Linear Algebra and its Applications 563 (2019) 193–206

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On maps preserving products of matrices

Louisa Catalano [a,*], Samuel Hsu [b], Regan Kapalko [c]

[a] *Department of Mathematical Sciences, Kent State University, Kent OH 44242, USA*
[b] *Department of Mathematics, University of California Berkeley, Berkeley CA 94720, USA*
[c] *Department of Mathematics, Miami University, Oxford OH 45056, USA*

ARTICLE INFO

Article history:
Received 27 July 2018
Accepted 30 October 2018
Available online 3 November 2018
Submitted by M. Bresar

MSC:
15A04
16W10

Keywords:
Maps preserving products
Jordan product

ABSTRACT

Let $\mathcal{D}$ be a division ring with characteristic different from 2, and let $\mathcal{R} = M_n(\mathcal{D})$. The first goal of this paper is to describe an additive map $f : \mathcal{R} \to \mathcal{R}$ satisfying the identity $f(x)f(y) = m$ for every $x, y \in \mathcal{R}$ such that $xy = k$, where $m, k \in \mathcal{R}$ are fixed invertible elements. Additionally, let $\mathcal{M} = M_n(\mathbb{C})$, the set of all $n\times n$ matrices with complex entries. We will describe a bijective linear map $g : \mathcal{M} \to \mathcal{M}$ satisfying $g(X) \circ g(Y) = M$ whenever $X \circ Y = K$ for every $X, Y \in \mathcal{M}$, where $M, K \in \mathcal{M}$ are fixed, and $\circ$ denotes the Jordan product.

1. Introduction

Let $\mathcal{A}$ be an associative ring. The Jordan product of any two elements $x, y \in \mathcal{A}$ is $x \circ y = xy + yx$. Throughout this paper, we will be discussing maps that preserve a certain product on $\mathcal{A}$; that is, if we let $*$ denote either the ordinary or the Jordan product, we will consider maps $\varphi : \mathcal{A} \to \mathcal{A}$ satisfying $\varphi(x) * \varphi(y) = \varphi(u) * \varphi(v)$ whenever

* Corresponding author.
E-mail addresses: lcatalan@math.kent.edu (L. Catalano), hsulacetheorem@berkeley.edu (S. Hsu), kapalkrc@miamioh.edu (R. Kapalko).

https://doi.org/10.1016/j.laa.2018.10.029
0024-3795/

$x * y = u * v = a$ for some fixed $a \in \mathcal{A}$. An example of a map φ satisfying this property is a homomorphism multiplied by a central element of $\mathcal{A}$. The question of interest becomes: when is this description the only possibility?

Perhaps the most obvious type of product preserving map is those that preserve the (ordinary) zero product; that is, maps satisfying

$$\theta(x)\theta(y) = 0 \text{ whenever } xy = 0. \tag{1}$$

A characterization of zero product preserving maps on various algebras can be found in [1,3,8]. Each of these results comes to the same conclusion: the map must be the product of a central element and a homomorphism. We say that a map ψ preserves square-zero matrices if $\psi(x)^2 = 0$ whenever $x^2 = 0$ for $x \in M_n(\mathcal{A})$, the ring of $n \times n$ matrices over $\mathcal{A}$. In [12], Šemrl studied such maps in the case that $\mathcal{A} = \mathbb{C}$, and in [4], his result was extended for matrices over commutative rings.

A logical extension of the zero product preserving maps is to consider when a map satisfies an identity similar to (1), but using the Jordan product in place of the usual product. Chebotar, Ke, Lee, and Zhang showed that a surjective additive map θ on $M_n(\mathcal{A})$ preserving the zero Jordan product (i.e. $\theta(x) \circ \theta(y) = 0$ whenever $x \circ y = 0$) must have the form $\theta(x) = \theta(1)\psi(x)$, where ψ is a Jordan homomorphism (that is, $\psi(x \circ y) = \psi(x) \circ \psi(y)$) and $\theta(1)$ is a central element of $M_n(\mathcal{A})$ [6]. More examples of maps satisfying similar properties with the Jordan product can be found in [10].

Another natural product preserving map to study is one preserving the identity product. For instance, Chebotar, Ke, Lee, and Shiao found that a bijective additive map α on a division ring $\mathcal{D}$ that satisfies $\alpha(a^{-1})\alpha(a) = \alpha(b^{-1})\alpha(b)$ for all nonzero $a, b \in \mathcal{D}$ must have the form $\alpha(x) = \alpha(1)\varphi(x)$, where φ is an automorphism or antiautomorphism, and $\alpha(1)$ is a central element of $\mathcal{D}$ [5]. Lin and Wong generalized this result to $M_n(\mathcal{D})$ [11].

