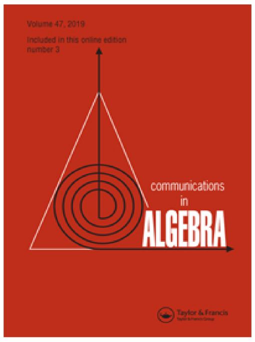

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On skew polynomial rings over locally nilpotent rings

Fei Yu Chen[a], Hannah Hagan[b], and Allison Wang[c]

[a]University of California, Berkeley, CA, USA; [b]Vanderbilt University, Nashville, Tennessee, USA; [c]California Institute of Technology, Pasadena, CA, USA

ABSTRACT
We will show that skew polynomial rings in several variables over locally nilpotent rings cannot contain nonzero idempotent elements. We will also prove that such rings are Brown–McCoy radical.

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1. Introduction

Recently, skew polynomial rings over locally nilpotent rings have received significant attention [1–7]. One of the most important results was the solution to a problem by Shestakov: Smoktunowicz and Ziembowski proved that the differential polynomial ring over a locally nilpotent ring need not be Jacobson radical [7]. In addition, Greenfeld, Smoktunowicz, and Ziembowski posed many intesting problems about skew polynomial rings in their paper [4]. Our first theorem is related to one of these problems, namely whether a skew Laurent polynomial ring over a locally nilpotent ring contains a nonzero idempotent. Though this problem was solved by Chebotar [2] by means of linear algebra, there is a more general and straightfoward solution. Specifically, the result is extended in Theorem 1 from one to several variables.

Our second theorem is motivated by the result due to Smoktunowicz [5, Proposition 3.1] that a differential polynomial ring over a locally nilpotent ring is Brown–McCoy radical. As above, we extend this result to the case of several variables and replace the derivation with skew derivations.

Let R be a ring. Let I be a set of indices, and let J be a subset of I. Then let $\Omega = \{\omega_i | i \in I\}$ be a set of endomorphisms of R and $\Psi = \{\psi_i \mid i \in I\}$ the set of corresponding ω_i-derivations. Let $\mathcal{X} = \{X_i \mid i \in I\}$ be a set of variables, and let $\mathcal{Y} = \{X_j^{-1} \mid j \in J\}$ be a set of inverses.

Define a word in $\mathcal{X}$ and $\mathcal{Y}$ to be a string $X_{i_1}^{\pm 1} X_{i_2}^{\pm 1} \dots X_{i_k}^{\pm 1}$, where X_{i_m} can have an exponent of -1 only if $i_m \in J$. Let $\mathcal{W}$ be the set of all words in $\mathcal{X}$ and $\mathcal{Y}$.

Define the (Ω, Ψ)-skew polynomial ring $R[\mathcal{X}, \mathcal{Y}; \Omega, \Psi]$ as the set of left polynomials in noncommutative variables in $\mathcal{X}$ and $\mathcal{Y}$. Elements are of the form $\sum_{w \in \mathcal{W}} a_w w$ where only finitely many $a_w \in R$ are nonzero. Addition is the usual addition of polynomials. Multiplication is associative, and we convert right polynomials to left polynomials by applying the rule

CONTACT Fei Yu Chen dennisfchen@berkeley.edu University of California, Berkeley, CA 94720, USA.

$X_ia = \omega_i(a)X_i + \psi_i(a)$, for all $i \in I$ and $a \in R$. We assume that $X_j^{-1}a$ is well-defined as a left polynomial, for all $j \in J$ and $a \in R$.

Example 1. (Differential polynomial rings). Let $\mathcal{X} = \{X\}$ and $\mathcal{Y} = \varnothing$ (here I contains only one element, and J is empty). Let ω be the identity map and ψ a derivation. Then $R[\mathcal{X}, \mathcal{Y}; \{\omega\}, \{\psi\}]$ is the differential polynomial ring $R[X; \psi]$.

Example 2. (Skew Laurent polynomial rings). Let $\mathcal{X} = \{X\}$ and $\mathcal{Y} = \{X^{-1}\}$ (here $I = J$ contains only one element). Let ω be any automorphism of R and ψ the zero map. Then $R[\mathcal{X}, \mathcal{Y}; \{\omega\}, \{\psi\}]$ is the skew Laurent polynomial ring $R[X, X^{-1}; \omega]$.

We are now ready to state the main results of the paper.

Theorem 1. Let $T = R[\mathcal{X}, \mathcal{Y}; \Omega, \Psi]$ be the (Ω, Ψ)-skew polynomial ring over a locally nilpotent ring R. Then T does not contain a nonzero idempotent.

Theorem 2. Let $T = R[\mathcal{X}, \mathcal{Y}; \Omega, \Psi]$ be the (Ω, Ψ)-skew polynomial ring over a locally nilpotent ring R. Then T cannot be mapped onto a ring with identity.

