frac{1}{\sum_{j \neq m} \frac{1}{D(f_j||g_j)}}. \tag{13}$$

The selection rule in (12) can be intuitively understood by noticing that $D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)})$ is the asymptotic increasing rate of $S_{m^{(1)}}(n)$ when cell $m^{(1)}$ is probed at each time. This is due to the fact that $m^{(1)}(n)$ is the true target after an initial phase (defined by the last passage time that $m^{(1)}(n)$ is an empty cell) which can be shown to have a bounded expected duration. Similarly, even though much more involved to prove, $\bar{F}_{m^{(1)}(n)}$ is the asymptotic rate at which $S_{m^{(2)}(n)}(n)$ decreases when cell $m^{(2)}(n)$ is probed at each time. To see the expression of $\bar{F}_m$ for any m as given in (13), consider the following analogy. Consider $M-1$ cars being driven by a single driver from 0 to $-\infty$. Car j ($j = 1, \ldots, M$, $j \neq m$) has a constant speed of $D(f_j||g_j)$. At each time, the car closest to the origin is chosen by the driver and driven by one unit of time. We are interested in the average moving speed of the position of the closest car to the origin. It is not difficult to see that it is given by $\bar{F}_m$ in (13). This analogy, concerned with deterministic processes, only serves as an intuitive explanation for the expression of $\bar{F}_m$. As detailed in Sec. IV, proving $\bar{F}_{m^{(1)}(n)}$ to be the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$ requires analyzing the trajectories of the M sum LLRs $\{S_m(n)\}_{m=1}^M$, which are stochastic processes with complex dependencies both in time and across processes.

B. DGFi Under Multiple Simultaneous Observations

Now we consider the case of $K > 1$. The stopping rule and the decision rule remains the same as given in (10), (11), whereas the selection rule requires a significant modification. The main reason is that when K cells can be observed simultaneously, the asymptotic increasing rate of $S_{m^{(1)}(n)}(n)$ and the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$ are much more involved to analyze.

The selection rule $\phi(n)$, at each time n, chooses either the K cells with the top K largest sum LLRs or those with the second to the $(K+1)^{th}$ largest sums LLRs as in (14), as shown at the bottom of this page, where

$$F_m(\kappa) \triangleq \min\{\kappa \bar{F}_m, \ \min_{j \neq m} D(f_j||g_j)\}. \tag{15}$$

Note that (15) reduces to (13) at $K = 1$ (i.e., $F_m(1) = \bar{F}_m$), in which case the minimum is always attained at the first term. Similar to the case with $K = 1$, the intuition behind the selection rule is to select K cells from which the observations increase $\Delta S(n)$ at the fastest rate. Specifically, $F_{m^{(1)}(n)}(K)$ is the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$ when K cells with the second largest to the $(K+1)$th largest sum LLRs are probed each time. When the cell with the top K largest sum LLRs are probed each time, the asymptotic increasing rate of $\Delta S(n)$ is $D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)}) + F_{m^{(1)}(n)}(K-1)$, where $D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)})$ is the asymptotic increasing rate of $S_{m^{(1)}(n)}(n)$ and $F_{m^{(1)}(n)}(K-1)$ is the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$ with $K-1$ drivers. It is easy to see that when $K = 1$, the policy reduces to the one described in section III-A.

The behavior of $F_m(\kappa)$ as a function of κ (extending κ to all positive real values) is crucial in understanding and analyzing the asymptotic optimality of DGFi for $K > 1$. It is easy to see that the first term in the right hand of (15) is a linearly increasing function of κ and the second term is a constant. This readily leads to the piecewise linear property of $F_m(\kappa)$ as illustrated in Fig. 2. Let $\tilde{K}_m$ denote the switching point

$$\phi(n) = \begin{cases} \left(m^{(1)}(n), m^{(2)}(n), \ldots, m^{(K)}(n)\right) & \text{if } D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)}) + F_{m^{(1)}(n)}(K-1) \geq F_{m^{(1)}(n)}(K) \\ \left(m^{(2)}(n), m^{(3)}(n), \ldots, m^{(K+1)}(n)\right) & \text{otherwise} \end{cases} \tag{14}$$

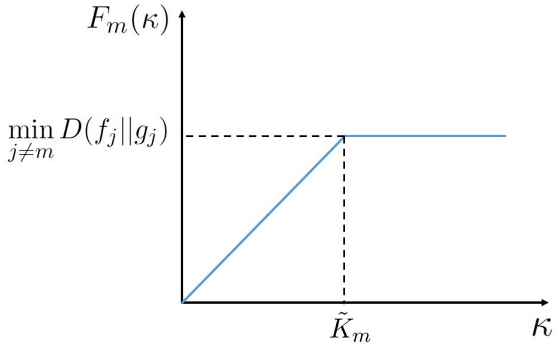

Fig. 2. The piecewise linear property of $F_m(\kappa)$.

between the increasing and constant regions, we have

$$\tilde{K}_m = \frac{\min_{j\neq m} D(f_j||g_j)}{\bar{F}_m} = \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)}. \quad (16)$$

The constant value of $F_m(\kappa)$ for $\kappa \geq \tilde{K}_m$ can be explained with the same car analogy. This constant value $\min_{j\neq m} D(f_j||g_j)$ is the speed of the slowest car among the $M-1$ cars (excluding the mth car). When the speed of the slowest car is sufficiently small, this car always lags behind even with a dedicated driver. This car becomes the bottleneck that caps the value of $F_m(\kappa)$ even when the number κ of drivers increases (note that each car can at most have one driver assigned). We refer to this case as the unbalanced case, which presents the most challenge in proving the asymptotic optimality of DGFi. The linearly increasing region of $\kappa < \tilde{K}_m$ is referred to as the balanced case, where $F_m(\kappa)$ is a weighted average among the $M-1$ cars.

IV. Performance Analysis

In this section, we establish the asymptotic optimality of the DGFi policy. While the intuitive exposition of DGFi given in Sec. III may make its asymptotic optimality seem expected, constructing a proof is much more involved. In particular, bounding the detection time of DGFi requires analyzing the trajectories of the M stochastic processes $\{S_m(n)\}_{m=1}^M$ which exhibit complex dependencies both over time and across processes as induced by the deterministic selection rule.

The asymptotic optimality of DGFi is established by comparing its Bayes risk (given in Theorem 1) with a lower bound on achievable Bayes risk (given in Theorem 2). We first analyze the rate function of DGFi. Define

$$I_m(\Gamma_{\text{DGFi}}) \triangleq \max\{D(g_m||f_m) + F_m(K-1), F_m(K)\}, \quad (17)$$

which is the increasing rate of $\Delta S(n)$ under hypothesis H_m when DGFi is employed. For a given *a priori* distribution $\{\pi_m\}_{m=1}^M$, define

$$I(\Gamma_{\text{DGFi}}) \triangleq \frac{1}{\sum_{m=1}^M \frac{\pi_m}{I_m(\Gamma_{\text{DGFi}})}}. \quad (18)$$

As shown in Theorem 1 below, $I(\Gamma_{\text{DGFi}})$ is the rate function of the Bayes risk of the DGFi policy.

Theorem 1: The Bayes risk $R(\Gamma_{\text{DGFi}})$ of the DGFi policy is given by

$$R(\Gamma_{\text{DGFi}}) \sim \frac{-c\log c}{I(\Gamma_{\text{DGFi}})}. \quad (19)$$

Proof: Here we provide a sketch of the proof. The detailed proof can be found in Appendix A. First, we show that when $\Delta S(\tau)$ is large, the probability of error is small, i.e. $P_e = O(c)$. As a result, by the definition of the Bayes risk, it suffices to show that the detection time is upper bounded by $-\log c/I(\Gamma_{\text{DGFi}})$. By the definition of $I(\Gamma_{\text{DGFi}})$ in (18), it suffices to show that the detection time is upper bounded by $-\log c/I_m\ (\Gamma_{\text{DGFi}})$ under hypothesis H_m. Since the decision maker might not complete to gather the required information from all the cells at the same time, we carry out the analysis by treating the balanced and the unbalanced cases separately. ■

Next we estabilsh a lower bound on the Bayes risk achievable by any policy. Define

$$I_m^* \triangleq \max_{u\in[0,1]} uD(g_m||f_m) + F_m(K-u). \quad (20)$$

$$I^* \triangleq \frac{1}{\sum_{m=1}^M \frac{\pi_m}{I_m^*}}. \quad (21)$$

Using the same car analogy, we can interpret I_m^* as the maximum increasing rate of $\Delta S(n)$ under hypothesis H_m with an optimal allocation of $u^* \in [0,1]$ driver to the target car. Comparing with the rate of DGFi under H_m in (17), we see that the deterministic nature of DGFi forces the allocation of drivers to the target to be either 0 or 1. As shown in Theorem 2 below, I^* is an upper bound on the rate function for any policy.

Theorem 2: Let $R(\Gamma)$ be the Bayes risk under an arbitrary policy Γ. We have

$$\inf_{\Gamma} R(\Gamma) \sim \frac{-c\log c}{I^*} \quad (22)$$

Proof: The outline of the proof is as follows. We first prove that if the Bayes risk is sufficiently small under strategy Γ, i.e., $R(\Gamma) = O(-c\log c)$, the difference between the largest sum LLRs and the second largest sum LLRs must be sufficiently large when the test terminates, i.e. $\Delta S(\tau) = \Omega(-\log c)$. Otherwise, it is not possible to achieve a risk $O(-c\log c)$ due to a large error probability. We then show that in order to make $\Delta S(n)$ sufficiently large, the sample size must be large enough, i.e., $\mathbf{E}[\tau|\Gamma] \geq \frac{-\log c}{I^*}$. Since each sample costs c, the total risk will be lower bounded by $\frac{-c\log c}{I^*}$ as desired. The detailed proof can be found in Appendix B. ■

Establishing the asymptotic optimality of DGFi rests on comparing its rate function $I(\Gamma_{\text{DGFi}})$ with the optimal rate function I^*. The key thus lies in analyzing the optimizer u_m^* in the right hand of (20) and showing whether and when it assumes integer values of 0 and 1 as used in DGFi. This is established in Lemma 1 that leads to the following necessary and sufficient condition for the asymptotic optimality of DGFi.

Theorem 3: A necessary and sufficient condition for the asymptotic optimality of the DGFi policy is that, for each $m = 1, \ldots, M$, at least one of the following three statements is true

(a) $D(g_m||f_m) \geq \bar{F}_m$.
(b) $K \leq \tilde{K}_m$.
(c) $K \geq \tilde{K}_m + 1$.

Proof: We first establish the following lemma on the maximizer u_m^* that attains I_m^* given in (20). The proof of this lemma is in Appendix C.

Lemma 1: Define

$$u_m^* \triangleq \arg\max_{u\in[0,1]} uD(g_m||f_m) + F_m(K-u). \tag{23}$$

Then,

$$u_m^* = \begin{cases} 1, & \text{if } D(g_m||f_m) \geq \bar{F}_m \\ \min\{\max\{K-\tilde{K}_m, 0\}, 1\}, & \text{if } D(g_m||f_m) < \bar{F}_m. \end{cases} \tag{24}$$

From (24) in Lemma 1, u_m^* takes the integer value of 0 or 1 if and only if at least one of the Statements (a), (b), (c) is true. Theorem 3 thus follows. ■

Corollary 1: The DGFi policy is asymptotically optimal except for at most three values of $K \in \{2, 3, \ldots, M\}$ for every given problem instance specified by $\{M, \{D(g_i||f_i), D(f_i||g_i)\}_{i=1}^M\}$.

