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Global Classical Solutions of Three Dimensional Viscous MHD System Without Magnetic Diffusion on Periodic Boxes

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Abstract

In this paper, we study the global existence of classical solutions to the three dimensional incompressible viscous magneto-hydrodynamical system without magnetic diffusion on periodic boxes, that is, with periodic boundary conditions. We work in Eulerian coordinates and employ a time-weighted energy estimate to prove the global existence result, under the assumptions that the initial magnetic field is close enough to an equilibrium state and the initial data have some symmetries.

1. Introduction

The equations of viscous magnetohydrodynamics (MHD) model the motion of electrically conducting fluids interacting with magnetic fields. When the fluids are strongly collisional plasmas, or the resistivity due to collisions is extremely small, the diffusion in a magnetic field is often neglected [4,8,13]. When magnetic diffusion is missing, it is extremely interesting to understand whether the fluid viscosity only could prevent singularity development from small smooth initial data in three dimensional physical space (in view of the strongly nonlinear coupling between fluids and the magnetic field). Mathematically, it is also close in structure to the model of dynamics of certain complex fluids, including the hydrodynamics of viscoelastic fluids, c.f. [16–20]. To this end, we investigate the global existence of smooth solutions to the following initial boundary value problem:

$$\begin{cases} B_t + u \cdot \nabla B = B \cdot \nabla u, \\ u_t + u \cdot \nabla u - \Delta u + \nabla p = B \cdot \nabla B, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(0,x) = u_0(x), \quad B(0,x) = B_0(x), \end{cases} \tag{1.1}$$

with periodic boundary conditions

$$x \in [-\pi, \pi]^3 = \mathbb{T}^3, \tag{1.2}$$

where $B = (B_1, B_2, B_3)$ denotes the magnetic field, $u = (u_1, u_2, u_3)$ the fluid velocity, and $p = q + \frac{1}{2}|B|^2$, where q denotes the scalar pressure of the fluid.

Impressive progress had been made in the past several decades for MHD systems. Indeed, according to the level of dissipations, there are roughly three different layers of models: inviscid and non-resistive (no viscosity, no magnetic diffusion, hence no dissipation); viscous and resistive (fully dissipative in fluids and in magnetic field); and partially dissipative (only viscosity or magnetic diffusion presents). On the one hand, it is natural to expect the global existence of classical solutions for viscous and resistive MHD at least for small initial data; this has been confirmed in classic papers by Duvaut and Lions [9] and by Sermange and Temam [23]. In 2008, Abidi and Paicu [1] generalized these results to the inhomogeneous MHD system with initial data in the so-called critical spaces. More recently, Cao and Wu [6] (also see [7]) proved the global well-posedness for any data in $H^2(\mathbb{R}^2)$ with mixed partial viscosity and magnetic diffusion in a two dimensional MHD system. On the other hand, it is somehow striking that Bardos et al. [3] proved that the inviscid and non-resistive MHD system also admits a unique global classical solution when the initial data is near a nontrivial equilibrium. It seems that dispersion and some coupling of nonlinearity between fluids and the magnetic field alone are sufficient to maintain the regularity from initial data. Very recently, the vanishing dissipation limit from a fully dissipative MHD system to an inviscid and non-resistive MHD system has been justified by He et al. [10], Cai and Lei [5], Wei and Zhang [25] under some structural conditions between viscosity and magnetic diffusion coefficients. Therefore, it is not a surprise that the remaining case, partially dissipative MHD, has attracted a lot of attention in the recent years. As documented in [6,15], the inviscid and resistive 2D MHD system admits a global H^1 weak solution, but the uniqueness of such a solution with higher order regularity is still not known.

In the case of our consideration, namely the incompressible MHD system with positive viscosity and zero resistivity, it is still an open problem whether or not there exists a global classical solution even in two dimensional space for generic smooth initial data. The main difficulty of studying these MHD systems lies in the non-resistivity of the magnetic equation. Some interesting results have been obtained for small smooth solutions. For a closely related model in three dimensions, the global well-posedness was established by Lin and Zhang [19], and a simpler proof was offered by Lin and Zhang [20]. With certain admissible conditions for initial data, Lin et al. [18] established the global existence in 2D for initial data close to an nontrivial equilibrium state and the three dimensional case was proved by Xu and Zhang [26]. Later, Ren et al. [21] removed the restriction in the 2D case (see Zhang [27] for a simplified proof). We also refer to another proof for the 2D incompressible case by Hu and Lin [11]. Hu [12] further established some results for the 2D compressible MHD system. Very recently, under the Lagrangian coordinate system, Abidi and Zhang [2] proved the global well-posedness for the three dimensional MHD system without the admissible restriction. For certain class of large data, Lei [14] proved the global regularity of some axially symmetric solutions in the three space dimensional MHD system. While all results down the line are about the Cauchy problem, an initial boundary value problem for the 2D case under the Eulerian coordinate in a strip domain $\mathbb{R} \times (0, 1)$ was done by Ren et al. [22] recently. The three dimensional case on $\mathbb{R}^2 \times (0, 1)$ for both compressible and incompressible fluids was considered by Tan and Wang [24] under Lagrangian coordinates. In the three dimensional case, these inspiring results, along with many innovative methods and estimates, made full use of partial dissipation offered by viscosity, dispersion of waves on an unbounded domain and the structure of the Lagrangian formulation (which contains a one time derivative already and helps capture the weak dissipation). It is then natural to explore two questions. The first of these is, is it possible to establish global existence of small smooth solutions on bounded domain, where the dispersion effect is limited? The second question is: can one work with Eulerian coordinates where the system takes a simpler form and thus involves the loss of one time derivative and the loss of possible time decay?

Our main aim in this paper is to offer answers to these questions. Indeed, as one step in this direction, we will establish the global existence of small smooth solutions to the three dimensional incompressible viscous magneto-hydrodynamical system without resistivity on periodic boxes, under the assumptions that the initial magnetic field is close enough to an equilibrium state and that the initial data have some symmetry structure. We will also avoid the use of Lagrangian formulation. The advantage of the Eulerian coordinates is that, if successful, things will be neat and simple.

To fix the idea, we adopt the following notations:

$$x_h = (x_1, x_2), \quad \nabla_h = (\partial_1, \partial_2), \quad B_h = (B_1, B_2)^\top,$$

as well as similar notations for other quantities, without causing further confusion.

We assume that

$$\begin{aligned} &u_{0,h}(x), \quad B_{0,3}(x) \quad \text{are even periodic with respect to } x_3, \\ &u_{0,3}(x), \quad B_{0,h}(x) \quad \text{are odd periodic with respect to } x_3, \end{aligned} \tag{1.3}$$

and moreover that

$$\int_{\mathbb{T}^3} u_0 \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^3} B_{0,3} \,\mathrm{d}x = \alpha \neq 0. \tag{1.4}$$

Our main result can be stated as follows:

Theorem 1.1. *Consider the three dimensional MHD system* (1.1)–(1.2) *with initial data that satisfies the conditions* (1.3)–(1.4). *Then there exists a small constant* $\varepsilon > 0$ *only depending on* α *such that the system* (1.1) *admits a global smooth solution provided that*

$$\|u_0\|_{H^{2s+1}} + \|\nabla B_0\|_{H^{2s}} \leqq \varepsilon,$$

where $s \geqq 5$ *is an integer.*

Remark 1.2. Our methods can be applied to other related models. Similar results for the compressible system will be presented in a forthcoming paper.

Without loss of generality, we assume that $\alpha = (2\pi)^3$, and following LIN AND ZHANG [19], we let

$$B_0 = b_0 + e_3,$$

where $e_3 = (0, 0, 1)^\top$. Hence, we have

$$\int_{\mathbb{T}^3} b_0 \,\mathrm{d}x = \int_{\mathbb{T}^3} u_0 \,\mathrm{d}x = 0. \tag{1.5}$$

Set $B = b + e_3$, so we get the system of pair (u, b) as follows:

$$\begin{cases} b_t + u \cdot \nabla b = b \cdot \nabla u + \partial_3 u, \\ u_t + u \cdot \nabla u - \Delta u + \nabla p = b \cdot \nabla b + \partial_3 b, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{1.6}$$

with initial data

$$u(0, x) = u_0(x), \quad b(0, x) = b_0(x),$$

and with the property of initial data (1.3) holding, that is,

$$\begin{aligned} &u_h(0, x), \quad b_3(0, x) \quad \text{are even periodic with respect to } x_3, \\ &u_3(0, x), \quad b_h(0, x) \quad \text{are odd periodic with respect to } x_3. \end{aligned} \tag{1.7}$$

Also, by the periodic boundary conditions (1.5) and system (1.6), we have

$$\int_{\mathbb{T}^3} b \,\mathrm{d}x = \int_{\mathbb{T}^3} u \,\mathrm{d}x = 0. \tag{1.8}$$

Remark 1.3. The property (1.7) will hold in the time evolution. Indeed, we can define $\bar{u}(t, x)$, $\bar{b}(t, x)$ as follows:

$$\begin{aligned} &\bar{u}_h(t, x_h, x_3) = u_h(t, x_h, -x_3), \quad \bar{u}_3(t, x_h, x_3) = -u(t, x_h, -x_3), \\ &\bar{b}_h(t, x_h, x_3) = -b_h(t, x_h, -x_3), \quad \bar{b}_3(t, x_h, x_3) = b_3(t, x_h, -x_3). \end{aligned}$$

Then, quantities $\bar{u}$, $\bar{b}$ satisfy the same system, (1.6), like u, b, and also have the same initial data. Hence, by the uniqueness of classical solution, we obtain $\bar{b}(t, x) = b(t, x)$ and $\bar{u}(t, x) = u(t, x)$. Therefore, we see property (1.7) persist.

Although property (1.7) (from (1.3)) could be realized physically initially and is preserved in time evolution as explained in the previous remark, its physical interpretation is not quite clear. It identifies a significant class of initial data on a periodic box admitting a global classical solution to (1.6) near a nontrivial magnetic equilibrium. Mathematically, it helps in analysis to allow us to use Poincaré inequality (Proposition 2.1) for some crucial terms in the estimates. On the other hand, this is also needed to rule out an extremely unclear situation related to the global regularity of 2D MHD without magnetic diffusion when the initial data is a small perturbation near the **trivial** equilibrium, which is a very difficult problem.