Our goal in this paper is to consider when more general products are preserved. Recently, Catalano was able to expand the result from Chebotar et al. regarding the identity preserving map, finding the form of α when it preserves an arbitrary fixed product; that is, $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ whenever $xy = uv = k$ for some fixed $k \in \mathcal{D}$ [2]. In this case, α has the form $\alpha(x) = \alpha(1)\varphi(x)$, where φ is an automorphism or antiautomorphism, but $\alpha(1)$ is not necessarily central. Our first result generalizes [2, Theorem 5] to a map on $M_n(\mathcal{D})$.

Theorem 1. *Let $\mathcal{D}$ be a division ring with characteristic different from 2. Let $\mathcal{R} = M_n(\mathcal{D})$ be the ring of $n \times n$ matrices with $n \geq 2$, and let $\mathcal{Z}$ be the center of $\mathcal{R}$. With $m, k \in \mathcal{R}$ invertible fixed elements, let $f : \mathcal{R} \to \mathcal{R}$ be a bijective additive map satisfying the identity*

$$f(x)f(y) = m \tag{2}$$

for every $x, y \in \mathcal{R}$ such that $xy = k$. Then $f(x) = f(1)\varphi(x)$ for all $x \in \mathcal{R}$, where $\varphi : \mathcal{R} \to \mathcal{R}$ is either an automorphism or an antiautomorphism. Moreover, we have

(1) *if* φ *is an automorphism, then* $f(1) \in \mathcal{Z}$*, and*
(2) *if* φ *is an antiautomorphism, then* $f(1) = f(k)^{-1}m$ *and* $f(k) \in \mathcal{Z}$*.*

Let $\mathcal{M} = M_n(\mathbb{C})$. For our next set of results, we will consider the natural extension of property (2), replacing the usual product with the Jordan product.

Theorem 2. *Let* $M, K \in \mathcal{M}$ *be fixed elements, where* K *has a square root. Let* $g : \mathcal{M} \to \mathcal{M}$ *be a linear map satisfying the property*

$$g(X) \circ g(Y) = M \tag{3}$$

for every $X, Y \in \mathcal{M}$ *such that* $X \circ Y = K$*. Then* g *preserves square-zero matrices.*

Theorem 2 is of particular interest when K is either diagonal, invertible, or idempotent, although [7] gives a complete description of matrices that have square roots. To the best of our knowledge, no one has considered a map preserving such an arbitrary product as we have in Theorem 2. However, we note that the most comparable results, such as those in [6] and [10], rely on the use of idempotent elements and matrix units in the proofs, and our method of proof differs significantly.

Our next result generalizes Theorem 2 in the case where K does not have a square root; nonetheless, the proof is considerably different and so will be presented separately.

Theorem 3. *Let* $M, K \in \mathcal{M}$ *be fixed elements, and let* $g : \mathcal{M} \to \mathcal{M}$ *be a linear map satisfying the property*

$$g(X) \circ g(Y) = M \tag{4}$$

for every $X, Y \in \mathcal{M}$ *such that* $X \circ Y = K$*. Then* g *preserves square-zero matrices.*

Šemrl's results on maps preserving square-zero matrices [12] allow us to give an explicit description of the maps described in Theorems 2 and 3.

Theorem 4. *Let* $M, K \in \mathcal{M}$ *be fixed elements, and let* $g : \mathcal{M} \to \mathcal{M}$ *be a bijective linear map satisfying the property*

$$g(X) \circ g(Y) = M \tag{5}$$

for every $X, Y \in \mathcal{M}$ *with* $X \circ Y = K$*. Then* g *is of one of the following forms:*

(1) $g(X) = cUXU^{-1}$*, or*
(2) $g(X) = cUX^TU^{-1}$*,*

for some invertible $U \in \mathcal{M}$ *and nonzero* $c \in \mathbb{C}$*, where* X^T *denotes the transpose of the matrix* X*.*