Proof: From Theorem 3, it is easy to see that for each m, there is only one possible $K = \lceil\tilde{K}_m\rceil$, which is the least integer greater than or equal to $\tilde{K}_m$, that makes I_m $(\Gamma_{\text{DGFi}}) < I_m^*$. Let $j' = \arg\min_j D(f_j||g_j)$. Since there is only one possible $K = \lceil\tilde{K}_{j'}\rceil$ that makes $I_{j'}(\Gamma_{\text{DGFi}}) < I_{j'}^*$, it remains to show that there are only two possible values of $K = \lceil\tilde{K}_m\rceil$ that makes I_m $(\Gamma_{\text{DGFi}}) < I_m^*$ when $m \neq j'$. Let

$$V \triangleq \sum_{j=1}^{M} \frac{D(f_{j'}||g_{j'})}{D(f_j||g_j)}.$$

Since $0 \leq \frac{D(f_{j'}||g_{j'})}{D(f_m||g_m)} \leq 1$, we have

$$\tilde{K}_m = \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)} = V - \frac{D(f_{j'}||g_{j'})}{D(f_m||g_m)} \in [V-1, V]$$

for all $m \neq j'$. This implies that $\lceil\tilde{K}_m\rceil$ $(m \neq j')$ can only take two possible integers as desired. ■

The above corollary also indicates that for $K = 1$, the DGFi policy is always asymptotically optimal. This can be easily seen since Statement (b) always holds for $K = 1$. To find those pathological values of K for which DGFi is not asymptotically optimal, we can compute $\lceil\tilde{K}_m\rceil$ defined in (16) for each $m = 1, 2, \ldots, M$. Since for each m, $\lceil\tilde{K}_m\rceil$ only requires $O(M)$ number of multiplication and summation, the computational complexity of finding those pathological values is $O(M^2)$.

V. Extension to Detecting Multiple Targets

In this section we extend the DGFi policy to the case with $L > 1$ targets. The number of hypotheses in this case is $\binom{M}{L}$. We consider first $K = 1$. The stopping rule and decision rule of DGFi for $L > 1$ are given below, similar in principle to those for $L = 1$ as described in Section III:

$$\tau = \inf\{n : \Delta S_L(n) \geq -\log c\}, \tag{25}$$
$$\delta = \{m^{(1)}(\tau), m^{(2)}(\tau), \ldots, m^{(L)}(\tau)\}, \tag{26}$$

where

$$\Delta S_L(n) \triangleq S_{m^{(L)}(n)}(n) - S_{m^{(L+1)}(n)}(n) \tag{27}$$

denotes the difference between the L^{th} and the $(L+1)^{th}$ largest observed sum LLRs at time n.

For the selection rule, define, for a given set $\mathcal{D} \subset \{1, 2, \ldots, M\}$ with $|\mathcal{D}| = L$,

$$\bar{F}_{\mathcal{D}} \triangleq \frac{1}{\sum_{j\notin\mathcal{D}} \frac{1}{D(f_j||g_j)}}. \tag{28}$$

Similar to $\bar{F}_m$ defined in (13), $F_{\mathcal{D}}$ can be viewed as the asymptotic increasing rate of $\Delta S_L(n)$ when the L targets are given by set $\mathcal{D}$ and we probe the cell with the $(L+1)^{th}$ largest sum LLR. We also define

$$\bar{G}_{\mathcal{D}} \triangleq \frac{1}{\sum_{j\in\mathcal{D}} \frac{1}{D(g_j||f_j)}}, \tag{29}$$

which can be viewed as the asymptotic increasing rate for $\Delta S_L(n)$ when we probe the cell with the L^{th} largest sum LLR.

The selection rule follows the same design principle of maximizing the asymptotic increasing rate of $\Delta S_L(n)$, and is given by

$$\phi(n) = \begin{cases} m^{(L)}(n), & \text{if } \bar{G}_{\mathcal{D}(n)} \geq \bar{F}_{\mathcal{D}(n)} \\ m^{(L+1)}(n), & \text{otherwise,} \end{cases} \tag{30}$$

where

$$\mathcal{D}(n) = \{m^{(1)}(n), m^{(2)}(n), \ldots, m^{(L)}(n)\}. \tag{31}$$

It is not difficult to see that when $L = 1$, the policy reduces to the one described in Section III.

Next, we establish the asymptotic optimality of the DGFi policy for $L > 1$ and $K = 1$. Let $\mathcal{D}$ denote a subset of L cells and $\pi_{\mathcal{D}}$ the prior probability of hypothesis $H_{\mathcal{D}}$ (i.e, the target cells are given by $\mathcal{D}$). Define

$$I_{\mathcal{D}} \triangleq \max\{\bar{F}_{\mathcal{D}}, \bar{G}_{\mathcal{D}}\},$$
$$I_L^* \triangleq \frac{1}{\sum_{\mathcal{D}} \frac{\pi_{\mathcal{D}}}{I_{\mathcal{D}}}}, \tag{32}$$

where I_L^* is again the optimal rate function of the Bayes risk as shown in the theorem below, and reduces to the one defined in (20) when $L = 1$.

Theorem 4: Let $R_L(\Gamma_{\text{DGFi}})$ and $R_L(\Gamma)$ be the Bayes risks under the DGFi policy and an arbitrary policy Γ, respectively. For $K = 1$, we have,

$$R_L(\Gamma_{\text{DGFi}}) \sim \frac{-c\log c}{I_L^*} \sim \inf_{\Gamma} R(\Gamma). \tag{33}$$

Proof: See Appendix D. ■

For $K > 1$, the stopping rule and the decision rule remain the same. For the selection rule, define

$$F_{\mathcal{D}}(\kappa) \triangleq \min\{\kappa\bar{F}_{\mathcal{D}}, \min_{j\notin\mathcal{D}} D(f_j||g_j)\}. \tag{34}$$

Similar to $F_m(\kappa)$ defined in (15), $F_{\mathcal{D}}(\kappa)$ can be viewed as the asymptotic increasing rate of $\Delta S_L(n)$ when the L targets are given by set $\mathcal{D}$ and we probe those κ cells with the $(L+1)^{th}$ to the $(L+\kappa)^{th}$ largest sum LLR. Similarly,

$$G_{\mathcal{D}}(\kappa) \triangleq \min\{\kappa \bar{G}_{\mathcal{D}}, \min_{j\in\mathcal{D}} D(g_j||f_j)\}, \tag{35}$$

which can be viewed as the asymptotic increasing rate of $\Delta S_L(n)$ when we probe the cells with the $(L-\kappa+1)^{th}$ to the L^{th} largest sum LLR.

Let

$$k^*_{\mathcal{D}} \triangleq \arg \max_{k=0,1,\ldots,K} F_{\mathcal{D}}(K-k) + G_{\mathcal{D}}(k), \tag{36}$$

which can be interpreted as the optimal number of target cells that should be probed at each time for maximizing the asymptotic increasing rate of $\Delta S_L(n)$. The selection rule of DGFi is thus given by

$$\phi(n) = \{m^{(L-k^*_{\mathcal{D}(n)}+1)}(n), \ldots, m^{(L-k^*_{\mathcal{D}(n)}+K)}(n)\}, \tag{37}$$

where

$$\mathcal{D}(n) = \{m^{(1)}(n), m^{(2)}(n), \ldots, m^{(L)}(n)\}. \tag{38}$$

The asymptotic optimality of DGFi for $L > 1$ and $K > 1$ remains open. Following the same insight in the single-target case, however, we have strong belief of the following conjecture.

Conjecture 1: The DGFi policy preserves its asymptotic optimality if

$$u^*_{\mathcal{D}} \triangleq \arg \max_{u\in[0,K]} F_{\mathcal{D}}(K-u) + G_{\mathcal{D}}(u) \tag{39}$$

is an integer for all $\mathcal{D}$, where we allow the domain of $F_{\mathcal{D}}(\cdot)$ and $G_{\mathcal{D}}(\cdot)$ to be real numbers.

VI. Comparison With the Chernoff Test

In this section, we compare the performance of the proposed DGFi policy and the Chernoff test in terms of both computational complexity and sample complexity.

A. The Chernoff Test

The Chernoff test has a randomized selection rule. Specifically, let $\mathbf{q}$ be a probability mass function over a set of ω available experiments $\{u_i\}_{i=1}^{\omega}$ that the decision maker can choose from. Note that in our case, $\omega = \binom{M}{K}$. For each hypothesis $m = 1, 2, \ldots, M$, the optimal action distribution is given by

$$\mathbf{q}^*_m = \arg \max_{\mathbf{q}} \min_{j\neq m} \sum_{u_i} q^{u_i} D(p_m^{u_i}||p_j^{u_i}), \tag{40}$$

where $p_j^{u_i}$ is the observation distribution under hypothesis j when action u_i is taken, and $\mathbf{q}^{u_i}$ is the ith element of $\mathbf{q}$ (i.e., the probability of choosing experiment u_i under $\mathbf{q}$). The rationale behind (40) is a zero-sum game formulation of the problem, and the optimal mixed strategy $\mathbf{q}^*_m$ leads to a random observation that best differentiates H_m from its closest alternative.

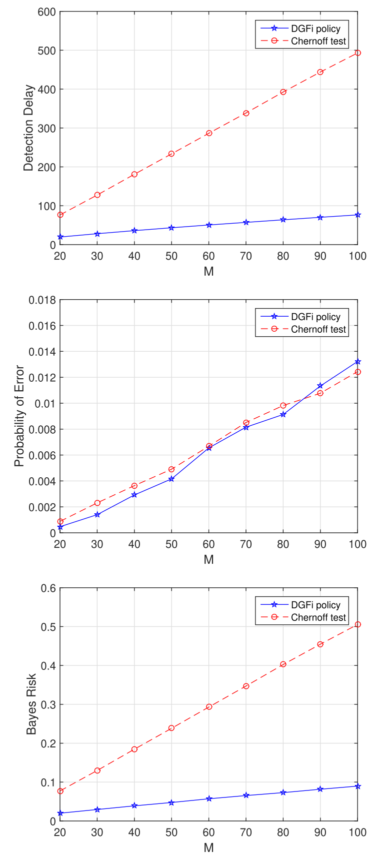

Fig. 3. Performance comparison ($K = 1, \lambda_g^{(m)} = 9 + m, \lambda_f^{(m)} = 0.0188$, $c = 10^{-3}$).

The action at time n under the Chernoff test is drawn from a distribution $\mathbf{q}^*_{\hat{i}(n)}$, where $\hat{i}(n)$ is the ML estimate of the true hypothesis at time n based on past actions and observations. The stopping rule and the decision rule are the same as in (10), (11).

The rate function of the Chernoff test Γ_C under hypothesis H_m is given by

$$I_m(\Gamma_C) = \min_{j\neq m} \sum_{u_i} q_m^{*u_i} D(p_m^{u_i}||p_j^{u_i}), \tag{41}$$

which is the increasing rate of $\Delta S(n)$ under hypothesis H_m when the Chernoff test is employed. The rate function of the Chernoff test under a given prior $\{\pi_m\}_{m=1}^M$ can be similary obtained as in (18).

We point out that in [2], while proving $I_m(\Gamma_C)$ equals the optimal rate I^*_m, Chernoff did not provide an explicit expression for I^*_m or $I_m(\Gamma_C)$. Both were given, as in (41), inexplicitly in terms of the optimizer $\mathbf{q}^*_m$ of the maximin problem in (40). Even for the problem studied here, a special case of that considered by Chernoff,[4] solving for $\mathbf{q}^*_m$ numerically is computationally expensive (see a detailed analysis on computational complexity in the next subsection). The explicit

[4]Note that the asymptotic optimality of the Chernoff test requires the assumption of positive KL diverence between every pair of hypotheses under every experiment. This does not hold for the problem at hand. However, it can be shown that the Chernoff test preserves its asymptotic optimality in this case.

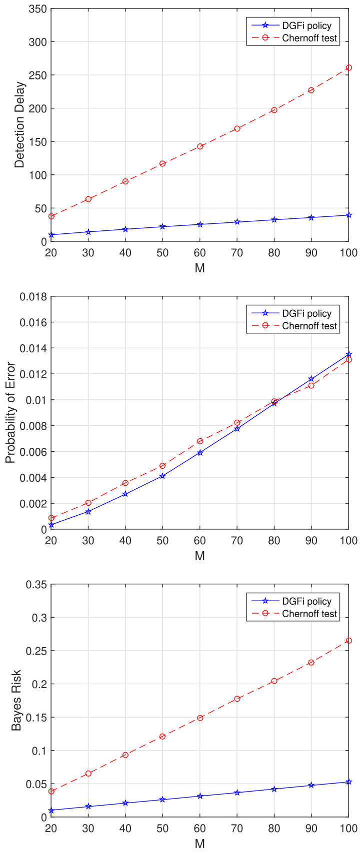

Fig. 4. Performance comparison ($K = 2, \lambda_g^{(m)} = 9 + m, \lambda_f^{(m)} = 0.0188$, $c = 10^{-3}$).