To help readers understand the situation, let's choose the following special class of initial data:

$$B_0(x) = \left(B_0^h(x_h), 1\right), u_0(x) = \left(u_0^h(x_h), 0\right), \nabla_h \cdot B_0^h = \nabla_h \cdot u_0^h = 0, \quad (1.9)$$

which reduces the original system (1.1) into the following 2D problem:

$$\begin{cases} B_t^h + u^h \cdot \nabla_h B^h = B^h \cdot \nabla_h u^h, \\ u_t^h + u^h \cdot \nabla_h u^h - \Delta_h u^h + \nabla_h P = B^h \cdot \nabla_h B^h, \\ \nabla_h \cdot B^h = \nabla_h \cdot u^h = 0, \\ B^h(0, x_h) = B_0^h(x_h), u^h(0, x_h) = u_0^h(x_h), \end{cases} \quad (1.10)$$

with initial data $(B_0^h(x_h), u_0^h(x_h))$ being a perturbation near a trivial equilibrium. This, however, is still a challenging open problem. Furthermore, we note that if $(B^h, u^h)(t, x_h)$ is a classical solution of (1.10), then $B(t, x) = (B^h(t, x_h), 1)$, $u(t, x) = (u^h(t, x_h), 0)$ is the corresponding solution of (1.1) with initial data (1.9). When the problem is considered in the whole space, by requiring finite energy in $\mathbb{R}^3$, one finds $B_0^h(x_h) = 0 = u_0^h(x_h)$, avoiding the complex situation successfully. However, on a periodic bounded domain, the finite energy condition is not sufficient to show that the solution of (1.10) is trivial. To avoid the unclear situation mentioned before, some additional conditions are needed. In this paper, we impose (1.7) (from (1.3)) to ensure that $B_0^h(x_h) = 0 = u_0^h(x_h)$, and thus system (1.10) has only a trivial solution. It would be interesting to explore other conditions for this purpose.

In order to prove Theorem 1.1, we only need to consider the system (1.6) instead.

In this paper, we have to face the difficulties from the bounded domain and the loss of weak dissipation without using the Lagrangian formulation. One of the major differences in analysis between the whole space and the bounded domain is the character of dissipation. For the whole space, although the system contains only the viscosity, it is possible to recover dissipative structure for all components of u and b, in addition to the advantage of wave dispersion. For the bounded domain, however, it is extremely difficult to recover dissipative structure for all components of u and b. Indeed, even with the help of condition (1.7) and Poincaré inequality (Proposition 2.1), we could not derive dissipation for b_3. We emphasize that the analysis of the whole space case is quite complicated and exhibits very different features compared with our case here. They are different in nature and difficult in various aspects. These challenges will be overcome through a carefully designed weighted energy method with the help of some observations on the structure of the system. One of the major observations is that the time derivative of b is essentially quadratic terms plus a derivative term in the good direction x_3 where dissipation kicks in. Another observation is that although the bounded domain pushes us away from the possible dispersion of waves, it does compensate us with Poincaré inequality. However, the high space dimensions, the lack of magnetic diffusion, and the strongly coupled nonlinearity of the problem make the mathematical analysis very challenging. Even with our carefully designed time-weighted energies, there are still many dedicated technical issues. One of our main obstacles is to derive the time dissipative estimate to the term $b \cdot \nabla b$, which behaves most wildly in the system. Writing $b \cdot \nabla b =$

$b_3 \cdot \partial_3 b + b_h \cdot \partial_h b$, we notice that $b_3 \cdot \partial_3 b$ contains one good quantity $\partial_3 b$ can be estimated relatively easily due to dissipation in the x_3 direction. Hence, we focus on the term $b_h \cdot \nabla_h b$ containing two bad terms. To overcome this difficulty, we make full use of the condition (1.7) and Poincaré inequality in x_3 direction. Thus the norm of b_h can be controlled by the norm of $\partial_3 b_h$. This specific choice of estimate avoids the presence of interaction between two wild quantities. Such an idea actually originates from the *null condition* in the theory of wave equations. However, we still have to come across other difficulties in the estimate process. For example, we cannot achieve the uniform bound of all higher order norms that we want. Instead, we turn to control the growth of such norms by the energy frame we construct in the next section. More detailed decay estimates will also be presented in Section 2.

2. Energy Estimate and the Proof of Main Result

2.1. Preliminary

In this subsection, we first introduce a useful proposition related to Poincaré inequality which plays an important role in our proof to the main theorem of this paper.

Proposition 2.1. *For any function $f(x) \in H^{k+1}(\mathbb{T}^3)$, $k \in \mathbb{N}$ satisfying the condition*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_h, x_3)\,\mathrm{d}x_3 = 0, \qquad \forall\, x_h \in \mathbb{T}^2, \tag{2.1}$$

it holds that

$$\|f\|_{H^k(\mathbb{T}^3)} \lesssim \|\partial_3 f\|_{H^k(\mathbb{T}^3)}.$$

Proof. First, we can write

$$\|f\|^2_{H^k(\mathbb{T}^3)} = \sum_{|\alpha|=0}^{k} \int_{\mathbb{T}^2}\int_{-\pi}^{\pi} |\partial^\alpha f(x_h, x_3)|^2 \mathrm{d}x_3 \mathrm{d}x_h. \tag{2.2}$$

Here, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

Notice the condition (2.1), where we have, for multi-index $\alpha = (\alpha_1, \alpha_2, 0)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \partial^\alpha f(x_h, x_3)\mathrm{d}x_3 = 0.$$

For multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_3 > 0$, by the periodic boundary condition, we also have

$$\int_{-\pi}^{\pi} \partial^\alpha f(x_h, x_3)\mathrm{d}x_3 = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3 - 1} f(x_h, \cdot)\Big|_{-\pi}^{\pi} = 0.$$

Therefore, the average value of $\partial^\alpha f(x_h, \cdot)$ in x_3 direction over $[-\pi, \pi]$ is zero. Hence, applying standard Poincaré inequality to $\partial^\alpha f(x_h, x_3)$ in the x_3 direction, we have $\forall x_h \in \mathbb{T}^2$ and,

$$\int_{-\pi}^{\pi} |\partial^\alpha f(x_h, x_3)|^2 \mathrm{d}x_3 \lesssim \int_{-\pi}^{\pi} |\partial^\alpha \partial_3 f(x_h, x_3)|^2 \mathrm{d}x_3.$$

According to the definition of $\|f\|_{H^k(\mathbb{T}^3)}$, that is (2.2), we finally obtain

$$\begin{aligned}\|f\|^2_{H^k(\mathbb{T}^3)} &= \sum_{|\alpha|=0}^{k} \int_{\mathbb{T}^2} \int_{-\pi}^{\pi} |\partial^\alpha f(x_h, x_3)|^2 \mathrm{d}x_3 \mathrm{d}x_h \\ &\lesssim \sum_{|\alpha|=0}^{k} \int_{\mathbb{T}^2} \int_{-\pi}^{\pi} |\partial^\alpha \partial_3 f(x_h, x_3)|^2 \mathrm{d}x_3 \mathrm{d}x_h \\ &= \|\partial_3 f\|^2_{H^k(\mathbb{T}^3)}.\end{aligned}$$

□

Now, let us introduce the energy frame that will enable us to achieve our desired estimate. Based on our discussion in Section 1, we define some time-weighted energies for the system (1.6). The energies below are defined on the domain $\mathbb{R}^+ \times \mathbb{T}^3$. For $s \in \mathbb{N}$ and $0 < \sigma < 1$, we set

$$\begin{aligned}E_0(t) =& \sup_{0 \leqq \tau \leqq t} (1+\tau)^{-\sigma} \left(\|u(\tau)\|^2_{H^{2s+1}} + \|b(\tau)\|^2_{H^{2s+1}} \right) \\ &+ \int_0^t (1+\tau)^{-1-\sigma} \left(\|u(\tau)\|^2_{H^{2s+1}} + \|b(\tau)\|^2_{H^{2s+1}} \right) \mathrm{d}\tau \\ &+ \int_0^t (1+\tau)^{-\sigma} \left(\|u(\tau)\|^2_{H^{2s+2}} + \|\partial_3 b(\tau)\|^2_{H^{2s}} \right) \mathrm{d}\tau, \\ G_0(t) =& \sup_{0 \leqq \tau \leqq t} (1+\tau)^{1-\sigma} \left(\|\partial_3 u(\tau)\|^2_{H^{2s}} + \|\partial_3 b(\tau)\|^2_{H^{2s}} \right) \\ &+ \int_0^t (1+\tau)^{1-\sigma} \|\partial_3 u(\tau)\|^2_{H^{2s+1}} \mathrm{d}\tau, \\ G_1(t) =& \sup_{0 \leqq \tau \leqq t} (1+\tau)^{3-\sigma} \left(\|\partial_3 u(\tau)\|^2_{H^{2s-2}} + \|\partial_3 b(\tau)\|^2_{H^{2s-2}} \right) \\ &+ \int_0^t (1+\tau)^{3-\sigma} \|\partial_3 u(\tau)\|^2_{H^{2s-1}} \mathrm{d}\tau, \\ E_1(t) =& \sup_{0 \leqq \tau \leqq t} (1+\tau)^{3-\sigma} \|u(\tau)\|^2_{H^{2s-2}} \\ &+ \int_0^t (1+\tau)^{3-\sigma} \left(\|u(\tau)\|^2_{H^{2s-1}} + \|\partial_3 b(\tau)\|^2_{H^{2s-3}} \right) \mathrm{d}\tau, \\ e_0(t) =& \sup_{0 \leqq \tau \leqq t} \|b(\tau)\|^2_{H^{2s}}.\end{aligned} \tag{2.3}$$

In the following, we will successively derive the estimate of each energy stated above. By (1.8) and Poincaré inequality, we only need to consider the highest order norms in each energy.

2.2. A Priori Estimate

First, we will deal with the highest order energy, that is, $E_0(t)$. It shows that the highest order norm $H^{2s+1}(\mathbb{T}^3)$ of $u(t,\cdot)$ and $b(t,\cdot)$ will grow in the time evolution.

Lemma 2.2. *Assume that $s \geqq 5$ and the energies are defined as in (2.3), then we have*

$$E_0(t) \lesssim E_0(0) + E_0(t)E_1^{1/2}(t) + E_1^{1/2}(t)e_0(t) + E_0^{5/6}(t)E_1^{1/6}(t)e_0^{1/2}(t) + E_0(t)e_0^{1/2}(t).$$

Proof. We divide the proof into two steps. Instead of deriving the estimate of E_0 directly, we shall first get the estimate of $E_{0,1}(t)$ defined by

$$\begin{aligned} E_{0,1}(t) \triangleq \sup_{0\leqq\tau\leqq t} & (1+\tau)^{-\sigma}(\|u(\tau)\|^2_{H^{2s+1}} + \|b(\tau)\|^2_{H^{2s+1}}) \\ &+ \int_0^t (1+\tau)^{-\sigma}\|u(\tau)\|^2_{H^{2s+2}}\,\mathrm{d}\tau \\ &+ \int_0^t (1+\tau)^{-1-\sigma}(\|u(\tau)\|^2_{H^{2s+1}} + \|b(\tau)\|^2_{H^{2s+1}})\,\mathrm{d}\tau. \end{aligned} \tag{2.4}$$