2. Proof of Theorem 1

Proof. Using identity (2), we know that $m = f(x)f(x^{-1}k)$ for every $x \in \mathcal{R}^\times$, the set of all invertible elements of $\mathcal{R}$. Let $y = a - aba$. A useful identity due to Hua states that

$$a - aba = \left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1},$$

so that $y^{-1} = a^{-1} + (b^{-1} - a)^{-1}$. Then

$$\begin{aligned} m &= f(y)f(y^{-1}k) \\ &= f(a - aba)f(a^{-1}k + (b^{-1} - a)^{-1}k) \\ &= \big(f(a) - f(aba)\big)\big(f(a^{-1}k) + f((b^{-1} - a)^{-1}k)\big) \\ &= f(a)f(a^{-1}k) + f(a)f\big((b^{-1} - a)^{-1}k\big) \\ &\quad - f(aba)f(a^{-1}k) - f(aba)f\big((b^{-1} - a)^{-1}k\big). \end{aligned}$$

However, since $m = f(a)f(a^{-1}k)$, we note the equality above simplifies to

$$0 = f(a)f\big((b^{-1} - a)^{-1}k\big) - f(aba)f(a^{-1}k) - f(aba)f\big((b^{-1} - a)^{-1}k\big).$$

For any $x \in \mathcal{R}^\times$, $m = f(x)f(x^{-1}k)$ is equivalent to $f(x^{-1}k) = f(x)^{-1}m$, and so we can see that

$$0 = f(a)f(b^{-1} - a)^{-1}m - f(aba)f(a)^{-1}m - f(aba)f(b^{-1} - a)^{-1}m. \tag{6}$$

Rearranging (6) and multiplying through by $m^{-1}f(b^{-1} - a)$ on the right hand side, we have

$$\begin{aligned} f(a) &= f(aba)f(a)^{-1}f(b^{-1} - a) + f(aba) \\ &= f(aba)f(a)^{-1}f(b^{-1}) - f(aba)f(a)^{-1}f(a) + f(aba) \\ &= f(aba)f(a)^{-1}f(b^{-1}), \end{aligned}$$

which can equivalently be written as

$$f(aba) = f(a)f(b^{-1})^{-1}f(a)$$

whenever $ab \neq 0, 1$. Therefore, for any $x \in \mathcal{R}^\times \setminus \{1\}$, we may let $a = 1, b = x$ to find that

$$f(x) = f(1)f(x^{-1})^{-1}f(1). \tag{7}$$

Now, define $\varphi(z) = f(1)^{-1}f(z)$ for all $z \in \mathcal{R}$. The additivity of f immediately yields the additivity of φ. Additionally, it is clear that $\varphi(1) = 1$. Using (7), we note that for any $x \in \mathcal{R}^\times \setminus \{1\}$,

$$\begin{aligned}\varphi(x) &= f(1)^{-1}f(x)\\ &= f(1)^{-1}f(1)f(x^{-1})^{-1}f(1)\\ &= f(x^{-1})^{-1}f(1),\end{aligned}$$

which implies that

$$\varphi(x^{-1}) = f(x)^{-1}f(1) = \big(f(1)^{-1}f(x)\big)^{-1} = \varphi(x)^{-1}.$$

That is, for every invertible $x \in \mathcal{R}$, $\varphi(x^{-1}) = \varphi(x)^{-1}$. Thus, [9] gives us that φ is an automorphism or an antiautomorphism.

Let x be an invertible element of $\mathcal{R}$. We can see that when φ is an automorphism,

$$\begin{aligned}f(1)f(k) &= f(x)f(x^{-1}k)\\ &= f(x)f(1)\varphi(x^{-1}k)\\ &= f(x)f(1)\varphi(x)^{-1}\varphi(k)\\ &= f(x)f(1)\varphi(x)^{-1}f(1)^{-1}f(1)\varphi(k)\\ &= f(x)f(1)\big(f(1)\varphi(x)\big)^{-1}f(1)\varphi(k)\\ &= f(x)f(1)f(x)^{-1}f(k);\end{aligned}$$

that is, $f(1)f(k) = f(x)f(1)f(x)^{-1}f(k)$. Multiplying through on the right by $f(k)^{-1}f(x)$, we get that $f(1)f(x) = f(x)f(1)$, and so $f(1)$ commutes with $f(x)$ for every $x \in \mathcal{R}^\times$, and in particular with every $x \in \mathcal{R}^\times$ such that $x - 1 \in \mathcal{R}^\times$. From this, we get that $f(1)$ is a central element [11, Lemma 3.2, Lemma 3.3(ii)].