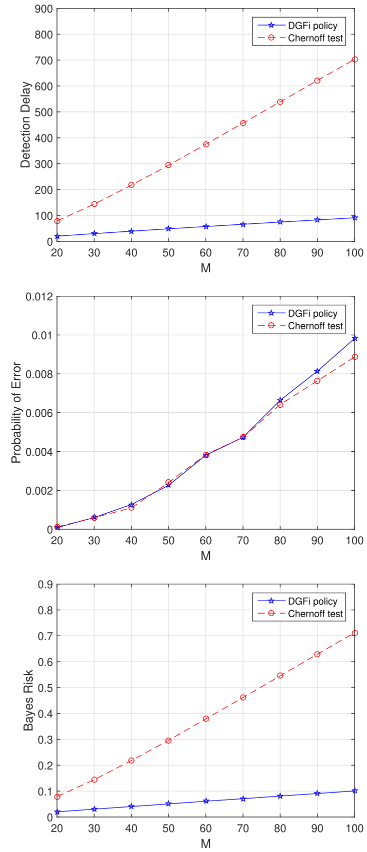

Fig. 5. Performance comparison ($L = 2, K = 1, \lambda_g^{(m)} = 9 + m, \lambda_f^{(m)} = 0.0188, c = 10^{-3}$).

characterization of I_m^* in (20), which equals to $I_m(\Gamma_{\text{DGFi}})$ in (17) under the necessary and sufficient condition given in Theorem 3, is a contribution of this work.

B. Comparison in Computational Complexity

While both the Chernoff test and the DGFi policy are asymptotically optimal, i.e., $I(\Gamma_{\text{DGFi}}) = I(\Gamma_C) = I^*$, they differ drastically in computational complexity. Specifically, the Chernoff test can be expensive to compute especially when the number of hypotheses or the number of experiments is large. Consider the case of a single target ($L = 1$). Computing the selection rule of the Chernoff test given in (40) requires solving M minimax problems, each corresponding to a particular value of the ML estimate $\hat{i}(n) \in \{1, \dots, M\}$. One efficient way of solving minimax problems is through linear programming, which takes polynomial time with respect to the number of variables and constraints. For this problem, the number of variables is $\binom{M}{K}$, which can be exponential in M in the worst case. Calculating the rate function given in (41) requires the optimal selection distribution $\mathbf{q}_m^*$ for all m, thus bears similar computational complexity. For multi-target detection, the number of hypotheses is $\binom{M}{L}$, further increasing the complexity.

The only computation involved in the selection rule of DGFi is (15), which requires M summations each with $M - 1$ elements. As a result, the computational time is $O(M^2)$, which is independent of K. Similarly, the computational complexity for calculating the rate function $I(\Gamma_{\text{DGFi}})$ is $O(M^2)$ as well.

C. Comparison in Sample Complexity

In this subsection, we compare the performance of DGFi with that of the Chernoff test in the finite regime (i.e., when the sample cost c is bounded away from 0).

Consider a uniform prior and exponentially distributed observations: $f_m \sim \exp(\lambda_f^{(m)})$ and $g_m \sim \exp(\lambda_g^{(m)})$. The KL divergences can be easily computed as follows.

$$D(g_m||f_m) = \log(\lambda_g^{(m)}) - \log(\lambda_f^{(m)}) + \frac{\lambda_f^{(m)}}{\lambda_g^{(m)}} - 1,$$
$$D(f_m||g_m) = \log(\lambda_f^{(m)}) - \log(\lambda_g^{(m)}) + \frac{\lambda_g^{(m)}}{\lambda_f^{(m)}} - 1.$$

Shown in Fig. 3 is the performance comparison between DGFi policy and Chernoff test for $L = 1$ and $K = 1$. The figure clearly demonstrates the significant reduction in detection delay and Bayes risk offered by the DGFi policy as compared with the Chernoff test. The performance gain increases drastically as M increases. The probability of errors for Chernoff test and DGFi policy are about the same order as shown. A similar comparison is observed in Fig. 4 with $L = 1, K = 2$. The performance comparison for a case with multiple targets is shown in Fig. 5 with $L = 2, K = 1$.

Next, we provide an intuition argument for the superior finite-time performance of DGFi. Consider a short horizon scenario where the sampling cost c is sufficiently high such that $D(f||g) > -\log c$. This implies that each empty cell can be distinguished from the target with, on the average, a single probing to achieve the required accuracy as determined by c. We can cast this as the coupon collection problem, where each empty cell is a coupon and the goal is to collect all $M-1$ coupons. Consider a special case where $K = 1$ and all f_i and g_i are identical, i.e., $f_i \equiv f$ and $g_i \equiv g$. Assume that $D(f||g) > (M-1)D(g||f)$. In this case, the DGFi policy chooses, at each time, the cell with the second largest sum LLR whereas the Chernoff test randomly and uniformly chooses a cell from all but the one with the largest sum LLR at each time (this can be shown by solving (40)). Since Chernoff test chooses empty cells with equal probability, based on results in coupon collectors problem, the expected probing time will be roughly $M \log M$. The DGFi policy, on the other hand, is deterministic and guaranteed to collect a new coupon at each time. The expected probing time is thus M.

VII. Conclusion

The problem of detecting anomalies among a large number of heterogeneous processes was considered. A low-complexity deterministic test was developed and shown to be asymptotically optimal. Its finite-time performance and computational complexity were shown to be superior to the classic Chernoff test for active hypothesis testing, especially when the problem size is large.

Appendix A
Proof of Theorem 1

For the ease of presentation, we first provide the proof for the case of $K = 1$.

Throughout this section, we use the following notations. Let

$$N_j(n) \triangleq \sum_{t=1}^{n} \mathbf{1}_j(t) \tag{42}$$

be the number of times that cell j has been observed up to time n. Let

$$\Delta S_{m,j}(n) \triangleq S_m(n) - S_j(n) \tag{43}$$

be the difference between the observed sum of LLRs of cells m and j. We also define

$$\Delta S_m(n) \triangleq \min_{j \neq m} \Delta S_{m,j}(n) \ . \tag{44}$$

As a result, we have:

$$\Delta S(n) = S_{m^{(1)}(n)}(n) - S_{m^{(2)}(n)}(n) = \max_m \Delta S_m(n) \ . \tag{45}$$

Without loss of generality we prove the theorem under hypothesis H_m. We define

$$\tilde{\ell}_k(i) = \begin{cases} \ell_k(i) - D(g_k||f_k), & \text{if } k = m, \\ \ell_k(i) + D(f_k||g_k), & \text{if } k \neq m, \end{cases} \tag{46}$$

which is a zero-mean random variable under hypothesis H_m.

We first bound the error probability of DGFi as given below.

Lemma 2: If DGFi policy is used, then the error probability is upper bounded by:

$$P_e \leq (M-1)c. \tag{47}$$

Proof: Let $\alpha_{m,j} = \mathbf{P}_m(\delta = j)$ for all $j \neq m$. Thus, $\alpha_m = \sum_{j \neq m} \alpha_{m,j}$. By the definition of the stopping rule under DGFi (see (10)), accepting H_j is done when $\Delta S_j(n) \geq -\log c$ which implies $\Delta S_{j,m} \geq -\log c$. Hence, for all $j \neq m$ we have:

$$\begin{aligned} \alpha_{m,j} &= \mathbf{P}_m(\delta = j) \\ &\leq \mathbf{P}_m\left(\Delta S_{j,m}(\tau) \geq -\log c\right) \\ &\leq c\mathbf{P}_j\left(\Delta S_{j,m}(\tau) \geq -\log c\right) \leq c, \end{aligned} \tag{48}$$

where changing the measure in the second inequality follows by the fact that $\Delta S_{j,m}(\tau) \geq -\log c$. As a result,

$$\alpha_m = \sum_{j \neq m} \alpha_{m,j} \leq \ (M-1)c$$

and (47) thus follows. ■

Next we show that the expected detection time of DGFi is bounded by $-\log c / I_m \ (\Gamma_{\text{DGFi}})$ under hypothesis H_m. To show this, we partition the detection process into three stages, all defined by certain last passage times. The first stage is defined by the last passage time, denoted by τ_1, that the maximum likelihood estimate is not the true hypothesis H_m. The second stage defined by a last passage time τ_2, indicates that the true hypothesis H_m can be distinguished from at least one false hypothesis with sufficiently high accuracy. The third stage defined by last passage time τ_3, indicates that H_m can be distinguished from all the other $M-1$ hypotheses with sufficient accuracy. The formal definitions of τ_1, τ_2, τ_3 are give below:

$$\begin{aligned} \tau_1 &\triangleq \min\{t : \forall j \neq m, \forall n \geq t, S_m(n) \geq S_j(n)\} \\ \tau_2 &\triangleq \min\{t : \exists j \neq m, \forall n \geq t, S_m(n) - S_j(n) \geq -\log c\} \\ \tau_3 &\triangleq \min\{t : \forall j \neq m, \forall n \geq t, S_m(n) - S_j(n) \geq -\log c\}. \end{aligned} \tag{49}$$

Here, we assume that the selection rule of DGFi policy is implemented indefinitely, which means we probe the cells according to the selection rule of DGFi as given in (14) indefinitely, while the stopping rule is disregarded. Note that τ_1, τ_2, τ_3 are not stopping times since they depend on the future.

Since $\tau \leq \tau_3$ based on the stopping rule of DGFi, it suffices to show τ_3 is bounded by $-\log c / I_m \ (\Gamma_{\text{DGFi}})$ under hypothesis H_m. Let $n_2 = \tau_2 - \tau_1$ and $n_3 = \tau_3 - \tau_2$. In Lemma 4 and Lemma 7, we show that τ_1 and n_3 are sufficiently small with high probability. In Lemma 5 we show that the probability that n_2 is greater than n decays exponentially with n when n is greater than $-\log c / I_m \ (\Gamma_{\text{DGFi}})$. Since $n_3 = \tau_1 + n_2 + n_3$, the expected detection time of DGFi is bounded by $-\log c / I_m \ (\Gamma_{\text{DGFi}})$ under hypothesis H_m as desired.

Lemma 3: There exist constants $C > 0$ and $\gamma > 0$ such that for any fixed $0 < q < 1$, under any arbitrary policy, the following statements hold:

$$\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) \geq qn\right) \leq Ce^{-\gamma n}, \tag{50}$$

and

$$\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_m(n) \geq qn\right) \leq Ce^{-\gamma n}, \tag{51}$$

for $m = 1, 2, \ldots, M$ and $j \neq m$.

Proof: We start with proving (50). Note that $N_j(n)$, $N_m(n)$ can take integer values $N_j(n) = \lceil qn \rceil, \lceil qn \rceil + 1, ...n$, and $N_m(n) = 0, ..., n$. Using the i.i.d. property of the observations across time yield:

$$\begin{aligned}
&\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) \geq qn\right) \\
&\quad \leq \sum_{r=\lceil qn \rceil}^{n} \sum_{k=0}^{n} \mathbf{P}_m\left(\sum_{i=1}^{r} \ell_j(i) + \sum_{i=1}^{k} -\ell_m(i) \geq 0\right) \\
&\quad \leq \sum_{r=\lceil qn \rceil}^{n} \sum_{k=0}^{n} \left[\mathbf{E}_m\left(e^{s\ell_j(1)}\right)\right]^r \left[\mathbf{E}_m\left(e^{s(-\ell_m(1))}\right)\right]^k
\end{aligned} \tag{52}$$

where we have used the following generic Chernoff bound for a random variable X:

$$\mathbf{P}(X \geq a) \leq \frac{\mathbf{E}[e^{\lambda X}]}{e^{\lambda a}}, \tag{53}$$

where it is assume that the moment generating function $\mathbf{E}[e^{\lambda X}]$ exists locally in an interval around $\lambda = 0$. Since the moment generating function is equal to one at $s = 0$ and $\mathbf{E}_m(\ell_j(1)) = -D(f_j||g_j) < 0$, $\mathbf{E}_m(-\ell_m(1)) = -D(g_m||f_m) < 0$ are strictly negative, differentiating the MGFs of $\ell_j(1), \ell_m(1)$ with respect to s yields strictly negative derivatives at $s = 0$. As a result, there exist $s > 0$ and $\gamma_1 > 0$ such that $\mathbf{E}_m\left(e^{s\ell_j(1)}\right)$, $\mathbf{E}_m\left(e^{s(-\ell_m(1))}\right)$ are strictly less than $e^{-\gamma_1} < 1$. Hence, there exist $C > 0$ and $\gamma = \gamma_1 q > 0$ such that

$$\begin{aligned}
&\mathbf{P}_m\left(S_j(n) - S_m(n) \geq 0, N_j(n) \geq qn\right) \\
&\qquad \leq \sum_{r=\lceil qn \rceil}^{n} e^{-\gamma_1 r} \sum_{k=0}^{n} e^{-\gamma_1 k} \leq Ce^{-\gamma n}.
\end{aligned} \tag{54}$$

Note that (51) can be proved with minor modifications. ■

Lemma 4: If the selection rule of DGFi is implemented indefinitely, there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m(\tau_1 > n) \leq Ce^{-\gamma n}, \tag{55}$$

for $m = 1, 2, \ldots, M$.

