

MEKLER'S CONSTRUCTION AND GENERALIZED STABILITY

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ABSTRACT. Mekler's construction gives an interpretation of any structure in a finite relational language in a group (nilpotent of class 2 and exponent $p > 2$, but not finitely generated in general). Even though this construction is not a bi-interpretation, it is known to preserve some model-theoretic tameness properties of the original structure including stability and simplicity. We demonstrate that k -dependence of the theory is preserved, for all $k \in \mathbb{N}$, and that NTP_2 is preserved. We apply this result to obtain first examples of strictly k -dependent groups (with no additional structure).

1. INTRODUCTION

Mekler's construction [15] provides a general method to interpret any structure in a finite relational language in a pure 2-nilpotent group of finite exponent (the resulting group is typically not finitely generated). This is not a bi-interpretation, however it tends to preserve various model-theoretic tameness properties. First Mekler proved that for any cardinal κ the constructed group is κ -stable if and only if the initial structure was [15]. Afterwards, it was shown by Baudisch and Pentzel that simplicity of the theory is preserved, and by Baudisch that, assuming stability, CM-triviality is also preserved [2]. See [12, Section A.3] for a detailed exposition of Mekler's construction.

The aim of this paper is to investigate further preservation of various generalized stability-theoretic properties from Shelah's classification program [19]. We concentrate on the classes of k -dependent and NTP_2 theories.

The classes of k -dependent theories (see Definition 4.1), for each $k \in \mathbb{N}$, were defined by Shelah in [20], and give a generalization of the class of NIP theories (which corresponds to the case $k = 1$). See [8, 11, 21] for some further results about k -dependent groups and fields and connections to combinatorics. In Theorem 4.7 we show that Mekler's construction preserves k -dependence. Our initial motivation was to obtain algebraic examples that witness the strictness of the k -dependence hierarchy. For $k \geq 2$, we will say that a theory is *strictly k -dependent* if it is k -dependent, but not $(k-1)$ -dependent. The usual combinatorial example of a strictly k -dependent theory is given by the random k -hypergraph. The first example of a strictly 2-dependent group was given in [11] (it was also considered in [22, Example 4.1.14]):

Example 1. Let G be $\oplus_{\omega} \mathbb{F}_p$, where \mathbb{F}_p is the finite field with p elements. Consider the structure $\mathcal{G} = (G, \mathbb{F}_p, 0, +, \cdot)$, where 0 is the neutral element, $+$ is addition in G , and \cdot is the bilinear form $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ from G to \mathbb{F}_p . This structure is not NIP, but is 2-dependent. In the case $p = 2$, \mathcal{G} is interpretable in an extra-special p -group $\mathcal{H} = (H, \cdot, 1)$, and conversely \mathcal{H} is interpretable in \mathcal{G} (see [14, Proposition

3.11] and the discussion around it, or the appendix in [16]). Hence \mathcal{H} provides an example of a strictly 2-dependent pure group.

In Corollary 4.8 we use Mekler's construction to show that for every k , there is a strictly k -dependent pure group.

The class of NTP_2 theories was defined in [18] (see Definition 5.1). It gives a common generalization of simple and NIP theories (along with containing many new important examples), and more recently it was studied in e.g. [3–5, 7]. In Theorem 5.6 we show that Mekler's construction preserves NTP_2 .

The paper is organized as follows. In Section 2 we review Mekler's construction and record some auxiliary lemmas, including the key lemma about type-definability of partial transversals and related objects (Proposition 2.18). In Section 3 we prove that NIP is preserved. In Section 4 we discuss indiscernible witnesses for k -dependence and give a proof that Mekler's construction preserves k -dependence. As an application, for each $k \geq 2$ we construct a strictly k -dependent pure group and discuss some related open problems. Finally, in Section 5 we prove that Mekler's construction preserves NTP_2 .

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2. PRELIMINARIES ON MEKLER'S CONSTRUCTION

We review Mekler's construction from [15], following the exposition and notation in [12, Section A.3] (to which we refer the reader for further details).

Definition 2.1. A graph (binary, symmetric relation without self-loops) is called *nice* if it satisfies the following two properties:

- (1) there are at least two vertices, and for any two distinct vertices a and b there is some vertex c different from a and b such that c is joined to a but not to b ;
- (2) there are no triangles or squares in the graph.

For any graph C and an odd prime p , we define a 2-nilpotent group of exponent p denoted by $G(C)$ which is generated freely in the variety of 2-nilpotent groups of exponent p by the vertices of C by imposing that two generators commute if and only if they are connected by an edge in C .

Now, let C be a nice graph and consider the group $G(C)$. Let G be any model of $\text{Th}(G(C))$. We consider the following \emptyset -definable equivalence relations on the elements of G .

Definition 2.2. Let g and h be elements of G , then

- $g \sim h$, if $C_G(g) = C_G(h)$.
- $g \approx h$ if there is some natural number r and c in $Z(G)$ such that $g = h^r \cdot c$.
- $g \equiv_Z h$ if $g \cdot Z(G) = h \cdot Z(G)$.

Note that $g \equiv_Z h$ implies $g \approx h$, which implies $g \sim h$.

Definition 2.3. Let g be an element of G and let q be a natural number. We say that g is of *type* q if there are q different \approx -equivalence classes in the \sim -class $[g]_\sim$

of g . Moreover, we say that g is *isolated* if all non central $h \in G$ which commute with g are \approx -equivalent to g .

All non-central elements of G can be partitioned into four different \emptyset -definable classes (see [12, Lemma A.3.6 - A.3.10] for the details):

- (1) elements of type 1 which are not isolated, also referred to as *elements of type 1^ν* (in $G(C)$ this class includes the elements given by the vertices of C),
- (2) elements of type 1 which are isolated, also referred to as *elements of type 1^t* ,
- (3) elements of type p , and
- (4) elements of type $p - 1$.

The elements of the latter two types are always non-isolated (it is easy to see from the definition that only an element of type 1 can be isolated).

By [12, Lemma A.3.8, (a) \Leftrightarrow (b)], for every element $g \in G$ of type p , the non-central elements of G which commute with g are precisely the elements \sim -equivalent to g , and an element b of type 1^ν together with the elements \sim -equivalent to b .

Definition 2.4. For every element $g \in G$ of type p , we call an element b of type 1^ν which commutes with g a *handle* of g .

Fact 2.5. By the above, we obtain immediately that a handle is definable from g up to \sim -equivalence.

Note here, that the center of G as well as the quotient $G/Z(G)$ are elementary abelian p -groups. Hence they can be viewed as \mathbb{F}_p -vector spaces. From now on, *independence* over some supergroup of $Z(G)$ will refer to linear independence in terms of the corresponding \mathbb{F}_p -vector space.