2. Proofs

Proof of Theorem 1. Suppose for some locally nilpotent ring R the corresponding (Ω, Ψ)-skew polynomial ring T contains a nonzero idempotent. Denote a nonzero idempotent by

$$e = \sum_{w \in \mathcal{W}} a_w w.$$

Since the multiplication on T is well-defined, each product wa_v is again a finite sum of the form

$$wa_v = \sum_{u \in \mathcal{W}} b_{w,v,u} u,$$

where $b_{w,v,u} \in R$. Then e^2 will be of the form

$$e^2 = \sum_{w,v \in \mathcal{W}} a_w w a_v v = \sum_{w,v,u \in \mathcal{W}} a_w b_{w,v,u} uv = \sum_{y \in \mathcal{W}} \left(\sum_{\substack{uv = y \\ w \in \mathcal{W}}} a_w b_{w,v,u} \right) y.$$

Now, let N be the ring generated by the finite set $\{a_w, b_{w,v,u}\}$. Then N is nilpotent, so we can find the smallest positive integer l such that $a_y \notin N^{l+1}$, for some $y \in \mathcal{W}$. Note $a_w \in N^l$ for all $w \in \mathcal{W}$.

As $e = e^2$, we obtain

$$a_y = \sum_{\substack{uv = y \\ w \in \mathcal{W}}} a_w b_{w,v,u}$$

by equating the coefficients of y. However, we know each $a_w \in N^l$ and each $b_{w,v,u} \in N$, so $a_y \in N^{l+1}$, a contradiction. □

Remark 3. In the statement of Theorem 1, the variables do not commute. However, the same argument works when the variables do commute. For example, let M be the ring of strictly upper triangular matrices over the field of real numbers. Let $R = M[z_1, ..., z_n]$ be the polynomial ring over M in commuting indeterminates $z_1, ..., z_n$. Let $R[X_1, ..., X_n; \delta_1, ..., \delta_n]$ be a differential polynomial ring with commuting variables such that $X_ia = aX_i + \delta_i(a)$ for all $a \in R$, where $\delta_i = \frac{\partial}{\partial z_i}$, the partial derivative with respect to z_i. Since R is a locally nilpotent ring, $R[X_1, \ldots, X_n; \delta_1, \ldots, \delta_n]$ does not contain a nonzero idempotent by the same reasoning as above.

Proof of Theorem 2. Suppose φ is a homomorphism from T onto a simple ring S with 1. Let $p = \sum_{w \in \mathcal{W}} a_w w$ be an element of T such that $\varphi(p) = 1$. Let $\widetilde{a}_w = \varphi(a_w)$ and $\widetilde{w} = \varphi(wp)$. Now,

$$\varphi(p) = \sum_{w \in \mathcal{W}} \widetilde{a}_w \widetilde{w} = 1. \tag{1}$$

Recall that the homomorphic image of a locally nilpotent ring is locally nilpotent. Therefore $\varphi(R)$ is locally nilpotent. Let N be the subring of $\varphi(R)$ generated by the finite set $\{\widetilde{a}_w\}$. So N is nilpotent, and there exists a positive integer m such that $N^m = 0$. Then

$$\varphi(p) = \sum_{w \in \mathcal{W}} \widetilde{a}_w \widetilde{w} = \sum_{w \in \mathcal{W}} \widetilde{a}_w \cdot 1 \cdot \widetilde{w} = \sum_{w \in \mathcal{W}} \widetilde{a}_w \cdot \left(\sum_{w \in \mathcal{W}} \widetilde{a}_w \widetilde{w} \right) \cdot \widetilde{w} = \sum_{u \in \mathcal{W}} \widetilde{b}_u \widetilde{u},$$

where each $\widetilde{b}_u$ is in N^2, and the third equality follows from Equation (1). In the same vein,

$$\varphi(p) = \sum_{u \in \mathcal{W}} \widetilde{b}_u \widetilde{u} = \sum_{u \in \mathcal{W}} \widetilde{b}_u \cdot 1 \cdot \widetilde{u} = \sum_{u \in \mathcal{W}} \widetilde{b}_u \cdot \left(\sum_{w \in \mathcal{W}} \widetilde{a}_w \widetilde{w} \right) \cdot \widetilde{u} = \sum_{v \in \mathcal{W}} \widetilde{c}_v \widetilde{v},$$

where each $\widetilde{c}_v$ is in N^3. Continuing in this fashion, we obtain

$$\varphi(p) = \sum_{y \in \mathcal{W}} \widetilde{z}_y \widetilde{y},$$

where each $\widetilde{z}_y$ belongs to N^m and thus is zero. So $1 = \varphi(p) = 0$, a contradiction. □

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