Proof: We focus on proving for $M > 2$. Proving for $M = 2$ is straightforward. Note that the event $\tau_1 > n$ implies that there exists a time instant t with $t \geq n$ such that $S_j(t) > S_m(t)$ for some $j \neq m$. Hence,

$$\begin{aligned}
\mathbf{P}_m(\tau_1 > n) &\leq \mathbf{P}_m\left(\max_{j\neq m} \sup_{t \geq n} \left(S_j(t) - S_m(t)\right) \geq 0\right) \\
&\leq \sum_{j \neq m} \sum_{t=n}^{\infty} \mathbf{P}_m\left(S_j(t) \geq S_m(t)\right).
\end{aligned} \tag{56}$$

Following (56), it suffices to show that there exist $C > 0$ and $\gamma > 0$ such that $\mathbf{P}_m\left(S_j(n) \geq S_m(n)\right) \leq Ce^{-\gamma n}$.

We next establish the required exponential decay. Let

$$\begin{aligned}
k_m &= \frac{\max_{j\neq m} D(f_j||g_j)}{\min_{j \neq m} D(f_j||g_j)}, \\
\underline{j_m} &= \arg\min_{j \neq m} D(f_j||g_j), \\
\rho_m &= \frac{1}{8(k_m+1)(M-2)}.
\end{aligned} \tag{57}$$

Note that $0 < \rho_m \leq 1/16$. Thus, we can write

$$\begin{aligned}
&\mathbf{P}_m\left(S_j(n) \geq S_m(n)\right) \\
&\quad \leq \mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) < \rho_m n, N_m(n) < \rho_m n\right) \\
&\qquad + \mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) \geq \rho_m n\right) \\
&\qquad + \mathbf{P}_m\left(S_j(n) \geq S_m(n), N_m(n) \geq \rho_m n\right).
\end{aligned} \tag{58}$$

The second and the third terms on the RHS of (58) decay exponentially with n by Lemma 3. Thus, it remains to show that the first term decays exponentially with n as well. Note that the event $(N_j(n) < \rho_m n, N_m(n) < \rho_m n)$ implies that at least $\tilde{n} = n - N_j(n) - N_m(n) \geq n(1 - 2\rho_m)$ times cells j, m are not probed. We define $\widetilde{N}_r(n)$ as the number of times in which cell $r \neq j, m$ has been probed and cells j, m have not been probed by time n. There exists a cell $r \neq j, m$ such that $\widetilde{N}_r(n) \geq \frac{\tilde{n}}{M-2} = \frac{n(1-2\rho_m)}{M-2}$. Hence, we can upper bound (58) as follows:

$$\begin{aligned}
&\mathbf{P}_m\left(S_j(n) \geq S_m(n)\right) \\
&\quad \leq \sum_{r \neq j,m} \mathbf{P}_m\Bigg(\tilde{N}_r(n) > \frac{n(1-2\rho_m)}{M-2}, \\
&\qquad\qquad N_j(n) < \rho_m n, N_m(n) < \rho_m n\Bigg) + 2De^{-\gamma_1 n},
\end{aligned} \tag{59}$$

where the second and third terms on the RHS of (58) are upper bounded by $De^{-\gamma_1 n}$ (there exist such $D > 0, \gamma_1 > 0$ by Lemma 3), and the first term on the RHS of (58) is upper bounded by the first term (i.e., the summation term) on the RHS of (59). Next, we show that each term in the summation decays exponentially with n to get the desired result.

Let $\tilde{t}^r_1, \tilde{t}^r_2, \ldots, \tilde{t}^r_{\tilde{N}_r(n)}$ be the indices for the time instants in which cell $r \neq j, m$ has been probed and cells j, m have not been probed by time n. Let

$$\zeta \triangleq \frac{1 - 2\rho_m}{2(M-2)}. \tag{60}$$

Note that the event $S_j(\tilde{t}^r_{\zeta n}) \leq S_r(\tilde{t}^r_{\zeta n})$ or $S_m(\tilde{t}^r_{\zeta n}) \leq S_r(\tilde{t}^r_{\zeta n})$ must occur (otherwise, cell j or m will be probed). Hence,[5]

$$\begin{aligned}
&\mathbf{P}_m\Bigg(\tilde{N}_r(n) > \frac{n(1-2\rho_m)}{M-2}, \\
&\qquad N_j(n) < \rho_m n, N_m(n) < \rho_m n\Bigg) \\
&\quad = \sum_{q=0}^{n-\zeta n} \sum_{n'=0}^{\rho_m n} \mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{\zeta n + q} \ell_r(i)\right) \\
&\qquad + \sum_{q=0}^{n-\zeta n} \sum_{n'=0}^{\rho_m n} \mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_m(i) \leq \sum_{i=1}^{\zeta n + q} \ell_r(i)\right).
\end{aligned} \tag{61}$$

[5]For the ease of presentation, throughout the proof we assume that ζn, $\rho_m n$ are integers. This assumption does not affect the exponential decay but only the exact value of $C > 0$ in (55) (since $\alpha n - 1 \leq \lfloor \alpha n \rfloor \leq \lceil \alpha n \rceil \leq \alpha n + 1$ holds for all $\alpha \geq 0$ for all $n = 0, 1, \ldots$).

For upper bounding the first term on the RHS of (61) we write the sum LLRs as follows:

$$\begin{aligned}\sum_{i=1}^{\zeta n+q} \ell_r(i) + \sum_{i=1}^{n'} -\ell_j(i) &= \sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} \tilde{\ell}_j(i) \\ &\quad -D(f_r||g_r)\,(\zeta n+q) + D(f_{n'}||g_{n'})n' \\ &\le \sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i) - D(f_{\underline{j_m}}||g_{\underline{j_m}})\left(\zeta n+q-k_m n'\right),\end{aligned} \tag{62}$$

and by the definitions of ζ, k_m, ρ_m in (57) and (60), we have

$$\begin{aligned}\zeta n+q-k_m n' &\ge \zeta n+q-k_m n' - (k_m+1)\left(\rho_m n - n'\right) \\ &= n\left(\zeta - (k_m+1)\rho_m\right) + q + n' \ge \frac{1}{4(M-2)} n + q + n' \\ &\ge \frac{1}{4(M-2)}(n+q+n'),\end{aligned}$$

for all $n' \le \rho_m n$. Therefore,

$$\sum_{i=1}^{\zeta n+q} \ell_r(i) + \sum_{i=1}^{n'} -\ell_j(i) \ge 0 \tag{63}$$

implies

$$\sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i) \ge C_1\left(n+q+n'\right), \tag{64}$$

where

$$C_1 = \frac{D(f_{\underline{j_m}}||g_{\underline{j_m}})}{4(M-2)} > 0. \tag{65}$$

Then we have

$$\begin{aligned}&\mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \le \sum_{i=1}^{\zeta n+q} \ell_r(i)\right) \\ &\le \mathbf{P}_m\left(\sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i) \ge C_1\left(n+q+n'\right)\right) \\ &\le \left[\mathbf{E}_m\left(e^{s\tilde{\ell}_r(1)}\right)\right]^{\zeta n+q} \left[\mathbf{E}_m\left(e^{s(-\tilde{\ell}_j(1))}\right)\right]^{n'} \\ &\quad \times e^{-sC_1\left(n+q+n'\right)} \\ &= \left[\mathbf{E}_m\left(e^{s\left(\tilde{\ell}_r(1)-C_1\right)}\right)\right]^{\zeta n+q} \left[\mathbf{E}_m\left(e^{s\left(-\tilde{\ell}_j(1)-C_1\right)}\right)\right]^{n'} \\ &\quad \times e^{-sC_1(n-\zeta n)}.\end{aligned} \tag{66}$$

for all $s > 0$.

Since $\mathbf{E}_m(\tilde{\ell}_r(1) - C_1) = -C_1 < 0$ and $\mathbf{E}_m(-\tilde{\ell}_j(1) - C_1) = -C_1 < 0$ are strictly negative, by applying a similar argument as at the end of the proof of Lemma 3, there exist $s > 0$ and $\gamma_2 > 0$ such that $\mathbf{E}_m\left(e^{(s\tilde{\ell}_r(1)-C_1)}\right)$, $\mathbf{E}_m\left(e^{s(-\tilde{\ell}_j(1)-C_1)}\right)$ and e^{-sC_1} are strictly less than $e^{-\gamma_2} < 1$. Hence,

$$\mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \le \sum_{i=1}^{\zeta n+q} \ell_r(i)\right) \le e^{-\gamma_2\left(n+q+n'\right)}, \tag{67}$$

and

$$\begin{aligned}&\sum_{q=0}^{n-\zeta n} \sum_{n'=0}^{\rho_m n} \mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \le \sum_{i=1}^{\zeta n+q} \ell_r(i)\right) \\ &\qquad \le e^{-\gamma_2 n} \sum_{q=0}^{n-\zeta n} e^{-\gamma_2 q} \sum_{n'=0}^{\rho_m n} e^{-\gamma_2 n'} \le C_2 e^{-\gamma_2 n},\end{aligned} \tag{68}$$

where $C_2 = \left(1 - e^{-\gamma_2}\right)^{-2}$.

A similar technique can be applied to upper bound the second term on the RHS of (61). ■

Lemma 5: If the selection rule of DGFi is implemented indefinitely, then for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m\left(n_2 > n\right) \le Ce^{-\gamma n} \quad \forall n > -(1+\epsilon)\log c / I_m(\Gamma_{\text{DGFi}}), \tag{69}$$

for all $m = 1, 2, \ldots, M$.

Proof: First, we consider the case where $I_m\ (\Gamma_{\text{DGFi}}) > D(g_m||f_m)$. Note that cell m is not observed for all $n \ge \tau_1$ in this case. Define $N'_j(\tau_1+t) = \sum_{i=\tau_1+1}^{\tau_1+t} 1_j(i)$ and $j^*(\tau_1+t) = \arg\max_j N'_j(\tau_1+t) D(f_j||g_j)$. Thus,

$$\begin{aligned}&\mathbf{P}_m(n_2 > n) \\ &\quad \le \mathbf{P}_m\left(\sup_{t\ge n} \sum_{i=\tau_1+1}^{\tau_1+t} \ell_{j^*(\tau_1+t)}(i) 1_{j^*(\tau_1+t)}(i) \ge \log c\right).\end{aligned} \tag{70}$$

Since t is the total number of observation from τ_1 to $\tau_1 + t$, by the definition of $j^*(t)$ we have

$$\begin{aligned}t &= \sum_{j\ne m} N'_j(\tau_1+t) = \sum_{j\ne m} \frac{N'_j(\tau_1+t) D(f_j||g_j)}{D(f_j||g_j)} \\ &\le \sum_{j\ne m} \frac{N'_{j^*(\tau_1+t)}(\tau_1+t) D(f_{j^*(\tau_1+t)}||g_{j^*(\tau_1+t)})}{D(f_j||g_j)}.\end{aligned} \tag{71}$$

Let $\epsilon_1 = I_m\ (\Gamma_{\text{DGFi}})\epsilon/(1+\epsilon)$. Since $I_m\ (\Gamma_{\text{DGFi}}) = \sum_{j\ne m} 1/D(f_j||g_j)$, we have

$$\epsilon_1 = \frac{\epsilon}{(1+\epsilon)\sum_{j\ne m} 1/D(f_j||g_j)}. \tag{72}$$

Then,

$$\begin{aligned}&\sum_{i=\tau_1+1}^{\tau_1+t} \ell_{j^*(\tau_1+t)}(i) 1_{j^*(\tau_1+t)}(i) - \log c \\ &\quad = \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i) 1_{j^*(\tau_1+t)}(i) \\ &\qquad - N'_{j^*(\tau_1+t)}(\tau_1+t) D(f_{j^*(\tau_1+t)}||g_{j^*(\tau_1+t)}) - \log c \\ &\quad \le \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i) 1_{j^*(\tau_1+t)}(i) \\ &\qquad - \frac{t}{\sum_{j\ne m} 1/D(f_j||g_j)} - \log c \\ &\quad \le \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i) 1_{j^*(\tau_1+t)}(i) - t I_m(\Gamma_{\text{DGFi}}) \\ &\qquad + t I_m(\Gamma_{\text{DGFi}})/(1+\epsilon) \\ &\quad \le \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i) 1_{j^*(\tau_1+t)}(i) - t\epsilon_1\end{aligned} \tag{73}$$

for all $t \geq n > -(1+\epsilon)\log c/I_m$ (Γ_{DGFi}). By applying the generic Chernoff bound given in (53), it can be shown that there exists $\gamma_1 > 0$ such that $\mathbf{P}_m(\sum_{\tau_1+1}^{\tau_1+t} -\tilde{\ell}_{j^*(\tau_1+t)}(i) \geq t\epsilon_1) < e^{-\gamma_1 t}$ for all $t \geq n > -(1+\epsilon)\log c/I_m$ (Γ_{DGFi}). Hence, there exist $C_1 > 0$ and $\gamma_1 > 0$ such that $\mathbf{P}_m(n_2 > n) \leq C_1 e^{-\gamma_1 n}$ for all $n > -(1+\epsilon)\log c/I_m$ (Γ_{DGFi}). A similar argument applies for case where I_m $(\Gamma_{\text{DGFi}}) \leq D(g_m||f_m)$. ■

To show that n_3 is sufficiently small, we define a random variable $\Psi(t)$ as the dynamic range between sum LLRs of empty cells:

$$\Psi(t) \triangleq \max_{j\neq m} S_j(t) - \min_{j\neq m} S_j(t). \tag{74}$$

Note that the dynamic range at time τ_2 can be viewed as a measure of the amount of information remains to gather in order to distinguish H_m from any other false hypothesis. Lemma 6 below shows that the dynamic range at time τ_2 is sufficiently small.

