



A singular Sphere Covering Inequality: uniqueness and symmetry of solutions to singular Liouville-type equations

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Abstract

We derive a singular version of the Sphere Covering Inequality which was recently introduced in Gui and Moradifam (Invent Math. <https://doi.org/10.1007/s00222-018-0820-2>, 2018) suitable for treating singular Liouville-type problems with superharmonic weights. As an application we deduce new uniqueness results for solutions of the singular mean field equation both on spheres and on bounded domains, as well as new self-contained proofs of previously known results, such as the uniqueness of spherical convex polytopes first established in Luo and Tian (Proc Am Math Soc 116(4):1119–1129, 1992). Furthermore, we derive new symmetry results for the spherical Onsager vortex equation.

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1 Introduction

We are concerned with a class of elliptic equations with exponential nonlinearities, namely the following Liouville-type equation,

$$\Delta u + h(x)e^u = f(x) \quad \text{in } \Omega, \quad (1)$$

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where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and $h(x)$ is a positive function. The latter equation (and its counterpart on manifolds, see (15) below) has been widely discussed in the last decades since it arises in several problems of mathematics and physics, such as Electroweak and Chern–Simons self-dual vortices [66,68,75], conformal geometry on surfaces [25,26,49,71], statistical mechanics of two-dimensional turbulence [21] and of self-gravitating systems [74] and cosmic strings [62], theory of hyperelliptic curves [23], Painlevé equations [28] and Moser–Trudinger inequalities [19,37,43,48,61]. There are by now many results concerning existence and multiplicity [3,7–9,11,16,17,22,31,33–36,56,58,59], uniqueness [13–15,45,47,53,55,67], blow-up phenomena [6,10,18,20,30,32,50,51,70,76] and entire solutions [2,29,64].

1.1 Singular Sphere Covering Inequality

A basic inequality related to (1) was recently introduced in [45], see Theorem A below, which yields sharp Moser–Trudinger inequalities and symmetry properties of Liouville type equations in $\mathbb{R}^2$ [45], symmetry properties of mean field equations on flat tori [46], uniqueness of solutions of mean field equation in bounded domains [47] and symmetry and uniqueness properties of Sinh-Gordon equation and Toda systems in bounded domains [44]. The inequality can be stated in the following form, see Theorem 3.1 in [45] and Theorem 1.1 in [47].

Theorem A [45] *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected domain and let $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$, $i = 1, 2$, satisfy,*

$$\Delta u_i + h(x)e^{u_i} = f_i(x) \text{ in } \Omega, \quad (2)$$

where $f_1, f_2, h = e^H \in C^2(\Omega) \cap C(\overline{\Omega})$ are such that,

$$f_2 \geq f_1 \geq -\Delta H \text{ in } \Omega. \quad (3)$$

Suppose that,

$$\begin{cases} u_2 \geq u_1, & u_2 \not\equiv u_1 \text{ in } \Omega, \\ u_2 = u_1 & \text{on } \partial\Omega. \end{cases}$$

Then it holds,

$$\int_{\Omega} (h(x)e^{u_1} + h(x)e^{u_2}) \, dx \geq 8\pi.$$

The latter result is based on symmetric rearrangements and the Alexandrov–Bol inequality, see the discussion in the sequel. Suppose for simplicity $h \equiv 1$ and $f_1 \equiv f_2 \equiv 0$ in (2). Then, by means of the substitution $\tilde{u}_i = \frac{u_i}{2} - \ln(2)$, Theorem A roughly asserts that the total area of two distinct neighbourhoods M_1, M_2 , with Gaussian curvature equal to 1, of possibly distinct surfaces, such that M_1 and M_2 admits local conformal charts $\Phi_i : M_i \rightarrow B_1, i = 1, 2$ where B_1 is the Euclidean unit disk, with the

same conformal factor on the boundary, is greater than that of the whole unit sphere, which is why one refers to the latter result as the Sphere Covering Inequality.

We point out that condition (3) on the weight $h(x)$ can not be dropped and indeed the result is false in general if we remove such assumption. Suppose for a moment $f_1 \equiv f_2 \equiv 0$ in Theorem A. With a small abuse of terminology, a weight h satisfying (3) with $f_1 \equiv f_2 \equiv 0$, i.e. $\Delta H \geq 0$, will be referred as a subharmonic weight. Analogously, we will refer to superharmonic weights whenever we have the reverse inequality $\Delta H \leq 0$. Such restriction on the weights, which in this case need to be subharmonic, prevents the application of the Sphere Covering Inequality in a large class of problems which are rather interesting and challenging, some of which we will address later on.

On the other hand, Theorem A is obtained in a smooth setting and thus is not suitable in treating singular problems. However, the presence of singular sources in (1) naturally arises both in geometry and in mathematical physics, typically as a sum of Dirac deltas which represent either conic points of the metric or vortex points of gauge or vorticity fields (see the references above). One of our aims is to address such kind of problems, see Sects. 1.2 and 1.3. However, the generalization of Theorem A to a weak setting is not straightforward and one needs to carry out a delicate argument.

Therefore, our first goal in this paper is to provide a singular version of the Sphere Covering Inequality suitable for treating both singular problems and superharmonic weights, see Theorem 1.2 below. In order to state the result let us introduce some notation first. Given $\alpha \in [0, 1)$, $\lambda > 0$ we set

$$U_{\lambda, \alpha}(x) = \ln \left(\frac{\lambda(1 - \alpha)}{1 + \frac{\lambda^2}{8} |x|^{2(1 - \alpha)}} \right)^2, \quad (4)$$

which satisfies

$$\Delta U_{\lambda, \alpha} + |x|^{-2\alpha} e^{U_{\lambda, \alpha}} = 0 \text{ in } \mathbb{R}^2 \setminus \{0\}. \quad (5)$$

Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain. Let $f \in L^s(\Omega)$ for some $s > 2$ and $h = e^H$ be given. Let F be the solution of

$$\Delta F = f \text{ in } \Omega, \quad F = 0 \text{ on } \partial\Omega,$$

so that (1) can be equivalently formulated as

$$\Delta v + e^{H+F} e^v = 0 \text{ in } \Omega,$$

where $v = u - F$. By the Riesz decomposition we have,

$$H + F = \mathcal{H}_+ - \mathcal{H}_-,$$

where $\mathcal{H}_+, \mathcal{H}_-$ are two superharmonic functions taking the form

$$\mathcal{H}_{\pm}(x) = \mathfrak{h}_{\pm}(x) + \int_{\Omega} G_x(y) d\mu_{\pm}(y), \quad (6)$$

where $\mathfrak{h}_{\pm}$ are harmonic functions in Ω , $\mu_{\pm}$ are non negative and mutually orthogonal measures of bounded total variation in Ω and G_p , $p \in \Omega$, is the Green's function,

$$\begin{cases} -\Delta G_p(y) = \delta_p & \text{in } \Omega, \\ G_p(y) = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Definition 1.1 Let $\omega \subseteq \Omega$ be any nonempty subdomain. We will denote by $\tilde{\omega}$ the interior of the closure of the union of ω with its “holes”, that is, with the bounded component of the complement of ω in $\mathbb{R}^2$. If ω is simply-connected or if it is the union of simply connected domains, then $\tilde{\omega} \equiv \omega$, while if the bounded component of the complement of ω in $\mathbb{R}^2$ is not empty, then $\omega \subset \tilde{\omega}$.

Definition 1.2 Let f, h be given as above and let μ_+ be defined as in (6). Let $\omega \subseteq \Omega$ be any nonempty subdomain and let $\tilde{\omega}$ be given as in Definition 1.1. We define $\alpha(\omega) = \alpha(\omega, h, f) \geq 0$ to be

$$\alpha(\omega) = \frac{1}{4\pi} \mu_+(\tilde{\omega}). \quad (8)$$

For reader's convenience let us show how the above construction applies to two model cases.

Example 1.1 Let us consider a smooth setting where $f, h \in C^2(\Omega) \cap C(\bar{\Omega})$ and suppose for simplicity $\tilde{\omega} = \omega$. Then we have,

$$\alpha(\omega) = \frac{1}{4\pi} \int_{E_+} -\Delta(H(x) + F(x)) dx = \frac{1}{4\pi} \int_{E_+} -(\Delta H(x) + f(x)) dx,$$

where

$$E_+ = \{x \in \omega : -(\Delta H(x) + f(x)) > 0\},$$

while clearly $\alpha(\omega) = 0$ whenever $E_+ = \emptyset$.

Example 1.2 Let us consider a singular weight $h = e^H$ in the form

$$H(x) = 4\pi \sum_{i=1}^m \alpha_i G_{p_i}(x) - 4\pi \sum_{j=1}^n \beta_j G_{q_j}(x),$$

where $G_p(x)$ is given in (7), p_i, q_j are distinct points in Ω and $\alpha_i, \beta_j > 0$. We have,

$$-\Delta H = 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i} - 4\pi \sum_{j=1}^n \beta_j \delta_{q_j}.$$

Suppose for simplicity $f = 0$ and $\tilde{\omega} = \omega$. Then we have $\mu_+ = 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i}$ and

$$\alpha(\omega) = \sum_{i \in I} \alpha_i,$$

where $I = \{i \in \{1, \dots, m\} : p_i \in \omega\}$.

Finally, from now on we will suppose that,

$$\alpha(\Omega) < 1 \text{ i.e. } \mu_+(\Omega) < 4\pi. \quad (9)$$

This is motivated by the case when all the measure μ_+ is collapsed to a singular Dirac delta, i.e. $\mu_+ = 4\pi\alpha_p\delta_p$, where we need $\alpha(\Omega) = \alpha_p < 1$ to ensure the integrability of e^{H+F} . From the geometric point of view this means that p is not a “cusp” but just a conical singularity, see [65].

Remark 1.1 In particular, since we are assuming $\mu_+(\Omega) < 4\pi$, then we deduce the following regularity property. Observe that there exists at most one point $x_0 \in \Omega$ such that $\mu_+(x_0) \geq 2\pi$. Thus, if we start from $he^u \in L^1(\Omega)$ where $u \in L^1(\Omega)$ is a solution of (1) in the sense of distributions, then it can be shown that $u \in W^{2,q}(\Omega)$ for some $q > 1$ and that for each $r > 0$ small enough there exists $s_r > 2$ such that $u \in W^{2,s_r}(\Omega \setminus B_r(x_0))$, see [5].

We will quote this regularity property by saying that $u \in W_{\text{loc}}^{2,s,\text{loc}}(\Omega \setminus \{x_0\})$ for some $s > 2$. In particular we conclude that u is a strong solution of (1).

Our first main result is the following singular Sphere Covering Inequality.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected domain. Let $x_0 \in \Omega$ be fixed as in Remark 1.1 and $u_i \in W_{\text{loc}}^{2,s,\text{loc}}(\Omega \setminus \{x_0\}) \cap W^{2,q}(\Omega) \cap C(\overline{\Omega})$ for some $s > 2$ and some $q > 1$, $i = 1, 2$, satisfy,*

$$\Delta u_i + h(x)e^{u_i} = f_i(x) \text{ in } \Omega, \quad (10)$$

where $h = e^H$ and $f_1, f_2 \in L^s(\Omega)$ are such that $\alpha(\Omega, h, f_i) < 1$, $i = 1, 2$, with $\alpha(\Omega, h, f_i)$ defined as in (8). Suppose that,

$$f_2 \geq f_1 \text{ a.e. in } \Omega,$$

and that there exists a smooth subdomain $\omega \subseteq \Omega$ such that,

$$\begin{cases} u_2 \geq u_1, & u_2 \not\equiv u_1 \text{ in } \omega, \\ u_2 = u_1 & \text{on } \partial\omega. \end{cases} \quad (11)$$

Let $\alpha(\omega) = \alpha(\omega, h, f_1)$. Then, it holds

$$\int_{\omega} (h(x)e^{u_1} + h(x)e^{u_2}) dx \geq 8\pi(1 - \alpha(\omega)). \quad (12)$$

Moreover, the equality holds if and only if (modulo conformal transformations) $\omega = B_{\delta}(0)$ for some $\delta > 0$, $f_1 \equiv f_2 := f$, $h(x)e^{u_i} \equiv |x|^{-2\alpha} e^{U_{\lambda_i, \alpha}}$, $i = 1, 2$, for some $\lambda_2 > \lambda_1$ where $U_{\lambda_i, \alpha}$ are defined as in (4), $\mu_+ = -\Delta H - f = 4\pi\alpha\delta_{p=0}$ in ω and $\alpha = \alpha(\omega)$.