Definition 2.6. Let G be a model of $\text{Th}(G(C))$. We define the following:

- A 1^ν -transversal of G is a set X^ν consisting of one representative for each \sim -class of elements of type 1^ν in G .
- An element is *proper* if it is not a product of any elements of type 1^ν in G .
- A p -transversal of G is a set X^p of pairwise \sim -inequivalent proper elements of type p in G which is maximal with the property that if Y is a finite subset of X^p and all elements of Y have the same handle, then Y is independent modulo the subgroup generated by all elements of type 1^ν in G and $Z(G)$.
- A 1^t -transversal of G is a set X^t of representatives of \sim -classes of proper elements of type 1^t in G which is maximal independent modulo the subgroup generated by all elements of types 1^ν and p in G , together with $Z(G)$.
- A set $X \subseteq G$ is a *transversal* of G if $X = X^\nu \sqcup X^p \sqcup X^t$, where X^ν , X^p and X^t are some transversals of the corresponding types.
- A subset Y of a transversal is called a *partial transversal* if it is closed under handles (i. e. for any element a of type p in Y there is an element of type 1^ν in Y which is a handle of a).
- For a given (partial) transversal X , we denote by X^ν , X^p , and X^t the elements in X of the corresponding types.

Lemma 2.7. Let $G \models \text{Th}(G(C))$. Given a small tuple of variables $\bar{x} = \bar{x}^\nu \frown \bar{x}^p \frown \bar{x}^t$, there is a partial type $\Phi(\bar{x})$ such that for any tuples \bar{a}^ν, \bar{a}^p and \bar{a}^t in G , we have that $G \models \Phi(\bar{a}^\nu, \bar{a}^p, \bar{a}^t)$ if and only if every element in \bar{a}^ν, \bar{a}^p and \bar{a}^t is of type $1^\nu, p$ and 1^t , respectively, and $\bar{a} = \bar{a}^\nu \frown \bar{a}^p \frown \bar{a}^t$ can be extended to a transversal of G .

Proof. By inspecting Definition 2.6. For example, let's describe the partial type $\Phi^p(\bar{x}^p)$ expressing that $\bar{x}^p = (x_i^p : i < \kappa)$ can be extended to a p -transversal (the conditions on \bar{x}^ν and \bar{x}^ι are expressed similarly). For $q \in \mathbb{N}$, let $\phi_q(x)$ be the formula defining the set of all elements of type q in G , let $\phi_\iota(x)$ define the set of isolated elements, and let $\phi_h(x_1, \dots, x_q)$ express that x_1, \dots, x_q have the same handle. The set of proper elements is defined by

$$\Phi_{\text{prop}}(x) := \{\forall y_1 \dots \forall y_{n-1} (\bigwedge_{i < n} \phi_1(y_i) \wedge \neg \phi_\iota(y_i)) \rightarrow (x \neq \prod_{i < n} y_i) : n \in \mathbb{N}\}.$$

The independence requirement in the definition of a p -transversal can be expressed as a partial type $\Phi_{\text{ind}}(\bar{x}^p)$ containing the formula

$$\begin{aligned} & \phi_h(x_{i_0}^p, \dots, x_{i_{n-1}}^p) \rightarrow \\ & \left(\forall y_0 \dots \forall y_{m-1} \forall z \bigwedge_{i < m} (\phi_1(y_i) \wedge \neg \phi_\iota(y_i) \wedge z \in Z(G)) \rightarrow z \cdot \prod_{i < m} y_i^{r_i} \neq \prod_{j < n} (x_{i_j}^p)^{q_j} \right) \\ & \text{for each } m, n \in \mathbb{N}, i_0 < \dots < i_{n-1} < \kappa \text{ and } q_0, \dots, q_{n-1}, r_0, \dots, r_{m-1} \in \mathbb{N}, 0 < q_j < p. \end{aligned}$$

Then we can take $\Phi^p(\bar{x}^p) := \{\phi_p(x_i^p) : i < \kappa\} \cup \{\neg(x_i^p \sim x_j^p) : i < j < \kappa\} \cup \bigcup_{i < \kappa} \Phi_{\text{prop}}(x_i^p) \cup \Phi_{\text{ind}}(\bar{x}^p)$. \square

The following can be easily deduced from [12, Corollary A.3.11].

Fact 2.8. *Let C be a nice graph. There is an interpretation Γ such that for any model G of $\text{Th}(G(C))$, we have that $\Gamma(G)$ is a model of $\text{Th}(C)$. More specifically, the graph $\Gamma(G) = (V, R)$ is given by the set of vertices $V = \{g \in G : g \text{ is of type } 1^\nu, g \notin Z(G)\} / \approx$ and the (well-defined) edge relation $([g]_\approx, [h]_\approx) \in R \iff g, h \text{ commute}$.*

The full set of transversal gives another important graph, a so called cover of a nice graph, which we define below.

Definition 2.9. (1) Let C be an infinite nice graph. A graph C^+ containing C as a subgraph is called a *cover* of C if for every vertex $b \in C^+ \setminus C$, one of the following two statements holds:

- there is a unique vertex a in C that is joined to b , and moreover a is in C and it has infinitely many adjacent vertices in C ;
- b is joined to no vertex in C .

(2) A cover C^+ of C is a λ -cover if

- for every vertex a in C the number of vertices in $C^+ \setminus C$ joined to a is λ if a is joined to infinitely many vertices in C , and zero otherwise;
- the number of new vertices in $C^+ \setminus C$ which are not joined to any other vertex in C^+ is λ .

Observe that a proper cover of a nice graph is never a nice graph.

Fact 2.10. [12, Corollary A.3.14(a) + proof of (c)]. *Given a 1^ν -transversal X^ν of G , we identify the elements of X^ν with the set of vertices of $\Gamma(G)$ by mapping $x \in X^\nu$ to its class $[x]_\approx$. Then a transversal X can be viewed as a cover of the nice graph given by the elements of type 1^ν in X , with the edge relation given by commutation. (See Remark 2.13 for an explanation.)*

Fact 2.11. [12, Theorem A.3.14, Corollary A.3.15] *Let G be a model of $\text{Th}(G(C))$ and let X be a transversal of G .*

- (1) *There is a subgroup of $Z(G)$ which we denote by K_X such that $G = \langle X \rangle \times K_X$.*
- (2) *The group K_X is an elementary abelian p -group, in particular $\text{Th}(K_X)$ is stable and eliminates quantifiers.*
- (3) *If G is saturated and uncountable, then both the graph $\Gamma(G)$ and the group K_X are also saturated (as $|K_X| = |G|$ and $\text{Th}(K_X)$ is uncountably categorical).*
- (4) *If G is a saturated model of $\text{Th}(G(C))$, then every automorphism of $\Gamma(G)$ can be lifted to an automorphism of G .*
- (5) *$\langle X \rangle \cong G(X)$ via an isomorphism which is the identity on the elements in X (where X is viewed as a graph as in Remark 2.10, note that it doesn't have to be nice).*
- (6) *If G is saturated then X (where X is viewed as a graph as in Remark 2.10) is a $|G|$ -cover of $\Gamma(G)$.*

Alternatively, one could work with a special model instead of a saturated one to avoid any set-theoretic issues.

Remark 2.12. The reason given for (3) in [12] appears to assume that K_X is definable, which is not the case, so we provide a proof.