If, alternatively, φ is an antiautomorphism, we have that

$$\begin{aligned}f(k)f(1) &= f(x)f(x^{-1}k)\\ &= f(x)f(1)\varphi(x^{-1}k)\\ &= f(x)f(1)\varphi(k)\varphi(x)^{-1}\\ &= f(x)f(1)\varphi(k)\varphi(x)^{-1}f(1)^{-1}f(1)\\ &= f(x)f(1)\varphi(k)\big(f(1)\varphi(x)\big)^{-1}f(1)\\ &= f(x)f(k)f(x)^{-1}f(1);\end{aligned}$$

that is, $f(k)f(1) = f(x)f(k)f(x)^{-1}f(1)$. Multiplying through on the right by $f(1)^{-1}f(x)$, we have $f(k)f(x) = f(x)f(k)$, and so $f(k)$ commutes with $f(x)$ for every $x \in \mathcal{R}^\times$. As before, using [11, Lemma 3.2, Lemma 3.3(ii)], we get that $f(k)$ is a central element. □

3. Proof of Theorem 2

Let $g : \mathcal{M} \to \mathcal{M}$ be a linear map satisfying property (3). Let $L \in \mathcal{M}$ be a square root of K (that is $L^2 = K$) and let $S \in \mathcal{M}$ be a square-zero matrix. We observe that

$$\tfrac{1}{2}L \circ L = L^2 = K,$$

and

$$\tfrac{1}{2}(L - S) \circ (L + S) = L^2 - S^2 = K.$$

From (3), we have that

$$g(\tfrac{1}{2}L) \circ g(L) = M = g(\tfrac{1}{2}(L - S)) \circ g(L + S).$$

Simplifying this, we can see that

$$g(L)^2 = g(L)^2 - g(S)^2,$$

which gives us that $g(S)^2 = 0$. Therefore, g preserves square-zero matrices.

4. Proof of Theorem 3

Observation 5. *Any linear map $g : \mathcal{M} \to \mathcal{M}$ satisfies the assumptions of Theorem 3 if and only if it satisfies the following property:*

$$2g(A)^2 - 2g(B)^2 = M \tag{8}$$

for every $A, B \in \mathcal{M}$ with $2A^2 - 2B^2 = K$.

This observation follows using the substitutions $X = A + B$ and $Y = A - B$ for the forward direction and using the substitutions $A = \frac{X+Y}{2}$ and $B = \frac{X-Y}{2}$ in the backward direction.

Let $\lambda \in \mathbb{C}$ be a fixed element such that $\lambda I - \frac{K}{2}$ is invertible. To get our desired result, we will also be using the following property for a map $g : \mathcal{M} \to \mathcal{M}$:

$$V^2 = W^2 = \lambda I \text{ implies } g(V)^2 = g(W)^2. \tag{9}$$

Lemma 6. *If $g : \mathcal{M} \to \mathcal{M}$ is a linear map satisfying (8), then g satisfies (9).*

Proof. Assume that g satisfies (8), and let $V, W \in \mathcal{M}$ be such that $V^2 = W^2 = \lambda I$. Since $\lambda I - \frac{K}{2}$ is invertible (and hence, has a square root), we know there exists $T \in \mathcal{M}$ such that $2V^2 - 2T^2 = 2W^2 - 2T^2 = K$. From this, we have that

$$2g(V)^2 - 2g(T)^2 = 2g(W)^2 - 2g(T)^2 = M.$$

That is, $g(V)^2 = \frac{M}{2} + g(T)^2 = g(W)^2$. $\square$

Lemma 7. *Let $g : \mathcal{M} \to \mathcal{M}$ be a linear map satisfying (9). Then g is a square-zero preserving map.*

Proof. Let S be an $n \times n$ square-zero matrix (i.e. $S^2 = 0$) and let J be the Jordan form of S. We know that J contains i Jordan blocks of size 2 of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for some $i \in \mathbb{Z}^+ \cup \{0\}$ with $i \leq \frac{n}{2}$, while all other entries are zero. Let P be the block diagonal matrix with i blocks of size 2 (corresponding to the 2×2 Jordan blocks of J) of the form

$$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{pmatrix},$$

the remaining $n - 2i$ diagonal entries are $\sqrt{\lambda}$, and all other entries are zero. It is easy to see that $P^2 = (J + P)^2 = (J - P)^2 = \lambda I$.