Remark 1.3 It will be evident from the proof of Theorem 1.2 that this theorem also holds for any smooth unbounded, and simply-connected domain Ω provided (11) is replaced by

$$\begin{cases} u_2 \geq u_1, & u_2 \not\equiv u_1 & \text{in } \omega, \\ u_2 = u_1 & & \text{on } \partial\omega, \\ \lim_{\substack{|x| \rightarrow \infty \\ x \in \omega}} (u_2 - u_1)(x) = 0. \end{cases}$$

Remark 1.4 We point out that if $f_1 \equiv f_2$, the condition $u_2 \geq u_1$, $u_2 \not\equiv u_1$ in the above theorem can be replaced just by $u_2 \not\equiv u_1$. We also observe that if $\alpha(\omega) = 0$ we recover the standard Sphere Covering Inequality of Theorem A, which however holds now in a weak setting.

Remark 1.5 The smoothness assumption in Theorem 1.2 about ω is not necessary and can be remarkably weakened as far as $\omega \subseteq \Omega$, see [5].

The latter result is obtained in the spirit of Theorem A with a non trivial adaptation to the weak setting by exploiting weighted symmetric rearrangements and the Alexandrov–Bol inequality, see Sect. 2 for full details. We stress that, according to the terminology introduced right after Theorem A, we are now able to cover the case of superharmonic weights.

Suppose for the moment that $f_1 \equiv f_2 \equiv 0$ in Theorem 1.2. Observe that the case $h(x)e^{u_i} \equiv |x|^{-2\alpha} e^{U_{\lambda_i, \alpha}}$, $i = 1, 2$, corresponding to the equality in Theorem 1.2 is a limiting case in which all the measure μ_+ is concentrated in a Dirac delta, i.e. $\mu_+ = 4\pi\alpha\delta_{p=0}$. In particular, the latter Dirac delta corresponds to a conical singularity of order $-\alpha$ on the surface related to (5), see for example [5]. In other words, by means of the substitution $\tilde{u}_i = \frac{u_i}{2} - \ln(2)$, in the terminology of singular surfaces (in the sense of Alexandrov [65]), Theorem 1.2 roughly asserts that the total area of two distinct neighbourhoods M_1, M_2 , with regular Gaussian curvature equal to 1, of possibly distinct singular surfaces, such that M_1 and M_2 admits local conformal charts $\Phi_i : M_i \rightarrow B_1$, $i = 1, 2$ where B_1 is the Euclidean unit disk, with the same conformal factor on the boundary, and the same local total (singular) curvature (which is $\frac{1}{4\pi}\mu = -\frac{1}{4\pi}\Delta H$ in B_1), is greater than that of a whole unit sphere with two antipodal conical singularities of order $-\alpha$ (where α is $\frac{1}{4\pi}\mu_+(B_1)$), namely an ‘American football’. We refer to [5] for more details concerning this geometric interpretation. Therefore we allude to Theorem 1.2 as the singular Sphere Covering Inequality.

Let us return to Eq. (10). If we further assume the two solutions u_1, u_2 have the same total mass in Ω , i.e. that (14) below holds, then we can argue as in [47] and improve Theorem 1.2. More precisely, we can relax the boundary condition in (11) and treat more general situations, see (13). This will be crucially used in proving uniqueness of solutions of singular Liouville equation (21) on bounded domains, see Theorem 1.10. Even though Theorem 1.6 shares some similarities with Theorem 1.2, its proof, based on a reversed Alexandrov–Bol’s inequality, substantially differs from the proof of Theorem 1.2, see Sect. 3. Our second main result is the following.

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected domain. Let $x_0 \in \Omega$ be fixed as in Remark 1.1 and $u_i \in W_{\text{loc}}^{2,s,\text{loc}}(\Omega \setminus \{x_0\}) \cap W^{2,q}(\Omega) \cap C(\overline{\Omega})$ for some $s > 2$ and some $q > 1$, $i = 1, 2$, satisfy,*

$$\Delta u_i + h(x)e^{u_i} = f_i(x) \text{ in } \Omega,$$

where $h = e^H$ and $f_1, f_2 \in L^s(\Omega)$ are such that $\alpha(\Omega, h, f_i) < 1$, $i = 1, 2$, where $\alpha(\Omega, h, f_i)$ are defined in (8). Suppose that,

$$f_2 \geq f_1 \text{ a.e. in } \Omega,$$

and that,

$$\begin{cases} u_1 \not\equiv u_2 & \text{in } \Omega, \\ u_2 - u_1 = c \text{ on } \partial\Omega, \end{cases} \quad (13)$$

for some $c \in \mathbb{R}$. Suppose moreover that,

$$\int_{\Omega} h(x)e^{u_1} dx = \int_{\Omega} h(x)e^{u_2} dx = \rho, \quad (14)$$

and set $\alpha(\Omega) = \alpha(\Omega, h, f_1)$. Then, it holds $\rho > 8\pi(1 - \alpha(\Omega))$.

Remark 1.7 It will be evident from the proof of Theorem 1.6 that this theorem also holds for any smooth unbounded, and simply-connected domain provided (13) is replaced by

$$\begin{cases} u_1 \not\equiv u_2 & \text{in } \Omega, \\ u_2 - u_1 = c & \text{on } \partial\Omega, \\ \lim_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} (u_2 - u_1)(x) = c. \end{cases}$$

It is worth to point out the following fact concerning the assumption about Ω being simply-connected.

Remark 1.8 Actually, both Theorems 1.2 and 1.6 hold for multiply-connected domains Ω provided the solutions of (10) take constant values on $\partial\Omega$, more precisely $u_i + H = c_i$ on $\partial\Omega$ for some $c_1, c_2 \in \mathbb{R}$. This follows from the fact that in the latter situation an

Alexandrov–Bol’s inequality related to (10) with Ω multiply-connected is available, see [14].

To motivate our studies and to see some applications of Theorems 1.2 and 1.6 we will address uniqueness issues concerning singular Liouville-type equations both on spheres and on bounded domains, and symmetry properties for the spherical Onsager vortex equation, see respectively Sects. 1.2–1.4 below. The argument will yield both to new results as well as to new self-contained proofs of previously known results, such as the uniqueness of spherical convex polytopes first established in [57]. We believe that Theorems 1.2 and 1.6 will have several other applications as it was for Theorem A (see the discussion before Theorem A). This will be the topic of our forthcoming papers.

1.2 The singular Liouville equation on $\mathbb{S}^2$

Let us start by considering the following equation

$$\Delta_g v + \rho \left(\frac{e^v}{\int_{\mathbb{S}^2} e^v dV_g} - \frac{1}{4\pi} \right) = 4\pi \sum_{j=1}^N \alpha_j \left(\delta_{p_j} - \frac{1}{4\pi} \right) \quad \text{on } \mathbb{S}^2, \quad (15)$$

$$\int_{\mathbb{S}^2} v dV_g = 0,$$

where ρ is a positive parameter, $\{p_1, \dots, p_N\} \subset \mathbb{S}^2$, $\alpha_j > -1$ for $j = 1, \dots, N$ and $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere with $|\mathbb{S}^2| = 4\pi$ equipped with its standard Riemannian metric g , Δ_g is the Laplace–Beltrami operator and dV_g is the volume form. Since the equation in (15) is invariant under translations $v \mapsto v + c$, then we can normalize solutions to have zero mean value.

Problem (15) is related to both mean field equations with vortex points and spherical metrics with conic singularities. We refer to the references in the beginning of the introduction for more details and some of the known results.

Equation (15) can be equivalently considered on the plane $\mathbb{R}^2$ via the stereographic projection: suppose without loss of generality that no one of the points p_i ’s coincides with the north pole $\mathcal{N} = (0, 0, 1) \in \mathbb{R}^3$ and let $\Pi : \mathbb{S}^2 \setminus \{\mathcal{N}\} \rightarrow \mathbb{R}^2$ be the stereographic projection with respect to $\mathcal{N}$, i.e.

$$\Pi(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right), \quad (16)$$

and define

$$w(x) = v(\Pi^{-1}(x)) - \ln \left(\int_{\mathbb{S}^2} e^v dV_g \right) \quad x \in \mathbb{R}^2.$$

With a small abuse of notation we will write $\sum_j \alpha_j$ to denote $\sum_{j=1}^N \alpha_j$. Then, w satisfies

$$\Delta w + \frac{4\rho}{(1 + |x|^2)^2} e^w = \frac{\rho - 4\pi \sum_j \alpha_j}{4\pi} \frac{4}{(1 + |x|^2)^2} + 4\pi \sum_{j=1}^N \alpha_j \delta_{q_j} \quad \text{on } \mathbb{R}^2,$$

for some $\{q_1, \dots, q_N\} \subset \mathbb{R}^2$. Letting further,

$$u(x) = w(x) - \frac{\rho - 4\pi \sum_j \alpha_j}{4\pi} \ln(1 + |x|^2) + \ln(4\rho),$$

we get

$$\Delta u + h(x)e^u = 4\pi \sum_{j=1}^N \alpha_j \delta_{q_j} \quad \text{on } \mathbb{R}^2, \quad (17)$$

where

$$h(x) = (1 + |x|^2)^{-l}, \quad l = \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi}, \quad (18)$$

and

$$\int_{\mathbb{R}^2} h(x)e^u dx = \rho.$$

Next, let us discuss what is the measure μ_+ defined in (6) corresponding to equation (17). To make the presentation simpler let us consider the regular case, i.e. $N = 0$, postponing the general case to Sect. 4. Recall the notation $h = e^H$. In order to apply either Theorem A or Theorem 1.2 we need first to consider the sign of

$$\Delta H(x) = \frac{\rho - 8\pi}{4\pi} \frac{4}{(1 + |x|^2)^2}. \quad (19)$$

Observe that the sign of this term depends on whether $\rho < 8\pi$ or $\rho \geq 8\pi$. The value $\rho = 8\pi$ plays an important role in Liouville-type problems alike (15): for example, it is related to the sharp Moser–Trudinger inequality which yields boundedness from below and coercivity of the energy functional associated to (15) for $\rho < 8\pi$ (we refer to the survey [58] for full details on this matter). For the latter range of the parameter ρ one expects the solution of (15) to be unique: this indeed holds true and it was first obtained in [52,53]. First, one carries out the moving plane method to show that all the solutions to (17) and (18) with $\rho < 8\pi$ (and $N = 0$) are radially symmetric with respect to the origin. Then, by considering the associated O.D.E. one can eventually prove uniqueness of such solutions.

However Theorem A does not apply in this framework since the term in (19) is negative and thus the weight $h(x)$ is superharmonic according to the terminology introduced right after Theorem A.

The argument in [52] applies also to the singular case $N = 1$ with $\alpha_1 > -1$. Letting $\alpha_- = \min\{\alpha_1, 0\}$ one deduces uniqueness of solution for $\rho < 4\pi(2 + \alpha_-)$. It seems the latter strategy can be extended to the case $N = 2$ of two antipodal singular points with $\alpha_1, \alpha_2 < 0$, yielding uniqueness for $\rho < 4\pi(2 + \sum_j \alpha_j)$, see for example the discussion in [64]. On the other hand, the case of multiple singular sources $N \geq 3$ is

almost completely open due to the fact that we can not rely on radial properties any more. One of the few exceptions concern metrics of constant Gaussian curvature with N -conical singularities at p_j of negative order $\alpha_j \in (-1, 0)$ on $\mathbb{S}^2$ (convex polytopes), that is, solutions of (15) with,

$$\rho = 4\pi \left(2 + \sum_j \alpha_j \right).$$

Based on an algebraic geometric approach, the authors in [57] showed (among other things) the uniqueness of such metrics in the subcritical case [71] (i.e. $4\pi(2 + \sum_j \alpha_j) < 8\pi(1 + \min_j \{\alpha_j, 0\})$), when $\alpha_j \in (-1, 0)$ for all $j = 1, \dots, N$ and $N \geq 3$. Here we will exploit the singular Sphere Covering Inequality of Theorem 1.2 to handle both superharmonic weights and the multiple singular sources in (17), see Sect. 4. As a consequence we obtain uniqueness results relevant for both the mean field theory of 2D turbulence [21,27] and the uniqueness of convex polytopes [57]. Our third main result is the following,

Theorem 1.9 *Let $\rho > 0$ and $\alpha_j \in (-1, 0)$ for $j = 1, \dots, N$, $N \geq 0$. Then we have:*

- (i) *If $N \geq 0$ and $\rho < 4\pi(2 + \sum_j \alpha_j)$, then (15) admits at most one solution;*
- (ii) *If $N \geq 3$ and $\rho = 4\pi(2 + \sum_j \alpha_j)$, then (15) admits at most one solution.*

The proof of Theorem 1.9 is rather delicate since, among other facts (see Remark 1.12), it requires a careful use of the characterization of the equality sign in the singular Sphere Covering Inequality (12). After all this is not surprising since part (ii) is somehow sharp. Indeed, for $N = 1$ solutions to (15) with $\rho = 4\pi(2 + \alpha_1)$ do not exist, as it is well known that the ‘tear drop’ (which is $\mathbb{S}^2$ with one conical singularity) does not admit constant curvature, see for example [2,41]. On the other hand, for $N = 0$ and $N = 2$ (still with $-1 < \alpha_1 \leq \alpha_2 < 0$ and $\rho = 4\pi(2 + \sum_j \alpha_j)$), solutions to (15) are classified and uniqueness does not hold, see [29,64,72]. In particular, uniqueness fails in general if some α_j is positive, see for example Remark 2 in [57] or Remark 3 in [38].