Assume that C is an infinite nice graph. Given $m, n \in \omega$, let $\psi_{m,n}$ be the sentence

$$\forall g_1, \dots, g_n \in Z \exists x \in Z \left(\forall z_1, \dots, z_{2m} \bigwedge \left(g_1^{\alpha_1} \cdots g_n^{\alpha_n} \cdot x^\alpha \neq \prod_{j=1}^m [z_{2j-1}, z_{2j}] \right) \right),$$

where the conjunction is over all $(\alpha_1, \dots, \alpha_{n-1}, \alpha) \in \{0, \dots, p-1\}^{n+1}$ with $\alpha \neq 0$.

We claim that $G(C) \models \psi_{m,n}$ for all $m, n \in \omega$. Indeed, by [12, Lemma A.3.2(b)+(d)], a basis for $G(C)' = Z(G(C))$ is given by the set of all elements of the form $[a, b]$ with a, b elements in C not connected by an edge. Since C is nice, given any vertices $v \neq w$ there is at most one vertex to which both v and w are connected. As C is infinite, we can find pairwise distinct elements a and $\{b_i : i < \omega\}$ in C such that a is not connected by an edge to any of the b_i 's. Indeed, take any pairwise distinct elements a', a'' , $(c_i : i < \omega)$ in C . For each i , at most one of a', a'' is connected to c_i , hence for one of a, a' there are infinitely many $i < \omega$ such that it is not connected to c_i . Hence $\{[a, b_i] : i < \omega\}$ is a linearly independent set and $Z(G(C))$ has infinite dimension. Now, given arbitrary m, n and $g_i \in Z(G(C))$, assume that each g_i is a linear combination of at most t vectors from the basis. If we take x to be a product of more than $nt + m$ non-trivial commutators, the equality above cannot hold (as $\alpha \neq 0$, and for any $l \in \omega$ the sum of l vectors in a basis cannot be written as a linear combination of less than l vectors in the same basis).

Now let G be a saturated model of $\text{Th}(G(C))$, $\lambda := |G|$. In view of Remark 2.16, it is enough to find a sequence $(h_i : i < \lambda)$ of elements in $Z(G)$ which are linearly independent in the elementary abelian p -group $Z(G)$ over G' . This can be done by transfinite induction using saturation of $G(C)$ and the fact that given any finitely many elements $g_1, \dots, g_n \in Z(G)$, we can find an element $x \in Z(G)$ independent from them over G' since $G \models \psi_{m,n}$ for all $m \in \omega$ (see Definition 2.17).

Remark 2.13. We provide some details concerning Fact 2.11(6). The fact that this is a cover in any model of $\text{Th}(G(C))$ doesn't seem to follow explicitly from the statement of Axiom 10, as stated in the proof of [12, Corollary A.3.14(c)]. However,

it follows by an argument similar to the proof of Remark 2.12. This time, working in the vector space $G/Z(G)$ and using [12, Lemma A.3.9], the proof of Axiom 10, and [12, Lemma A.3.2(c)] in $G(C)$, all the relevant sets in Definition 2.9 are infinite, so by saturation can be demonstrated to be of size $|G|$ in a saturated model G .

The following lemma is a refinement of Fact 2.11(4) and [2, Lemma 4.12].

Lemma 2.14. *Let G be a saturated model of $\text{Th}(G(C))$, X be a transversal, and $K_X \leq Z(G)$ be such that $G = \langle X \rangle \times K_X$. Let Y and Z be two small subsets of X and let \bar{h}_1, \bar{h}_2 be two tuples in K_X . Suppose that*

- *there is a bijection f between Y and Z which respects the 1^ν -, p -, and 1^t -parts, the handles, and $\text{tp}_\Gamma(Y^\nu) = \text{tp}_\Gamma(f(Y^\nu))$,*
- *$\text{tp}_{K_X}(\bar{h}_1) = \text{tp}_{K_X}(\bar{h}_2)$.*

Then there is an automorphism of G coinciding with f on Y and sending \bar{h}_1 to \bar{h}_2 .

Proof. By Remark 2.10, we identify $\Gamma(G)$ with X^ν . By saturation of $\Gamma(G)$, $f \upharpoonright Y_\nu$ extends to an automorphism σ of the graph X^ν . As X is a $|G|$ -cover of X^ν by saturation of G , and f respects the 1^ν -, p -, and 1^t -parts and the handles, σ extends to an automorphism τ of the graph X agreeing with f using a straightforward back-and-forth argument. By Fact 2.11(5), we have that $\langle X \rangle \cong G(X)$ and τ lifts to an automorphism of the group $G(X)$, hence to an automorphism $\tilde{\tau}$ of $\langle X \rangle$ extending f by construction. As K_X is saturated by Fact 2.11(3), there is an automorphism ρ of K_X which maps \bar{h}_1 to \bar{h}_2 . We can now take the cartesian product of $\tilde{\tau} \times \rho$ to obtain an automorphism of G which extends f and maps \bar{h}_1 to \bar{h}_2 . \square

Next, we observe that in Fact 2.11 the choice of a transversal and an elementary abelian subgroup of the center in the decomposition of G can be made entirely independently of each other.

Lemma 2.15. *Let G be any model of $\text{Th}(G(C))$ and let X be a transversal of G . Then we have $G' = \langle X \rangle'$.*

Proof. Let K be a subgroup of $Z(G)$ as in Fact 2.11, such that $G = \langle X \rangle \times K$. It is enough to show that for all $g, g' \in G$, we have that $[g, g']$ is in $\langle X \rangle'$. We choose $x, y \in \langle X \rangle$ and $h, k \in K$ such that $g = x \cdot h$ and $g' = y \cdot k$. Then, using that G is 2-nilpotent, we have

$$[g, g'] = [x \cdot h, y \cdot k] = [x, y],$$

which is in $\langle X \rangle'$. \square

Remark 2.16. Let X and Y be two transversals, and let K_X be such that $G = \langle X \rangle \times K_X$. Then the proof of [12, Theorem A.3.14(d)] and Lemma 2.15 imply that $G = \langle Y \rangle \times K_X$. Furthermore, that proof also implies that for any set B of elements in $Z(G)$ which are linearly independent in the elementary abelian p -group $Z(G)$ over G' (seen as an \mathbb{F}_p -vector space), there is a subgroup K of G containing B such that for any transversal X of G , we have that $G = \langle X \rangle \times K$.

Definition 2.17. For any small cardinal κ , let $\Psi((y_i)_{i \in \kappa})$ be the partial type consisting of the formulas “ $y_i \in Z(G)$ ” for all $i < \kappa$, and the formulas

$$\forall z_0, \dots, z_{2m-1} \left(\bigwedge_{(\alpha_0, \dots, \alpha_{n-1}) \in \{0, \dots, p-1\}^n \setminus (0, \dots, 0)} \left(y_{i_0}^{\alpha_0} \cdots y_{i_{n-1}}^{\alpha_{n-1}} \neq \prod_{j=0}^{m-1} [z_{2j}, z_{2j+1}] \right) \right)$$

for all $m, n \in \mathbb{N}$ and $i_0, \dots, i_n \in \kappa$.