Step 1

Applying ∇^{2s+1} derivative on the system (1.6). Then, taking inner product with $\nabla^{2s+1}b$ for the first equation of system (1.6) and taking inner product with $\nabla^{2s+1}u$ for the second equation of system (1.6). Adding them up and multiplying the time weight $(1+t)^{-\sigma}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{-\sigma}\left(\|u\|^2_{\dot H^{2s+1}} + \|b\|^2_{\dot H^{2s+1}}\right) + \frac{\sigma}{2}(1+t)^{-1-\sigma}\left(\|u\|^2_{\dot H^{2s+1}} + \|b\|^2_{\dot H^{2s+1}}\right) \\ &\quad + (1+t)^{-\sigma}\|u\|^2_{\dot H^{2s+2}} = I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{2.5}$$

where,

$$\begin{aligned} I_1 = &-(1+t)^{-\sigma}\int_{\mathbb{T}^3} u\cdot\nabla\nabla^{2s+1}u\;\nabla^{2s+1}u + u\cdot\nabla\nabla^{2s+1}b\;\nabla^{2s+1}b\,\mathrm{d}x \\ &+ (1+t)^{-\sigma}\int_{\mathbb{T}^3} b\cdot\nabla\nabla^{2s+1}b\;\nabla^{2s+1}u + b\cdot\nabla\nabla^{2s+1}u\;\nabla^{2s+1}b\,\mathrm{d}x \\ &+ (1+t)^{-\sigma}\int_{\mathbb{T}^3} \nabla^{2s+1}\partial_3 u\;\nabla^{2s+1}b + \nabla^{2s+1}\partial_3 b\;\nabla^{2s+1}u\,\mathrm{d}x, \\ I_2 = &-(1+t)^{-\sigma}\sum_{k=1}^{2s+1}\int_{\mathbb{T}^3}\nabla^k u\cdot\nabla\nabla^{2s+1-k}u\;\nabla^{2s+1}u\,\mathrm{d}x, \end{aligned}$$

$$I_3 = (1+t)^{-\sigma} \sum_{k=1}^{s} \int_{\mathbb{T}^3} \left(\nabla^k b \cdot \nabla \nabla^{2s+1-k} u - \nabla^k u \cdot \nabla \nabla^{2s+1-k} b \right) \nabla^{2s+1} b \, \mathrm{d}x$$
$$+ (1+t)^{-\sigma} \sum_{k=s+1}^{2s+1} \int_{\mathbb{T}^3} \left(\nabla^k b \cdot \nabla \nabla^{2s+1-k} u - \nabla^k u \cdot \nabla \nabla^{2s+1-k} b \right) \nabla^{2s+1} b \, \mathrm{d}x,$$
$$I_4 = - (1+t)^{-\sigma} \sum_{k=1}^{s} \int_{\mathbb{T}^3} \nabla^k b \cdot \nabla \nabla^{2s+1-k} b \; \nabla^{2s+1} u \, \mathrm{d}x$$
$$- (1+t)^{-\sigma} \sum_{k=s+1}^{2s+1} \int_{\mathbb{T}^3} \nabla^k b \cdot \nabla \nabla^{2s+1-k} b \; \nabla^{2s+1} u \, \mathrm{d}x.$$

We shall estimate each term on the right hand side of (2.5). First, for the term I_1, using integration by parts and the divergence free condition, we have

$$I_1 = 0. \tag{2.6}$$

The main idea for the next estimates is that we will carefully derive the bound of each term so that it can be controlled by the combination of energies defined in (2.3). By Hölder inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned} |I_2| &\lesssim (1+t)^{-\sigma} \|u\|_{W^{s+1,\infty}} \|u\|_{H^{2s+1}}^2 \\ &\lesssim (1+t)^{-\sigma} \|u\|_{H^{s+3}} \|u\|_{H^{2s+1}}^2 \\ &\lesssim (1+t)^{-\sigma} \|u\|_{H^{2s-1}} \|u\|_{H^{2s+1}}^2, \end{aligned}$$

provided that $s \geqq 4$. Hence,

$$\int_0^t |I_2(\tau)| \, \mathrm{d}\tau \lesssim \sup_{0 \leqq \tau \leqq t} (1+\tau)^{-\sigma} \|u\|_{H^{2s+1}}^2 \int_0^t \|u\|_{H^{2s-1}} \, \mathrm{d}\tau \lesssim E_0(t) E_1^{1/2}(t). \tag{2.7}$$

Similarly, for the first part of I_3 (we denote the first term on the right hand as $I_{3,1}$ and the second term as $I_{3,2}$), we see that

$$\begin{aligned} |I_{3,1}| &\lesssim (1+t)^{-\sigma} \left(\|b\|_{W^{s,\infty}} \|u\|_{H^{2s+1}} \|b\|_{H^{2s+1}} + \|u\|_{W^{s,\infty}} \|b\|_{H^{2s+1}}^2 \right) \\ &\lesssim (1+t)^{-\sigma} \left(\|b\|_{H^{s+2}} \|u\|_{H^{2s+1}} \|b\|_{H^{2s+1}} + \|u\|_{H^{s+2}} \|b\|_{H^{2s+1}}^2 \right) \\ &\lesssim (1+t)^{-\sigma} \left(\|b\|_{H^{2s}} \|u\|_{H^{2s+1}} \|b\|_{H^{2s+1}} + \|u\|_{H^{2s-1}} \|b\|_{H^{2s+1}}^2 \right), \end{aligned}$$

provided that $s \geqq 3$. And for the second part of I_3, we have

$$\begin{aligned} |I_{3,2}| &\lesssim (1+t)^{-\sigma} \left(\|b\|_{H^{2s+1}}^2 \|u\|_{W^{s+1,\infty}} + \|u\|_{H^{2s+1}} \|b\|_{W^{s+1,\infty}} \|b\|_{H^{2s+1}} \right) \\ &\lesssim (1+t)^{-\sigma} \left(\|b\|_{H^{2s+1}}^2 \|u\|_{H^{s+3}} + \|u\|_{H^{2s+1}} \|b\|_{H^{s+3}} \|b\|_{H^{2s+1}} \right) \\ &\lesssim (1+t)^{-\sigma} \left(\|b\|_{H^{2s+1}}^2 \|u\|_{H^{2s-1}} + \|u\|_{H^{2s+1}} \|b\|_{H^{2s}} \|b\|_{H^{2s+1}} \right), \end{aligned}$$

provided that $s \geqq 4$. Hence, combining $I_{3,1}$ and $I_{3,2}$ and using Hölder inequality, we get

$$\begin{aligned}\int_0^t |I_3(\tau)|\,\mathrm{d}\tau &\lesssim \sup_{0\leqq\tau\leqq t} \|b\|_{H^{2s}} \left(\int_0^t (1+\tau)^{-1-\sigma}\|b\|^2_{H^{2s+1}}\,\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t (1+\tau)^{1-\sigma}\|u\|^2_{H^{2s+1}}\,\mathrm{d}\tau\right)^{\frac12}\\ &\quad+\sup_{0\leqq\tau\leqq t}(1+\tau)^{-\sigma}\|b\|^2_{H^{2s+1}}\int_0^t \|u\|_{H^{2s-1}}\mathrm{d}\tau. \end{aligned}\tag{2.8}$$

Using Gagliardo–Nirenberg interpolation inequality and Hölder inequality, we can bound

$$\begin{aligned}\int_0^t (1+\tau)^{1-\sigma}\|u\|^2_{H^{2s+1}}\,\mathrm{d}\tau &\lesssim \int_0^t \left[(1+\tau)^{3-\sigma}\|u\|^2_{H^{2s-1}}\right]^{\frac13}\left[(1+\tau)^{-\sigma}\|u\|^2_{H^{2s+2}}\right]^{\frac23}\mathrm{d}\tau\\ &\lesssim E_0^{2/3}(t)E_1^{1/3}(t).\end{aligned}\tag{2.9}$$

Thus, combining (2.8) with (2.9), we finally obtain the estimate of I_3

$$\int_0^t |I_3(\tau)|\,\mathrm{d}\tau \lesssim E_0^{5/6}(t)E_1^{1/6}(t)e_0^{1/2}(t) + E_1^{1/2}(t)e_0(t).\tag{2.10}$$

Next, for the last term, I_4, we use the same method as above and obtain

$$\begin{aligned}|I_4| &\lesssim (1+t)^{-\sigma}\|b\|_{H^{2s+1}}\|b\|_{W^{s+1,\infty}}\|u\|_{H^{2s+1}}\\ &\lesssim (1+t)^{-\sigma}\|b\|_{H^{2s+1}}\|b\|_{H^{2s}}\|u\|_{H^{2s+1}},\end{aligned}$$

provided that $s \geqq 3$. Hence,

$$\int_0^t |I_4(\tau)|\,\mathrm{d}\tau \lesssim E_0^{5/6}(t)E_1^{1/6}(t)e_0^{1/2}(t).\tag{2.11}$$

Summing up the estimates for I_1–I_4, that is, (2.6), (2.7), (2.10) and (2.11), and integrating (2.5) in time, we can get the estimate of $E_{0,1}(t)$ which is defined in (2.4)

$$E_{0,1}(t) \lesssim E_0(0) + E_0(t)E_1^{1/2}(t) + E_1^{1/2}(t)e_0(t) + E_0^{5/6}(t)E_1^{1/6}(t)e_0^{1/2}(t).\tag{2.12}$$

Here, we have used the Poincaré inequality to consider the highest order norms only.

Step 2

Now, let us work for the remaining term in $E_0(t)$. Applying ∇^{2s} on the second equation of system (1.6) and taking inner product with $\nabla^{2s}\partial_3 b$, then multiplying the time weight $(1+t)^{-\sigma}$ we get

$$(1+t)^{-\sigma}\|\partial_3 b\|^2_{\dot H^{2s}} = I_5 + I_6 + I_7 + I_8,\tag{2.13}$$

where

$$I_5 = (1+t)^{-\sigma} \int_{\mathbb{T}^3} \nabla^{2s}(u \cdot \nabla u)\ \nabla^{2s} \partial_3 b \, \mathrm{d}x - (1+t)^{-\sigma} \int_{\mathbb{T}^3} \nabla^{2s} \Delta u \ \nabla^{2s} \partial_3 b \, \mathrm{d}x,$$

$$I_6 = -(1+t)^{-\sigma} \sum_{k=0}^{s} \int_{\mathbb{T}^3} \nabla^k b_h \cdot \nabla_h \nabla^{2s-k} b\ \nabla^{2s} \partial_3 b + \nabla^k b_3 \cdot \nabla_3 \nabla^{2s-k} b\ \nabla^{2s} \partial_3 b \, \mathrm{d}x$$

$$- (1+t)^{-\sigma} \sum_{k=s+1}^{2s} \int_{\mathbb{T}^3} \nabla^k b_h \cdot \nabla_h \nabla^{2s-k} b\ \nabla^{2s} \partial_3 b$$

$$+ \nabla^k b_3 \cdot \nabla_3 \nabla^{2s-k} b\ \nabla^{2s} \partial_3 b \, \mathrm{d}x,$$

$$I_7 = \frac{\mathrm{d}}{\mathrm{d}t} (1+t)^{-\sigma} \int_{\mathbb{T}^3} \nabla^{2s} u\ \nabla^{2s} \partial_3 b \, \mathrm{d}x + \sigma (1+t)^{-1-\sigma} \int_{\mathbb{T}^3} \nabla^{2s} u\ \nabla^{2s} \partial_3 b \, \mathrm{d}x,$$

$$I_8 = (1+t)^{-\sigma} \int_{\mathbb{T}^3} \nabla^{2s} \partial_3 u\ \nabla^{2s} \partial_t b \, \mathrm{d}x.$$

Like the process in Step 1, we shall derive the estimate of each term on the right hand side of (2.13). First, using Hölder inequality and the Sobolev imbedding theorem, we can bound I_5 as follows:

$$\begin{aligned} |I_5| &\lesssim (1+t)^{-\sigma} \|u\|_{H^{2s+1}} \|u\|_{H^{s+2}} \|\partial_3 b\|_{H^{2s}} + (1+t)^{-\sigma} \|u\|_{H^{2s+2}} \|\partial_3 b\|_{H^{2s}} \\ &\lesssim (1+t)^{-\sigma} \|u\|_{H^{2s+1}} \|b\|_{H^{2s+1}} \|u\|_{H^{2s-1}} + (1+t)^{-\sigma} \|u\|_{H^{2s+2}} \|\partial_3 b\|_{H^{2s}}, \end{aligned}$$

provided that $s \geqq 3$. Thus, we have

$$\int_0^t |I_5(\tau)| \, \mathrm{d}\tau \lesssim E_0(t) E_1^{1/2}(t) + E_{0,1}^{1/2}(t) \left(\int_0^t (1+\tau)^{-\sigma} \|\partial_3 b\|_{H^{2s}}^2 \mathrm{d}\tau \right)^{1/2}. \tag{2.14}$$

Next, we turn to the estimate of I_6. Notice that I_6 is the most wild term in our proof, due to the bad behaviour of $b \cdot \nabla b$. Although we have already divided this term into $b_h \cdot \nabla_h b$ and $b_3 \cdot \nabla_3 b$ two terms, the estimate for $b_h \cdot \nabla_h b$ is still nontrivial. Thanks to the Proposition 2.1 that we have proved at the beginning of this section, we can overcome this problem using the following strategy.