Moreover, Theorem 1.9 also covers most of the previously known results and gives a new self-contained proof of them. Indeed, for $N = 0$ and $\rho < 8\pi$, we get the sharp uniqueness result of [52,53], for $N \leq 2$ and $\rho < 4\pi(2 + \sum_j \alpha_j)$, we obtain the sharp result of [52], while part (ii) covers the uniqueness of convex polytopes [57].

Finally, we remark that whenever the subcriticality condition $\rho < 8\pi(1 + \min_j \{\alpha_j, 0\})$ is also satisfied, then we have existence [71] and uniqueness in (i) and (ii). The fact that the uniqueness threshold $4\pi(2 + \sum_j \alpha_j)$ may be larger than the subcritical threshold seems to suggest another possible application of Theorem 1.9 to the non existence issue for (15) in the supercritical regime $\rho \in (8\pi(1 + \min_j \{\alpha_j, 0\}), 4\pi(2 + \sum_j \alpha_j))$. Indeed, the recent evaluation of the topological degree d_ρ associated to (15) in [33] shows that if $N \geq 3$ and $-1 < \alpha_1 < \dots < \alpha_N < 0$, then $d_\rho = 0$ for $\rho \in (8\pi(1 + \alpha_1), 4\pi(2 + \sum_j \alpha_j))$. If we knew that any such a solution is non-

degenerate, then we would conclude by Theorem 1.9 that solutions do not exist in this supercritical region. This motivates the following:

Open problem Is it true that if $N \geq 3$, $-1 < \alpha_1 < \dots < \alpha_N < 0$ and $\rho \in (8\pi(1 + \alpha_1), 4\pi(2 + \sum_j \alpha_j))$, then (15) has no solutions?

We point out that some non existence results in this direction were obtained in [2,16,69] only for the case $N \leq 2$ by using Pohozaev-type identities.

We conclude this part by mentioning some related results. The uniqueness of solutions to (15) for $N = 0$ was recently improved to the case $\rho \in (8\pi, 16\pi]$ in [45] by using Theorem A. In [38], exploiting arguments from complex analysis the author shows uniqueness of metrics of constant Gaussian curvature with 3-conical singularities (i.e. $\rho = 4\pi(2 + \sum_j \alpha_j)$) provided all α_j 's are not integers, see also [39,40] for further discussions. Finally, a nondegeneracy result for solutions of (15) for $N \geq 3$ with only one or two negative α_j was recently obtained in [73].

1.3 The singular Liouville equation on bounded domains

Next let us consider the counterpart of (15) on bounded domains, that is,

$$\begin{cases} \Delta u + \rho \frac{e^u}{\int_{\Omega} e^u dx} = 4\pi \sum_{j=1}^N \alpha_j \delta_{p_j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (20)$$

where $\Omega \subset \mathbb{R}^2$ is a smooth open bounded domain, ρ is a positive parameter, $\{p_1, \dots, p_N\} \subset \Omega$ and $\alpha_j > -1$ for $j = 1, \dots, N$. The latter equation is related to mean field equations of turbulent Euler flows and we refer to the references above for more details about this point.

As for (15) in Sect. 1.2 one expects uniqueness of solutions to (20) below a certain level of ρ . Indeed, for the regular case (i.e. $N = 0$), uniqueness was first proved in [67] for Ω simply-connected and $\rho < 8\pi$, then improved in [24] for $\rho = 8\pi$ and finally generalized to the case of Ω multiply-connected in [14,47] for the case of more general boundary conditions. The argument is mainly based on the Alexandrov–Bol inequality and the study of the linearized equation, and it was generalized in [13] to cover the singular case where $\rho \leq 8\pi$ and $\alpha_j > 0$ for all $j = 1, \dots, N$, $N > 0$. More recently, in [73] the authors considered the case of one negative singularity, i.e. $\alpha_1 \in (-1, 0)$ and $\alpha_j > 0$ for all $j = 2, \dots, N$, proving uniqueness of solutions provided $\rho \leq 8\pi(1 + \alpha_1)$.

On the other hand, the case of multiple negative singular sources is completely open and we will apply the singular Sphere Covering Inequality of Theorem 1.6 to handle this situation. Indeed, suppose that there exist u_1, u_2 satisfying (20). We set,

$$w_i(x) = u_i(x) - \ln \left(\int_{\Omega} e^{u_i} dx \right) + \ln(\rho), \quad i = 1, 2,$$

so that, for $i = 1, 2$, we have,

$$\begin{cases} \Delta w_i + e^{w_i} = 4\pi \sum_{j=1}^N \alpha_j \delta_{p_j} & \text{in } \Omega, \\ w_i = c_i & \text{on } \partial\Omega, \end{cases} \quad (21)$$

where

$$c_i = -\ln \left(\int_{\Omega} e^{u_i} dx \right) + \ln(\rho), \quad i = 1, 2,$$

and

$$\int_{\Omega} e^{w_1} dx = \int_{\Omega} e^{w_2} dx = \rho.$$

Next, let us define,

$$J = \{j \in \{1, \dots, N\} : \alpha_j \in (-1, 0)\},$$

and let $\alpha = \alpha(\Omega, h, f) > 0$, satisfying (9), be the measure associated to (21) as defined in (8), that is

$$\alpha = -\sum_{j \in J} \alpha_j.$$

Then, Theorem 1.6 readily yields our fourth main result.

Theorem 1.10 *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain and fix $\rho \leq 8\pi(1 - \alpha)$. Then (20) admits at most one solution.*

We remark that (20) admits a variational formulation and that the corresponding functional is well known to be coercive for $\rho < 8\pi(1 + \min_j \{\alpha_j, 0\})$. Therefore we have existence and uniqueness in Theorem 1.10 if either $|J| \geq 2$ and $\rho \leq 8\pi(1 - \alpha)$ or if $|J| = 1$ and $\rho < 8\pi(1 - \alpha) \equiv 8\pi(1 + \alpha_1)$ or if $|J| = 0$ and $\rho < 8\pi$.

Remark 1.11 Since Theorem 1.6 applies to multiply-connected domains, see Remark 1.8 for more details, we conclude that Theorem 1.10 holds for Ω multiply-connected as well. Moreover, we can also treat the case where e^u is replaced by $e^H e^u$ with H subharmonic and where $u = g$ on $\partial\Omega$ with $g \in C(\partial\Omega)$.

We point out that Theorem 1.10 covers all the previously known results. Indeed, for $N = 0$ and $\rho \leq 8\pi$, we get the sharp uniqueness result of [14, 24] (i.e. the regular case). For $N > 0$, $\alpha_j > 0$ for all $j = 1, \dots, N$ and $\rho \leq 8\pi$, we get the sharp result of [13], while in the case $N \geq 2$ of only one negative singularity $\alpha_1 \in (-1, 0)$ with $\rho \leq 8\pi(1 + \alpha_1)$ we recover the sharp result of [73].

Remark 1.12 It is worth to make a remark about the discrepancy of the uniqueness thresholds as obtained in Theorems 1.9 and 1.10. It turns out that a rather elementary but still crucial point in the proof of Theorem 1.9 is that, since the equation is solved

on $\mathbb{S}^2$, then one has an upper bound on the total (positive) singular curvature, see (59). This estimate, in turn, allows one to adopt an optimization trick which rules out the case where the singular Sphere Covering Inequality would yield no information, that is when $\alpha(\omega) \geq 1$ in Theorem 1.2. This is not anymore possible on a bounded domain and we come up with a threshold which, for $|J| \geq 2$ and unlike the case of $\mathbb{S}^2$, is always lower than the subcritical existence threshold $8\pi(1 + \min_j \{\alpha_j, 0\})$. Actually, it seems that this uniqueness result could have been obtained also by an adaptation of the argument in [13] and it is a challenging open problem to understand whether or not uniqueness still holds for $\rho \in (8\pi(1 - \alpha), 8\pi(1 + \min_j \{\alpha_j, 0\}))$ with $|J| \geq 2$.

Actually, Theorem 1.6 yields uniqueness of solutions to a much more general problem than the one considered in (20) and the following holds true.

Theorem 1.13 *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected domain. Let $x_0 \in \Omega$ be fixed as in Remark 1.1 and consider solutions $u \in W_{\text{loc}}^{2,s,\text{loc}}(\Omega \setminus \{x_0\}) \cap W^{2,q}(\Omega) \cap C(\overline{\Omega})$ for some $s > 2$ and some $q > 1$, to,*

$$\begin{cases} \Delta u + \rho \frac{h(x)e^u}{\int_{\Omega} h(x)e^u dx} = f(x) & \text{in } \Omega, \\ u = g(x) & \text{on } \partial\Omega, \end{cases} \quad (22)$$

where $g \in C(\partial\Omega)$, $h = e^H$ and $f \in L^s(\Omega)$ are such that $\alpha = \alpha(\Omega, h, f) < 1$, with $\alpha(\Omega, h, f)$ defined as in (8). Then, for any $\rho \leq 8\pi(1 - \alpha)$ there exists at most one solution for (22).

We point out that the non-degeneracy of solutions to (22) was recently studied in [12].

1.4 The Onsager mean field equation on the sphere

Let us consider the equation,

$$\begin{aligned} \Delta_g v(y) + \frac{\exp(\beta v(y) - \gamma \langle n, y \rangle)}{\int_{\mathbb{S}^2} \exp(\beta v(y) - \gamma \langle n, y \rangle) dV_g} - \frac{1}{4\pi} &= 0 \quad \text{on } \mathbb{S}^2, \\ \int_{\mathbb{S}^2} v dV_g &= 0, \end{aligned} \quad (23)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^3$, $n = \vec{n} \in \mathbb{R}^3$ is a unit vector, $\beta \geq 0$ and $\gamma \in \mathbb{R}$. Since γ can be changed to $-\gamma$ by replacing the north pole with the south pole, there is no loss of generality in assuming that $\gamma \geq 0$. Observe that the equation in (23) is invariant under the addition of a constants, which is why we can impose the condition of zero mean value. Equation (23) is the mean field equation arising from the spherical Onsager vortex theory, see [27,60,63].

Let us briefly list the known results concerning (23). By a moving plane argument, it is shown in [53] that if $\beta < 8\pi$, then for any $\gamma \geq 0$ the Eq. (23) has a unique solution which is axially symmetric with respect to $\vec{n}$. Moreover, the author made the following conjecture.

Conjecture B *Let $\gamma > 0$ and $\beta \leq 16\pi$. Then every solution of (23) is axially symmetric with respect to $\vec{n}$.*

In this direction the following results for $\beta > 8\pi$ has been proved in [54].

Theorem C [54] *For every $\gamma > 0$, there exists $\beta_0 = \beta_0(\gamma) > 8\pi$ such that, for $8\pi < \beta \leq \beta_0$, any solution of (23) is axially symmetric with respect to $\vec{n}$.*

Theorem D [54] *Let $\{v_i\}_i$ be a sequence of solutions of (23) with $\gamma = 0$ and $\beta_i \rightarrow 16\pi$. Suppose that $\lim_{i \rightarrow \infty} \sup_{\mathbb{S}^2} v_i(\gamma) = +\infty$. Then v_i is axially symmetric with respect to some direction $\vec{n}_i$ in $\mathbb{R}^3$ for i large enough.*

Recently, in [45] the authors applied the standard Sphere Covering inequality, see Theorem A, to prove the following result.

Theorem E [45] *Suppose $8\pi < \beta \leq 16\pi$ and*

$$0 \leq \gamma \leq \frac{\beta}{8\pi} - 1.$$

Then every solution of (23) is axially symmetric with respect to $\vec{n}$.

The aim here is to use the singular Sphere Covering Inequality, Theorem 1.2, to get a new symmetry result. Our fifth main result is the following.

Theorem 1.14 *Suppose $8\pi < \beta \leq 16\pi$ and*

$$0 \leq \gamma \leq 3 - \frac{\beta}{8\pi} + 2\sqrt{2 - \frac{\beta}{8\pi}} = \left(\sqrt{2 - \frac{\beta}{8\pi}} + 1 \right)^2. \quad (24)$$

Then every solution of (23) is evenly symmetric with respect to a plane passing through the origin and containing the vector $\vec{n}$.