An easy inspection yields that for any tuple \bar{b} , $\bar{b} \models \Psi(\bar{y})$ if and only if \bar{b} is a tuple of central elements linearly independent in the elementary abelian p -group $Z(G)$ over G' (seen as an \mathbb{F}_p -vector space).

Let now $\Phi(\bar{x})$, with $\bar{x} = \bar{x}^\nu \frown \bar{x}^p \frown \bar{x}^t$, be the partial type given by Lemma 2.7. Consider the partial type

$$\pi(\bar{x}, \bar{y}) = \Phi(\bar{x}) \cup \Psi(\bar{y}).$$

Proposition 2.18. *Let G be a model of $\text{Th}(G(C))$. Then $G \models \pi(\bar{a}, \bar{b})$ if and only if we can extend \bar{a} to a transversal X of G and find a subset $H \subseteq Z(G)$ containing \bar{b} which is linearly independent over G' , so that $G = \langle X \rangle \times \langle H \rangle$.*

Proof. Combining this with Remark 2.16, there is a subgroup K of G containing \bar{b} such that for any transversal X of G , we have that $G = \langle X \rangle \times K$. Combining this with Lemma 2.7, we can conclude. \square

3. PRESERVATION OF NIP

We begin with the simplest case demonstrating that NIP is preserved. Recall the following basic characterization of NIP.

Fact 3.1. (see e.g. [1]) *Let T be a complete first-order theory and let $\mathbb{M} \models T$ be a monster model. Let κ be the regular cardinal $|T|^+$. Then the following are equivalent.*

- (1) *T is NIP.*
- (2) *For every indiscernible sequence $I = (\bar{a}_i : i \in \kappa)$ of finite tuples and a finite tuple \bar{b} in \mathbb{M} , there is some $\alpha < \kappa$ such that $\text{tp}(\bar{b}\bar{a}_i) = \text{tp}(\bar{b}\bar{a}_j)$ for all $i, j > \alpha$.*

As in Section 2, let C be a nice graph and let $G(C)$ be the 2-nilpotent group of exponent p which is freely generated (in the variety of 2-nilpotent groups) by the vertices of C by imposing that two generators commute if and only if they are connected by an edge in C .

Theorem 3.2. *$\text{Th}(C)$ is NIP if and only if $\text{Th}(G(C))$ is NIP.*

Proof. If $\text{Th}(G(C))$ is NIP, then $\text{Th}(C)$ is also NIP as C is interpretable in $G(C)$.

Now, we want to prove the converse. Let $G \models \text{Th}(G(C))$ be a saturated model, and assume that $\text{Th}(G(C))$ has IP but $\text{Th}(C)$ is NIP. Fix κ to be $(\aleph_0)^+$. Then there is some formula $\phi(\bar{x}, \bar{y}) \in L_G$, and a sequence $I = (\bar{a}_i : i \in \kappa)$ in G shattered by $\phi(\bar{x}, \bar{y})$, i.e. such that for every $S \subseteq \kappa$, there is some \bar{b}_S in G satisfying $G \models \phi(\bar{b}_S, \bar{a}_i)$ if and only if $i \in S$.

Let X be a transversal for G and $H \subseteq Z(G)$ a set of elements linearly independent over G' and such that $G = \langle X \rangle \times \langle H \rangle$. Then for each $i \in \kappa$ we have, slightly abusing notation, $\bar{a}_i = t_i(\bar{x}_i, \bar{h}_i)$ for some L_G -term t_i and some finite tuples $\bar{x}_i = \bar{x}_i^\nu \frown \bar{x}_i^p \frown \bar{x}_i^t$ from X where $\bar{x}_i^\nu, \bar{x}_i^p, \bar{x}_i^t$ list all of the elements of type $1^\nu, p, 1^t$ in \bar{x}_i , respectively, and \bar{h}_i from H . After adding some elements of type 1^ν to the beginning of the tuple and changing the term t_i accordingly, we may assume that for each $i \in \kappa$ and $j < |\bar{x}_i^p|$, the handle of the j -th element of \bar{x}_i^p is the j -th element of \bar{x}_i^ν (there might be some repetitions of elements of type 1^ν as different elements of type p might have the same handle). As $\kappa > |L_G| + \aleph_0$, passing to a cofinal subsequence and reordering the tuples if necessary, we may assume that:

- (1) $t_i = t \in L_G$ and $|\bar{x}_i|$ and $|\bar{h}_i|$ are constant for all $i \in \kappa$,

- (2) $|\bar{x}_i^\nu|, |\bar{x}_i^p|, |\bar{x}_i^t|$ are constant for all $i \in \kappa$.

Consider the L_G -formula $\phi'(\bar{x}, \bar{y}') = \phi(\bar{x}; t(\bar{y}_1, \bar{y}_2))$ with $\bar{y}' := \bar{y}_1 \hat{\smallfrown} \bar{y}_2$ and $|\bar{y}_1| = |\bar{x}_i|$ and $|\bar{y}_2| = |\bar{h}_i|$. Let $\bar{a}'_i := \bar{x}_i \hat{\smallfrown} \bar{h}_i$. Then the sequence $I' := (\bar{a}'_i : i \in \kappa)$ is shattered by $\phi'(\bar{x}, \bar{y}')$. Note however that I' is generally not indiscernible.

To fix this, let $J = ((\bar{x}'_i)^\nu \hat{\smallfrown} \bar{h}'_i : i \in \kappa)$ be an L_G -indiscernible sequence of tuples in G with the same EM-type as I' . Then we have:

- (1) J is still shattered by $\phi'(\bar{x}, \bar{y}')$,
- (2) for each $i \in \kappa$ and $j < |\bar{x}'_i|^p$, we have that the handle of the j -th element of $(\bar{x}'_i)^p$ is the j -th element of $(\bar{x}'_i)^\nu$ (since being a handle is a definable condition, see Definition 2.5, and the corresponding property was true on all elements in I').
- (3) The set of all elements of G appearing in the sequence $(\bar{x}'_i : i \in \kappa)$ still can be extended to some transversal X' of G .
- (4) The set of all elements of G appearing in the sequence $(\bar{h}'_i : i \in \kappa)$ can be extended to some set $H' \subseteq Z(G)$ linearly independent over G' and such that $G = \langle X' \rangle \times \langle H' \rangle$.

The last two conditions hold as the sets of all elements appearing in the sequences $(\bar{x}_i : i \in \kappa)$ and $(\bar{h}_i : i \in \kappa)$ satisfied the respective conditions, these conditions are type-definable by Proposition 2.18 and J has the same EM-type as I' .

Now let $\bar{b} \in G$ be such that both sets $\{i \in \kappa : G \models \phi'(\bar{b}, \bar{a}'_i)\}$ and $\{i \in \kappa : G \models \neg\phi'(\bar{b}, \bar{a}'_i)\}$ are cofinal in κ . Then $\bar{b} = s(\bar{z}, \bar{k})$ for some term $s \in L_G$ and some finite tuples \bar{z} in X' and \bar{k} in H' . Write $\bar{z} = \bar{z}^\nu \hat{\smallfrown} \bar{z}^p \hat{\smallfrown} \bar{z}^t$, with $\bar{z}^\nu, \bar{z}^p, \bar{z}^t$ listing the elements of the corresponding types in \bar{z} . In the same way as extending \bar{x}_i , we may add elements to the tuple \bar{z} and assume that the handle of the j -th element of \bar{z}^p is the j -th element of \bar{z}^ν .