Notice the property (1.7). We easily know that in the x_3 direction, the average value of function $b_h(x_h, \cdot)$ over $[-\pi, \pi]$ equals zero. Thus, using the Proposition 2.1, Hölder inequality and the Sobolev imbedding theorem, we get

$$\begin{aligned} |I_6| &\lesssim (1+t)^{-\sigma} \left(\|b_h\|_{W^{s,\infty}} \|b\|_{H^{2s+1}} \|\partial_3 b\|_{H^{2s}} + \|b_3\|_{W^{s,\infty}} \|\partial_3 b\|_{H^{2s}}^2 \right) \\ &\quad + (1+t)^{-\sigma} \left(\|b_h\|_{H^{2s}} \|b\|_{W^{s,\infty}} \|\partial_3 b\|_{H^{2s}} + \|b_3\|_{H^{2s}} \|\partial_3 b\|_{W^{s-1,\infty}} \|\partial_3 b\|_{H^{2s}} \right) \\ &\lesssim (1+t)^{-\sigma} \left(\|\partial_3 b_h\|_{H^{s+2}} \|b\|_{H^{2s+1}} \|\partial_3 b\|_{H^{2s}} + \|b_3\|_{H^{s+2}} \|\partial_3 b\|_{H^{2s}}^2 \right) \\ &\quad + (1+t)^{-\sigma} \left(\|\partial_3 b_h\|_{H^{2s}} \|b\|_{H^{s+2}} \|\partial_3 b\|_{H^{2s}} + \|b_3\|_{H^{2s}} \|\partial_3 b\|_{H^{s+1}} \|\partial_3 b\|_{H^{2s}} \right) \\ &\lesssim (1+t)^{-\sigma} \left(\|\partial_3 b\|_{H^{2s-3}} \|b\|_{H^{2s+1}} \|\partial_3 b\|_{H^{2s}} + \|b\|_{H^{2s}} \|\partial_3 b\|_{H^{2s}}^2 \right), \end{aligned}$$

provided that $s \geqq 5$. Hence,

$$\int_0^t |I_6(\tau)| \, \mathrm{d}\tau \lesssim E_0(t) E_1^{1/2}(t) + E_0(t) e_0^{1/2}(t). \tag{2.15}$$

For the next term I_7, using Hölder inequality, it is straightforward to see

$$\Big|\int_0^t I_7(\tau)\,\mathrm{d}\tau\Big| \lesssim E_{0,1}(t). \tag{2.16}$$

For the last term I_8, using the first equation of system (1.6) and integrating by parts, we find

$$\begin{aligned} I_8 &= (1+t)^{-\sigma}\int_{\mathbb{T}^3} \nabla^{2s}\partial_3 u \nabla^{2s}(\partial_3 u + b\cdot\nabla u - u\cdot\nabla b)\,\mathrm{d}x \\ &= -(1+t)^{-\sigma}\int_{\mathbb{T}^3} \nabla^{2s+1}\partial_3 u \nabla^{2s-1}(\partial_3 u + b\cdot\nabla u - u\cdot\nabla b)\,\mathrm{d}x. \end{aligned}$$

Thus, by Hölder inequality and Sobolev imbedding theorem, we have

$$|I_8| \lesssim (1+t)^{-\sigma}(\|u\|^2_{H^{2s+2}} + \|u\|^2_{H^{2s+2}}\|b\|_{H^{2s}}),$$

provided that $s \geqq 2$. Hence, we arrive at

$$\int_0^t |I_8(\tau)|\,\mathrm{d}\tau \lesssim E_{0,1}(t) + E_0(t)e_0^{1/2}(t). \tag{2.17}$$

Summing up the estimates for I_5–I_8, that is, (2.14), (2.15), (2.16) and (2.17), and integrating (2.13) in time, using Young inequality and Poincaré inequality we can easily bound

$$\int_0^t (1+\tau)^{-\sigma}\|\partial_3 b\|^2_{H^{2s}}\,\mathrm{d}\tau \lesssim E_{0,1} + E_0(t)E_1^{1/2}(t) + E_0(t)e_0^{1/2}(t). \tag{2.18}$$

This gives the estimate for the last term in $E_0(t)$. Now, multiplying (2.12) by suitable large number and plus (2.18), we then complete the proof of this lemma. □

Next, we work with the lower order energies defined in (2.3), especially we want to derive the decay estimate and get the uniform bound of lower order norms of magnetic field.

Lemma 2.3. *Assume that $s \geqq 5$ and the energies are defined as in (2.3), then we have*

$$\begin{aligned} G_0(t) \lesssim\ & E_0(t) + G_0(t)E_1^{1/2}(t) + E_0^{1/2}(t)G_0^{1/2}(t)\Big[G_1^{1/2}(t) + E_1^{1/2}(t)\Big] \\ & + E_0^{1/2}(t)G_0^{1/4}(t)G_1^{1/4}(t)e_0^{1/2}(t). \end{aligned}$$

Proof. First, applying $\nabla^{2s}\partial_3$ derivative on the system (1.6). Then, taking inner product with $\nabla^{2s}\partial_3 b$ for the first equation of system (1.6) and taking inner product with $\nabla^{2s}\partial_3 u$ for the second equation of system (1.6). Summing them up and multiplying the time weight $(1+t)^{1-\sigma}$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{1-\sigma}\Big(\|\partial_3 u\|^2_{\dot H^{2s}} + \|\partial_3 b\|^2_{\dot H^{2s}}\Big) + (1+t)^{1-\sigma}\|\partial_3 u\|^2_{\dot H^{2s+1}} = \sum_{i=1}^{6} J_i, \tag{2.19}$$

where,

$$
\begin{aligned}
J_1 =& \frac{1-\sigma}{2}(1+t)^{-\sigma}\left(\|\partial_3 u\|_{\dot H^{2s}}^2 + \|\partial_3 b\|_{\dot H^{2s}}^2\right),\\
J_2 =& -(1+t)^{1-\sigma}\int_{\mathbb{T}^3} u\cdot\nabla\nabla^{2s}\partial_3 u\ \nabla^{2s}\partial_3 u + u\cdot\nabla\nabla^{2s}\partial_3 b\ \nabla^{2s}\partial_3 b\,\mathrm{d}x\\
&+(1+t)^{1-\sigma}\int_{\mathbb{T}^3} b\cdot\nabla\nabla^{2s}\partial_3 b\ \nabla^{2s}\partial_3 u + b\cdot\nabla\nabla^{2s}\partial_3 u\ \nabla^{2s}\partial_3 u\,\mathrm{d}x\\
&+(1+t)^{1-\sigma}\int_{\mathbb{T}^3} \nabla^{2s}\partial_3^2 u\ \nabla^{2s}\partial_3 b + \nabla^{2s}\partial_3^2 b\ \nabla^{2s}\partial_3 u\,\mathrm{d}x,\\
J_3 =& -(1+t)^{1-\sigma}\sum_{k=1}^{2s}\int_{\mathbb{T}^3}\nabla^k u\cdot\nabla\nabla^{2s-k}\partial_3 u\ \nabla^{2s}\partial_3 u\,\mathrm{d}x\\
&-(1+t)^{1-\sigma}\sum_{k=0}^{2s}\int_{\mathbb{T}^3}\nabla^k\partial_3 u\cdot\nabla\nabla^{2s-k} u\ \nabla^{2s}\partial_3 u\,\mathrm{d}x,\\
J_4 =& -(1+t)^{1-\sigma}\sum_{k=1}^{2s}\int_{\mathbb{T}^3}\nabla^k u\cdot\nabla\nabla^{2s-k}\partial_3 b\ \nabla^{2s}\partial_3 b\,\mathrm{d}x\\
&-(1+t)^{1-\sigma}\sum_{k=0}^{2s}\int_{\mathbb{T}^3}\nabla^k\partial_3 u\cdot\nabla\nabla^{2s-k} b\ \nabla^{2s}\partial_3 b\,\mathrm{d}x,\\
J_5 =& (1+t)^{1-\sigma}\sum_{k=0}^{2s}\int_{\mathbb{T}^3}\nabla^k\partial_3 b\cdot\nabla\nabla^{2s-k} u\ \nabla^{2s}\partial_3 b\,\mathrm{d}x\\
&+(1+t)^{1-\sigma}\sum_{k=1}^{2s}\int_{\mathbb{T}^3}\nabla^k b\cdot\nabla\nabla^{2s-k}\partial_3 u\ \nabla^{2s}\partial_3 b\,\mathrm{d}x,\\
J_6 =& (1+t)^{1-\sigma}\sum_{k=1}^{2s}\int_{\mathbb{T}^3}\nabla^k b\cdot\nabla\nabla^{2s-k}\partial_3 b\ \nabla^{2s}\partial_3 u\,\mathrm{d}x\\
&+(1+t)^{1-\sigma}\sum_{k=0}^{2s}\int_{\mathbb{T}^3}\nabla^k\partial_3 b\cdot\nabla\nabla^{2s-k} b\ \nabla^{2s}\partial_3 u\,\mathrm{d}x.
\end{aligned}
$$