Remark 1.15 We point out that for $8\pi < \beta \leq 16\pi$ it holds,

$$3 - \frac{\beta}{8\pi} \geq \frac{\beta}{8\pi} - 1,$$

and thus Theorem 1.14 covers a wider range of parameters compared to Theorem E.

This paper is organized as follows. In Sect. 2 we introduce the argument which yields to the proof of the singular Sphere Covering Inequality of Theorem 1.2, in Sect. 3 we deduce the improved version of it under same total mass condition, see Theorem 1.6, in Sect. 4 we prove the uniqueness result for the singular Liouville equation on $\mathbb{S}^2$, see Theorem 1.9, and in Sect. 5 we finally derive symmetry of solutions for the spherical Onsager vortex equation, i.e. Theorem 1.14.

2 The singular Sphere Covering Inequality

In this section we derive the singular Sphere Covering Inequality of Theorem 1.2. The argument is mainly based on weighted symmetric rearrangements and the Alexandrov–Bol inequality in the spirit of Theorem A.

Let us start by recalling the following version of the Alexandrov–Bol inequality, first proved in the analytical framework in [1] and more recently generalized to the weak setting in [4,5].

Proposition 2.1 [4,5] *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected domain. Let $x_0 \in \Omega$ be fixed as in Remark 1.1 and let $u \in W_{\text{loc}}^{2,s,\text{loc}}(\Omega \setminus \{x_0\}) \cap W_{\text{loc}}^{2,q}(\Omega)$ for some $s > 2$ and some $q > 1$, satisfy*

$$\Delta u + h(x)e^u = f(x) \text{ in } \Omega,$$

where $h = e^H$ and $f \in L_{\text{loc}}^s(\Omega)$ are such that $\alpha(\Omega, h, f)$ (as defined in (8)) satisfies $\alpha(\Omega, h, f) < 1$. Let $\omega \subseteq \Omega$ be a smooth subdomain and let $\alpha(\omega) = \alpha(\omega, h, f)$. Then it holds,

$$\begin{aligned} & \left(\int_{\partial\omega} (h(x)e^u)^{\frac{1}{2}} d\sigma \right)^2 \\ & \geq \frac{1}{2} \left(\int_{\omega} h(x)e^u dx \right) \left(8\pi(1 - \alpha(\omega)) - \int_{\omega} h(x)e^u dx \right). \end{aligned} \quad (25)$$

Moreover, the equality holds if and only if (modulo conformal transformations) $\omega = B_{\delta}(0)$ for some $\delta > 0$, $h(x)e^u \equiv |x|^{-2\alpha} e^{U_{\lambda,\alpha}}$ for some $\lambda > 0$ where $U_{\lambda,\alpha}$ is defined in (4), $\mu_+ = -\Delta H - f = 4\pi\alpha\delta_{p=0}$ in ω and $\alpha = \alpha(\omega)$. In particular, if ω is not simply-connected, then the inequality is always strict.

Remark 2.2 The smoothness assumption about ω is not necessary and can be remarkably weakened as far as $\omega \Subset \Omega$, see [5]. Moreover, we point out that if $\mu_+ = 0$ then $\alpha(\omega, h, f) = 0$ and we recover the standard Alexandrov–Bol inequality.

We will need in the sequel the following counterpart of the Alexandrov–Bol inequality in the radial setting which is derived in the spirit of [45,67].

Proposition 2.3 *Let $\alpha \in [0, 1)$, $R > 0$ and $\psi \in C(\overline{B_R(0)}) \cap W^{1,p}(B_R(0))$ for some $p > 2$, be a strictly decreasing radial function satisfying,*

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq \int_{B_r(0)} |x|^{-2\alpha} e^{\psi} dx \text{ for a.e. } r \in (0, R). \quad (26)$$

Then,

$$\begin{aligned} & \left(\int_{\partial B_R(0)} (|x|^{-2\alpha} e^{\psi})^{\frac{1}{2}} d\sigma \right)^2 \\ & \geq \frac{1}{2} \left(\int_{B_R(0)} |x|^{-2\alpha} e^{\psi} dx \right) \left(8\pi(1 - \alpha) - \int_{B_R(0)} |x|^{-2\alpha} e^{\psi} dx \right). \end{aligned}$$

Moreover, if $\int_{\partial B_r(0)} |\nabla \psi| d\sigma \not\equiv \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx$ in $(0, R)$, then the inequality is strict.

Proof We start by letting $\beta = \psi(R)$ and

$$k(t) = \int_{\{\psi > t\}} |x|^{-2\alpha} e^\psi dx, \quad \mu(t) = \int_{\{\psi > t\}} |x|^{-2\alpha} dx, \quad t > \beta.$$

Clearly, k and μ are absolutely continuous and by using the co-area formula we find that,

$$-k'(t) = \int_{\{\psi=t\}} \frac{|x|^{-2\alpha} e^\psi}{|\nabla \psi|} d\sigma = -e^t \mu'(t), \quad (27)$$

for a.e. $t > \beta$. Therefore, by using (26) and then the Cauchy–Schwarz inequality we deduce that,

$$\begin{aligned} -k(t)k'(t) &= \left(\int_{\{\psi > t\}} |x|^{-2\alpha} e^\psi dx \right) \left(\int_{\{\psi=t\}} \frac{|x|^{-2\alpha} e^\psi}{|\nabla \psi|} d\sigma \right) \\ &\geq \left(\int_{\{\psi=t\}} |\nabla \psi| d\sigma \right) \left(\int_{\{\psi=t\}} \frac{|x|^{-2\alpha} e^\psi}{|\nabla \psi|} d\sigma \right) \\ &= \left(\int_{\{\psi=t\}} \left(|x|^{-2\alpha} e^\psi \right)^{\frac{1}{2}} d\sigma \right)^2 = e^t \left(\int_{\{\psi=t\}} |x|^{-\alpha} d\sigma \right)^2, \end{aligned}$$

for a.e. $t > \beta$. Moreover we have,

$$\left(\int_{\{\psi=t\}} |x|^{-\alpha} d\sigma \right)^2 = 4\pi(1-\alpha) \int_{\{\psi > t\}} |x|^{-2\alpha} dx = 4\pi(1-\alpha)\mu(t),$$

and recalling (27) it follows that,

$$\frac{d}{dt} \left(e^t \mu(t) - k(t) + \frac{1}{8\pi(1-\alpha)} k^2(t) \right) = e^t \mu(t) + \frac{1}{4\pi(1-\alpha)} k'(t)k(t) \leq 0,$$

for a.e. $t > \beta$. Therefore, by integrating the latter equation, we deduce that,

$$\left[e^t \mu(t) - k(t) + \frac{1}{8\pi(1-\alpha)} k^2(t) \right]_\beta^{+\infty} = - \left(e^\beta \mu(\beta) - k(\beta) + \frac{1}{8\pi(1-\alpha)} k^2(\beta) \right) \leq 0,$$

namely,

$$e^\beta \mu(\beta) \geq k(\beta) \left(1 - \frac{1}{8\pi(1-\alpha)} k(\beta) \right).$$

To conclude the proof it is enough to observe that,

$$k(\beta) = \int_{B_R(0)} |x|^{-2\alpha} e^\psi dx$$

and that,

$$\begin{aligned} e^\beta \mu(\beta) &= e^\beta \int_{B_R(0)} |x|^{-2\alpha} dx = e^\beta \frac{1}{4\pi(1-\alpha)} \left(\int_{\partial B_R(0)} |x|^{-\alpha} d\sigma \right)^2 \\ &= \frac{1}{4\pi(1-\alpha)} \left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^\psi \right)^{\frac{1}{2}} d\sigma \right)^2. \end{aligned}$$

Furthermore, going back through the argument it is clear that if $\int_{\partial B_r(0)} |\nabla \psi| d\sigma \not\equiv \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx$ in $(0, R)$, then the inequality in Proposition 2.3 is strict. $\square$

With a similar argument it is possible to prove the following reversed Alexandrov–Bol inequality.

Proposition 2.4 *Let $\alpha \in [0, 1)$, $R > 0$ and $\psi \in C(\mathbb{R}^2 \setminus B_R(0)) \cap W_{\text{loc}}^{1,p}(\mathbb{R}^2 \setminus B_R(0))$ for some $p > 2$, be a strictly decreasing radial function satisfying,*

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq 8\pi(1-\alpha) - \int_{\mathbb{R}^2 \setminus B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r \in (R, +\infty) \quad (28)$$

and $\int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx < 8\pi(1-\alpha)$. Then,

$$\begin{aligned} &\left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^\psi \right)^{\frac{1}{2}} d\sigma \right)^2 \\ &\leq \frac{1}{2} \left(\int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx \right) \left(8\pi(1-\alpha) - \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx \right). \end{aligned}$$

Moreover, if $\int_{\partial B_r(0)} |\nabla \psi| d\sigma \not\equiv 8\pi(1-\alpha) - \int_{\mathbb{R}^2 \setminus B_r(0)} |x|^{-2\alpha} e^\psi dx$ in $(R, +\infty)$, then the inequality is strict.

Proof We let $\beta = \psi(R)$ and

$$k(t) = 8\pi(1-\alpha) - \int_{\{\psi < t\}} |x|^{-2\alpha} e^\psi dx, \quad \mu(t) = \int_{\{\psi > t\}} |x|^{-2\alpha} dx + \frac{\pi}{1-\alpha} R^{2(1-\alpha)},$$

for $t < \beta$. The argument follows then the same steps of the proof of Proposition 2.3 and we refer to [46] for further details. $\square$

Next, as in [45], we relate the strictly decreasing radial function ψ satisfying (26) with the functions $U_{\lambda_1, \alpha}, U_{\lambda_2, \alpha}$ defined in (4) with $\lambda_2 > \lambda_1$ and $\alpha \in [0, 1)$, such that $\psi = U_{\lambda_1, \alpha} = U_{\lambda_2, \alpha}$ on $\partial B_R(0)$.

Lemma 2.5 Let $U_{\lambda_1, \alpha}, U_{\lambda_2, \alpha}$ be defined as in (4) with $\lambda_2 > \lambda_1$ and $\alpha \in [0, 1)$. Let $\psi \in C(\overline{B_R(0)}) \cap W^{1,p}(B_R(0))$ for some $p > 2$, be a strictly decreasing radial function satisfying,

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r \in (0, R) \quad (29)$$

and $\psi = U_{\lambda_1, \alpha} = U_{\lambda_2, \alpha}$ on $\partial B_R(0)$. Then, either

$$\begin{aligned} \int_{B_R(0)} |x|^{-2\alpha} e^\psi dx &\leq \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx \\ \text{or} \quad \int_{B_R(0)} |x|^{-2\alpha} e^\psi dx &\geq \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} dx. \end{aligned}$$

If $\int_{\partial B_r(0)} |\nabla \psi| d\sigma \not\equiv \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx$ in $(0, R)$, then the above inequalities are strict.

Moreover, we have

$$\int_{B_R(0)} \left(|x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} + |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} \right) dx = 8\pi(1 - \alpha).$$

Proof Let us set,

$$\begin{aligned} m_i &= \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_i, \alpha}} dx \quad i = 1, 2, \\ m &= \int_{B_R(0)} |x|^{-2\alpha} e^\psi dx, \end{aligned}$$

and, recalling that $\psi = U_{\lambda_1, \alpha} = U_{\lambda_2, \alpha}$ on $\partial B_R(0)$,

$$\begin{aligned} \beta &= \left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^\psi \right)^{\frac{1}{2}} d\sigma \right)^2 \\ &= \left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} \right)^{\frac{1}{2}} d\sigma \right)^2 = \left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} \right)^{\frac{1}{2}} d\sigma \right)^2. \end{aligned}$$

By Proposition 2.3 we know that,

$$\beta \geq \frac{1}{2} m (8\pi(1 - \alpha) - m), \quad (30)$$

and by Proposition 2.1 and Remark 2.2 we have,

$$\beta = \frac{1}{2} m_1 (8\pi(1 - \alpha) - m_1) = \frac{1}{2} m_2 (8\pi(1 - \alpha) - m_2).$$

It follows that m_1, m_2 are the roots of the following equation:

$$y^2 - 8\pi(1 - \alpha)y + 2\beta = 0.$$

On the other hand, by (30) we have,

$$m^2 - 8\pi(1 - \alpha)m + 2\beta \geq 0.$$

Then, either $m \leq m_1$ or $m \geq m_2$ which proves the alternative of Lemma 2.5. It is clear that if $\int_{\partial B_r(0)} |\nabla \psi| d\sigma \not\equiv \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx$ in $(0, R)$, then the latter inequalities are strict.