Let $C \in \mathcal{M}$ be an invertible matrix such that $S = CJC^{-1}$. Consider the matrix $Q = CPC^{-1}$. Using the fact that scalar matrices are central elements of $\mathcal{M}$, we can see that

$$Q^2 = CP^2C^{-1} = C(\lambda I)C^{-1} = \lambda I.$$

Similarly, we have that

$$(S + Q)^2 = (S - Q)^2 = \lambda I.$$

Let $g : \mathcal{M} \to \mathcal{M}$ be a linear map satisfying (9). This implies that $g(S + Q)^2 = g(S - Q)^2$. The linearity of g gives us that

$$g(S)^2 + g(S) \circ g(Q) + g(Q)^2 = g(S)^2 - g(S) \circ g(Q) + g(Q)^2,$$

and this simplifies to

$$g(S) \circ g(Q) = 0. \tag{10}$$

Additionally, we know that $g(Q)^2 = g(S + Q)^2$, and again, linearity yields

$$g(Q)^2 = g(S)^2 + g(S) \circ g(Q) + g(Q)^2.$$

Subtracting $g(Q)^2$ from each side of the equality and using (10), we can see that $g(S)^2 = 0$. That is, g preserves square-zero matrices. □

At this point, we can see that Observation 5, Lemma 6, and Lemma 7 together give us that a map $g : \mathcal{M} \to \mathcal{M}$ satisfying the assumptions of Theorem 3 is also a square-zero preserving map.

5. Proof of Theorem 4

As before, let $\lambda \in \mathbb{C}$ be a fixed element such that $\lambda I - \frac{K}{2}$ is invertible. Furthermore, we will let e_{ij} denote the matrix unit with 1 in the (i, j) position and 0 elsewhere. We will begin by defining sets of matrices. When $n = 2m + 1$ for $m \in \mathbb{N}$, let

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & & & \cdots & & 0 \\ & a_1 & b_1 & & & \\ \vdots & c_1 & -a_1 & & & \vdots \\ & & & \ddots & & 0 \\ & & & & a_m & b_m \\ 0 & & \cdots & 0 & c_m & -a_m \end{pmatrix} : a_i^2 + b_i c_i = \lambda \text{ for } 1 \leq i \leq m \right\},$$

and

$$\mathcal{B} = \left\{ \begin{pmatrix} a_1 & b_1 & 0 & \cdots & & 0 \\ c_1 & -a_1 & & & & \\ 0 & & \ddots & & & \vdots \\ \vdots & & & a_m & b_m & \\ & & & c_m & -a_m & \\ 0 & & \cdots & & & 0 \end{pmatrix} : a_i^2 + b_i c_i = \lambda \text{ for } 1 \leq i \leq m \right\}.$$

Additionally, when $n = 2m$ for $m \in \mathbb{N}$, let

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 & & & & \cdots & & 0 \\ & 0 & & & \cdots & & 0 \\ & & a_1 & b_1 & & & \\ & & c_1 & -a_1 & & & \vdots \\ \vdots & \vdots & & & \ddots & & 0 \\ & & & & & a_{m-1} & b_{m-1} \\ 0 & 0 & & \cdots & 0 & c_{m-1} & -a_{m-1} \end{pmatrix} : a_i^2 + b_i c_i = \lambda \text{ for } 1 \leq i \leq m-1 \right\},$$

and

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 & & & \cdots & & & 0 \\ & a_1 & b_1 & & & & \\ & c_1 & -a_1 & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & a_{m-1} & b_{m-1} & \\ & & & & c_{m-1} & -a_{m-1} & \\ 0 & & & \cdots & & & 0 \end{pmatrix} : a_i^2 + b_i c_i = \lambda \text{ for } 1 \leq i \leq m-1 \right\}.$$

Lemma 8. *Let $n = 2m + 1$ for $m \in \mathbb{N}$. Let $X, Y \in \mathcal{M}$ such that*

$$X \circ A = 0 \textit{ for every } A \in \mathcal{A}$$

and

$$Y \circ B = 0 \textit{ for every } B \in \mathcal{B}.$$

Then $X = ae_{11}$ and $Y = be_{nn}$ for some $a, b \in \mathbb{C}$.

Lemma 9. *Let $n = 2m$ for $m \in \mathbb{N}$. Let $X, Y \in \mathcal{M}$ such that*

$$X \circ C = 0 \textit{ for every } C \in \mathcal{C}$$

and

$$Y \circ D = 0 \textit{ for every } D \in \mathcal{D}.$$

Then

$$X = \begin{pmatrix} x_{11} & x_{12} & 0 & \cdots & 0 \\ x_{21} & x_{22} & 0 & & \\ 0 & 0 & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix},$$

and

$$Y = \begin{pmatrix} x_{11} & 0 & \cdots & 0 & x_{1n} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ x_{n1} & 0 & \cdots & 0 & x_{nn} \end{pmatrix},$$

where $x_{ij} \in \mathbb{C}$ for $i, j \in \{1, 2, n\}$.

The proofs of these lemmas are technical and straightforward, and so we shall omit them.

Proof of Theorem 4. We will consider three cases: the first when n is odd, the second when $n \geq 4$ is even, and the third when $n = 2$.