Like the proof in Lemma 2.2, we shall now estimate each term on the right hand side of (2.19). First, for the term J_1, it is easy to see that

$$\int_0^t |J_1(\tau)|\,\mathrm{d}\tau \lesssim E_0(t). \tag{2.20}$$

Using integration by parts and divergence free condition, it is clear that

$$J_2 = 0. \tag{2.21}$$

For each term in J_3, we divide it into two parts: $k \leqq s$ and $k > s$. We treat these two cases respectively and estimate as follows:

$$
\begin{aligned}
|J_3| &\lesssim (1+t)^{1-\sigma}\left(\|u\|_{W^{s,\infty}}\|\partial_3 u\|_{H^{2s}}^2 + \|u\|_{H^{2s}}\|\partial_3 u\|_{W^{s,\infty}}\|\partial_3 u\|_{H^{2s}}\right)\\
&\quad + (1+t)^{1-\sigma}\left(\|\partial_3 u\|_{W^{s,\infty}}\|u\|_{H^{2s+1}}\|\partial_3 u\|_{H^{2s}} + \|\partial_3 u\|_{H^{2s}}^2\|u\|_{W^{s,\infty}}\right)\\
&\lesssim (1+t)^{1-\sigma}\left(\|u\|_{H^{2s-1}}\|\partial_3 u\|_{H^{2s}}^2 + \|u\|_{H^{2s+1}}\|\partial_3 u\|_{H^{2s-1}}\|\partial_3 u\|_{H^{2s}}\right),
\end{aligned}
$$

provided that $s \geqq 3$. Thus, we have

$$
\begin{aligned}
\int_0^t |J_3(\tau)|\,\mathrm{d}\tau \lesssim G_0(t)\cdot \int_0^t \|u\|_{H^{2s-1}}\,\mathrm{d}\tau&\\
+ E_0^{1/2}(t)G_0^{1/2}(t)&\int_0^t (1+\tau)^{1/2}\|\partial_3 u\|_{H^{2s-1}}\,\mathrm{d}\tau\\
\lesssim G_0(t)E_1^{1/2}(t) &+ E_0^{1/2}(t)G_0^{1/2}(t)G_1^{1/2}(t).
\end{aligned}
\tag{2.22}
$$

The term J_4 can be estimated by the same method as in J_3, as follows:

$$
\begin{aligned}
|J_4| &\lesssim (1+t)^{1-\sigma}\left(\|u\|_{W^{s,\infty}}\|\partial_3 b\|_{H^{2s}}^2 + \|u\|_{H^{2s}}\|\partial_3 b\|_{W^{s,\infty}}\|\partial_3 b\|_{H^{2s}}\right)\\
&\quad + (1+t)^{1-\sigma}\left(\|\partial_3 u\|_{W^{s,\infty}}\|b\|_{H^{2s+1}}\|\partial_3 b\|_{H^{2s}} + \|\partial_3 u\|_{H^{2s}}\|b\|_{W^{s,\infty}}\|\partial_3 b\|_{H^{2s}}\right)\\
&\lesssim (1+t)^{1-\sigma}\left(\|u\|_{H^{2s-1}}\|\partial_3 b\|_{H^{2s}}^2 + \|u\|_{H^{2s}}\|\partial_3 b\|_{H^{2s-3}}\|\partial_3 b\|_{H^{2s}}\right)\\
&\quad + (1+t)^{1-\sigma}\left(\|\partial_3 u\|_{H^{2s-1}}\|b\|_{H^{2s+1}}\|\partial_3 b\|_{H^{2s}} + \|\partial_3 u\|_{H^{2s}}\|b\|_{H^{2s}}\|\partial_3 b\|_{H^{2s}}\right),
\end{aligned}
$$

provided that $s \geqq 5$. Now,

$$
\begin{aligned}
\int_0^t |J_4(\tau)|\,\mathrm{d}\tau &\lesssim G_0(t)\int_0^t \|u\|_{H^{2s-1}}\,\mathrm{d}\tau + E_0^{1/2}(t)G_0^{1/2}(t)\int_0^t (1+\tau)^{1/2}\|\partial_3 b\|_{H^{2s-3}}\,\mathrm{d}\tau\\
&\quad + E_0^{1/2}(t)G_0^{1/2}(t)\int_0^t (1+\tau)^{1/2}\|\partial_3 u\|_{H^{2s-1}}\,\mathrm{d}\tau\\
&\quad + e_0^{1/2}(t)\Big(\int_0^t (1+\tau)^{-\sigma}\|\partial_3 b\|_{H^{2s}}^2\,\mathrm{d}\tau\Big)^{1/2}\Big(\int_0^t (1+\tau)^{2-\sigma}\|\partial_3 u\|_{H^{2s}}^2\,\mathrm{d}\tau\Big)^{1/2}\\
&\lesssim G_0(t)E_1^{1/2}(t) + E_0^{1/2}(t)G_0^{1/2}(t)E_1^{1/2}(t) + E_0^{1/2}(t)G_0^{1/4}(t)G_1^{1/4}(t)e_0^{1/2}(t),
\end{aligned}
\tag{2.23}
$$

where, we have used the following inequality

$$
\begin{aligned}
\int_0^t (1+\tau)^{2-\sigma}\|\partial_3 u\|_{H^{2s}}^2\,\mathrm{d}\tau &\lesssim \int_0^t (1+\tau)^{\frac{1-\sigma}{2}}\|\partial_3 u\|_{H^{2s+1}}(1+\tau)^{\frac{3-\sigma}{2}}\|\partial_3 u\|_{H^{2s-1}}\,\mathrm{d}\tau\\
&\lesssim G_0^{1/2}(t)G_1^{1/2}(t).
\end{aligned}
\tag{2.24}
$$

Similarly, we can estimate J_5 as follows:

$$
\begin{aligned}
|J_5| &\lesssim (1+t)^{1-\sigma}\left(\|\partial_3 b\|_{W^{s,\infty}}\|u\|_{H^{2s+1}}\|\partial_3 b\|_{H^{2s}} + \|\partial_3 b\|_{H^{2s}}^2\|u\|_{W^{s,\infty}}\right)\\
&\quad + (1+t)^{1-\sigma}\left(\|b\|_{W^{s,\infty}}\|\partial_3 u\|_{H^{2s}}\|\partial_3 b\|_{H^{2s}} + \|b\|_{H^{2s}}\|\partial_3 u\|_{W^{s,\infty}}\|\partial_3 b\|_{H^{2s}}\right)\\
&\lesssim (1+t)^{1-\sigma}\left(\|\partial_3 b\|_{H^{2s-3}}\|u\|_{H^{2s+1}}\|\partial_3 b\|_{H^{2s}} + \|\partial_3 b\|_{H^{2s}}^2\|u\|_{H^{2s-1}}\right)\\
&\quad + (1+t)^{1-\sigma}\|b\|_{H^{2s}}\|\partial_3 u\|_{H^{2s}}\|\partial_3 b\|_{H^{2s}},
\end{aligned}
$$

provided that $s \geqq 5$. Hence, using (2.24), we easily get

$$\begin{aligned}
&\int_0^t |J_5(\tau)|\,\mathrm{d}\tau \\
&\lesssim E_0^{1/2}(t)G_0^{1/2}(t)\int_0^t (1+\tau)^{1/2}\|\partial_3 b\|_{H^{2s-3}}\,\mathrm{d}\tau + G_0(t)\int_0^t \|u\|_{H^{2s-1}}\,\mathrm{d}\tau \\
&\quad + e_0^{1/2}E_0^{1/2}\left(\int_0^t (1+\tau)^{2-\sigma}\|\partial_3 u\|_{H^{2s}}^2\,\mathrm{d}\tau\right)^{1/2} \\
&\lesssim E_0^{1/2}(t)G_0^{1/2}(t)E_1^{1/2}(t) + G_0(t)E_1^{1/2}(t) + E_0^{1/2}(t)G_0^{1/4}(t)G_1^{1/4}(t)e_0^{1/2}(t).
\end{aligned} \tag{2.25}$$

In the same manner, we can estimate the last term J_6. Indeed,

$$\begin{aligned}
|J_6| &\lesssim (1+t)^{1-\sigma}(\|b\|_{W^{s,\infty}}\|\partial_3 b\|_{H^{2s}}\|\partial_3 u\|_{H^{2s}} + \|b\|_{H^{2s}}\|\partial_3 b\|_{W^{s,\infty}}\|\partial_3 u\|_{H^{2s}}) \\
&\quad + (1+t)^{1-\sigma}(\|\partial_3 b\|_{W^{s,\infty}}\|b\|_{H^{2s+1}}\|\partial_3 u\|_{H^{2s}} + \|\partial_3 b\|_{H^{2s}}\|b\|_{W^{s,\infty}}\|\partial_3 u\|_{H^{2s}}) \\
&\lesssim (1+t)^{1-\sigma}\big(\|b\|_{H^{2s}}\|\partial_3 b\|_{H^{2s}}\|\partial_3 u\|_{H^{2s}} + \|\partial_3 b\|_{H^{2s-3}}\|b\|_{H^{2s+1}}\|\partial_3 u\|_{H^{2s}}\big),
\end{aligned}$$

provided that $s \geqq 5$. Then, it is clear that

$$\begin{aligned}
\int_0^t |J_6(\tau)|\,\mathrm{d}\tau &\lesssim e_0^{1/2}E_0^{1/2}\left(\int_0^t (1+\tau)^{2-\sigma}\|\partial_3 u\|_{H^{2s}}^2\,\mathrm{d}\tau\right)^{1/2} \\
&\quad + E_0^{1/2}(t)G_0^{1/2}(t)\int_0^t (1+\tau)^{1/2}\|\partial_3 b\|_{H^{2s-3}}\,\mathrm{d}\tau \\
&\lesssim E_0^{1/2}(t)G_0^{1/4}(t)G_1^{1/4}(t)e_0^{1/2}(t) + E_0^{1/2}(t)G_0^{1/2}(t)E_1^{1/2}(t).
\end{aligned} \tag{2.26}$$

Finally, summing up the estimates for J_1–J_6, that is, (2.20), (2.21), (2.22), (2.23), (2.25) and (2.26), and integrating (2.19) in time, using Poincaré inequality we can complete the proof of this lemma. □

Lemma 2.4. *Assume that $s \geqq 4$ and the energies are defined as in (2.3), then we have*

$$G_1(t) \lesssim E_0(0)+G_0(t)+E_0^{1/3}(t)E_1^{2/3}(t)+G_1(t)E_1^{1/2}(t)+G_1^{1/2}(t)E_1^{1/2}(t)e_0^{1/2}(t).$$