Lemma 6: If the selection rule of DGFi is implemented indefinitely. Then, for every fixed $\epsilon_1 > 0, \epsilon_2 > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n) \leq Ce^{-\gamma n}, \quad \forall n > -(1+\epsilon_2)\log c/I_m(\Gamma_{\text{DGFi}}) \tag{75}$$

for all $m = 1, 2, \ldots, M$.

Proof: Note that

$$\mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n) \leq \mathbf{P}_m(\tau_2 > n) + \mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n) \tag{76}$$

Since $\tau_2 = \tau_1 + n_2$, applying Lemmas 4, 5 implies that the first term on the RHS of (76) decreases exponentially with n for all $n > -(1+\epsilon_2)\log c/I_m$ (Γ_{DGFi}) for every fixed $\epsilon_2 > 0$. It remains to show that the second term on the RHS of (76) decreases exponentially with n. Let $\bar{j} = \arg\max_{j\neq m} S_j(\tau_2)$, $\underline{j} = \arg\min_{j\neq m} S_j(\tau_2)$. Let t_0 be the smallest integer such that $S_{\underline{j}}(t) \leq S_{\bar{j}}(t)$ for all $t_0 < t \leq \tau_2$. As a result, $\Psi(\tau_2) > \epsilon_1 n$ implies

$$\sum_{t=t_0}^{\tau_2} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \sum_{t=t_0}^{\tau_2} \ell_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t) > \epsilon_1 n.$$

Note that the second term on the RHS of (76) can be rewritten as:

$$\begin{aligned}&\mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n)\\ &\quad = \mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n, t_0 \geq \tau_1)\\ &\qquad + \mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n, t_0 < \tau_1)\end{aligned} \tag{77}$$

First, we upper bound the first term on the RHS of (77). Note that for all $\tau_1 \leq t_0 < t \leq \tau_2$, we have $\mathbf{1}_{\underline{j}}(t) = 0$. Hence,

$$\begin{aligned}&\sum_{t=t_0}^{\tau_2} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \sum_{t=t_0}^{\tau_2} \ell_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t) = \sum_{t=t_0}^{\tau_2} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t)\\ &\quad = \sum_{t=t_0}^{\tau_2} \tilde{\ell}_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - D(f_{\bar{j}}||g_{\bar{j}}) \leq \sum_{t=t_0}^{\tau_2} \tilde{\ell}_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t)\end{aligned} \tag{78}$$

Then, applying the generic Chernoff bound given in (53) completes the proof for this case.

Next, we upper bound the second term on the RHS of (77). Let $\epsilon_3 \triangleq \frac{\epsilon_1}{4\max_j D(f_j||g_j)} > 0$. Note that

$$\begin{aligned}&\mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n, t_0 < \tau_1)\\ &\quad \leq \mathbf{P}_m(\tau_1 > \epsilon_3 n)\\ &\qquad + \mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n, t_0 < \tau_1, \tau_1 \leq \epsilon_3 n).\end{aligned} \tag{79}$$

The first term on the RHS of (79) decreases exponentially with n by Lemma 4. Thus, it remains to show that the second term on the RHS of (79) decreases exponentially with n. Note that $\Psi(\tau_2) > \epsilon_1 n$ implies $\sum_{t=t_0}^{\tau_1} \ell_{\bar{j}}\mathbf{1}_{\bar{j}}(t) + \sum_{t=\tau_1+1}^{\tau_2} \ell_{\bar{j}}\mathbf{1}_{\bar{j}}(t) > \epsilon_1 n$. Therefore, the second term on the RHS of (79) can be rewritten as:

$$\begin{aligned}&\mathbf{P}_m(\Psi(\tau_2) > \epsilon_1 n, \tau_2 \leq n, t_0 < \tau_1, \tau_1 \leq \epsilon_3 n)\\ &\leq \mathbf{P}_m\left(\sum_{t=t_0}^{\tau_1} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) > \frac{\epsilon_1 n}{2}, \tau_2 \leq n, t_0 < \tau_1, \tau_1 \leq \epsilon_3 n\right)\\ &\quad + \mathbf{P}_m\left(\sum_{t=\tau_1+1}^{\tau_2} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) > \frac{\epsilon_1 n}{2}, \tau_2 \leq n, t_0 < \tau_1, \tau_1 \leq \epsilon_3 n\right)\end{aligned} \tag{80}$$

The second term on the RHS of (80) decreases exponentially with n using a similar argument as in (78). Next, it remains to show that the first term on the RHS of (80) decreases exponentially with n. Note that

$$\begin{aligned}&\sum_{t=t_0}^{\tau_1} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \sum_{t=t_0}^{\tau_1} \ell_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t)\\ &\quad \leq \sum_{t=t_0}^{\tau_1} \tilde{\ell}_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \sum_{t=t_0}^{\tau_1} \tilde{\ell}_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t) + \max_j D(f_j||g_j)\tau_1\\ &\quad \leq \sum_{t=t_0}^{\tau_1} \left[\tilde{\ell}_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \tilde{\ell}_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t)\right] + \frac{\epsilon_1}{4}n\end{aligned} \tag{81}$$

for all $\tau_1 \leq \epsilon_3 n$.

As a result,

$$\sum_{t=t_0}^{\tau_1} \ell_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \ell_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t) > \frac{\epsilon_1}{2}n \tag{82}$$

implies

$$\sum_{t=t_0}^{\tau_1} \left[\tilde{\ell}_{\bar{j}}(t)\mathbf{1}_{\bar{j}}(t) - \tilde{\ell}_{\underline{j}}(t)\mathbf{1}_{\underline{j}}(t)\right] > \frac{\epsilon_1}{4}n \tag{83}$$

for all $\tau_1 \leq \epsilon_3 n$. Applying the generic Chernoff bound given in (53), we arrive at the lemma. ■

Lemma 7: If the selection rule of DGFi is implemented indefinitely, then for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m(n_3 > n) \leq Ce^{-\gamma n} \quad \forall n > -\epsilon\log c/I_m(\Gamma_{\text{DGFi}}), \tag{84}$$

for all $m = 1, 2, \ldots, M$.

Proof: To prove the Lemma, we first define $\tau_3^j \triangleq \max\{t : \forall n \geq t, S_m(n) - S_j(n) \geq -\log c\}$ and N_3^j as the total number of observations that the decision maker collected from cell j between τ_2 and τ_3^j. Since $n_3 \leq \sum_j N_3^j$ and $\tau_3 = \max_j \tau_3^j$, we only need to show that $\mathbf{P}_m(N_3^j > n)$ decays exponentially with n. We can write $\mathbf{P}_m(N_3^j > n)$ as follows:

$$\mathbf{P}_m(N_3^j > n) \leq \mathbf{P}_m\left(\Psi(\tau_2) > n\frac{\min_j D(f_j||g_j)}{2}\right) + \mathbf{P}_m\left(N_3^j > n|\Psi(\tau_2) \leq n\frac{\min_j D(f_j||g_j)}{2}\right) \tag{85}$$

Lemma 6 provides the desired decay for the first term on the RHS. We next show the desired decay for the second term. Let $t_1, t_2, \ldots$ denote the time indices when cell j is observed between τ_2 and τ_3^j. We can write:

$$\begin{aligned}\mathbf{P}_m&\left(N_3^j > n|\Psi(\tau_2) \leq n\frac{\min_j D(f_j||g_j)}{2}\right)\\ &\leq \mathbf{P}_m\left(\inf_{r>n}\sum_{i=1}^{r} -\ell_j(t_i) < n\frac{\min_j D(f_j||g_j)}{2}\right)\\ &\leq \mathbf{P}_m\left(\sum_{i=1}^{r}\tilde{\ell}_j(t_i) > r\frac{\min_j D(f_j||g_j)}{2}\right).\end{aligned} \tag{86}$$

Using the i.i.d. property of $\tilde{\ell}_j(t_i)$ yields:

$$\mathbf{P}_m\left(\sum_{i=1}^{n}\tilde{\ell}_j(t_i) > n\frac{\min_j D(f_j||g_j)}{2}\right) < C_3 e^{-\gamma n} \tag{87}$$

for some C_3, γ_3 which completes the proof. ■

The following Lemma provides an upper bound on the detection time when DGFi policy is implemented.

Lemma 8: If DGFi policy is implemented, then the expected detection time τ is upper bounded by:

$$\mathbf{E}_m(\tau) \leq -(1+o(1))\frac{\log(c)}{I_m(\Gamma_{\text{DGFi}})}, \tag{88}$$

for $m = 1, \ldots, M$.

Proof: Since the actual detection time under DGFi is upper bounded by: $\tau \leq \tau_3 = \tau_1 + n_2 + n_3$, combining Lemmas 4, 5 and 7 proves the statement. ■

Combining Lemma 2 and Lemma 8, Theorem 1 follows for the case of $K = 1$.

The proof for $K > 1$ follows with similar structure except for Lemma 6, which involves sum LLR analysis of the heterogeneous empty cells with balanced case and unbalanced case as described below. The detailed proof for $K > 1$ can be found in [31].

For the balanced case, the key to bounding the detection time in this case is to show that the dynamic range of the $M-1$ sum LLRs corresponding to the $M-1$ empty cells are sufficiently small such that the increasing rate of $\Delta S(n)$ is given by a certain averaging among the heterogeneous processes.

For the unbalanced case, there is a process with a sufficiently small information acquisition rate $D(f_j||g_j)$ such that it becomes the bottleneck of the detection process and determines the asymptotic increasing rate of $\Delta S(n)$. Directly bounding the dynamic range of all sum LLR trajectories is no longer tractable. Instead, the proof is built upon the analysis of the trajectory of the sum LLR with the smallest expected increment. In particular, we recognize that the key in handling the imbalance in the information acquisition rates among empty cells is to define a last passage time as the last time at which the empty cell with the smallest $D(f_j||g_j)$ is not probed and then analyze, separately, the detection process before and after this last passage time.

Appendix B
Proof of Theorem 2

First we show that in order to achieve a small order of Bayes Risk, $\Delta S_m(\tau)$ defined in (44) need to be sufficient large.

Lemma 9: Assume that $\alpha_j(\Gamma) = O(-c\log c)$ for all $j = 1, \ldots, M$. Let $0 < \epsilon < 1$. Then:

$$\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c \mid \Gamma\right) = O(-c^{\epsilon}\log c), \tag{89}$$

for all $m = 1, \ldots, M$.