Consider all of the elements in \bar{z}^ν and $((\bar{x}'_i)^\nu : i \in \kappa)$ as elements in $\Gamma(G)$ — a saturated model of $\text{Th}(C)$, and note that as $\Gamma(G)$ is interpretable in G we have that the sequence $((\bar{x}'_i)^\nu : i \in \kappa)$ is also indiscernible in $\Gamma(G)$. As $\text{Th}(\Gamma(G))$ is NIP, by Fact 3.1 there is some $\alpha < \kappa$ such that $\text{tp}_\Gamma(\bar{z}^\nu, (\bar{x}'_i)^\nu) = \text{tp}_\Gamma(\bar{z}^\nu, (\bar{x}'_j)^\nu)$ for all $i, j > \alpha$. Moreover, using indiscernibility of the sequence (\bar{x}'_i) and possibly throwing away finitely many elements from the sequence, we have that

$$(\bar{x}'_i)^p \cap \bar{z}^p = (\bar{x}'_j)^p \cap \bar{z}^p, (\bar{x}'_i)^t \cap \bar{z}^t = (\bar{x}'_j)^t \cap \bar{z}^t \text{ (as tuples)}$$

and $\bar{x}'_i \cap \bar{x}'_j$ is constant, for all $i, j \in \kappa$. Thus, for any $i, j > \alpha$, the bijection $\sigma_{i,j}$ sending $\bar{x}'_i \bar{z}$ to $\bar{x}'_j \bar{z}$ and preserving the order of the elements satisfies:

- (1) $\text{tp}_\Gamma((\bar{x}'_i)^\nu \bar{z}^\nu) = \text{tp}_\Gamma(\sigma_{i,j}((\bar{x}'_i)^\nu \bar{z}^\nu))$,
- (2) the map $\sigma_{i,j}$ fixes \bar{z} ,
- (3) the map $\sigma_{i,j}$ respects the 1^ν -, p - and 1^t -parts and the handles (since the handle of the j -th element of $(\bar{x}'_i)^p$ is the j -th element of $(\bar{x}'_i)^\nu$).

Now consider \bar{k} and $(\bar{h}'_i : i \in \kappa)$ as tuples of elements in $\langle H' \rangle$, which is a model of the stable theory $\text{Th}(\langle H' \rangle)$. Moreover, as $(\bar{h}'_i : i \in \kappa)$ is L_G -indiscernible and $\text{Th}(\langle H' \rangle)$ eliminates quantifiers, $(\bar{h}'_i : i \in \kappa)$ is also indiscernible in the sense of $\text{Th}(\langle H' \rangle)$. Hence, by stability, there is some $\beta \in \kappa$ such that $\text{tp}_{\langle H' \rangle}(\bar{k} \bar{h}'_i) = \text{tp}_{\langle H' \rangle}(\bar{k} \bar{h}'_j)$ for all $i, j > \beta$.

Now, Lemma 2.14 gives us an automorphism of G sending $\bar{x}'_i \bar{h}'_i \bar{z} \bar{k}$ to $\bar{x}'_j \bar{h}'_j \bar{z} \bar{k}$, so $\text{tp}_G(\bar{x}'_i \bar{h}'_i / \bar{z} \bar{k}) = \text{tp}_G(\bar{x}'_j \bar{h}'_j / \bar{z} \bar{k})$ for all $i, j > \max\{\alpha, \beta\}$. This contradicts the choice of $\bar{b} = s(\bar{z}, \bar{k})$. \square

An alternative argument for NIP. An alternative proof can be provided relying on the previous work of Mekler and set-theoretic absoluteness.

Recall that the *stability spectrum* of a complete theory T is defined as the function

$$f_T(\kappa) := \sup\{|S_1(M)| : M \models T, |M| = \kappa\}$$

for all infinite cardinals κ . Furthermore, for every infinite cardinal κ , let

$$\text{ded } \kappa := \sup\{\lambda : \text{exists a linear order of size } \leq \kappa \text{ with } \lambda\text{-many cuts}\}.$$

See [10] and [6, Section 6] for a general discussion of the function $\text{ded } \kappa$ and its connection to NIP. We will only need the following two facts.

Fact 3.3 (Shelah [19]). *Let T be a theory in a countable language.*

- (1) *If T is NIP, then $f_T(\kappa) \leq (\text{ded } \kappa)^{\aleph_0}$ for all infinite cardinals κ .*
- (2) *If T has IP, then $f_T(\kappa) = 2^\kappa$ for all infinite cardinals κ .*

It is possible that in a model of ZFC, $\text{ded } \kappa = 2^\kappa$ for all infinite cardinals κ (e.g. in a model of the Generalized Continuum Hypothesis). However, there are models of ZFC in which these two functions are different.

Fact 3.4 (Mitchell [17]). *For every cardinal κ of uncountable cofinality, there is a cardinal preserving Cohen extension such that $(\text{ded } \kappa)^{\aleph_0} < 2^\kappa$.*

In the original paper of Mekler [15] it is demonstrated that if C is a nice graph and $\text{Th}(C)$ is stable, then $\text{Th}(G(C))$ is stable. More precisely, the following result is established (in ZFC).

Fact 3.5. *Let C be a nice graph. Then $f_{\text{Th}(G(C))}(\kappa) \leq f_{\text{Th}(C)}(\kappa) + \kappa$ for all infinite cardinals κ .*

Finally, note that the property “ T is NIP” is a finitary formula-by-formula statement, hence set-theoretically absolute. Thus in order to prove Theorem 3.2, it is enough to prove it in *some* model of ZFC. Working in Mitchell’s model for some κ of uncountable cofinality (hence $(\text{ded } \kappa)^{\aleph_0} + \kappa < 2^\kappa$), it follows immediately by combining Facts 3.3 and 3.5.

4. PRESERVATION OF k -DEPENDENCE

We are following the notation from [8], and begin by recalling some of the facts there.

Definition 4.1. A formula $\varphi(x; y_0, \dots, y_{k-1})$ has the *k -independence property* (with respect to a theory T), if in some model there is a sequence $(a_{0,i}, \dots, a_{k-1,i})_{i \in \omega}$ such that for every $s \subseteq \omega^k$ there is b_s such that

$$\models \phi(b_s; a_{0,i_0}, \dots, a_{k-1,i_{k-1}}) \Leftrightarrow (i_0, \dots, i_{k-1}) \in s.$$

Here x, y_0, \dots, y_{k-1} are tuples of variables. Otherwise we say that $\varphi(x, y_0, \dots, y_{k-1})$ is *k -dependent*. A theory is *k -dependent* if it implies that every formula is k -dependent.

To characterize k -dependence in a formula-free way, we have to work with a more complicated form of indiscernibility.