Proof. First, taking $\nabla^{2s-2}\partial_3$ derivative on the system (1.6). Then, taking inner product with $\nabla^{2s-2}\partial_3 b$ for the first equation of system (1.6) and taking inner product with $\nabla^{2s-2}\partial_3 u$ for the second equation of system (1.6). Summing them up and multiplying the time weight $(1+t)^{3-\sigma}$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{3-\sigma}\left(\|\partial_3 u\|_{\dot H^{2s-2}}^2 + \|\partial_3 b\|_{\dot H^{2s-2}}^2\right) + (1+t)^{3-\sigma}\|\partial_3 u\|_{\dot H^{2s-1}}^2 = \sum_{i=1}^6 N_i, \tag{2.27}$$

where,

$$
\begin{aligned}
N_1 &= \frac{3-\sigma}{2}(1+t)^{2-\sigma}\left(\|\partial_3 u\|_{\dot H^{2s-2}}^2 + \|\partial_3 b\|_{\dot H^{2s-2}}^2\right),\\
N_2 &= -(1+t)^{3-\sigma}\int_{\mathbb{T}^3} u\cdot\nabla\nabla^{2s-2}\partial_3 u\ \nabla^{2s-2}\partial_3 u + u\cdot\nabla\nabla^{2s-2}\partial_3 b\ \nabla^{2s-2}\partial_3 b\,\mathrm{d}x\\
&\quad + (1+t)^{3-\sigma}\int_{\mathbb{T}^3} b\cdot\nabla\nabla^{2s-2}\partial_3 b\ \nabla^{2s-2}\partial_3 u + b\cdot\nabla\nabla^{2s-2}\partial_3 u\ \nabla^{2s-2}\partial_3 b\,\mathrm{d}x\\
&\quad + (1+t)^{3-\sigma}\int_{\mathbb{T}^3} \nabla^{2s-2}\partial_3^2 u\ \nabla^{2s-2}\partial_3 b + \nabla^{2s-2}\partial_3^2 b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x,\\
N_3 &= -(1+t)^{3-\sigma}\sum_{k=1}^{2s-2}\int_{\mathbb{T}^3}\nabla^k u\cdot\nabla\nabla^{2s-2-k}\partial_3 u\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x\\
&\quad - (1+t)^{3-\sigma}\sum_{k=0}^{2s-2}\int_{\mathbb{T}^3}\nabla^k \partial_3 u\cdot\nabla\nabla^{2s-2-k} u\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x,\\
N_4 &= (1+t)^{3-\sigma}\sum_{k=0}^{2s-2}\int_{\mathbb{T}^3}\nabla^k \partial_3 b\cdot\nabla\nabla^{2s-2-k} u\ \nabla^{2s-2}\partial_3 b\,\mathrm{d}x\\
&\quad - (1+t)^{3-\sigma}\sum_{k=1}^{2s-2}\int_{\mathbb{T}^3}\nabla^k u\cdot\nabla\nabla^{2s-2-k}\partial_3 b\ \nabla^{2s-2}\partial_3 b\,\mathrm{d}x,\\
N_5 &= (1+t)^{3-\sigma}\sum_{k=1}^{2s-2}\int_{\mathbb{T}^3}\nabla^k b\cdot\nabla\nabla^{2s-2-k}\partial_3 u\ \nabla^{2s-2}\partial_3 b\,\mathrm{d}x\\
&\quad - (1+t)^{3-\sigma}\sum_{k=0}^{2s-2}\int_{\mathbb{T}^3}\nabla^k \partial_3 u\cdot\nabla\nabla^{2s-2-k} b\ \nabla^{2s-2}\partial_3 b\,\mathrm{d}x,\\
N_6 &= (1+t)^{3-\sigma}\sum_{k=1}^{2s-2}\int_{\mathbb{T}^3}\nabla^k b\cdot\nabla\nabla^{2s-2-k}\partial_3 b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x\\
&\quad + (1+t)^{3-\sigma}\sum_{k=0}^{2s-2}\int_{\mathbb{T}^3}\nabla^k \partial_3 b\cdot\nabla\nabla^{2s-2-k} b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x.
\end{aligned}
$$

The first term N_1 can be bounded as follows:

$$
\begin{aligned}
|N_1| &\lesssim (1+t)^{2-\sigma}\left(\|\partial_3 u\|_{H^{2s}}^2 + \|\partial_3 b\|_{H^{2s-2}}^2\right)\\
&\lesssim (1+t)^{\frac{1-\sigma}{2}}\|\partial_3 u\|_{H^{2s+1}}(1+t)^{\frac{3-\sigma}{2}}\|\partial_3 u\|_{H^{2s-1}}\\
&\quad + \left[(1+t)^{-\sigma/2}\|\partial_3 b\|_{H^{2s}}\right]^{2/3}\left[(1+t)^{\frac{3-\sigma}{2}}\|\partial_3 b\|_{H^{2s-3}}\right]^{4/3},
\end{aligned}
$$

and thus

$$
\int_0^t |N_1(\tau)|\,\mathrm{d}\tau \lesssim G_0^{1/2}(t)G_1^{1/2}(t) + E_0^{1/3}(t)E_1^{2/3}(t). \tag{2.28}
$$

Using integration by parts and the divergence free condition, we find

$$N_2 = 0. \tag{2.29}$$

For the term N_3, thanks to Hölder inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}\int_0^t |N_3(\tau)|\,\mathrm{d}\tau &\lesssim \int_0^t (1+\tau)^{3-\sigma} \|\partial_3 u\|_{H^{2s-2}}^2 \|u\|_{H^{2s-1}}\,\mathrm{d}\tau \\ &\lesssim G_1(t) \int_0^t \|u\|_{H^{2s-1}}\,\mathrm{d}\tau \\ &\lesssim G_1(t) E_1^{1/2}(t),\end{aligned} \tag{2.30}$$

provided that $s \geqq 3$.

Then, we turn to the term N_4. For each term in N_4, we divide it into two parts: $k \leqq s-1$ and $k \geqq s$. We treat these two cases respectively and estimate as follows:

$$\begin{aligned}|N_4| &\lesssim (1+t)^{3-\sigma} \left(\|\partial_3 b\|_{W^{s-1,\infty}} \|u\|_{H^{2s-1}} \|\partial_3 b\|_{H^{2s-2}} + \|\partial_3 b\|_{H^{2s-2}} \|u\|_{W^{s-1,\infty}} \|\partial_3 b\|_{H^{2s-2}}\right) \\ &\quad + (1+t)^{3-\sigma} \left(\|u\|_{W^{s-1,\infty}} \|\partial_3 b\|_{H^{2s-2}}^2 + \|\partial_3 b\|_{W^{s-1,\infty}} \|u\|_{H^{2s-2}} \|\partial_3 b\|_{H^{2s-2}}\right) \\ &\lesssim (1+t)^{3-\sigma} \|u\|_{H^{2s-1}} \|\partial_3 b\|_{H^{2s-2}}^2,\end{aligned}$$

provided that $s \geqq 3$. Indeed,

$$\begin{aligned}\int_0^t |N_4(\tau)|\,\mathrm{d}\tau &\lesssim G_1(t) \int_0^t \|u\|_{H^{2s-1}}\,\mathrm{d}\tau \\ &\lesssim G_1(t) E_1^{1/2}(t).\end{aligned} \tag{2.31}$$

Also for the next term N_5, we divide each term into two parts: $k \leqq s-1$ and $k \geqq s$. Using Hölder inequality and the Sobolev inequality respectively, we can bound

$$\begin{aligned}|N_5| &\lesssim (1+t)^{3-\sigma} \left| \sum_{k=1}^{2s-2} \int_{\mathbb{T}^3} \nabla \left(\nabla^k b \cdot \nabla \nabla^{2s-2-k} \partial_3 u\right) \nabla^{2s-3} \partial_3 b\,\mathrm{d}x \right| \\ &\quad + (1+t)^{3-\sigma} \left| \sum_{k=0}^{2s-2} \int_{\mathbb{T}^3} \nabla \left(\nabla^k \partial_3 u \cdot \nabla \nabla^{2s-2-k} b\right) \nabla^{2s-3} \partial_3 b\,\mathrm{d}x \right| \\ &\lesssim (1+t)^{3-\sigma} (\|b\|_{W^{s,\infty}} \|\partial_3 u\|_{H^{2s-1}} \|\partial_3 b\|_{H^{2s-3}} + \|b\|_{H^{2s-1}} \|\partial_3 u\|_{W^{s,\infty}} \|\partial_3 b\|_{H^{2s-3}}) \\ &\quad + (1+t)^{3-\sigma} (\|\partial_3 u\|_{W^{s,\infty}} \|b\|_{H^{2s}} \|\partial_3 b\|_{H^{2s-3}} + \|\partial_3 u\|_{H^{2s-1}} \|b\|_{W^{s,\infty}} \|\partial_3 b\|_{H^{2s-3}}) \\ &\lesssim (1+t)^{3-\sigma} \|\partial_3 u\|_{H^{2s-1}} \|\partial_3 b\|_{H^{2s-3}} \|b\|_{H^{2s}},\end{aligned}$$

provided that $s \geqq 3$. Hence,

$$\int_0^t |N_5(\tau)|\,\mathrm{d}\tau \lesssim G_1^{1/2}(t) E_1^{1/2}(t) e_0^{1/2}(t). \tag{2.32}$$

We divide the last term N_6 into two parts as follows:

$$
\begin{aligned}
N_6 = &-(1+t)^{3-\sigma}\left\{\sum_{k=2}^{2s-2}\int_{\mathbb{T}^3}\nabla^k b\cdot\nabla\nabla^{2s-2-k}\partial_3 b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x\right.\\
&\left.+\sum_{k=0}^{2s-3}\int_{\mathbb{T}^3}\nabla^k\partial_3 b\cdot\nabla\nabla^{2s-2-k} b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x\right\}\\
&-(1+t)^{3-\sigma}\left\{\int_{\mathbb{T}^3}\nabla b\cdot\nabla\nabla^{2s-3}\partial_3 b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x\right.\\
&\left.+\int_{\mathbb{T}^3}\nabla^{2s-2}\partial_3 b\cdot\nabla b\ \nabla^{2s-2}\partial_3 u\,\mathrm{d}x\right\}\\
\triangleq\ & N_{6,1}+N_{6,2}.
\end{aligned}
$$

For the first part $N_{6,1}$, using Hölder inequality and the Sobolev inequality, we easily get

$$|N_{6,1}|\lesssim(1+t)^{3-\sigma}\|b\|_{H^{2s-1}}\|\partial_3 b\|_{H^{2s-3}}\|\partial_3 u\|_{H^{2s-2}},$$

provided that $s\geqq 4$. Then for the second part $N_{6,2}$, using integration by parts, we can bound

$$
\begin{aligned}
|N_{6,2}|&\lesssim(1+t)^{3-\sigma}\|b\|_{W^{2,\infty}}\|\partial_3 b\|_{H^{2s-3}}\|\partial_3 u\|_{H^{2s-1}}\\
&\lesssim(1+t)^{3-\sigma}\|b\|_{H^{2s}}\|\partial_3 b\|_{H^{2s-3}}\|\partial_3 u\|_{H^{2s-1}},
\end{aligned}
$$

provided that $s\geqq 2$. Combining the estimates of $N_{6,1}$ and $N_{6,2}$, we finally obtain

$$\int_0^t|N_6(\tau)|\,\mathrm{d}\tau\lesssim G_1^{1/2}(t)E_1^{1/2}(t)e_0^{1/2}(t). \tag{2.33}$$

As with the process in the above lemmas, according to (2.28), (2.29), (2.30), (2.31), (2.32) and (2.33), we complete the proof of this lemma. □

Lemma 2.5. *Assume that $s\geqq 4$ and the energies are defined as in (2.3), then we have*

$$
\begin{aligned}
E_1(t)\lesssim\ & E_0(t)+G_0(t)+G_1(t)+E_1^{3/2}(t)+E_1(t)e_0^{1/2}(t)\\
&+G_1^{1/2}(t)E_1^{1/2}(t)e_0^{1/2}(t).
\end{aligned}
$$

Proof. Like the proof in Lemma 2.2, we divide the proof into two steps. We first deal with $E_{1,1}(t)$ which is defined as follows:

$$E_{1,1}(t):=\sup_{0\leqq\tau\leqq t}(1+\tau)^{3-\sigma}\|u(\tau)\|^2_{H^{2s-2}}+\int_0^t(1+\tau)^{3-\sigma}\|u(\tau)\|^2_{H^{2s-1}}\,\mathrm{d}\tau. \tag{2.34}$$