We are left with the last equality of Lemma 2.5. This is a standard evaluation and we derive it here for the sake of completeness. First of all we have,

$$\int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_i, \alpha}} dx = 8\pi(1 - \alpha) \frac{\lambda_i^2 R^{2(1-\alpha)}}{8 + \lambda_i^2 R^{2(1-\alpha)}}, \quad i = 1, 2. \quad (31)$$

Since $U_{\lambda_1, \alpha} = U_{\lambda_2, \alpha}$ on $\partial B_R(0)$ we also find that,

$$\frac{\lambda_1}{1 + \frac{\lambda_1^2}{8} R^{2(1-\alpha)}} = \frac{\lambda_2}{1 + \frac{\lambda_2^2}{8} R^{2(1-\alpha)}} = C,$$

for some $C > 0$. It follows that λ_1, λ_2 are the roots of the following equation:

$$y^2 - \frac{8}{CR^{2(1-\alpha)}}y + \frac{8}{R^{2(1-\alpha)}} = 0, \quad (32)$$

and thus,

$$\lambda_1 + \lambda_2 = \frac{8}{CR^{2(1-\alpha)}}, \quad \lambda_1 \lambda_2 = \frac{8}{R^{2(1-\alpha)}}. \quad (33)$$

By using (32) and then (33) we conclude that,

$$\begin{aligned} \int_{B_R(0)} \left(|x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} + |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} \right) dx &= 8\pi(1 - \alpha) \left(\frac{\lambda_1^2 R^{2(1-\alpha)}}{8 + \lambda_1^2 R^{2(1-\alpha)}} + \frac{\lambda_2^2 R^{2(1-\alpha)}}{8 + \lambda_2^2 R^{2(1-\alpha)}} \right) \\ &= 8\pi(1 - \alpha) \left(\frac{\lambda_1^2 R^{2(1-\alpha)}}{\frac{8\lambda_1}{C}} + \frac{\lambda_2^2 R^{2(1-\alpha)}}{\frac{8\lambda_2}{C}} \right) \\ &= 8\pi(1 - \alpha) \left(\frac{CR^{2(1-\alpha)}}{8} (\lambda_1 + \lambda_2) \right) \\ &= 8\pi(1 - \alpha). \end{aligned}$$

The proof of Lemma 2.5 is now complete. $\square$

On the other hand, an analogous argument with obvious modifications based on Proposition 2.4 yields the following result (see [46] for full details).

Lemma 2.6 *Let $U_{\lambda_1, \alpha}, U_{\lambda_2, \alpha}$ be defined as in (4) with $\lambda_2 > \lambda_1$ and $\alpha \in [0, 1)$. Let $\psi \in C(\mathbb{R}^2 \setminus B_R(0)) \cap W_{\text{loc}}^{1,p}(\mathbb{R}^2 \setminus B_R(0))$ for some $p > 2$, be a strictly decreasing radial function satisfying,*

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq 8\pi(1 - \alpha) - \int_{\mathbb{R}^2 \setminus B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r \in (R, +\infty)$$

and $\int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx < 8\pi(1 - \alpha)$. Suppose $\psi = U_{\lambda_1, \alpha} = U_{\lambda_2, \alpha}$ on $\partial B_R(0)$. Then,

$$\int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} dx \leq \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx \leq \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx.$$

If $\int_{\partial B_r(0)} |\nabla \psi| d\sigma \not\equiv \int_{\mathbb{R}^2 \setminus B_r(0)} |x|^{-2\alpha} e^\psi dx$ in $(R, +\infty)$, then the above inequalities are strict.

Finally, let us introduce some known facts about weighted symmetric rearrangements with respect to two measures: under the assumptions of Proposition 2.1, for a given function $\phi \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ for some $p > 2$, such that $\phi = C$ on $\partial\omega$, $\omega \subseteq \Omega$, we will consider its equimeasurable rearrangement in ω with respect to the measures $h(x)e^u dx$ and $|x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx$, where $\alpha = \alpha(\omega, h, f)$ and $U_{\lambda, \alpha}$ are defined in (4) and (8), and u, h, f are given in Proposition 2.1. More exactly, for $t > t_0 = \min_{x \in \overline{\omega}} \phi(x)$ let

$$\begin{aligned} \{\phi = t\} &= \{x \in \omega : \phi(x) = t\} \subseteq \omega, \\ \omega_t &= \{x \in \omega : \phi(x) > t\} \subseteq \omega, \end{aligned} \quad (34)$$

and let $\mathcal{B}_t^*$ be the ball centered at the origin such that

$$\int_{\mathcal{B}_t^*} |x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx = \int_{\omega_t} h(x)e^u dx.$$

Then, $\phi^* : \mathcal{B}_{t_0}^* \rightarrow \mathbb{R}$ defined by $\phi^*(x) = \sup\{t \in \mathbb{R} : x \in \mathcal{B}_t^*\}$ is a radial, decreasing, equimeasurable rearrangement of ϕ with respect to the measures $h(x)e^u dx$ and $|x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx$, i.e. $\{\phi^* > t\} \equiv \mathcal{B}_t^*$ and,

$$\int_{\{\phi^* > t\}} |x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx = \int_{\omega_t} h(x)e^u dx, \quad (35)$$

for all $t > \min_{x \in \overline{\omega}} \phi(x)$. Elementary arguments show that ϕ^* is a BV function. By exploiting Proposition 2.1 we get the following property.

Lemma 2.7 *Under the assumptions of Proposition 2.1, let $\phi \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ for some $p > 2$ be such that $\phi = C$ on $\partial\omega$, $\omega \subseteq \Omega$. If ϕ^* is the equimeasurable symmetric*

rearrangement of ϕ in ω with respect to the measures $h(x)e^u dx$ and $|x|^{-2\alpha}e^{U_{\lambda,\alpha}} dx$ defined above with $\alpha = \alpha(\omega, h, f)$, then

$$\int_{\{\phi^*=t\}} |\nabla \phi^*| d\sigma \leq \int_{\{\phi=t\}} |\nabla \phi| d\sigma,$$

for a.e. $t > \min_{x \in \overline{\omega}} \phi(x)$.

Proof The argument is standard so we will be sketchy and refer to [4,5,13,24,45] for full details. We first apply the Cauchy–Schwarz inequality and then the co-area formula to get that,

$$\begin{aligned} \int_{\{\phi=t\}} |\nabla \phi| d\sigma &\geq \left(\int_{\{\phi=t\}} (h(x)e^u)^{\frac{1}{2}} d\sigma \right)^2 \left(\int_{\{\phi=t\}} \frac{h(x)e^u}{|\nabla \phi|} d\sigma \right)^{-1} \\ &= \left(\int_{\{\phi=t\}} (h(x)e^u)^{\frac{1}{2}} d\sigma \right)^2 \left(-\frac{d}{dt} \int_{\omega_t} h(x)e^u dx \right)^{-1}, \end{aligned}$$

for a.e. t . Then, in view of the Alexandrov–Bol inequality of Proposition 2.1 we see that,

$$\begin{aligned} &\left(\int_{\{\phi=t\}} (h(x)e^u)^{\frac{1}{2}} d\sigma \right)^2 \left(-\frac{d}{dt} \int_{\omega_t} h(x)e^u dx \right)^{-1} \\ &\geq \frac{1}{2} \left(\int_{\omega_t} h(x)e^u dx \right) \left(8\pi(1 - \alpha(\omega)) - \int_{\omega_t} h(x)e^u dx \right) \left(-\frac{d}{dt} \int_{\omega_t} h(x)e^u dx \right)^{-1}. \end{aligned}$$

Since ϕ^* is an equimeasurable rearrangement of ϕ in ω with respect to the measures $h(x)e^u dx$, $|x|^{-2\alpha}e^{U_{\lambda,\alpha}} dx$, and since $|x|^{-2\alpha}e^{U_{\lambda,\alpha}}$ realizes the equality in Proposition 2.1, then we can argue in the other way around,

$$\begin{aligned} &\frac{1}{2} \left(\int_{\omega_t} h(x)e^u dx \right) \left(8\pi(1 - \alpha(\omega)) - \int_{\omega_t} h(x)e^u dx \right) \left(-\frac{d}{dt} \int_{\omega_t} h(x)e^u dx \right)^{-1} \\ &= \frac{1}{2} \left(\int_{B_t^*} |x|^{-2\alpha} e^{U_{\lambda,\alpha}} dx \right) \left(8\pi(1 - \alpha(\omega)) - \int_{B_t^*} |x|^{-2\alpha} e^{U_{\lambda,\alpha}} dx \right) \\ &\quad \times \left(-\frac{d}{dt} \int_{B_t^*} |x|^{-2\alpha} e^{U_{\lambda,\alpha}} dx \right)^{-1} \\ &= \left(\int_{\{\phi^*=t\}} (|x|^{-2\alpha} e^{U_{\lambda,\alpha}})^{\frac{1}{2}} d\sigma \right)^2 \left(-\frac{d}{dt} \int_{B_t^*} |x|^{-2\alpha} e^{U_{\lambda,\alpha}} dx \right)^{-1} \\ &= \int_{\{\phi^*=t\}} |\nabla \phi^*| d\sigma, \end{aligned}$$

as claimed, where in the last equality we used the co-area formula for BV functions, see [42]. $\square$

At this point we are ready to derive the main result of this section, namely the singular Sphere Covering Inequality.

Proof of Theorem 1.2 Let u_1, u_2 and $\omega \subseteq \Omega$ be as in the statement of Theorem 1.2. Let $\alpha = \alpha(\omega, h, f_1)$, where $\alpha(\omega, h, f_1)$ is given in (8), and let $U_{\lambda_1, \alpha}, U_{\lambda_2, \alpha}$ be as defined in (4) for some $\lambda_2 > \lambda_1$. Take λ_1, λ_2 such that $U_{\lambda_1, \alpha} = U_{\lambda_2, \alpha}$ on $\partial B_1(0)$ and,

$$\int_{\omega} h(x) e^{u_1} dx = \int_{B_1(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx.$$

Let $\phi = u_2 - u_1$ and let us assume without loss of generality that,

$$\begin{cases} \phi > 0 & \text{in } \omega, \\ \phi = 0 & \text{on } \partial\omega. \end{cases}$$

Since u_1 satisfies,

$$\Delta u_1 + h(x) e^{u_1} = f_1(x) \text{ in } \Omega,$$

and Ω is simply-connected, then, as in the discussion after Lemma 2.6, we can define the radial, decreasing, equimeasurable rearrangement ϕ^* of ϕ in ω with respect to the two measures $h(x) e^{u_1} dx$ and $|x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx$. Let $\{\phi = t\}$ and $\omega_t \subseteq \omega$ be defined as in (34). In particular we have,

$$\int_{\{\phi^* > t\}} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx = \int_{\omega_t} h(x) e^{u_1} dx,$$

for $t \geq 0$. Observe now that due to the assumption $f_2 \geq f_1$ a.e. in Ω , we also have,

$$\Delta(u_2 - u_1) + h(x) e^{u_2} - h(x) e^{u_1} = f_2(x) - f_1(x) \geq 0 \text{ a.e. in } \Omega. \quad (36)$$

We first estimate the gradient of the rearrangement by applying Lemma 2.7 and then use Eq. (36) to obtain,

$$\begin{aligned} \int_{\{\phi^* = t\}} |\nabla \phi^*| d\sigma &\leq \int_{\{\phi = t\}} |\nabla(u_2 - u_1)| d\sigma \\ &\leq \int_{\omega_t} (h(x) e^{u_2} - h(x) e^{u_1}) dx, \end{aligned}$$

for a.e. $t > 0$. By using the equimeasurability with the Fubini theorem and then also the Eq. (5) satisfied by $U_{\lambda_1, \alpha}$, we deduce that,

$$\begin{aligned} \int_{\omega_t} (h(x) e^{u_2} - h(x) e^{u_1}) dx &= \int_{\{\phi^* > t\}} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha} + \phi^*} dx - \int_{\{\phi^* > t\}} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx \\ &= \int_{\{\phi^* > t\}} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha} + \phi^*} dx - \int_{\{\phi^* = t\}} |\nabla U_{\lambda_1, \alpha}| d\sigma, \end{aligned}$$

for a.e. $t > 0$. Therefore, we conclude that,

$$\int_{\{\phi^*=t\}} |\nabla(U_{\lambda_1,\alpha} + \phi^*)| d\sigma \leq \int_{\{\phi^*>t\}} |x|^{-2\alpha} e^{U_{\lambda_1,\alpha}+\phi^*} dx,$$

for a.e. $t > 0$. Let $\psi = U_{\lambda_1,\alpha} + \phi^*$. Since ϕ^* is decreasing by construction then ψ is a strictly decreasing function. Moreover, the above estimate can be rewritten as,