Definition 4.2. Fix a language $L_{\text{opg}}^k = \{R(x_0, \dots, x_{k-1}), <, P_0(x), \dots, P_{k-1}(x)\}$. An *ordered k -partite hypergraph* is an L_{opg}^k -structure $\mathcal{A} = (A; <, R, P_0, \dots, P_{k-1})$ such that:

- (1) A is the (pairwise disjoint) union $P_0^A \sqcup \dots \sqcup P_{k-1}^A$,
- (2) R^A is a symmetric relation so that if $(a_0, \dots, a_{k-1}) \in R^A$ then $P_i \cap \{a_0, \dots, a_{k-1}\}$ is a singleton for every $i < k$,
- (3) $<^A$ is a linear ordering on A with $P_0(A) < \dots < P_{k-1}(A)$.

Fact 4.3. [8, Fact 4.4 + Remark 4.5] *Let \mathcal{K} be the class of all finite ordered k -partite hypergraphs. Then \mathcal{K} is a Fraïssé class, and its limit is called the ordered k -partite random hypergraph, which we will denote by $G_{k,p}$. An ordered k -partite hypergraph \mathcal{A} is a model of $\text{Th}(G_{k,p})$ if and only if:*

- $(P_i(A), <)$ is a model of DLO for each $i < k$,
- for every $j < k$, any finite disjoint sets $A_0, A_1 \subset \prod_{i < k, i \neq j} P_i(A)$ and $b_0 < b_1 \in P_j(A)$, there is $b_0 < b < b_1$ such that: $R(b, \bar{a})$ holds for every $\bar{a} \in A_0$ and $\neg R(b, \bar{a})$ holds for every $\bar{a} \in A_1$.

We denote by $O_{k,p}$ the reduct of $G_{k,p}$ to the language $L_{\text{op}}^k = \{<, P_0(x), \dots, P_{k-1}(x)\}$.

Definition 4.4. Let T be a theory in the language L , and let \mathbb{M} be a monster model of T .

- (1) Let I be a structure in the language L_0 . We say that $\bar{a} = (a_i)_{i \in I}$, with a_i a tuple in \mathbb{M} , is I -indiscernible over a set of parameters $C \subseteq \mathbb{M}$ if for all $n \in \omega$ and all i_0, \dots, i_n and j_0, \dots, j_n from I we have:

$$\text{qftp}_{L_0}(i_0, \dots, i_n) = \text{qftp}_{L_0}(j_0, \dots, j_n) \Rightarrow$$

$$\text{tp}_L(a_{i_0}, \dots, a_{i_n}/C) = \text{tp}_L(a_{j_0}, \dots, a_{j_n}/C).$$

- (2) For L_0 -structures I and J , we say that $(b_i)_{i \in J}$ is *based on* $(a_i)_{i \in I}$ over a set of parameters $C \subseteq \mathbb{M}$ if for any finite set Δ of $L(C)$ -formulas, and for any finite tuple (j_0, \dots, j_n) from J there is a tuple (i_0, \dots, i_n) from I such that:
 - $\text{qftp}_{L_0}(j_0, \dots, j_n) = \text{qftp}_{L_0}(i_0, \dots, i_n)$ and
 - $\text{tp}_\Delta(b_{j_0}, \dots, b_{j_n}) = \text{tp}_\Delta(a_{i_0}, \dots, a_{i_n})$.

The following fact gives a method for finding $G_{k,p}$ -indiscernibles using structural Ramsey theory.

Fact 4.5. [8, Corollary 4.8] *Let $C \subseteq \mathbb{M}$ be a small set of parameters.*

- (1) *For any $\bar{a} = (a_g)_{g \in O_{k,p}}$, there is some $(b_g)_{g \in O_{k,p}}$ which is $O_{k,p}$ -indiscernible over C and is based on \bar{a} over C .*
- (2) *For any $\bar{a} = (a_g)_{g \in G_{k,p}}$, there is some $(b_g)_{g \in G_{k,p}}$ which is $G_{k,p}$ -indiscernible over C and is based on \bar{a} over C .*

Fact 4.6. [8, Proposition 6.3] *Let T be a complete theory and let $\mathbb{M} \models T$ be a monster model. For any $k \in \mathbb{N}$, the following are equivalent:*

- (1) *T is k -dependent.*
- (2) *For any $(a_g)_{g \in G_{k,p}}$ and b with a_g, b finite tuples in \mathbb{M} , if $(a_g)_{g \in G_{k,p}}$ is $G_{k,p}$ -indiscernible over b and $O_{k,p}$ -indiscernible (over \emptyset), then it is $O_{k,p}$ -indiscernible over b .*

We are ready to prove the main theorem of the section.

Theorem 4.7. *For any $k \in \mathbb{N}$ and a nice graph C , $\text{Th}(C)$ is k -dependent if and only if $\text{Th}(G(C))$ is k -dependent.*

Proof. Let $G \models \text{Th}(G(C))$ be a saturated model, let X be a transversal, and let H be a set in $Z(G)$ which is linearly independent over G' such that $G = \langle X \rangle \times \langle H \rangle$. Moreover, fix κ to be \aleph_0^+ .

As in the NIP case, if $\text{Th}(G(C))$ is k -dependent, then $\text{Th}(C)$ is also k -dependent as C is interpretable in $G(C)$.

Now suppose that $\text{Th}(C)$ is k -dependent but $\text{Th}(G(C))$ has the k -independence property witnessed by the formula $\varphi(x; y_0, \dots, y_{k-1}) \in L_G$. By compactness we can find a sequence $(a_{0,\alpha}, \dots, a_{k-1,\alpha})_{\alpha \in \kappa}$ such that for any $s \subseteq \kappa^k$ there is some b_s such that

$$\models \phi(b_s; a_{0,\alpha_0}, \dots, a_{k-1,\alpha_{k-1}}) \Leftrightarrow (\alpha_0, \dots, \alpha_{k-1}) \in s.$$