Using Theorem 3, we have that since the map g satisfies (5), g will preserve square-zero matrices. Let sl_n be the set of all matrices with trace 0. We can see that $g : sl_n \to sl_n$, and a result by Šemrl [12, Corollary 2] gives that when restricted to sl_n, g is of the form

$$g(Z) = cUZU^{-1}, \tag{11}$$

or

$$g(Z) = cUZ^TU^{-1}, \tag{12}$$

for every $Z \in sl_n$, where $U \in \mathcal{M}$ is invertible and $c \in \mathbb{C}$ is nonzero.

Notice that for any $X \in \mathcal{M}$, we may write $X = X_0 + \text{tr}(X)e_{11}$, where $X_0 \in sl_n$, and $\text{tr}(X)$ denotes the trace of X. If we can show that $g(e_{11}) = cUe_{11}U^{-1}$, then we will have either

$$g(X) = g(X_0) + \text{tr}(X)g(e_{11}) = cUX_0U^{-1} + \text{tr}(X)cUe_{11}U^{-1} = cUXU^{-1},$$

if g is of the form of (11) or, similarly,

$$g(X) = cUX^TU^{-1},$$

if g is of the form of (12), which is our desired result.

We define $\psi(X) = c^{-1}U^{-1}g(X)U$ for every $X \in \mathcal{M}$. It is clear that ψ is a linear bijective map. Since g satisfies (9), we also have that ψ satisfies (9). Additionally, we can see that for $Z \in sl_n$, $\psi(Z) = Z$ (if g is of the form of (11)) or $\psi(Z) = Z^T$ (if g is of the form of (12)). Finally, it is straightforward to see that $g(e_{11}) = cUe_{11}U^{-1}$ if and only if $\psi(e_{11}) = e_{11}$, so showing $\psi(e_{11}) = e_{11}$ will be our goal throughout cases 1 and 2, and part of case 3.

Case 1: $n = 2m + 1$ for some $m \in \mathbb{N}$. Let $A \in \mathcal{A}$ and observe that $(\sqrt{\lambda}e_{11} + A)^2 = (\sqrt{\lambda}e_{11} - A)^2 = \lambda I$, so by (9) we have

$$\psi(\sqrt{\lambda}e_{11} + A)^2 = \psi(\sqrt{\lambda}e_{11} - A)^2.$$

Using the linearity of ψ, this equation yields

$$\lambda\psi(e_{11})^2 + \sqrt{\lambda}\psi(e_{11}) \circ \psi(A) + \psi(A)^2 = \lambda\psi(e_{11})^2 - \sqrt{\lambda}\psi(e_{11}) \circ \psi(A) + \psi(A)^2,$$

and cancellation gives us $\psi(e_{11}) \circ \psi(A) = 0$. Since $\text{tr}(A) = 0$, ψ acts as either the identity map or the transpose map on A. In either case, $\psi(A) \in \mathcal{A}$. Therefore, we have that $\psi(e_{11}) \circ A = 0$ for every $A \in \mathcal{A}$. Lemma 8 and the bijectivity of ψ give us that $\psi(e_{11}) = ae_{11}$, where $a \in \mathbb{C} \setminus \{0\}$.

Now, let $B \in \mathcal{B}$, and notice that $(\sqrt{\lambda}e_{nn} + B)^2 = (\sqrt{\lambda}e_{nn} - B)^2 = \lambda I$, and again using (9), we have

$$\psi(\sqrt{\lambda}e_{nn} + B)^2 = \psi(\sqrt{\lambda}e_{nn} - B)^2.$$

Expanding and canceling as before, we get that $\psi(e_{nn}) \circ \psi(B) = 0$, and since $\psi(B) \in \mathcal{B}$, we have that $\psi(e_{nn}) \circ B = 0$ for every $B \in \mathcal{B}$. Therefore, once more using Lemma 8 and the bijectivity of ψ, we must have $\psi(e_{nn}) = be_{nn}$, where $b \in \mathbb{C} \setminus \{0\}$.

Since $e_{11} - e_{nn}$ is a symmetric matrix of trace 0, we know $\psi(e_{11} - e_{nn}) = e_{11} - e_{nn}$. However, $\psi(e_{11} - e_{nn}) = \psi(e_{11}) - \psi(e_{nn}) = ae_{11} - be_{nn}$, and thus $a = b = 1$. Hence, we have that $\psi(e_{11}) = e_{11}$.