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r > 0, \quad (37)$$

which shows that $\psi \in W^{1,p}(B_1(0))$ for some $p > 2$. Furthermore, recalling that $\phi > 0$ in ω we have $\phi^* \geq 0$, $\phi^* \not\equiv 0$, which implies that,

$$\int_{B_1(0)} |x|^{-2\alpha} e^\psi dx = \int_{B_1(0)} |x|^{-2\alpha} e^{U_{\lambda_1,\alpha}+\phi^*} dx > \int_{B_1(0)} |x|^{-2\alpha} e^{U_{\lambda_1,\alpha}} dx.$$

Observing that $\phi = 0$ on $\partial\omega$ we have $\phi^* = 0$ on $\partial B_1(0)$ and $\psi = U_{\lambda_1,\alpha} = U_{\lambda_2,\alpha}$ on $\partial B_1(0)$. By (37) and the above estimate we can exploit the alternative of Lemma 2.5 about ψ , to obtain,

$$\int_{B_1(0)} |x|^{-2\alpha} e^{U_{\lambda_1,\alpha}+\phi^*} dx = \int_{B_1(0)} |x|^{-2\alpha} e^\psi dx > \int_{B_1(0)} |x|^{-2\alpha} e^{U_{\lambda_2,\alpha}} dx.$$

Therefore, by using Lemma 2.5 once more, we conclude that,

$$\begin{aligned} \int_{\omega} (h(x)e^{u_1} + h(x)e^{u_2}) dx &= \int_{B_1(0)} \left(|x|^{-2\alpha} e^{U_{\lambda_1,\alpha}} + |x|^{-2\alpha} e^{U_{\lambda_1,\alpha}+\phi^*} \right) dx \\ &\geq \int_{B_1(0)} \left(|x|^{-2\alpha} e^{U_{\lambda_1,\alpha}} + |x|^{-2\alpha} e^{U_{\lambda_2,\alpha}} \right) dx \\ &= 8\pi(1 - \alpha). \end{aligned} \quad (38)$$

Moreover, going back through the proof, we see that the equality holds only if we have equality in (36), i.e. $f_1 \equiv f_2 := f$, and in (37) for a.e. $r > 0$. It follows that in Lemma 2.7 we have equality in the estimate of the gradient of the rearrangement and hence the equality in the Alexandrov–Bol inequality of Proposition 2.1 for u_1 in Ω . Therefore, (modulo conformal transformations) $\omega = B_\delta(0)$ for some $\delta > 0$, $h(x)e^{u_1} \equiv |x|^{-2\alpha} e^{U_{\tilde{\lambda}_1,\alpha}}$ for some $\tilde{\lambda}_1 > 0$ and

$$\mu_+ = -\Delta H(x) - f(x) = 4\pi\alpha\delta_{p=0} \text{ in } B_\delta(0), \quad (39)$$

where we recall that $h = e^H$. In particular we have

$$U_{\tilde{\lambda}_1,\alpha} \equiv u_1 + 2\alpha \ln |x| + H(x). \quad (40)$$

Since $\omega = B_\delta(0)$, and by setting $w_2 = u_2 + 2\alpha \ln |x| + H(x)$, then (10) and (39) imply that w_2 satisfies,

$$\Delta w_2 + |x|^{-2\alpha} e^{w_2} = 0 \text{ in } B_\delta(0).$$

Since $u_2 = u_1$ on $\partial B_\delta(0)$ by assumption, then (40) implies that $w_2 = U_{\tilde{\lambda}_1, \alpha}$ on $\partial B_\delta(0)$. Furthermore, observe that, as a consequence of the equality sign in (38), we have

$$\int_{B_\delta(0)} |x|^{-2\alpha} e^{w_2} dx = \int_{B_\delta(0)} h(x) e^{u_2} dx \equiv \int_{B_1(0)} |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} dx,$$

which is already known by (31). It is then not difficult to see that $w_2 \equiv U_{\tilde{\lambda}_2, \alpha}$ for some $\tilde{\lambda}_2 > \tilde{\lambda}_1$, i.e. $h(x) e^{u_2} \equiv |x|^{-2\alpha} e^{U_{\tilde{\lambda}_2, \alpha}}$, as claimed. The proof is complete. $\square$

3 The singular Sphere Covering Inequality with same total mass

In this section we will deduce the improved singular Sphere Covering Inequality of Theorem 1.6 under the same total mass condition, i.e. (14). This is done in the spirit of [47].

First of all, let us derive the following lemma which is a simple consequence of the radial Alexandrov–Bol inequality in Proposition 2.3.

Lemma 3.1 *Let $U_{\lambda, \alpha}$ be defined as in (4) with $\lambda > 0$ and $\alpha \in [0, 1)$. Let $\psi \in C(\overline{B_R(0)}) \cap W^{1,p}(B_R(0))$ for some $p > 2$, be a strictly decreasing radial function satisfying,*

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r \in (0, R).$$

Suppose

$$\int_{B_R(0)} |x|^{-2\alpha} e^\psi dx = \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx < 8\pi(1 - \alpha).$$

Then $U_{\lambda, \alpha}(R) \leq \psi(R)$.

Proof We start by using the radial Alexandrov–Bol inequality in Proposition 2.3 with $U_{\lambda, \alpha}$, exploit the assumptions of the lemma and then apply Proposition 2.3 once more to ψ to deduce

$$\begin{aligned}
\left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^{U_{\lambda, \alpha}} \right)^{\frac{1}{2}} d\sigma \right)^2 &= \frac{1}{2} \left(\int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx \right) \\
&\quad \times \left(8\pi(1-\alpha) - \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda, \alpha}} dx \right) \\
&= \frac{1}{2} \left(\int_{B_R(0)} |x|^{-2\alpha} e^{\psi} dx \right) \\
&\quad \times \left(8\pi(1-\alpha) - \int_{B_R(0)} |x|^{-2\alpha} e^{\psi} dx \right) \\
&\leq \left(\int_{\partial B_R(0)} \left(|x|^{-2\alpha} e^{\psi} \right)^{\frac{1}{2}} d\sigma \right)^2.
\end{aligned}$$

Therefore, it readily follows that $U_{\lambda, \alpha}(R) \leq \psi(R)$. $\square$

We will now prove the main result of this section, Theorem 1.6.

Proof of Theorem 1.6 The idea is to proceed as in the proof of the singular Sphere Covering Inequality of Theorem 1.2 and exploit the extra information (14). Let $\alpha = \alpha(\Omega, h, f_1) > 0$ be as defined in (8). We discuss two cases separately.

Case 1. We argue by contradiction and suppose first that $\rho < 8\pi(1-\alpha)$. Take $\lambda_1 > 0$ and $R > 0$ such that

$$\int_{\Omega} h(x) e^{u_1} dx = \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx, \quad (41)$$

where $U_{\lambda_1, \alpha}$ is defined in (4). Let $\phi = u_2 - u_1$, which, since $u_1 \neq u_2$ in Ω and in view of (14), obviously changes sign in Ω . Since Ω is simply-connected, then, as right after Lemma 2.5, we can define the radial, decreasing, equimeasurable rearrangement ϕ^* of ϕ in Ω with respect to the two measures $h(x) e^{u_1} dx$ and $|x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx$. Moreover, observe that

$$\Delta(u_2 - u_1) + h(x) e^{u_2} - h(x) e^{u_1} = f_2(x) - f_1(x) \geq 0 \text{ a.e. in } \Omega. \quad (42)$$

By arguing exactly as in the proof of Theorem 1.2 we deduce that,

$$\int_{\{\phi^* = t\}} |\nabla(U_{\lambda_1, \alpha} + \phi^*)| d\sigma \leq \int_{\{\phi^* > t\}} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha} + \phi^*} dx,$$

for a.e. $t > 0$. Let $\psi = U_{\lambda_1, \alpha} + \phi^*$. Since ϕ^* is decreasing by construction then ψ is a strictly decreasing function. Moreover, the above estimate can be rewritten as,

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq \int_{B_r(0)} |x|^{-2\alpha} e^{\psi} dx \quad \text{for a.e. } r > 0. \quad (43)$$

On the other hand, by (14) and (41) we have,

$$\begin{aligned}\int_{B_R(0)} |x|^{-2\alpha} e^\psi dx &= \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha} + \phi^*} dx = \int_{\Omega} h(x) e^{u_2} dx \\ &= \int_{\Omega} h(x) e^{u_1} dx = \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx.\end{aligned}$$

Therefore, we observe that,

$$\begin{aligned}\int_{B_R(0)} |x|^{-2\alpha} e^\psi dx &= \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx \\ &= \int_{\Omega} h(x) e^{u_1} dx = \rho < 8\pi(1 - \alpha),\end{aligned}\quad (44)$$

by assumption.

Once we have (43) and (44) we may apply Lemma 3.1 to get $U_{\lambda_1, \alpha}(R) \leq \psi(R)$. However, since $\phi < 0$ on a subset of Ω with positive measure, then $\phi^*(R) < 0$, and thus,

$$\psi(R) = U_{\lambda_1, \alpha}(R) + \phi^*(R) < U_{\lambda_1, \alpha}(R),$$

which is the desired contradiction.

Case 2. Next we suppose that $\rho = 8\pi(1 - \alpha)$. Fix $\lambda_1 > 0$ and observe that

$$\int_{\Omega} h(x) e^{u_1} dx = \rho = 8\pi(1 - \alpha) = \int_{\mathbb{R}^2} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx.$$

As above we set $\phi = u_2 - u_1$ and consider its equimeasurable rearrangement ϕ^* with respect to the two measures $h(x) e^{u_1} dx$ and $|x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx$. Reasoning as in CASE 1 we deduce that,

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq \int_{B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r > 0, \quad (45)$$

where $\psi = U_{\lambda_1, \alpha} + \phi^*$ and

$$\int_{\mathbb{R}^2} |x|^{-2\alpha} e^\psi dx = \int_{\mathbb{R}^2} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx = 8\pi(1 - \alpha). \quad (46)$$

In particular, we also have,

$$\int_{\partial B_r(0)} |\nabla \psi| d\sigma \leq 8\pi(1 - \alpha) - \int_{\mathbb{R}^2 \setminus B_r(0)} |x|^{-2\alpha} e^\psi dx \quad \text{for a.e. } r > R, \quad (47)$$

where $R > 0$ is any fixed number. By the estimate (46) we see there exists $R > 0$ such that $\psi(R) = U_{\lambda_1, \alpha}(R)$. Take now $\lambda_2 \neq \lambda_1$ such that $U_{\lambda_2, \alpha}(R) = U_{\lambda_1, \alpha}(R) = \psi(R)$.

Since ϕ^* is decreasing, by the definition of ψ and R we conclude $\psi \leq U_{\lambda_1, \alpha}$ in $\mathbb{R}^2 \setminus B_R(0)$, $\psi \not\equiv U_{\lambda_1, \alpha}$. It follows that,

$$\int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx < \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx.$$

Since (47) holds true, then we may apply Lemma 2.6 and deduce on one side that $\lambda_1 < \lambda_2$ and on the other side that,

$$\int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx \geq \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} dx. \quad (48)$$

On the other hand, we observe that $\psi \geq U_{\lambda_1, \alpha}$ in $B_R(0)$, $\psi \not\equiv U_{\lambda_1, \alpha}$. It follows that,

$$\int_{B_R(0)} |x|^{-2\alpha} e^\psi dx > \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_1, \alpha}} dx,$$

and hence, since (45) holds true, Lemma 2.5 yields,

$$\int_{B_R(0)} |x|^{-2\alpha} e^\psi dx \geq \int_{B_R(0)} |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} dx. \quad (49)$$

Finally, exploiting (46) and summing up (48) and (49) we end up with,

$$\begin{aligned} 8\pi(1 - \alpha) &= \int_{\mathbb{R}^2} |x|^{-2\alpha} e^\psi dx = \int_{B_R(0)} |x|^{-2\alpha} e^\psi dx + \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{-2\alpha} e^\psi dx \\ &\geq \int_{\mathbb{R}^2} |x|^{-2\alpha} e^{U_{\lambda_2, \alpha}} dx = 8\pi(1 - \alpha). \end{aligned} \quad (50)$$

In order to obtain the desired contradiction we are left with showing that the latter inequality is strict. We first observe the equality holds only if we have equality in (42), i.e. $f_1 \equiv f_2$. Next, we may suppose without loss of generality $u_2 - u_1 = c \geq 0$ on $\partial\Omega$. The equality in (50) holds if and only if, in particular, we have equality in (45). It follows that in Lemma 2.7 we have equality in the estimate of the gradient of the rearrangement ϕ^* of $\phi = u_2 - u_1$ and hence the equality in the Alexandrov–Bol inequality of Proposition 2.1 in $\Omega_t = \{x \in \Omega : \phi(x) > t\}$ for a.e. $t > \min_{\overline{\Omega}} \phi$. Therefore, Ω_t must be simply-connected for a.e. $t > \min_{\overline{\Omega}} \phi$. Recalling that $\phi \geq 0$ on $\partial\Omega$, this is impossible since Ω_t is not simply-connected for $\min_{\overline{\Omega}} \phi < t < 0$. We conclude the inequality in (50) is strict and the proof is completed. $\square$

4 Uniqueness of solutions of the singular Liouville equation on $\mathbb{S}^2$

In this section we show a first application of the singular Sphere Covering Inequality of Theorem 1.2 by proving the uniqueness of solutions of the singular Liouville equation (15) on $\mathbb{S}^2$, see Theorem 1.9 and the discussion in Sect. 1.2.