Step 1

Applying ∇^{2s-2} on the second equation of system (1.6). Then, taking inner product with $\nabla^{2s-2}u$ and multiplying the time weight $(1+t)^{3-\sigma}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{3-\sigma}\|u\|_{\dot{H}^{2s-2}}^2+(1+t)^{3-\sigma}\|u\|_{\dot{H}^{2s-1}}^2=F_1+F_2+F_3+F_4, \tag{2.35}$$

where,

$$\begin{aligned}
F_1 &= \frac{3-\sigma}{2}(1+t)^{2-\sigma}\|u\|_{\dot{H}^{2s-2}}^2,\\
F_2 &= -(1+t)^{3-\sigma}\Bigg(\int_{\mathbb{T}^3} u\cdot\nabla\nabla^{2s-2}u\nabla^{2s-2}u\,\mathrm{d}x\\
&\quad+\sum_{k=1}^{2s-2}\int_{\mathbb{T}^3}\nabla^k u\cdot\nabla\nabla^{2s-2-k}u\nabla^{2s-2}u\,\mathrm{d}x\Bigg),\\
F_3 &= (1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-2}\partial_3 b\nabla^{2s-2}u\,\mathrm{d}x,\\
F_4 &= (1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-2}(b\cdot\nabla b)\nabla^{2s-2}u\,\mathrm{d}x.
\end{aligned}$$

Similarly, we shall estimate each term on right hand side of (2.35). First, for the term F_1, by Gagliardo–Nirenberg interpolation inequality, we have

$$\begin{aligned}
|F_1| &\lesssim (1+t)^{2-\sigma}\|u\|_{H^{2s}}^2\\
&\lesssim \left[(1+t)^{-\sigma}\|u\|_{H^{2s+2}}^2\right]^{1/3}\left[(1+t)^{3-\sigma}\|u\|_{H^{2s-1}}^2\right]^{2/3}.
\end{aligned}$$

Hence,

$$\int_0^t |F_1(\tau)|\,\mathrm{d}\tau \lesssim E_0^{1/3}(t)E_1^{2/3}(t). \tag{2.36}$$

For the term F_2, integrating by parts and using the divergence free condition, we directly know that the first part of F_2 equals 0. Hence, by Hölder inequality and the Sobolev imbedding theorem, we get

$$\begin{aligned}
\int_0^t |F_2(\tau)|\,\mathrm{d}\tau &\lesssim \int_0^t (1+\tau)^{3-\sigma}\|u\|_{W^{s-1,\infty}}\|u\|_{H^{2s-2}}^2\,\mathrm{d}\tau\\
&\lesssim \sup_{0\leqq\tau\leqq t}(1+\tau)^{3-\sigma}\|u\|_{H^{2s-2}}^2\int_0^t\|u\|_{H^{2s-1}}\,\mathrm{d}\tau\\
&\lesssim E_1^{3/2}(t),
\end{aligned} \tag{2.37}$$

provided that $s\geqq 2$.

Next, we turn to the estimate of F_3 and F_4 which are the wildest terms, due to the bad behaviour of b. Thanks to the Proposition 2.1, we can use the same strategy as the estimate of I_6 in Lemma 2.2 to solve this problem.

For the term F_3, using integration by parts and Proposition 2.1, we get

$$\begin{aligned}
|F_3| &\lesssim (1+t)^{3-\sigma}\left|\int_{\mathbb{T}^3}\nabla^{2s-3}b_h\nabla^{2s-1}\partial_3 u_h-\nabla^{2s-3}\partial_3 b_3\nabla^{2s-1}u_3\,\mathrm{d}x\right|\\
&\lesssim (1+t)^{3-\sigma}\left(\|b_h\|_{H^{2s-3}}\|\partial_3 u_h\|_{H^{2s-1}}+\|\partial_3 b_3\|_{H^{2s-3}}\|u_3\|_{H^{2s-1}}\right)\\
&\lesssim (1+t)^{\frac{3-\sigma}{2}}\|\partial_3 b\|_{H^{2s-3}}(1+t)^{\frac{3-\sigma}{2}}\|\partial_3 u\|_{H^{2s-1}}.
\end{aligned}$$

Hence,

$$\int_0^t |F_3(\tau)|\,\mathrm{d}\tau \lesssim G_1^{1/2}(t)E_1^{1/2}(t). \tag{2.38}$$

Also, for the term F_4, using integration by parts and dividing the term into four parts, we have

$$\begin{aligned} F_4 =& -(1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-3}(b\cdot\nabla b)\nabla^{2s-1}u\,\mathrm{d}x \\ =& -(1+t)^{3-\sigma}\sum_{k=0}^{s-1}\int_{\mathbb{T}^3}\left(\nabla^k b_h\cdot\nabla_h\nabla^{2s-3-k}b+\nabla^k b_3\cdot\nabla_3\nabla^{2s-3-k}b\right)\nabla^{2s-1}u\,\mathrm{d}x \\ & -(1+t)^{3-\sigma}\sum_{k=s}^{2s-3}\int_{\mathbb{T}^3}\left(\nabla^k b_h\cdot\nabla_h\nabla^{2s-3-k}b+\nabla^k b_3\cdot\nabla_3\nabla^{2s-3-k}b\right)\nabla^{2s-1}u\,\mathrm{d}x. \end{aligned}$$

Using Hölder inequality, the Sobolev imbedding theorem and Proposition 2.1, we get

$$\begin{aligned} |F_4| \lesssim& (1+t)^{3-\sigma}\big(\|b_h\|_{W^{s-1,\infty}}\|b\|_{H^{2s-2}}\|u\|_{H^{2s-1}}+\|b_3\|_{W^{s-1,\infty}}\|\partial_3 b\|_{H^{2s-3}}\|u\|_{H^{2s-1}} \\ &+\|b_h\|_{H^{2s-3}}\|b\|_{W^{s-2,\infty}}\|u\|_{H^{2s-1}}+\|b_3\|_{H^{2s-3}}\|\partial_3 b\|_{W^{s-3,\infty}}\|u\|_{H^{2s-1}}\big) \\ \lesssim& (1+t)^{3-\sigma}\big(\|\partial_3 b\|_{H^{s+1}}\|b\|_{H^{2s-2}}\|u\|_{H^{2s-1}}+\|b_3\|_{H^{s+1}}\|\partial_3 b\|_{H^{2s-3}}\|u\|_{H^{2s-1}} \\ &+\|\partial_3 b\|_{H^{2s-3}}\|b\|_{H^{s}}\|u\|_{H^{2s-1}}+\|b_3\|_{H^{2s-3}}\|\partial_3 b\|_{H^{s-1}}\|u\|_{H^{2s-1}}) \\ \lesssim& (1+t)^{3-\sigma}\|b\|_{H^{2s-1}}\|\partial_3 b\|_{H^{2s-3}}\|u\|_{H^{2s-1}}, \end{aligned}$$

provided that $s \geqq 4$. Hence,

$$\begin{aligned} &\int_0^t |F_4(\tau)|\,\mathrm{d}\tau \\ &\lesssim \sup_{0\leqq\tau\leqq t}\|b\|_{H^{2s-1}}\Big(\int_0^t(1+\tau)^{3-\sigma}\|\partial_3 b\|_{H^{2s-3}}^2\,\mathrm{d}\tau\Big)^{1/2} \\ &\quad\Big(\int_0^t(1+\tau)^{3-\sigma}\|u\|_{H^{2s-1}}^2\,\mathrm{d}\tau\Big)^{1/2} \\ &\lesssim E_1(t)e_0^{1/2}(t). \end{aligned} \tag{2.39}$$

Summing up the estimates for $F_1 \sim F_4$, that is, (2.36), (2.37), (2.38) and (2.39), and integrating (2.35) in time, we can get the estimate of $E_{1,1}(t)$ which is defined in (2.34):

$$E_{1,1}(t) \lesssim E_1(0)+E_0(t)^{1/3}E_1^{2/3}(t)+G_1^{1/2}(t)E_1^{1/2}(t)+E_1^{3/2}(t)+E_1(t)e_0^{1/2}(t). \tag{2.40}$$

Here, we have used the Poincaré inequality to consider the highest order norms only.

Step 2

Now, let us work for the remaining term in $E_1(t)$. Applying ∇^{2s-3} derivative on the second equation of system (1.6) and taking inner product with $\nabla^{2s-3}\partial_3 b$, multiplying the time-weight $(1+t)^{3-\sigma}$ we get

$$(1+t)^{3-\sigma}\|\partial_3 b\|^2_{\dot{H}^{2s-3}} = F_5 + F_6 + F_7 + F_8, \tag{2.41}$$

where

$$\begin{aligned}
F_5 &= (1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-3}(u\cdot\nabla u)\nabla^{2s-3}\partial_3 b\,\mathrm{d}x\\
&\quad-(1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-3}\Delta u\nabla^{2s-3}\partial_3 b\,\mathrm{d}x,\\
F_6 &= -(1+t)^{3-\sigma}\sum_{k=0}^{s-1}\int_{\mathbb{T}^3}\nabla^k b_h\cdot\nabla_h\nabla^{2s-3-k}b\nabla^{2s-3}\partial_3 b\\
&\quad+\nabla^k b_3\cdot\nabla_3\nabla^{2s-3-k}b\nabla^{2s-3}\partial_3 b\,\mathrm{d}x\\
&\quad-(1+t)^{3-\sigma}\sum_{k=s}^{2s-3}\int_{\mathbb{T}^3}\nabla^k b_h\cdot\nabla_h\nabla^{2s-3-k}b\nabla^{2s-3}\partial_3 b\\
&\quad+\nabla^k b_3\cdot\nabla_3\nabla^{2s-3-k}b\nabla^{2s-3}\partial_3 b\,\mathrm{d}x,\\
F_7 &= \frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-3}u\nabla^{2s-3}\partial_3 b\,\mathrm{d}x\\
&\quad-(3-\sigma)(1+t)^{2-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-3}u\nabla^{2s-3}\partial_3 b\,\mathrm{d}x,\\
F_8 &= (1+t)^{3-\sigma}\int_{\mathbb{T}^3}\nabla^{2s-3}\partial_3 u\nabla^{2s-3}\partial_t b\,\mathrm{d}x.
\end{aligned}$$

Similar to the process in Step 1, we shall drive the estimate of each term on the right hand side of (2.41). First, using Hölder inequality and Sobolev imbedding theorem, we get

$$|F_5| \lesssim (1+t)^{3-\sigma}\|u\|_{H^{2s-2}}\|u\|_{H^{s+2}}\|\partial_3 b\|_{H^{2s-3}} + (1+t)^{3-\sigma}\|u\|_{H^{2s-1}}\|\partial_3 b\|_{H^{2s-3}}.$$

Hence, for $s \geqq 3$,

$$\int_0^t |F_5(\tau)|\,\mathrm{d}\tau \lesssim E_1^{3/2}(t) + E_{1,1}^{1/2}(t)\left[\int_0^t (1+\tau)^{3-\sigma}\|\partial_3 b\|^2_{H^{2s-3}}\,\mathrm{d}\tau\right]^{1/2}. \tag{2.42}$$