Case 2: $n = 2m$ for some $m \in \mathbb{N}$ with $m \geq 2$. Let $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Observe that $(\sqrt{\lambda}(e_{11} + e_{22}) + C)^2 = (\sqrt{\lambda}(e_{11} + e_{22}) - C)^2 = \lambda I$, so by (9) we have

$$\psi(\sqrt{\lambda}(e_{11} + e_{22}) + C)^2 = \psi(\sqrt{\lambda}(e_{11} + e_{22}) - C)^2.$$

Using the linearity of ψ, this equation yields

$$\begin{aligned}\lambda\psi(e_{11} + e_{22})^2 + {} & \sqrt{\lambda}\psi(e_{11} + e_{22}) \circ \psi(C) + \psi(C)^2 \\ & = \lambda\psi(e_{11} + e_{22})^2 - \sqrt{\lambda}\psi(e_{11} + e_{22}) \circ \psi(C) + \psi(C)^2,\end{aligned}$$

and cancellation gives us $\psi(e_{11} + e_{22}) \circ \psi(C) = 0$. This results in

$$\psi(e_{11}) \circ \psi(C) + \psi(e_{22}) \circ \psi(C) = 0. \tag{13}$$

Similarly, we can show

$$\psi(e_{11}) \circ \psi(C) - \psi(e_{22}) \circ \psi(C) = 0, \tag{14}$$

$$\psi(e_{11}) \circ \psi(D) + \psi(e_{nn}) \circ \psi(D) = 0, \tag{15}$$

and

$$\psi(e_{11}) \circ \psi(D) - \psi(e_{nn}) \circ \psi(D) = 0. \tag{16}$$

We can add (13) to (14) to obtain $\psi(e_{11}) \circ \psi(C) = 0$. Analogously, we can add (15) to (16) to get $\psi(e_{11}) \circ \psi(D) = 0$. Since $\psi(C) \in \mathcal{C}$, we have that $\psi(e_{11}) \circ C = 0$ for every $C \in \mathcal{C}$. Therefore, by Lemma 9, $\psi(e_{11})$ must have the form

$$\begin{pmatrix} x_{11} & x_{12} & 0 & \cdots & 0 \\ x_{21} & x_{22} & 0 & & \\ 0 & 0 & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix},$$

where $x_{ij} \in \mathbb{C}$ for $i, j \in \{1, 2\}$. In addition, since $\psi(D) \in \mathcal{D}$, we have that $\psi(e_{11}) \circ D = 0$ for every $D \in \mathcal{D}$. Hence, by Lemma 9, $\psi(e_{11})$ is equal to

$$\begin{pmatrix} x_{11} & 0 & \cdots & 0 & x_{1n} \\ 0 & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & 0 \\ x_{n1} & 0 & \cdots & 0 & x_{nn} \end{pmatrix},$$

where $x_{ij} \in \mathbb{C}$ for $i, j \in \{1, n\}$. Combining the two forms of $\psi(e_{11})$ results in $\psi(e_{11}) = d_1 e_{11}$ for some $d_1 \in \mathbb{C} \setminus \{0\}$.

We may use a similar approach to prove that $\psi(e_{nn}) = d_n e_{nn}$ for some nonzero $d_n \in \mathbb{C}$. Since $e_{11} - e_{nn}$ has trace 0, we know that $\psi(e_{11} - e_{nn}) = e_{11} - e_{nn}$. However, we also know that $\psi(e_{11} - e_{nn}) = \psi(e_{11}) - \psi(e_{nn}) = d_1 e_{11} - d_n e_{nn}$, and so we must have that $d_1 = d_n = 1$. Therefore, $\psi(e_{11}) = e_{11}$, as desired.

Case 3: $n = 2$. Note that $(\sqrt{\lambda} e_{11} + \sqrt{\lambda} e_{22})^2 = (\sqrt{\lambda} e_{11} - \sqrt{\lambda} e_{22})^2 = \lambda I$, so by (9) we have

$$\psi(e_{11} + e_{22})^2 = \psi(e_{11} - e_{22})^2.$$

Next, we know that $e_{11} - e_{22}$ has trace zero and is symmetric, so $\psi(e_{11} - e_{22}) = e_{11} - e_{22}$. This implies that

$$\psi(I)^2 = \psi(e_{11} + e_{22})^2 = \psi(e_{11} - e_{22})^2 = (e_{11} - e_{22})^2 = I,$$

which in turn yields

$$(\psi(I) - I)(\psi(I) + I) = 0. \tag{17}$$

Assume both $\psi(I) - I$ and $\psi(I) + I$ are singular. Let

$$\psi(I) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

so that

$$\det(\psi(I) - I) = (w_{11} - 1)(w_{22} - 1) - w_{12} w_{21} = 0,$$
$$\det(\psi(I) + I) = (w_{11} + 1)(w_{22} + 1) - w_{12} w_{21} = 0.$$