Proof of Theorem 1.9 Suppose by contradiction that there exist v_1, v_2 satisfying (15) such that $v_1 \not\equiv v_2$. We define,

$$\tilde{v}_i(x) = v_i(x) - \ln \left(\int_{\mathbb{S}^2} e^{v_i} dV_g \right), \quad x \in \mathbb{S}^2, \quad i = 1, 2,$$

and observe that,

$$\int_{\mathbb{S}^2} e^{\tilde{v}_1} dV_g = \int_{\mathbb{S}^2} e^{\tilde{v}_2} dV_g$$

and $\tilde{v}_1 \not\equiv \tilde{v}_2$. We choose then a point $\mathcal{N} \in S^2$ such that $\tilde{v}_1(\mathcal{N}) = \tilde{v}_2(\mathcal{N})$. As discussed in Sect. 1.2 we let $\Pi : \mathbb{S}^2 \setminus \{\mathcal{N}\} \rightarrow \mathbb{R}^2$ be the stereographic projection with respect to the north pole $\mathcal{N}$ defined in (16) and we define,

$$u_i(x) = \tilde{v}_i(\Pi^{-1}(x)) - \frac{\rho - 4\pi \sum_j \alpha_j}{4\pi} \ln(1 + |x|^2) + \ln(4\rho), \quad x \in \mathbb{R}^2, \quad i = 1, 2,$$

where we recall that $\alpha_j \in (-1, 0)$ for all $j = 1, \dots, N$, $N \geq 0$ and $\sum_j \alpha_j$ stands for $\sum_{j=1}^N \alpha_j$. Then, u_1, u_2 satisfy,

$$\Delta u_i + h(x)e^{u_i} = 4\pi \sum_{j=1}^N \alpha_j \delta_{q_j} \quad \text{on } \mathbb{R}^2, \quad i = 1, 2, \quad (51)$$

for some $\{q_j, \dots, q_N\} \subset \mathbb{R}^2$, where,

$$h(x) = (1 + |x|^2)^{-l}, \quad l = \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi}, \quad (52)$$

and,

$$\int_{\mathbb{R}^2} h(x)e^{u_i} dx = \rho, \quad i = 1, 2. \quad (53)$$

Moreover, we have $u_1 \not\equiv u_2$. Since $\tilde{v}_1 = \tilde{v}_2$ at the north pole,

$$\lim_{|x| \rightarrow \infty} (u_2 - u_1)(x) = 0,$$

and hence $u_2 - u_1$ changes sign at infinity. Thus there must exist at least two disjoint simply-connected regions $\Omega_i \subset \mathbb{R}^2, i = 1, 2$ (not necessarily bounded) such that there exist $\omega_i \subseteq \Omega_i, i = 1, 2$, with

$$\begin{cases} u_1 > u_2 & \text{in } \omega_1, & u_2 > u_1 & \text{in } \omega_2, \\ u_1 = u_2 & \text{on } \partial\omega_1 \cup \partial\omega_2, \\ \lim_{\substack{|x| \rightarrow \infty \\ x \in \omega_1 \cup \omega_2}} (u_2 - u_1)(x) = 0 \end{cases}$$

Remark 4.1 We will not discuss here the regularity of ω_1, ω_2 since Theorem 1.2 applies under very weak assumptions on ω_1, ω_2 as far as $\omega_i \in \Omega_i, i = 1, 2$, see Remark 1.5. If $\omega_i \notin \Omega_i$ for some i , we may always choose a larger and smooth $O_i \subset \mathbb{R}^2$ such that $\omega_i \in O_i$ and then apply Theorem 1.2 with Ω_i replaced by O_i .

The idea is then to apply the singular Sphere Covering Inequality of Theorem 1.2 in (at least one of) the Ω_i (see Remark 1.3). To this end we recall the notation $h = e^H$ and observe that,

$$-\Delta H(x) = l \frac{4}{(1 + |x|^2)^2} = \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} \frac{4}{(1 + |x|^2)^2}. \quad (54)$$

For fixed $\omega \subseteq \mathbb{R}^2$ we define,

$$I(\omega) = \frac{1}{4\pi} \int_{\omega} \frac{4}{(1 + |x|^2)^2} dx.$$

Recalling the Definition 1.1 of $\tilde{\omega}$, we then set,

$$I_s(\omega) = \begin{cases} I(\omega) & \text{if } \omega \text{ is simply-connected,} \\ I(\tilde{\omega}) & \text{if } \omega \text{ is not simply-connected.} \end{cases} \quad (55)$$

We will prove (i) and (ii) of Theorem 1.9 separately.

Proof of (i) We have $N \geq 0$ and $\rho < 4\pi(2 + \sum_j \alpha_j)$. Observe that in this case $-\Delta H > 0$ in $\mathbb{R}^2$. Thus, by the definition of $\alpha(\omega_i) = \alpha(\omega_i, h, f) \geq 0$ in (8), for (51) we have,

$$\alpha(\omega_i) = \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} I_s(\omega_i) - \sum_{j \in J_i} \alpha_j, \quad i = 1, 2, \quad (56)$$

where $J_i = \{j \in \{1, \dots, N\} : q_j \in \tilde{\omega}_i\}$. Moreover, let us define,

$$\alpha = \alpha(\omega_1) + \alpha(\omega_2).$$

We point out that obviously,

$$I_s(\mathbb{R}^2) = 1 = \frac{|\mathbb{S}^2|}{4\pi}. \quad (57)$$

Therefore, we have

$$\begin{aligned}
 \alpha &= \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} (I_s(\omega_1) + I_s(\omega_2)) - \sum_{j \in J_1 \cup J_2} \alpha_j \\
 &\leq \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} I_s(\mathbb{R}^2) - \sum_{j \in J_1 \cup J_2} \alpha_j \\
 &= \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} - \sum_{j \in J_1 \cup J_2} \alpha_j \\
 &\leq \frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} - \sum_j \alpha_j \\
 &= \frac{8\pi - \rho}{4\pi},
 \end{aligned} \tag{58}$$

and then, in particular, it holds,

$$\alpha < 2. \tag{59}$$

We claim that

$$2\rho > 16\pi - 8\pi\alpha, \tag{60}$$

and discuss two cases separately.

Case 1. Suppose first that $\alpha(\omega_i) < 1$ for all $i = 1, 2$. By applying Theorem 1.2 in Ω_1 and in Ω_2 we find that,

$$\int_{\omega_i} (h(x)e^{u_1} + h(x)e^{u_2}) dx \geq 8\pi(1 - \alpha(\omega_i)), \quad i = 1, 2. \tag{61}$$

Observe that the above inequalities are strict. Indeed, let us focus for example on $i = 1$. In view of Theorem 1.2, the equality holds if and only if, in suitable coordinates, $\omega_1 = B_\delta(0)$ for some $\delta > 0$ and $h(x)e^{u_i} \equiv |x|^{-2\alpha}e^{U_{\lambda_i, \alpha}}$ for some $\lambda_i > 0$, $i = 1, 2$, where $\alpha = \frac{1}{4\pi}\mu_+(\omega_1)$ and $U_{\lambda_i, \alpha}$ is defined in (4). Moreover, $\mu_+(\omega_1) = -\Delta H - f = 4\pi\alpha\delta_{p=0}$. Therefore, we have $u_i + H(x) \equiv U_{\lambda_i, \alpha} - 2\alpha \ln |x|$ and in particular

$$\Delta u_i + \Delta H(x) = \Delta U_{\lambda_i, \alpha} - 4\pi\alpha\delta_{p=0}, \text{ in } B_\delta(0).$$

By using first (5) and (51) and then $h(x)e^{u_i} \equiv |x|^{-2\alpha}e^{U_{\lambda_i, \alpha}}$ we come up with,

$$4\pi \sum_{j \in J_1} \alpha_j \delta_{q_j} + \Delta H(x) = -4\pi\alpha\delta_{p=0}, \text{ in } B_\delta(0),$$

where J_1 is defined after (56). By (54) we have,

$$\frac{4\pi(2 + \sum_j \alpha_j) - \rho}{4\pi} \frac{4}{(1 + |x|^2)^2} = 4\pi\alpha\delta_{p=0} + 4\pi \sum_{j \in J_1} \alpha_j \delta_{q_j}, \text{ in } B_\delta(0),$$

which is impossible since we also have $\rho < 4\pi(2 + \sum_j \alpha_j)$. We conclude that the inequalities in (61) are strict.

At this point we recall that (53) holds true and that Ω_1, Ω_2 are disjoint regions. Therefore, since $\omega_i \subseteq \Omega_i, i = 1, 2$, by summing up the above inequalities we deduce that,

$$\begin{aligned} 2\rho &= \int_{\mathbb{R}^2} (h(x)e^{u_1} + h(x)e^{u_2}) dx \geq \sum_{i=1}^2 \int_{\omega_i} (h(x)e^{u_1} + h(x)e^{u_2}) dx \\ &> 16\pi - 8\pi(\alpha(\omega_1) + \alpha(\omega_2)), \end{aligned}$$

which proves the claim.

Case 2. We assume without loss of generality that $\alpha(\omega_1) \geq 1$. Then, we have

$$\alpha(\omega_2) = \alpha - \alpha(\omega_1) \leq \alpha - 1.$$

Moreover, in view of (59) we deduce that $\alpha(\omega_2) < 1$. By applying Theorem 1.2 in ω_2 and by the latter estimate we obtain

$$\begin{aligned} 2\rho &= \int_{\mathbb{R}^2} (h(x)e^{u_1} + h(x)e^{u_2}) dx > \int_{\omega_2} (h(x)e^{u_1} + h(x)e^{u_2}) dx \\ &\geq 8\pi(1 - \alpha(\omega_2)) \geq 16\pi - 8\pi\alpha, \end{aligned}$$

which proves the claim.

By using (60) together with (58) we end up with

$$2\rho > 16\pi - 8\pi\alpha > 2\rho,$$

which is the desired contradiction. The proof of (i) is concluded. $\square$

Proof of (ii). Here we have $N \geq 3$ and $\rho = 4\pi(2 + \sum_j \alpha_j)$. Therefore, in view of (53), we see that a necessary condition for the existence of a solution is,

$$\sum_j \alpha_j > -2. \quad (62)$$

Therefore, from now on we assume without loss of generality that (62) holds true. In view of (54), we find that $-\Delta H = 0$ in $\mathbb{R}^2$. Hence, by the definition of $\alpha(\omega_i) =$

$\alpha(\omega_i, h, f) \geq 0$ in (8), for the Eq. (51) we have,

$$\alpha(\omega_i) = - \sum_{j \in J_i} \alpha_j, \quad i = 1, 2, \quad (63)$$

where $J_i = \{j \in \{1, \dots, N\} : q_j \in \tilde{\omega}_i\}$. As above we set,

$$\alpha = \alpha(\omega_1) + \alpha(\omega_2).$$

Observe that, in view of (62), we have,

$$\alpha = - \sum_{j \in J_1 \cup J_2} \alpha_j \leq - \sum_j \alpha_j < 2. \quad (64)$$

We claim that

$$2\rho > 16\pi + 8\pi \sum_j \alpha_j. \quad (65)$$

By arguing as in Case 2 of (i) above and by using (64) we find that,

$$2\rho > 16\pi - 8\pi\alpha \geq 16\pi + 8\pi \sum_j \alpha_j,$$

whenever $\alpha(\omega_i) \geq 1$ for some $i \in \{1, 2\}$, where $\alpha(\omega_i)$ is defined in (63). Therefore, we may restrict to the case $\alpha(\omega_i) < 1$ for all $i = 1, 2$. By applying Theorem 1.2 in Ω_1 and in Ω_2 we find that,

$$\int_{\omega_i} (h(x)e^{u_1} + h(x)e^{u_2}) dx \geq 8\pi(1 - \alpha(\omega_i)), \quad i = 1, 2. \quad (66)$$