Proof: Note that:

$$\begin{aligned}&\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c|\Gamma\right)\\ &= \mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right)\\ &\quad + \mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\,,\ \delta \neq m|\Gamma\right)\\ &\leq \mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right) + \alpha_m(\Gamma),\end{aligned} \tag{90}$$

where $\alpha_m(\Gamma) = O(-c\log c)$ by assumption. In what follows, we upper bound

$$\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right).$$

Similar to [2, Lemma 4] we can show that for all $j \neq m$ there exists $G > 0$ such that:

$$\begin{aligned}-Gc\log c &\geq \mathbf{P}_j\left(\delta \neq j|\Gamma\right) \geq \mathbf{P}_j\left(\delta = m|\Gamma\right)\\ &\geq \mathbf{P}_j\left(\Delta S_{m,j}(\tau) \leq -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right)\\ &\geq c^{1-\epsilon}\mathbf{P}_m\left(\Delta S_{m,j}(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right),\end{aligned} \tag{91}$$

where the last inequality holds by changing the measure as in [2, Lemma 4]. Thus,

$$\mathbf{P}_m\left(\Delta S_{m,j}(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right) = O\left(-c^{\epsilon}\log c\right) \quad \forall j \neq m. \tag{92}$$

As a result,

$$\begin{aligned}&\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right)\\ &\leq \sum_{j\neq m}\mathbf{P}_m\left(\Delta S_{m,j}(\tau) < -(1-\epsilon)\log c\,,\ \delta = m|\Gamma\right)\\ &= O\left(-c^{\epsilon}\log c\right).\end{aligned} \tag{93}$$

Finally,

$$\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c|\Gamma\right) = O\left(-c^{\epsilon}\log c\right). \tag{94}$$

■

Lemma 10: Assume that

$$D(g_m||f_m) \geq \frac{1}{\sum_{j\neq m}\frac{1}{D(f_j||g_j)}}. \tag{95}$$

Then, the function:

$$d(t) \triangleq t\left[D(g_m||f_m) + \frac{\frac{n}{t}-1}{\sum_{j\neq m}\frac{1}{D(f_j||g_j)}}\right] \quad (96)$$

is monotonically increasing with t for $0 \leq t \leq n$.

Proof: Differentiation $d(t)$ with respect to t yields:

$$\frac{\partial d(t)}{\partial t} = D(g_m||f_m) - \frac{1}{\sum_{j\neq m}\frac{1}{D(f_j||g_j)}} \geq 0,$$

which completes the proof. ■

For the next lemma we define

$$j^*(t) \triangleq \arg\min_{j\neq m} N_j(t)D(f_j||g_j), \quad (97)$$

and

$$W_m^*(t) \triangleq \sum_{i=1}^{t}\tilde{\ell}_m(i)\mathbf{1}_m(i) - \sum_{i=1}^{t}\tilde{\ell}_{j^*(t)}(i)\mathbf{1}_{j^*(t)}(i), \quad (98)$$

which is a sum of zero-mean random variable

Lemma 11: For every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m\left(\max_{1\leq t\leq n} W_m^*(t) \geq n\epsilon|\Gamma\right) \leq Ce^{-\gamma n} \quad (99)$$

for all $m = 1, \ldots, M$ and for any policy Γ.

Proof: We upper bound (99) by summing over any possible values that $N_m(t), N_{j^*(t)}(t)$ can take and using the generic Chernoff bound given in (53):

$$\begin{aligned}
&\mathbf{P}_m\left(\max_{1\leq t\leq n} W_m^*(t) \geq n\epsilon|\Gamma\right)\\
&= \sum_{t=1}^{n}\sum_{i=0}^{t}\sum_{j=0}^{t}\mathbf{P}_m\Bigg(\sum_{r=1}^{t}\tilde{\ell}_m(r)\mathbf{1}_m(r)\\
&\quad + \sum_{r=1}^{t}-\tilde{\ell}_{j^*(t)}(r)\mathbf{1}_{j^*(t)}(r) \geq n\epsilon, N_m(t)=i, N_{j^*(t)}=j|\Gamma\Bigg)\\
&\leq \sum_{t=1}^{n}\sum_{i=0}^{t}\sum_{j=0}^{t}\left[\mathbf{E}_m\left(e^{s(\tilde{\ell}_m(1)-\epsilon/2)}\right)\right]^i\\
&\quad \times\left[\mathbf{E}_m\left(e^{s(-\tilde{\ell}_{j^*(t)}(1)-\epsilon/2)}\right)\right]^j \times \exp\left\{-s\frac{\epsilon}{2}(2n-i-j)\right\},
\end{aligned} \quad (100)$$

for all $s > 0$.

Since $\mathbf{E}_m(\tilde{\ell}_m(1)-\epsilon/2) = -\epsilon/2 < 0$ and $\mathbf{E}_m(-\tilde{\ell}_{j^*(t)}(1)-\epsilon/2) = -\epsilon/2 < 0$ are strictly negative, using a similar argument as at the end of the proof of Lemma 3, there exist $s > 0$ and $\gamma' > 0$ such that $\mathbf{E}_m\left(e^{s(\tilde{\ell}_m(1)-\epsilon/2)}\right)$, $\mathbf{E}_m\left(e^{s(-\tilde{\ell}_{j^*(t)}(1)-\epsilon/2)}\right)$ and $e^{-s\epsilon/2}$ are strictly less than $e^{-\gamma'} < 1$. Since $2n-i-j \geq 0$, there exist $C > 0$ and $\gamma > 0$, such that summing over t, i, j yields (99). ■

Lemma 12: For any fixed $\epsilon > 0$,

$$\mathbf{P}_m\left(\max_{1\leq t\leq n}\Delta S_m(t) \geq n\left(I_m^*+\epsilon\right) \mid \Gamma\right) \to 0 \text{ as } n\to\infty, \quad (101)$$

for all $m = 1, \ldots, M$ and for any policy Γ.

Proof: We next show exponential decay of (101) (which is stronger than the polynomial decay shown under the binary composite hypothesis testing case in [2, Lemma 5]). Let

$$\Delta S_m^*(t) \triangleq S_m(t) - S_{j^*(t)}(t).$$

Note that $\Delta S_m(t) \leq \Delta S_m^*(t)$ for all m and t. As a result,

$$\begin{aligned}
&\mathbf{P}_m\left(\max_{1\leq t\leq n}\Delta S_m(t) \geq n\left(I_m^*+\epsilon\right)|\Gamma\right)\\
&\qquad \leq \mathbf{P}_m\left(\max_{1\leq t\leq n}\Delta S_m^*(t) \geq n\left(I_m^*+\epsilon\right)|\Gamma\right).
\end{aligned} \quad (102)$$

We next prove the lemma for the case where $I_m^* = F_m(K)$ and $u_m^* = 0$. Proving the lemma for the cases where $u_m^* > 0$ applies with minor modifications.

Note that:

$$\begin{aligned}
\Delta S_m^*(t) &= W_m^*(t) + N_m(t)D(g_m||f_m)\\
&\quad + N_{j^*(t)}(t)D(f_{j^*(t)}||g_{j^*(t)})\\
&\leq W_m^*(t) + N_m(t)\cdot\frac{1}{\sum_{j\neq m}1/D(f_j||g_j)}\\
&\quad + N_{j^*(t)}(t)D(f_{j^*(t)}||g_{j^*(t)}).
\end{aligned} \quad (103)$$

Since that $j^*(t) = \arg\min_{j\neq m} N_j(t)D(f_j||g_j)$ and $Kt - N_m(t)$ is the total number of observations taken from $M-1$ cells $j \neq m$, we have:

$$\sum_{j\neq m}\frac{N_{j^*(t)}D(f_{j^*(t)}||g_{j^*(t)})}{D(f_j||g_j)} \leq Kt - N_m(t) \leq Kn - N_m(t). \quad (104)$$

Hence,

$$\begin{aligned}
\Delta S_m^*(t) &\leq W_m^*(t) + Kn\frac{1}{\sum_{j\neq m}1/D(f_j||g_j)}\\
&= W_m^*(t) + nI_m^*.
\end{aligned} \quad (105)$$

Therefore,

$$\Delta S_m^*(t) \geq n\left(I_m^*+\epsilon\right)$$

implies

$$W_m^*(t) \geq n\epsilon.$$

By Lemma 11 we have:

$$\begin{aligned}
&\mathbf{P}_m\left(\max_{1\leq t\leq n}\Delta S_m(t) \geq n\left(I_m^*+\epsilon\right)\right)\\
&\leq \mathbf{P}_m\left(\max_{1\leq t\leq n} W_m^*(t) \geq n\epsilon\right)\\
&\leq Ce^{-\gamma n} \to 0 \text{ as } n\to\infty.
\end{aligned} \quad (106)$$

■

Finally, we show that the Bayes risk cannot be made smaller than $\frac{-c\log(c)}{I_m^*}$:

Lemma 13: Any policy Γ that satisfies $R_j(\Gamma) = O(-c\log c)$ for all $j = 1, \ldots, M$ must satisfy:

$$R_m(\Gamma) \geq -(1+o(1))\frac{c\log(c)}{I_m^*}. \quad (107)$$

for all $m = 1, \ldots, M$.

Proof: For any $\epsilon > 0$ let $n_c = -(1-\epsilon)\frac{\log c}{I_m^* + \epsilon}$. Note that

$$\begin{aligned}
&\mathbf{P}_m\left(\tau \le n_c \mid \Gamma\right) \\
&\quad = \mathbf{P}_m\left(\tau \le n_c\ ,\ \Delta S_m(\tau) \ge -(1-\epsilon)\log c \mid \Gamma\right) \\
&\qquad + \mathbf{P}_m\left(\tau \le n_c\ ,\ \Delta S_m(\tau) < -(1-\epsilon)\log c \mid \Gamma\right) \\
&\quad \le \mathbf{P}_m\left(\max_{t \le n_c} \Delta S_m(t) \ge -(1-\epsilon)\log c \mid \Gamma\right) \\
&\qquad + \mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c \mid \Gamma\right). \qquad (108)
\end{aligned}$$

Both terms on the RHS approaches zero as $c \to 0$ by Lemmas 9, 12. Hence,

$$\begin{aligned}
\mathbf{E}_m(\tau|\Gamma) &\ge \sum_{n=n_c+1}^{\infty} n\mathbf{P}_m\left(\tau = n|\Gamma\right) \\
&\ge n_c \mathbf{P}_m\left(\tau \ge n_c + 1|\Gamma\right) \to n_c \quad \text{as} \quad c \to 0 \qquad (109)
\end{aligned}$$

Since $\epsilon > 0$ is arbitrarily small we have $\mathbf{E}_m(\tau|\Gamma) \ge -(1+o(1))\log(c)/I_m^*$. As a result, $R_m(\Gamma) \ge c\mathbf{E}_m(\tau|\Gamma) \ge -(1+o(1))\,c\log(c)/I_m^*$. ■

Appendix C
Proof of Lemma 1

Define

$$h_m(u) = uD(g_m||f_m) + F_m(K-u). \qquad (110)$$

By taking the derivative of $h_m(u)$, we have

$$h_m'(u) = D(g_m||f_m) - F_m'(K-u), \qquad (111)$$

where

$$F_m'(v) = \begin{cases} \frac{1}{\sum_{j \ne m} \frac{1}{D(f_j||g_j)}}, & \text{if } v \le \tilde{K}_m \\ 0, & \text{if } v > \tilde{K}_m. \end{cases} \qquad (112)$$

Since $F_m'(v)$ is piecewise constant with a breakpoint $\tilde{K}_m$, $h_m'(u)$ is piecewise constant with a breakpoint $K - \tilde{K}_m$. Therefore,

1) If $D(g_m||f_m) \ge \bar{F}_m$, then $h_m'(u) > 0$ and $u_m^* = 1$.
2) If $K > \tilde{K}_m + 1$, then $h_m'(u) = D(f_j||g_j) > 0$ is a positive constant and $u_m^* = 1$
3) If $D(g_m||f_m) < \bar{F}_m$ and $K < \tilde{K}_m$ then $h_m'(u) = D(g_m||f_m) < \bar{F}_m < 0$ is a negative constant and $u_m^* = 0$
4) If none of the above is true, then $h_m'(u) > 0$ for $u < K - \tilde{K}_m$ and $h_m'(u) > 0$ for $u < K - \tilde{K}_m$. Therefore, $u_m^* = K - \tilde{K}_m$.