By the choice of X and H , for each $i < k$ and $\alpha \in \kappa$, there is some term $t_{i,\alpha} \in L_G$ and some finite tuples $\bar{x}_{i,\alpha}$ from X and $\bar{h}_{i,\alpha}$ from H such that $a_{i,\alpha} = t_{i,\alpha}(\bar{x}_{i,\alpha}, \bar{h}_{i,\alpha})$. As $\kappa > |L_G| + \aleph_0$, passing to a subsequence of length κ for each $i < k$ we may assume that $t_{i,\alpha} = t_i$ and $\bar{x}_{i,\alpha} = \bar{x}_{i,\alpha}^\nu \widehat{\bar{x}_{i,\alpha}^p} \bar{x}_{i,\alpha}^t$ with $\bar{x}_{i,\alpha}^\nu, \bar{x}_{i,\alpha}^p, \bar{x}_{i,\alpha}^t$ listing all elements of the corresponding type in $\bar{x}_{i,\alpha}$ and $|\bar{x}_{i,\alpha}^\nu|, |\bar{x}_{i,\alpha}^p|, |\bar{x}_{i,\alpha}^t|$ constant for all $i < j$ and $\alpha \in \kappa$. Moreover, in the same way as in the NIP case, we add the handles of the elements in the tuple $\bar{x}_{i,\alpha}^p$ to the beginning of $\bar{x}_{i,\alpha}^\nu$. Taking $\psi(x; y'_0, \dots, y'_{k-1}) := \phi(x; t_0(y'_0), \dots, t_{k-1}(y'_{k-1}))$, we see that the sequence $(\bar{x}_{0,\alpha} \widehat{\bar{h}_{0,\alpha}}, \dots, \bar{x}_{k-1,\alpha} \widehat{\bar{h}_{k-1,\alpha}} : \alpha \in \kappa)$ is shattered by ψ , i. e. for each $A \subset \kappa^k$ there is some \bar{b} such that $G \models \psi(\bar{b}; \bar{x}_{0,i_0} \widehat{\bar{h}_{0,i_0}}, \dots, \bar{x}_{k-1,i_{k-1}} \widehat{\bar{h}_{k-1,i_{k-1}}})$ if and only if $(i_0, \dots, i_{k-1}) \in A$. We define an L_{op} -structure on κ by interpreting each of the $P_i, i < k$ as some countable disjoint subsets of κ , and choosing any ordering isomorphic to $(\mathbb{Q}, <)$ on each of the P_i 's. We pass to the corresponding subsequences of $(\bar{x}_{i,\alpha} \widehat{\bar{h}_{i,\alpha}} : \alpha \in \kappa)$, namely for each $i \in k$, we consider the sequence given by $(\bar{x}_{i,\alpha} \widehat{\bar{h}_{i,\alpha}} : \alpha \in P_i)$. Taking these k different sequences together we obtain the sequence $(\bar{x}_g \widehat{\bar{h}_g} : g \in O_{k,p})$ indexed by $O_{k,p}$. This sequence is shattered in the following sense: for each $A \subset P_0 \times \dots \times P_{k-1}$ there is some $\bar{b} \in G$ such that $G \models \psi(\bar{b}; \bar{x}_{g_0} \widehat{\bar{h}_{g_0}}, \dots, \bar{x}_{g_{k-1}} \widehat{\bar{h}_{g_{k-1}}})$ if and only if $(g_0, \dots, g_{k-1}) \in A$.

By Fact 4.5(1), let $(\bar{y}_g \widehat{\bar{m}_g} : g \in O_{k,p})$ be an $O_{k,p}$ -indiscernible in G based on $(\bar{x}_g \widehat{\bar{h}_g} : g \in O_{k,p})$. Observe that, using Proposition 2.18 as in the proof of Theorem 3.2, we still have:

- (1) $(\bar{y}_g \widehat{\bar{m}_g} : g \in O_{k,p})$ is shattered by ψ ,
- (2) the handle for each j th element in the tuple \bar{y}_g^p is the j th element of the tuple \bar{y}_g^ν ,
- (3) the set of all elements of G appearing in $(\bar{y}_g : g \in O_{k,p})$ is a partial transversal, hence can be extended to a transversal Y of G ,
- (4) the set of all elements of G appearing in $(\bar{m}_g : g \in O_{k,p})$ is still a set of elements in $Z(G)$ linearly independent over G' , hence can be extended to a linearly independent set M such that $G = \langle Y \rangle \times \langle M \rangle$.

We can expand $O_{k,p}$ to $G_{k,p}$ (see Fact 4.3). As $(\bar{y}_g \widehat{\bar{m}_g} : g \in O_{k,p})$ is shattered by ψ , we can find an element $b \in G$ such that $G \models \psi(b; \bar{y}_{g_0} \widehat{\bar{m}_{g_0}}, \dots, \bar{y}_{g_{k-1}} \widehat{\bar{m}_{g_{k-1}}}) \iff G_{k,p} \models R(g_0, \dots, g_{k-1})$, for all $g_i \in P_i$. We can write $b = s(\bar{z}, \bar{\ell})$ for some term $s \in L_G$ and some finite tuples $\bar{z} = \bar{z}^\nu \widehat{\bar{z}^p} \bar{z}^t$ in Y and $\bar{\ell}$ in M . As usual, extending \bar{z}^ν if necessary, we may assume that \bar{z} is closed under handles. Taking $\theta(x'; y'_0, \dots, y'_{k-1}) := \psi(s(x'); y'_0, \dots, y'_{k-1})$, we still have that

$$G \models \theta(\bar{z} \widehat{\bar{\ell}}; \bar{y}_{g_0} \widehat{\bar{m}_{g_0}}, \dots, \bar{y}_{g_{k-1}} \widehat{\bar{m}_{g_{k-1}}}) \iff G_{k,p} \models R(g_0, \dots, g_{k-1})$$

for all $g_i \in P_i$.

By Fact 4.5(2), we can find $(\bar{z}_g \bar{\ell}_g : g \in G_{k,p})$ which is $G_{k,p}$ -indiscernible over $\bar{z} \bar{\ell}$ and is based on $(\bar{y}_g \bar{m}_g : g \in G_{k,p})$ over $\bar{z} \bar{\ell}$. Then we have:

- (1) $G \models \theta(\bar{z} \bar{\ell}; \bar{z}_{g_0} \bar{\ell}_{g_0}, \dots, \bar{z}_{g_{k-1}} \bar{\ell}_{g_{k-1}}) \iff G_{k,p} \models R(g_0, \dots, g_{k-1})$, for all $g_i \in P_i$;
- (2) for $\bar{z}_g = \bar{z}_g^\nu \bar{z}_g^p \bar{z}_g^t$ we have that:
 - all of these tuples are of fixed length and list elements of the corresponding type,
 - the handle of the j -th element of \bar{z}_g^p is the j -th element of \bar{z}_g^ν ;
- (3) the set of all elements of G appearing in \bar{z} and $(\bar{z}_g : g \in G_{k,p})$ is a partial transversal, hence can be extended to some transversal Z of G ;
- (4) the set of all elements of G appearing in $\bar{\ell}$ and $(\bar{\ell}_g : g \in G_{k,p})$ is still a set of elements in $Z(G)$ linearly independent over G' , hence can be extended to a linearly independent set L such that $G = \langle Z \rangle \times \langle L \rangle$;
- (5) $(\bar{z}_g \bar{\ell}_g : g \in G_{k,p})$ is $O_{k,p}$ -indiscernible over \emptyset (follows since $(\bar{z}_g \bar{\ell}_g : g \in G_{k,p})$ is based on $(\bar{y}_g \bar{m}_g : g \in G_{k,p})$, which was $O_{k,p}$ -indiscernible, as in the proof of [8, Lemma 6.2]).