Next, for the most wild term F_6, similar to the estimate of I_6 in Lemma 2.2, we use property (1.7) and Proposition 2.1 to obtain

$$\begin{aligned}
|F_6| &\lesssim (1+t)^{3-\sigma}\left(\|b_h\|_{W^{s-1,\infty}}\|b\|_{H^{2s-2}}\|\partial_3 b\|_{H^{2s-3}} + \|b_3\|_{W^{s-1,\infty}}\|\partial_3 b\|^2_{H^{2s-3}}\right)\\
&\quad+(1+t)^{3-\sigma}\left(\|b_h\|_{H^{2s-3}}\|b\|_{W^{s-2,\infty}}\|\partial_3 b\|_{H^{2s-3}}\right.\\
&\quad\left.+\|b_3\|_{H^{2s-3}}\|\partial_3 b\|_{W^{s-3,\infty}}\|\partial_3 b\|_{H^{2s-3}}\right)\\
&\lesssim (1+t)^{3-\sigma}\left(\|\partial_3 b_h\|_{H^{s+1}}\|b\|_{H^{2s-2}}\|\partial_3 b\|_{H^{2s-3}} + \|b_3\|_{H^{s+1}}\|\partial_3 b\|^2_{H^{2s-3}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ (1+t)^{3-\sigma} \left(\|\partial_3 b_h\|_{H^{2s-3}} \|b\|_{H^s} \|\partial_3 b\|_{H^{2s-3}} \right.\\
&\left. + \|b_3\|_{H^{2s-3}} \|\partial_3 b\|_{H^{s-1}} \|\partial_3 b\|_{H^{2s-3}} \right)\\
&\lesssim (1+t)^{3-\sigma} \|\partial_3 b\|^2_{H^{2s-3}} \|b\|_{H^{2s-2}},
\end{aligned}
$$

provided that $s \geqq 4$. Hence,

$$
\int_0^t |F_6(\tau)| \,\mathrm{d}\tau \lesssim E_1(t) e_0^{1/2}(t). \tag{2.43}
$$

And, for the term F_7, by Hölder inequality, we can get

$$
\begin{aligned}
\int_0^t |F_7(\tau)| \,\mathrm{d}\tau \lesssim\ & G_1^{1/2}(t) E_1^{1/2}(t)\\
&+ \int_0^t (1+\tau)^{\frac{1-\sigma}{2}} \|u\|_{H^{2s-3}} (1+\tau)^{\frac{3-\sigma}{2}} \|\partial_3 b\|_{H^{2s-3}} \,\mathrm{d}\tau\\
&\lesssim G_1^{1/2}(t) E_1^{1/2}(t) + E_1(t)^{1/2} \left(\int_0^t (1+\tau)^{1-\sigma} \|u\|^2_{H^{2s+1}} \,\mathrm{d}\tau \right)^{1/2}\\
&\lesssim G_1^{1/2}(t) E_1^{1/2}(t) + E_0^{1/3}(t) E_1^{2/3}(t).
\end{aligned} \tag{2.44}
$$

For the last term F_8, using the first equation of system (1.6), we can write

$$
F_8 = (1+t)^{3-\sigma} \int_{\mathbb{T}^3} \nabla^{2s-3} \partial_3 u \nabla^{2s-3} (\partial_3 u + b \cdot \nabla u - u \cdot \nabla b) \,\mathrm{d}x.
$$

By Hölder inequality and the Sobolev imbedding theorem, we have

$$
|F_8| \lesssim (1+t)^{3-\sigma} \left(\|\partial_3 u\|^2_{H^{2s-3}} + \|\partial_3 u\|_{H^{2s-3}} \|b\|_{H^{2s-2}} \|u\|_{H^{2s-2}} \right),
$$

provided that $s \geqq 3$. Hence,

$$
\int_0^t |F_8(\tau)| \,\mathrm{d}\tau \lesssim E_{1,1}(t) + e_0(t)^{1/2} E_1(t)^{1/2} G_1(t)^{1/2}. \tag{2.45}
$$

Summing up the estimates for $F_5 \sim F_8$, that is, (2.42), (2.43), (2.44) and (2.45), and integrating (2.41) in time, and using Young inequality, we easily get

$$
\begin{aligned}
&\int_0^t (1+\tau)^{3-\sigma} \|\partial_3 b\|^2_{H^{2s-3}} \,\mathrm{d}\tau\\
&\lesssim E_{1,1}(t) + E_1^{3/2}(t) + E_1(t) e_0^{1/2}(t) + G_1^{1/2}(t) E_1^{1/2}(t)\\
&\quad + E_0^{1/3}(t) E_1^{2/3}(t) + G_1^{1/2}(t) E_1^{1/2}(t) e_0^{1/2}(t).
\end{aligned} \tag{2.46}
$$

This gives the estimate for the last term in $E_1(t)$. Now, multiplying (2.40) by a suitable large number, plus (2.46), and using Young inequality, we complete the proof of this lemma. □

Lemma 2.6. *Assume that $s \geqq 3$ and the energies are defined as in (2.3), then we have*

$$\begin{aligned} e_0(t) &\lesssim E_0(0) + G_0(t) + G_1(t) + E_0^{1/6}(t)E_1^{1/3}(t)e_0(t) \\ &\quad + E_0^{1/2}(t)E_1^{1/2}(t)e_0^{1/2}(t) + G_0^{1/4}(t)G_1^{1/4}(t)e_0(t). \end{aligned}$$

Proof. Taking ∇^{2s} derivative on the first equation of system (1.6). Then, taking inner product with $\nabla^{2s}b$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{\dot{H}^{2s}}^2 = M_1 + M_2 + M_3, \tag{2.47}$$

where,

$$\begin{aligned} M_1 &= \sum_{k=1}^{s}\int_{\mathbb{T}^3}\left(\nabla^k b\cdot\nabla\nabla^{2s-k}u - \nabla^k u\cdot\nabla\nabla^{2s-k}b\right)\,\nabla^{2s}b\,\mathrm{d}x \\ &\quad + \sum_{k=s+1}^{2s}\int_{\mathbb{T}^3}\left(\nabla^k b\cdot\nabla\nabla^{2s-k}u - \nabla^k u\cdot\nabla\nabla^{2s-k}b\right)\,\nabla^{2s}b\,\mathrm{d}x, \\ M_2 &= \int_{\mathbb{T}^3}\left(b_h\cdot\nabla_h\nabla^{2s}u + b_3\cdot\nabla_3\nabla^{2s}u\right)\nabla^{2s}b\,\mathrm{d}x, \\ M_3 &= \int_{\mathbb{T}^3}\nabla^{2s}\partial_3 u\;\nabla^{2s}b\,\mathrm{d}x. \end{aligned}$$

Now we will estimate each term on the right hand side of (2.47) line by line.

First, by Hölder inequality and the Sobolev imbedding theorem, we easily get

$$\begin{aligned} |M_1| &\lesssim \|b\|_{W^{s,\infty}}\|u\|_{H^{2s}}\|b\|_{H^{2s}} + \|u\|_{W^{s,\infty}}\|b\|_{H^{2s}}^2 \\ &\quad + \|b\|_{H^{2s}}\|u\|_{W^{s,\infty}}\|b\|_{H^{2s}} + \|u\|_{H^{2s}}\|b\|_{W^{s,\infty}}\|b\|_{H^{2s}} \\ &\lesssim \|u\|_{H^{2s}}\|b\|_{H^{2s}}^2, \end{aligned}$$

provided that $s \geqq 2$. Hence, using the Gagliardo–Nirenberg interpolation inequality and Hölder inequality, we can bound

$$\int_0^t |M_1(\tau)|\,\mathrm{d}\tau \lesssim e_0(t)\int_0^t \|u\|_{H^{2s+2}}^{1/3}\|u\|_{H^{2s-1}}^{2/3}\,\mathrm{d}\tau \lesssim E_0^{1/6}(t)E_1^{1/3}(t)e_0(t). \tag{2.48}$$

For the next term M_2, using the same method as above, we directly obtain

$$\begin{aligned} |M_2| &\lesssim \|b_h\|_{L^\infty}\|u\|_{H^{2s+1}}\|b\|_{H^{2s}} + \|b_3\|_{L^\infty}\|\partial_3 u\|_{H^{2s}}\|b\|_{H^{2s}} \\ &\lesssim \|b_h\|_{H^{2s-3}}\|u\|_{H^{2s+1}}\|b\|_{H^{2s}} + \|\partial_3 u\|_{H^{2s}}\|b\|_{H^{2s}}^2, \end{aligned}$$

provided that $s \geqq 3$. According to the Proposition 2.1, we can use the same strategy as the estimate of I_6 in Lemma 2.2, and obtain

$$|M_2| \lesssim \|\partial_3 b_h\|_{H^{2s-3}}\|u\|_{H^{2s+1}}\|b\|_{H^{2s}} + \|\partial_3 u\|_{H^{2s}}\|b\|_{H^{2s}}^2.$$

Using (2.24) and Hölder inequality, we get

$$\begin{aligned}\int_0^t |M_2(\tau)|\, d\tau &\lesssim E_0^{1/2}(t)e_0^{1/2}(t)\int_0^t (1+\tau)^{\sigma/2}\|\partial_3 b\|_{H^{2s-3}}\, d\tau \\ &\quad + e_0(t)\int_0^t \|\partial_3 u\|_{H^{2s}}\, d\tau \\ &\lesssim E_0^{1/2}(t)E_1^{1/2}(t)e_0^{1/2}(t) + G_0^{1/4}(t)G_1^{1/4}(t)e_0(t).\end{aligned} \tag{2.49}$$

For the last term, M_3, we also have

$$\int_0^t |M_3(\tau)|\, d\tau \lesssim e_0^{1/2}(t)\cdot\int_0^t \|\partial_3 u\|_{H^{2s}}\, d\tau \lesssim G_0^{1/4}(t)G_1^{1/4}(t)e_0^{1/2}(t). \tag{2.50}$$

Combining (2.48), (2.49) and (2.50) together, we now complete the proof of this lemma by using Young's inequality. □

2.3. Proof of the Theorem 1.1

Now, let us combine the above *a priori* estimates of all the energies defined in (2.3) together, and finally give the proof of Theorem 1.1. First, we define the total energy as follows:

$$E_{\text{total}}(t) = E_0(t) + G_0(t) + G_1(t) + E_1(t) + e_0(t).$$

Then, multiplying each inequality in the above five lemmas by different suitable number, and summing them up, we can obtain the following inequality:

$$E_{\text{total}}(t) \leqq C_1 E_0(0) + C_1 E_{\text{total}}^{3/2}(t), \tag{2.51}$$

for some positive constant C_1.

According to the setting of initial data in Theorem 1.1, there exists a positive constant C_2 such that $E_{\text{total}}(0) + C_1 E_0(0) \leqq C_2\varepsilon$. Due to the local existence result, which can be achieved through a basic energy method, there exists a positive time T such that

$$E_{\text{total}}(t) \leqq 2C_2\varepsilon, \quad \forall\, t \in [0, T]. \tag{2.52}$$

Let T^* be the largest possible time of T for what (2.52) holds, then we only need to show $T^* = \infty$ while completing the proof of Theorem 1.1. Notice the estimate (2.51); we can use a standard continuation argument to show that $T^* = \infty$ provided that ϵ is small enough. We omit the details here. Hence, we finish the proof of Theorem 1.1.

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