Appendix D
Proof of Theorem 4

We now focus on proving asymptotic optimality for $L > 1$, and $K = 1$. For $L > 1$, we define τ_1 as the smallest integer such that $S_m(n) > S_j(n)$ for all $m \in D$, $j \ne D$ and $n \ge \tau_1$. Note that when $K = 1$ and $n \ge \tau_1$ the decision maker always probe the consistent cell (target or not depending on the order of $\bar{G}_D$ and $\bar{F}_D$) for making the difference between the L^{th} and $(L+1)^{th}$ largest sum LLRs greater than the threshold $-\log c$. As a result, the decision maker can always balance the detection time so that the difference between the largest sum LLR and the sum LLRs of any other cell exceeds the threshold $-\log c$ approximately at the same time as $c \to 0$. Thus, proving the asymptotic optimality of DGFi for $L > 1$ and $K = 1$ follows similar arguments as in the balanced case in the proof of Theorem 1 given in Appendix B, and we focus here only on the key modifications. Let

$$\Delta S_{\mathcal{D}}(n) \triangleq \min_{m \in \mathcal{D}, j \notin \mathcal{D}} \Delta S_{m,j}(n), \qquad (113)$$

where $\Delta S_{m,j}(n)$ is defined in (43). Without loss of generality we prove the theorem when set $\mathcal{D}$ contains all the targets. We define

$$\tilde{\ell}_k(i) = \begin{cases} \ell_k(i) - D(g_k||f_k), & \text{if } k \in \mathcal{D}, \\ \ell_k(i) + D(f_k||g_k), & \text{if } k \notin \mathcal{D}, \end{cases} \qquad (114)$$

which is a zero-mean random variable.

We start by showing the upper bound on the Bayes risk obtained by DGFi. Similar to Lemma 2, we can show that the error probability under DGFi is $O(c)$. Specifically, we can show that the error probability is upper bounded by:

$$P_e \le (M-L)L \cdot c. \qquad (115)$$

We can show this by letting $\alpha_{\mathcal{D}} = \mathbf{P}_{\mathcal{D}}(\delta \ne \mathcal{D})$ and $\alpha_{\mathcal{D},j} = \mathbf{P}_{\mathcal{D}}(j \in \delta)$ for all $j \notin \mathcal{D}$, where the subscript D denotes the measure when set D contains all the targets. Thus, $\alpha_{\mathcal{D}} \le \sum_{j \notin \mathcal{D}} \alpha_{\mathcal{D},j}$. By the stopping rule, accepting $j \in \delta$ implies $\Delta S_{j,m} \ge -\log c$ for some $m \in \mathcal{D}$. Hence, for all $j \notin \mathcal{D}$ we have:

$$\begin{aligned}
\alpha_{\mathcal{D},j} &= \mathbf{P}_{\mathcal{D}}\left(j \in \mathcal{D}\right) \\
&\le \sum_{m \in \mathcal{D}} \mathbf{P}_{\mathcal{D}}\left(\Delta S_{j,m}(\tau) \ge -\log c\right) \\
&\le \sum_{m \in \mathcal{D}} c\mathbf{P}_{\mathcal{D} \cup j \setminus m}\left(\Delta S_{j,m}(\tau) \ge -\log c\right) \le L \cdot c, \quad (116)
\end{aligned}$$

where we changed the measure in the second inequality. As a result,

$$\alpha_{\mathcal{D}} \le \sum_{j \notin \mathcal{D}} \alpha_{\mathcal{D},j} \le \ (M-L)L \cdot c,$$

which yields (115).

Here we consider the case where $I_{\mathcal{D}} = \bar{G}_{\mathcal{D}}$, the case $I_{\mathcal{D}} = \bar{F}_{\mathcal{D}}$ applies with minor modifications. For showing that τ_1 is sufficiently small we need to show first the following Lemmas:

Lemma 14: For all $j \notin D$, $\forall 0 < q < 1$, there exist C, $\gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}(N_j(n) > qn) < Ce^{-\gamma n} \qquad (117)$$

Proof: For each j, define $t^j(n)$ as the time when cell j is observed for the n^{th} time. By DGFi selection rule, if cell j is observed at time t, then there exists $m \in \mathcal{D}$ such that $S_j(t) \ge S_m(t)$. Hence,

$$\begin{aligned}
&\mathbf{P}_{\mathcal{D}}(N_j(n) > qn) \\
&\quad \le \sum_{t=1}^{n} \mathbf{P}_{\mathcal{D}}(N_j(t) > qn, \exists m \in \mathcal{D} : S_j(t) > S_m(t)) \\
&\qquad \times \mathbf{P}_{\mathcal{D}}(t^j(\lceil qn \rceil) = t). \qquad (118)
\end{aligned}$$

It suffices to show that there exist constants C, γ such that

$$\mathbf{P}_{\mathcal{D}}(N_j(t) > qn, \exists m \in \mathcal{D} : S_j(t) > S_m(t)) \le Ce^{-\gamma n} \quad (119)$$

for all $t \le n$.

First we have

$$\mathbf{P}_{\mathcal{D}}(N_j(t) > qn, \exists m \in \mathcal{D} : S_j(t) > S_m(t)) \le \sum_{m \in \mathcal{D}} \mathbf{P}_{\mathcal{D}}(N_j(t) > qn, S_j(t) > S_m(t)). \quad (120)$$

Fix m, then we have

$$\begin{aligned}&\mathbf{P}_{\mathcal{D}}(N_j(t) > qn, S_j(t) > S_m(t))\\ &\le \sum_{r=\lceil qn \rceil}^{n} \sum_{k=0}^{n} \mathbf{P}_{\mathcal{D}}\left(\sum_{i=1}^{n} \ell_j(i) + \sum_{k=1}^{k} -\ell_m(i) \ge 0\right)\\ &\le C_m e^{-\gamma_m n}. \quad (121)\end{aligned}$$

The last inequality can be shown using the generic Chernoff bound given in (53).

To show (119), we let $C = \sum_m C_m, \gamma = \min_m \gamma_m$, which completes the proof. ∎

Lemma 15: For all $m \in D$, and $\epsilon > 0$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}\left(N_m(n) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n\right) \le Ce^{-\gamma n} \quad (122)$$

Proof: For each m, define $t^m(n)$ as the time when cell m is observed for the n^{th} time. By DGFi selection rule, if cell m is observed at time t, either there exists $j \notin D$ such that $S_j(n) > S_m(n)$ or $S_{m'}(n) > S_m(n)$ for all $m' \in D$. Similar to (118), it suffices to show that

$$\mathbf{P}_{\mathcal{D}}\left(N_m(t) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \exists j \notin \mathcal{D} : S_j(t) > S_m(t)\right) \le Ce^{-\gamma n} \quad (123)$$

and

$$\mathbf{P}_{\mathcal{D}}\left(N_m(t) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \forall m' \in \mathcal{D} : S_{m'}(t) > S_m(t)\right) \le Ce^{-\gamma n} \quad (124)$$

for all $t < n$.

Since (123) can be shown similarly as in (119), it remains to show (124). By the definition of $\bar{G}_{\mathcal{D}}$, if $N_m(t) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m)-\epsilon} \cdot n$, there exists $m' \in D$ and $\epsilon' > 0$ such that $N_{m'}(t) < \frac{\bar{G}_{\mathcal{D}}}{D(g'_m||f'_m)+\epsilon'} \cdot t$. Hence,

$$\begin{aligned}&\mathbf{P}_{\mathcal{D}}\left(N_m(t) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \forall m' \in D : S_{m'}(t) > S_m(t)\right)\\ &\le \sum_{m' \in D} \mathbf{P}_{\mathcal{D}}\left(N_m(t) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, S_{m'}(t) > S_m(t),\right.\\ &\qquad \left. N'_m(t) < \frac{\bar{G}_{\mathcal{D}}}{D(g'_m||f'_m) + \epsilon'} \cdot t\right). \quad (125)\end{aligned}$$

Fix m', and let $s_1 = \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m)-\epsilon}$, $s_2 = \frac{\bar{G}_{\mathcal{D}}}{D(g_{m'}||f_{m'})+\epsilon}$. Then, we have

$$\begin{aligned}&\mathbf{P}_{\mathcal{D}}\left(N_m(t) > \frac{\bar{G}_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, S_{m'}(t) > S_m(t),\right.\\ &\qquad \left. N'_m(t) < \frac{\bar{G}_{\mathcal{D}}}{D(g_{m'}||f_{m'}) + \epsilon'} \cdot t\right)\\ &\le \sum_{r=\lceil s_1 n \rceil}^{n} \sum_{k=0}^{\lfloor s_2 t \rfloor} \mathbf{P}_{\mathcal{D}}\left(\sum_{i=1}^{r} -\ell_m(i) + \sum_{i=1}^{k} \ell_{m'}(i) \ge 0\right)\\ &\le \sum_{r=\lceil s_1 n \rceil}^{n} \sum_{k=0}^{\lfloor s_2 t \rfloor} \mathbf{P}_{\mathcal{D}}\left(\sum_{i=1}^{r} D(g_m||f_m) - \epsilon - \ell_m(i)\right.\\ &\qquad \left. + \sum_{i=1}^{k} \ell_{m'}(i) - D(g_{m'}||f_{m'}) - \epsilon' \ge 0\right)\\ &\le \sum_{r=\lceil s_1 n \rceil}^{n} \sum_{k=0}^{\lfloor s_2 t \rfloor} \left[\mathbf{E}_{\mathcal{D}}\left(e^{s\ (-\tilde{\ell}_m(1)-\epsilon)}\right)\right]^r \left[\mathbf{E}_{\mathcal{D}}\left(e^{s(\ell_{m'}(1)-\epsilon')}\right)\right]^k\\ &\le C_{m'} e^{-\gamma_{m'} n} \quad (126)\end{aligned}$$

The last inequality can be shown using the generic Chernoff bound given in (53). To show (125), we let $C = \sum_{m'} C_{m'}, \gamma = \min_{m'} \gamma_{m'}$, which completes the proof. ∎

Lemma 16: For all $m \in D$, $\forall \epsilon > 0$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}\left(N_m(n) < (\frac{\bar{G}_{\mathcal{D}}}{2D(g_m||f_m)})n\right) \le Ce^{-\gamma n}. \quad (127)$$

Proof: By choosing q_j and ϵ'_m in Lemma 14 and Lemma 15 such that $\sum_j q_j + \sum'_m \epsilon'_m = \frac{\bar{G}_{\mathcal{D}}}{2D(g_m||f_m)}$, we have

$$\begin{aligned}&\mathbf{P}_{\mathcal{D}}\left(N_m(n) < (\frac{\bar{G}_{\mathcal{D}}}{2D(g_m||f_m)})n\right)\\ &\le \sum_{j \notin \mathcal{D}} \mathbf{P}_{\mathcal{D}}\left(N_j(n) > q_j n\right)\\ &\quad + \sum_{m' \in \mathcal{D}} \mathbf{P}_{\mathcal{D}}\left(N'_m(n) > (\frac{\bar{G}_{\mathcal{D}}}{D(g'_m||f'_m)} + \epsilon'_m)n\right) \le C_{m'} e^{-\gamma n} \quad (128)\end{aligned}$$

as desired. ∎

Next, similar to Lemma 4, we can show that the probability that τ_1 is greater than n decreases exponentially with n. This result is used when evaluating the asymptotic expected search time to show that it is not affected by τ_1. We can show this by noting that

$$\begin{aligned}\mathbf{P}_{\mathcal{D}}(\tau_1 > n) &\le \mathbf{P}_{\mathcal{D}}\left(\max_{j \notin \mathcal{D}, m \in \mathcal{D}} \sup_{t \ge n} \left(S_j(t) - S_m(t)\right) \ge 0\right)\\ &\le \sum_{j \notin \mathcal{D}, m \in \mathcal{D}} \sum_{t=n}^{\infty} \mathbf{P}_{\mathcal{D}}\left(S_j(t) \ge S_m(t)\right). \quad (129)\end{aligned}$$

Following (129), it suffices to show that $\mathbf{P}_{\mathcal{D}}\left(S_j(n) \geq S_m(n)\right)$ decays exponentially with n. Note that

$$\begin{aligned}\mathbf{P}_{\mathcal{D}}&\left(S_j(n) \geq S_m(n)\right)\\ &\leq \mathbf{P}_{\mathcal{D}}\left(S_j(n) \geq S_m(n), N_m(n) \geq (\frac{\bar{G}_{\mathcal{D}}}{2D(g_m||f_m)})n\right)\\ &\quad+\mathbf{P}_{\mathcal{D}}\left(N_m(n) < (\frac{\bar{G}_{\mathcal{D}}}{2D(g_m||f_m)})n\right)\end{aligned} \quad (130)$$

The first term decays exponentially with n by Lemma 3 (with minor modifications). The second term decays exponentially with n by Lemma 16.