Next, in view of (53), and since Ω_1, Ω_2 are disjoint regions and $\omega_i \subseteq \Omega_i, i = 1, 2$, by summing up the above inequalities we deduce that,

$$\begin{aligned} 2\rho &= \int_{\mathbb{R}^2} (h(x)e^{u_1} + h(x)e^{u_2}) dx \geq \sum_{i=1}^2 \int_{\omega_i} (h(x)e^{u_1} + h(x)e^{u_2}) dx \\ &\geq 16\pi - 8\pi(\alpha(\omega_1) + \alpha(\omega_2)) = 16\pi - 8\pi\alpha \\ &\geq 16\pi + 8\pi \sum_j \alpha_j, \end{aligned}$$

where we have used (64) in the last step. To prove the claim (65) it is enough to prove one of the above inequalities is strict. The equality in the last step holds if and only if $\alpha = - \sum_j \alpha_j$, that is, by the definition of $\alpha(\omega_i)$ in (63), if and only if

$$\{q_1, \dots, q_N\} \subset \tilde{\omega}_1 \cup \tilde{\omega}_2.$$

Since by assumption $N \geq 3$ we conclude that at least one of $\tilde{\omega}_1, \tilde{\omega}_2$ must contain more than one of the points q_j 's. Without loss of generality we assume

$$|J_1| > 1,$$

where J_i is defined after (63). We shall prove that in this case the inequality in (66) is strict for $i = 1$. Indeed, in view of Theorem 1.2, the equality holds if and only if, in suitable coordinates, $\tilde{\omega}_1 = \omega_1 = B_\delta(0)$ for some $\delta > 0$ and $h(x)e^{u_i} \equiv |x|^{-2\alpha} e^{U_{\lambda_i, \alpha}}$ for some $\lambda_i > 0, i = 1, 2$, where $\alpha = \frac{1}{4\pi} \mu_+(\omega_1)$ and $U_{\lambda_i, \alpha}$ is defined in (4). Moreover, $\mu_+(\omega_1) = -\Delta H - f = 4\pi\alpha\delta_{p=0}$. Therefore, we have $u_i + H(x) \equiv U_{\lambda_i, \alpha} - 2\alpha \ln |x|$ and in particular

$$\Delta u_i + \Delta H(x) = \Delta U_{\lambda_i, \alpha} - 4\pi\alpha\delta_{p=0}, \text{ in } B_\delta(0).$$

By using first (5), (51) and then $h(x)e^{u_i} \equiv |x|^{-2\alpha} e^{U_{\lambda_i, \alpha}}$ we find that,

$$4\pi \sum_{j \in J_1} \alpha_j \delta_{q_j} + \Delta H(x) = -4\pi\alpha\delta_{p=0}, \text{ in } B_\delta(0).$$

Recalling that $-\Delta H = 0$ in $\mathbb{R}^2$ and that $|J_1| > 1$, we readily conclude that the latter equality is impossible. Therefore, in particular the inequality in (66) is strict for $i = 1$, which proves the claim (65).

At this point, since $\rho = 4\pi(2 + \sum_j \alpha_j)$, then, by using (65), we deduce that,

$$2\rho > 16\pi + 8\pi \sum_j \alpha_j = 2\rho,$$

which is the desired contradiction. The proof is completed. $\square$

5 Symmetry for spherical Onsager vortex equation

In this section we provide the proof of the symmetry result for the spherical Onsager vortex equation (23) of Theorem 1.14, see Sect. 1.4. The argument is based on the singular Sphere Covering Inequality, Theorem 1.2, jointly with some ideas introduced in [45].

Proof of Theorem 1.14 Without loss of generality we may assume that $\vec{n}$ coincides with the north pole, i.e. $\vec{n} = \mathcal{N} = (0, 0, 1)$. Let $\Pi : \mathbb{S}^2 \setminus \{\mathcal{N}\} \rightarrow \mathbb{R}^2$ be the stereographic projection with respect to the north pole defined in (16) and v be a solution of (23) with $8\pi < \beta \leq 16\pi$ and γ as in (24). By setting,

$$u(x) = v(\Pi^{-1}(x)) \text{ for } x \in \mathbb{R}^2,$$

then u satisfies,

$$\Delta u + \frac{J^2(x) \exp(\beta u - \gamma \psi(x))}{\int_{\mathbb{R}^2} J^2(x) \exp(\beta u - \gamma \psi(x)) dx} - \frac{J^2(x)}{4\pi} = 0 \quad \text{in } \mathbb{R}^2, \quad (67)$$

where

$$J(x) = \frac{2}{1 + |x|^2} \quad \text{and} \quad \psi(x) = \frac{|x|^2 - 1}{|x|^2 + 1}.$$

As in [45,53] we define,

$$w(x) = \beta \left(u(x) - \frac{1}{4\pi} \ln(1 + |x|^2) \right) - c,$$

with

$$c = \gamma + \ln \left(\frac{2}{\beta} \int_{\mathbb{R}^2} J^2(x) e^{\beta u - \gamma \psi} dx \right).$$

Then, we have,

$$\Delta w + h(x) e^w = 0 \quad \text{in } \mathbb{R}^2, \quad (68)$$

and,

$$\int_{\mathbb{R}^2} h(x) e^w dx = \beta,$$

where,

$$h(x) = h(|x|) = 8(1 + |x|^2)^{\left(-2 + \frac{\beta}{4\pi}\right)} e^{\gamma J(x)}, \quad (69)$$

which, letting $h = e^H$, satisfies,

$$\Delta H(x) = \frac{4 \left(-2 + \frac{\beta}{4\pi}\right)}{(1 + |x|^2)^2} + \frac{8\gamma(|x|^2 - 1)}{(1 + |x|^2)^3}. \quad (70)$$

Observe that $-\Delta H \leq 0$ in $\mathbb{R}^2$ if and only if $\gamma \leq \frac{\beta}{8\pi} - 1$. For the latter range of γ we already know from Theorem E that every solution to (23) is axially symmetric with respect to $\vec{n}$. Therefore, we suppose from now on that $\gamma > \frac{\beta}{8\pi} - 1$. Then, $-\Delta H > 0$ in $B_r(0)$, where

$$r^2 = \frac{\gamma + 1 - \frac{\beta}{8\pi}}{\gamma - 1 + \frac{\beta}{8\pi}}. \quad (71)$$

Our goal is to show that the solution v to (23) is evenly symmetric with respect to a plane passing through the origin and containing the vector $\vec{n}$, i.e. that w is evenly symmetric with respect to a line passing through the origin and a point $p \in \mathbb{R}^2$. First of all, observe that $\lim_{|x| \rightarrow +\infty} w(x) = -\infty$ and hence w has a maximum point which we denote by $p \in \mathbb{R}^2$. Without loss of generality we may assume that p lies on x_1 -axis, and then define $w^*(x_1, x_2) = w(x_1, -x_2)$ and

$$\tilde{w} = w - w^*.$$

With these notations we are left to prove that $\tilde{w} \equiv 0$ in $\mathbb{R}^2$ and we assume by contradiction this is not the case. Observe that $\tilde{w}$ satisfies,

$$\Delta \tilde{w} + c(x)\tilde{w} = 0 \quad \text{in } \mathbb{R}^2, \quad c(x) = h(x) \frac{e^w - e^{w^*}}{w - w^*}. \quad (72)$$

On the other hand $\tilde{w}(x_1, 0) = 0$ for all $x_1 \in \mathbb{R}$. Moreover, $\tilde{w}$ has a critical point at p that lies on the x_1 -axis and thus $\tilde{w}_{x_2}(p) = 0$.

We claim that $\tilde{w}$ changes sign in $\mathbb{R}_+^2$. Indeed, if this were not the case, then we could assume that $\tilde{w} < 0$ in any $B_R^+(p) = \{x \in B_R(p) : x_2 > 0\}$. However, the latter fact jointly with $\tilde{w}_{x_2}(p) = 0$ contradicts the thesis of Hopf's Lemma when applied to the Eq. (72) at the point p in the domain $B_R^+(p)$.

As a consequence, we conclude that the nodal line of $w - w^*$ divides a neighborhood of p into at least four regions. Thus there exist at least two disjoint simply-connected regions $\Omega_i \subset \mathbb{R}_+^2$, $i = 1, 2$ (not necessarily bounded) such that there exist a pair of open subsets $\omega_i \subseteq \Omega_i$, $i = 1, 2$, such that,

$$\begin{cases} w > w^* & \text{in } \omega_1, & w^* > w & \text{in } \omega_2, \\ w = w^* & \text{on } \partial\omega_1 \cup \partial\omega_2, \\ \lim_{\substack{|x| \rightarrow \infty \\ x \in \omega_1 \cup \omega_2}} (w - w^*)(x) = 0. \end{cases}$$

As for the regularity of ω_1, ω_2 we refer to Remark 4.1. By applying Theorem 1.2 in both ω_i , $i = 1, 2$ (see Remark 1.3), and summing up we get,

$$\begin{aligned} \beta &= \int_{\mathbb{R}^2} h(x)e^w dx = \int_{\mathbb{R}_+^2} \left(h(x)e^w + h(x)e^{w^*} \right) dx \\ &\geq \sum_{i=1}^2 \int_{\omega_i} \left(h(x)e^w + h(x)e^{w^*} \right) dx > 8\pi (2 - \alpha(\omega_1) - \alpha(\omega_2)). \end{aligned}$$

One can show that the last inequality is strict by using the same argument as in the proof of Theorem 1.9. Hence,

$$\beta + 8\pi(\alpha(\omega_1) + \alpha(\omega_2)) > 16\pi. \quad (73)$$

On the other hand, recalling the definition of α in (8) (see also Example 1.1), the expression in (70) and the discussion right above (71), we find that,

$$\begin{aligned}
 8\pi(\alpha(\omega_1) + \alpha(\omega_2)) &\leq -2 \int_{B_r^+(0)} \left(\frac{4\left(-2 + \frac{\beta}{4\pi}\right)}{(1 + |x|^2)^2} + \frac{8\gamma(|x|^2 - 1)}{(1 + |x|^2)^3} \right) dx \\
 &= - \int_{B_r(0)} \left(\frac{4\left(-2 + \frac{\beta}{4\pi}\right)}{(1 + |x|^2)^2} + \frac{8\gamma(|x|^2 - 1)}{(1 + |x|^2)^3} \right) dx \\
 &= - \left(4\left(-2 + \frac{\beta}{4\pi}\right) + 8\gamma \right) \int_{B_r(0)} \frac{dx}{(1 + |x|^2)^2} + 16\gamma \int_{B_r(0)} \frac{dx}{(1 + |x|^2)^3} \\
 &= -8\pi \left(-1 + \frac{\beta}{8\pi} + \gamma \right) \left(1 - \frac{1}{1 + r^2} \right) + 8\pi\gamma \left(1 - \frac{1}{(1 + r^2)^2} \right) \\
 &= 8\pi \left(1 - \frac{1}{1 + r^2} \right) \left(-\left(-1 + \frac{\beta}{8\pi} + \gamma\right) + \gamma \left(1 + \frac{1}{1 + r^2} \right) \right).
 \end{aligned}$$

In view of (71) we also have,

$$\begin{aligned}
 8\pi(\alpha(\omega_1) + \alpha(\omega_2)) &\leq 8\pi \left(1 - \frac{1}{1 + r^2} \right) \left(-\left(-1 + \frac{\beta}{8\pi} + \gamma\right) + \gamma \left(1 + \frac{1}{1 + r^2} \right) \right) \\
 &= 8\pi \left(\frac{\gamma + 1 - \frac{\beta}{8\pi}}{2\gamma} \right) \left(-\left(-1 + \frac{\beta}{8\pi} + \gamma\right) + \frac{3\gamma - 1 + \frac{\beta}{8\pi}}{2} \right) \\
 &= 8\pi \frac{\left(\gamma + 1 - \frac{\beta}{8\pi}\right)^2}{4\gamma}.
 \end{aligned}$$

Inserting the latter estimate into (73) we end up with,

$$\frac{\beta}{8\pi} + \frac{(\gamma + 1 - \frac{\beta}{8\pi})^2}{4\gamma} > 2.$$

The above inequality can be rewritten as

$$\gamma^2 + 2\gamma \left(\frac{\beta}{8\pi} - 3 \right) + \left(\frac{\beta}{8\pi} - 1 \right)^2 > 0,$$

which contradicts the assumption (24) on γ . It follows that w is evenly symmetric about the line passing through the origin and the point $p \in \mathbb{R}^2$, as claimed. $\square$

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