This system of equations gives us $w_{11} + w_{22} = 0$. Hence, tr($\psi(I)$)=0. However, since ψ is bijective and maps sl_n to itself, ψ must send matrices of nonzero trace to matrices with nonzero trace, and so we have a contradiction. Therefore, one of $\psi(I) - I$ or $\psi(I) + I$ is invertible, so from (17), either $\psi(I) - I = 0$ or $\psi(I) + I = 0$.

If $\psi(I) = I$, then

$$\psi(I) = \psi(e_{11} + e_{22}) = e_{11} + e_{22}.$$

Furthermore, we know

$$\psi(e_{11} - e_{22}) = e_{11} - e_{22}.$$

Adding these two equations and using the linearity of ψ, we see that $\psi(e_{11}) = e_{11}$, as desired.

If $\psi(I) = -I$, then

$$\psi(I) = \psi(e_{11} + e_{22}) = -e_{11} - e_{22}.$$

Moreover, we know

$$\psi(e_{11} - e_{22}) = e_{11} - e_{22}.$$

Adding these two equations and using the linearity of ψ, we have $\psi(e_{11}) = -e_{22}$ and $\psi(e_{22}) = -e_{11}$. Additionally, for any $r, s \in \mathbb{C}$, we know $re_{12} + se_{21}$ has trace 0, so that either

$$\psi(re_{12} + se_{21}) = re_{12} + se_{21},$$

or

$$\psi(re_{12} + se_{21}) = se_{12} + re_{21}.$$

This gives us a complete description of ψ. In particular, let $X \in \mathcal{M}$ be arbitrary and let $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $E^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have that

$$\psi(X) = -EXE^{-1},$$

or

$$\psi(X) = -EX^T E^{-1}.$$

Using the definition of ψ, we find that either

$$\begin{aligned} g(X) &= -cUEXE^{-1}U^{-1} \\ &= -c(UE)X(UE)^{-1}, \end{aligned}$$

or

$$g(x) = -c(UE)X^T(UE)^{-1}.$$

In either case, we have that g is of the desired form. $\square$

Acknowledgement

The authors would like to thank Dr. Mikhail Chebotar for his guidance and assistance. Additionally, we would like to thank the Department of Mathematical Sciences at Kent State University for its hospitality. The authors are supported in part by the NSF grant DMS-1653002.

References

[1] M. Brešar, Characterizing homomorphisms, derivations and multipliers in rings with idempotents, Proc. Roy. Soc. Edinburgh 137A (2007) 9–21.
[2] L. Catalano, On maps characterized by action on equal products, J. Algebra 511 (2018) 148–154.
[3] M.A. Chebotar, W.-F. Ke, P.-H. Lee, Maps characterized by action on zero products, Pacific J. Math. 216 (2004) 217–228.
[4] M.A. Chebotar, W.-F. Ke, P.-H. Lee, On maps preserving square-zero matrices, J. Algebra 289 (2005) 421–445.
[5] M.A. Chebotar, W.-F. Ke, P.-H. Lee, L.-S. Shiao, On maps preserving products, Canad. Math. Bull. 48 (2005) 355–369.
[6] M.A. Chebotar, W.-F. Ke, P.-H. Lee, R. Zhang, On maps preserving zero Jordan products, Monatsh. Math. 149 (2006) 91–101.
[7] G.W. Cross, P. Lancaster, Square roots of complex matrices, Linear Multilinear Algebra 1 (1974) 289–293.
[8] J. Cui, J. Hou, Linear maps on Von Neumann algebras preserving zero products or TR-rank, Bull. Aust. Math. Soc. 65 (2002) 79–91.
[9] H. Essannouni, A. Kaidi, Le théorème de Hua pour les algeèbres artiniennes simples, Linear Algebra Appl. 291 (1999) 9–22.
[10] W. Huang, J. Li, J. He, Characterizations of Jordan mappings on some rings and algebras through zero products, Linear Multilinear Algebra 66 (2018) 334–346.
[11] Y.-F. Lin, T.-L. Wong, A note on 2-local maps, Proc. Edinb. Math. Soc. 49 (2006) 701–708.
[12] P. Šemrl, Linear mappings preserving square-zero matrices, Bull. Aust. Math. Soc. 48 (1993) 365–370.