Note that we obtained that the expectation of τ_1 is bounded, and we can use similar arguments as in the balanced case of Theorem 1 in Appendix B to obtain the detection rate I_D for $n \geq \tau_1$. Combining these results yields that the expected detection time τ under the DGFi policy is upper bounded by:

$$\mathbf{E}_{\mathcal{D}}(\tau) \leq -(1+o(1))\frac{\log(c)}{I_{\mathcal{D}}}, \quad (131)$$

for $m = 1, \ldots, M$.

Finally, showing that the asymptotic Bayes risk is lower bounded by $-c\log c/I_L^*$ follows a similar outline as in Appendix B. Specifically, similar to Lemma 9, if $\alpha_{\mathcal{D}}(\Gamma) = O(-c\log c)$ for all $\mathcal{D}$, and we let $0 < \epsilon < 1$, then:

$$\mathbf{P}_{\mathcal{D}}\left(\Delta S_m(\tau) < -(1-\epsilon)\log c \mid \Gamma\right) = O(-c^{\epsilon}\log c), \quad (132)$$

for all $\mathcal{D}$ and $m \in \mathcal{D}$.

Then, we define:

$$j^*(t) \triangleq \arg\min_{j \notin \mathcal{D}} N_j(t)D(f_j||g_j), \quad (133)$$

$$m^*(t) \triangleq \arg\min_{m \in \mathcal{D}} N_{m^*(t)}(t)D(g_m||f_m), \quad (134)$$

and

$$W_{\mathcal{D}}^*(t) \triangleq \sum_{i=1}^{t}\tilde{\ell}_{m^*(t)}(i)\mathbf{1}_{m^*(t)}(i) - \sum_{i=1}^{t}\tilde{\ell}_{j^*(t)}(i)\mathbf{1}_{j^*(t)}(i), \quad (135)$$

where $W_{\mathcal{D}}^*(t)$ is a sum of zero-mean random variable. Using these definitions, similar to Lemma 11, we can show that for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}\left(\max_{1\leq t\leq n} W_{\mathcal{D}}^*(t) \geq n\epsilon|\Gamma\right) \leq Ce^{-\gamma n} \quad (136)$$

for all $\mathcal{D}$ and for any policy Γ.

Next, similar to Lemma 12 we can show that for any fixed $\epsilon > 0$,

$$\mathbf{P}_{\mathcal{D}}\left(\max_{1\leq t\leq n}\Delta S_{\mathcal{D}}(t) \geq n\left(I_{\mathcal{D}}+\epsilon\right) \mid \Gamma\right) \to 0 \quad \text{as } n \to \infty, \quad (137)$$

for all $\mathcal{D}$ and for any policy Γ.

Finally, similar to Lemma 13, we can show that any policy Γ that satisfies $R_{\mathcal{D}}(\Gamma) = O(-c\log c)$ for all $\mathcal{D}$ must satisfy:

$$R_{\mathcal{D}}(\Gamma) \geq -(1+o(1))\frac{c\log(c)}{I_{\mathcal{D}}}. \quad (138)$$

for all $\mathcal{D}$.

References

[1] Q. Zhao and B. M. Sadler, "A survey of dynamic spectrum access," *IEEE Signal Process. Mag.*, vol. 24, no. 3, pp. 79–89, May 2007.

[2] H. Chernoff, "Sequential design of experiments," *Ann. Math. Statist.*, vol. 30, no. 3, pp. 755–770, 1959.

[3] A. Wald, *Sequential Analysis*. Hoboken, NJ, USA: Wiley, 1947.

[4] S. A. Bessler, "Theory and applications of the sequential design of experiments, k-actions and infinitely many experiments: Part 1—Theory," Dept. Statist., Stanford Univ., Stanford, CA, USA, Tech. Rep. 55, 1960.

[5] S. Nitinawarat, G. K. Atia, and V. V. Veeravalli, "Controlled sensing for hypothesis testing," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process. (ICASSP)*, Mar. 2012, pp. 5277–5280.

[6] S. Nitinawarat, G. K. Atia, and V. V. Veeravalli, "Controlled sensing for multihypothesis testing," *IEEE Trans. Autom. Control*, vol. 58, no. 10, pp. 2451–2464, Oct. 2013.

[7] S. Nitinawarat and V. V. Veeravalli, "Controlled sensing for sequential multihypothesis testing with controlled Markovian observations and non-uniform control cost," *Design Methods Appl.*, vol. 34, no. 1, pp. 1–24, Oct. 2013.

[8] M. Naghshvar and T. Javidi, "Active sequential hypothesis testing," *Ann. Statist.*, vol. 41, no. 6, pp. 2703–2738, 2013.

[9] M. Naghshvar and T. Javidi, "Sequentiality and adaptivity gains in active hypothesis testing," *IEEE J. Sel. Topics Signal Process.*, vol. 7, no. 5, pp. 768–782, Oct. 2013.

[10] D. A. Castanon, "Optimal search strategies in dynamic hypothesis testing," *IEEE Trans. Syst., Man and*, vol. 25, no. 7, pp. 1130–1138, Jul. 1995.

[11] K. Cohen, Q. Zhao, and A. Swami, "Optimal index policies for anomaly localization in resource-constrained cyber systems," *IEEE Trans. Signal Process.*, vol. 62, no. 16, pp. 4224–4236, Aug. 2014.

[12] N. K. Vaidhiyan and R. Sundaresan. (2015). "Learning to detect an oddball target." [Online]. Available: https://arxiv.org/pdf/1508.05572

[13] K. Leahy and M. Schwager, "Always choose second best: Tracking a moving target on a graph with a noisy binary sensor," in *Proc. Eur. Control Conf. (ECC)*, Jun./Jul. 2016, pp. 1715–1721.

[14] J. Heydari, A. Tajer, and H. V. Poor, "Quickest linear search over correlated sequences," *IEEE Trans. Inf. Theory*, vol. 62, no. 10, pp. 5786–5808, Oct. 2016.

[15] K. Cohen and Q. Zhao, "Active hypothesis testing for anomaly detection," *IEEE Trans. Inf. Theory*, vol. 61, no. 3, pp. 1432–1450, Mar. 2015.

[16] K. S. Zigangirov, "On a problem in optimal scanning," *Theory Probab. Appl.*, vol. 11, no. 2, pp. 294–298, 1966.

[17] A. Tajer and H. V. Poor, "Quick search for rare events," *IEEE Trans. Inf. Theory*, vol. 59, no. 7, pp. 4462–4481, Jul. 2013.

[18] K. Cohen and Q. Zhao, "Asymptotically optimal anomaly detection via sequential testing," *IEEE Trans. Signal Process.*, vol. 63, no. 11, pp. 2929–2941, Jun. 2015.

[19] G. Fellouris, G. V. Moustakides, and V. V. Veeravalli, "Multistream quickest change detection: Asymptotic optimality under a sparse signal," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process. (ICASSP)*, Mar. 2017, pp. 6444–6447.

[20] L. Lai, H. V. Poor, Y. Xin, and G. Georgiadis, "Quickest search over multiple sequences," *IEEE Trans. Inf. Theory*, vol. 57, no. 8, pp. 5375–5386, Aug. 2011.

[21] S. Nitinawarat and V. V. Veeravalli, "Universal scheme for optimal search and stop," in *Proc. Inf. Theory Appl. Workshop (ITA)*, 2015, pp. 322–328.

[22] B. Hemo, K. Cohen, and Q. Zhao, "Asymptotically optimal search of unknown anomalies," in *Proc. 16th IEEE Symp. Signal Process. Inf. Technol. (ISSPIT)*, Limassol, Cyprus, Dec. 2016, pp. 75–80.

[23] M. L. Malloy, G. Tang, and R. D. Nowak, "Quickest search for a rare distribution," in *Proc. IEEE 46th Annu. Conf. Inf. Sci. Syst. (CISS)*, Mar. 2012, pp. 1–6.

[24] Y. Pei, Y.-C. Liang, K. C. Teh, and K. H. Li, "Energy-efficient design of sequential channel sensing in cognitive radio networks: Optimal sensing strategy, power allocation, and sensing order," *IEEE J Sel Areas Commun.*, vol. 29, no. 8, pp. 1648–1659, Sep. 2011.

[25] R. Caromi, Y. Xin, and L. Lai, "Fast multiband spectrum scanning for cognitive radio systems," *IEEE Trans. Commun.*, vol. 61, no. 1, pp. 63–75, Jan. 2013.

[26] L. Ferrari, Q. Zhao, and A. Scaglione, "Utility maximizing sequential sensing over a finite horizon," *IEEE Trans. Signal Process.*, vol. 65, no. 13, pp. 3430–3445, Jul. 2017.

[27] M. Egan, J.-M. Gorce, and L. Cardoso, “Fast initialization of cognitive radio systems,” in *Proc. IEEE Int. Workshop Signal Process. Adv. Wireless Commun.*, Jul. 2017, pp. 1–5.

[28] A. Tajer, V. V. Veeravalli, and H. V. Poor, “Outlying sequence detection in large data sets: A data-driven approach,” *IEEE Signal Process. Mag.*, vol. 31, no. 5, pp. 44–56, Sep. 2014.

[29] V. Chandola, A. Banerjee, and V. Kumar, “Anomaly detection: A survey,” *ACM Comput. Surv.*, vol. 41, no. 3, p. 15, 2009.

[30] M. H. Bhuyan, D. K. Bhattacharyya, and J. K. Kalita, “Network anomaly detection: Methods, systems and tools,” *IEEE Commun. Surveys Tuts.*, vol. 16, no. 1, pp. 303–336, 1st Quart., 2014.

[31] B. Huang, K. Cohen, and Q. Zhao. (2017). “Active anomaly detection in heterogeneous processes.” [Online]. Available: https://arxiv.org/abs/1704.00766

Boshuang Huang received the B.S. degree in electronic engineering from Tsinghua University, Beijing, China in 2015, and the M.S. degree in electrical and computer engineering from Cornell University, Ithaca, NY, USA, in 2018. He is currently working toward the Ph.D. degree at Cornell University, Ithaca, NY, USA. His research interests include sequential decision theory, active learning, and statistical inference.

Kobi Cohen received the B.Sc. and Ph.D. degrees in electrical engineering from Bar-Ilan University, Ramat Gan, Israel, in 2007 and 2013, respectively. In October 2015, he joined the Department of Electrical and Computer Engineering at Ben-Gurion University of the Negev (BGU), Beer Sheva, Israel, as a Senior Lecturer (a.k.a. Assistant Professor in the US). He is also a member of the Cyber Security Research Center, and the Data Science Research Center at BGU. Before joining BGU, he was with the Coordinated Science Laboratory at the University of Illinois at Urbana-Champaign (08/2014–07/2015) and the Department of Electrical and Computer Engineering at the University of California, Davis (11/2012–07/2014) as a postdoctoral research associate. His main research interests include decision theory, stochastic optimization, and statistical inference and learning, with applications in large-scale systems, cyber systems, wireless and wireline networks. Dr. Cohen received several awards including the Best Paper Award in the International Symposium on Modeling and Optimization in Mobile, Ad hoc and Wireless Networks (WiOpt) 2015, the Feder Family Award (second prize), granted by the Advanced Communication Center at Tel Aviv University (2011), and President’s Fellowship (2008–2012) and honor list’s prizes (2006, 2010, 2011) from Bar-Ilan University.

Qing Zhao (F’13) is a Professor and Gordon Lankton Sesquicentennial Faculty Fellow in the School of Electrical and Computer Engineering at Cornell University, Ithaca, NY. Prior to joining Cornell University in 2015, she was a Professor at University of California, Davis. She received her Ph.D. degree in Electrical Engineering from Cornell University in 2001. Her research interests include sequential decision theory and stochastic optimization, machine learning, statistical inference, and algorithmic theory with applications in infrastructure and communication systems and social economic networks. She is a Fellow of IEEE. She received the 2010 IEEE Signal Processing Magazine Best Paper Award and the 2000 Young Author Best Paper Award from the IEEE Signal Processing Society. While on the faculty of UC Davis, she held the title of UC Davis Chancellor’s Fellow. She was recently awarded a Marie Skłodowska-Curie Fellowship by the European Union’s Horizon 2020 research and innovation program, and a Jubilee Chair Professorship of Chalmers University, Sweden.