Consider now all of the elements in \bar{z}^ν and $(\bar{z}_g^\nu : g \in G_{k,p})$ as elements in $\Gamma(G)$, a saturated model of $\text{Th}(C)$, and note that as $\Gamma(G)$ is interpretable in G , we have that the sequence $(\bar{z}_g^\nu : g \in G_{k,p})$ is also $G_{k,p}$ -indiscernible over \bar{z}^ν and is $O_{k,p}$ -indiscernible over \emptyset , both in $\Gamma(G)$. As $\text{Th}(C)$ is k -dependent, it follows by Fact 4.6 that $(\bar{z}_g^\nu : g \in G_{k,p})$ is $O_{k,p}$ -indiscernible over \bar{z}^ν in $\Gamma(G)$. Hence for any finite tuples $g_0, \dots, g_n, q_0, \dots, q_n \in G_{k,p}$ such that $\text{tp}_{L_{\text{op}}^k}(\bar{g}) = \text{tp}_{L_{\text{op}}^k}(\bar{q})$, we have that $\text{tp}_\Gamma(\bar{z}_{g_0}^\nu, \dots, \bar{z}_{g_n}^\nu / \bar{z}^\nu)$ is equal to $\text{tp}_\Gamma(\bar{z}_{q_0}^\nu, \dots, \bar{z}_{q_n}^\nu / \bar{z}^\nu)$. Now, using $O_{k,p}$ -indiscernibility and that \bar{z} is finite, for each $i < k$ there is some finite $\lambda_i \subseteq P_i$ such that for all $g \neq q \in P_i$ with $g, q > \lambda_i$ (i.e. $g > h$ for every element $h \in \lambda_i$, and the same for q) we have

$$\bar{z}_g^p \cap \bar{z}^p = \bar{z}_q^p \cap \bar{z}^p, \bar{z}_g^t \cap \bar{z}^t = \bar{z}_q^t \cap \bar{z}^t \text{ (as tuples)}$$

and $\bar{z}_g \cap \bar{z}_q$ is constant. Thus, for any $g_0, \dots, g_{k-1}, q_0, \dots, q_{k-1}$ such that $g_i, q_i > \lambda_i$ and $g_i, q_i \in P_i$, we get that mapping $\bar{z}_{g_0}, \dots, \bar{z}_{g_{k-1}}, \bar{z}$ to $\bar{z}_{q_0}, \dots, \bar{z}_{q_{k-1}}, \bar{z}$ preserving the positions of the elements in the tuples defines a bijection $\sigma_{\bar{g}, \bar{q}}$ such that:

- (1) $\text{tp}_\Gamma(\bar{z}_{g_0}^\nu, \dots, \bar{z}_{g_{k-1}}^\nu, \bar{z}^\nu) = \text{tp}_\Gamma(\sigma_{\bar{g}, \bar{q}}(\bar{z}_{g_0}^\nu, \dots, \bar{z}_{g_{k-1}}^\nu, \bar{z}^\nu))$,
- (2) the map $\sigma_{\bar{g}, \bar{q}}$ fixes \bar{z} ,
- (3) the map $\sigma_{\bar{g}, \bar{q}}$ respects the 1^ν -, p - and 1^t -parts and the handles.

Next we consider all the elements in $\bar{\ell}$ and $(\bar{\ell}_g : g \in G_{k,p})$ as elements in $\langle L \rangle$, a saturated model of the stable theory $\text{Th}(\langle L \rangle)$. By quantifier elimination, we still have that $(\bar{\ell}_g : g \in G_{k,p})$ is both $O_{k,p}$ -indiscernible and $G_{k,p}$ -indiscernible over $\bar{\ell}$ in $\langle L \rangle$. As $\langle L \rangle$ is stable, so in particular k -dependent, by Fact 4.6, $(\bar{\ell}_g : g \in G_{k,p})$ is $O_{k,p}$ -indiscernible over $\bar{\ell}$.

Now let $\bar{g}, \bar{q} \in G_{k,p}$ be such that $g_i, q_i > \lambda_i$ and $g_i, q_i \in P_i$ for all $i < k$, and such that $G_{k,p} \models R(g_0, \dots, g_{k-1}) \wedge \neg R(q_0, \dots, q_{k-1})$ holds. Then by the choice of $\bar{z} \bar{\ell}$ we have that $G \models \theta(\bar{z} \bar{\ell}; \bar{z}_{g_0} \bar{\ell}_{g_0}, \dots, \bar{z}_{g_{k-1}} \bar{\ell}_{g_{k-1}}) \wedge \neg \theta(\bar{z} \bar{\ell}; \bar{z}_{q_0} \bar{\ell}_{q_0}, \dots, \bar{z}_{q_{k-1}} \bar{\ell}_{q_{k-1}})$. On the other hand, combining the last two paragraphs and using Lemma 2.14, we find an automorphism of G sending $(\bar{z}_{g_0} \bar{\ell}_{g_0}, \dots, \bar{z}_{g_{k-1}} \bar{\ell}_{g_{k-1}})$ to $(\bar{z}_{q_0} \bar{\ell}_{q_0}, \dots, \bar{z}_{q_{k-1}} \bar{\ell}_{q_{k-1}})$ and fixing $\bar{z} \bar{\ell}$ — a contradiction. \square

Corollary 4.8. *For every $k \geq 2$, there is a strictly k -dependent pure group G . Moreover, we can find such a G with a simple theory.*

Proof. For each $k \geq 2$, let A_k be the random k -hypergraph. It is well-known that $\text{Th}(A_k)$ is simple. Moreover, A_k is clearly not $(k-1)$ -dependent, as witnessed by the edge relation, and it is easy to verify that A_k is k -dependent (as it eliminates quantifiers and all relation symbols are at most k -ary, see e.g. [8, Proposition 6.5]).

Now A_k , as well as any other structure in a finite relational language, is bi-interpretable with some nice graph C_k by [12, Theorem 5.5.1 + Exercise 5.5.9], so C_k also has all of the aforementioned properties. Then Mekler's construction produces a group $G(C_k)$ with all of the desired properties, by Theorem 4.7 and preservation of simplicity from [2]. \square

This corollary gives first examples of strictly k -dependent groups, however many other questions about the existence of strictly k -dependent algebraic structures remain.

Problem 4.9. (1) Are there pseudofinite strictly k -dependent groups, for $k > 2$?
 (2) Are there \aleph_0 -categorical strictly k -dependent groups, for $k > 2$?

We note that the strictly 2-dependent group in Example 1 is both pseudofinite and \aleph_0 -categorical (see [14, Proposition 3.11] and the discussion around it). However, Mekler's construction does not preserve \aleph_0 -categoricity in general (this is mentioned in [2, Introduction]), e.g. because the proof in Remark 2.12 shows that if C is an infinite nice graph, then in $G(C)$ there are infinitely many pairwise inequivalent formulas $\phi_n(x)$ expressing that x is a product of at most n commutators.

Problem 4.10. Are there strictly k -dependent fields, for any $k \geq 2$? We conjecture that there aren't any with a simple theory. It is proved in [11] that any k -dependent PAC field is separably closed, and there are no known examples of fields with a simple theory which are not PAC.

5. PRESERVATION OF NTP_2

We recall the definition of NTP_2 (and refer to [4] for further details).

Definition 5.1. (1) A formula $\phi(x, y)$, with x, y tuples of variables, has TP_2 if there is an array $(a_{i,j} : i, j \in \omega)$ of tuples in $\mathbb{M} \models T$ and some $k \in \omega$ such that:
 (a) for all $i \in \omega$, the set $\{\phi(x, a_{i,j}) : j \in \omega\}$ is k -inconsistent.
 (b) for all $f : \omega \rightarrow \omega$, the set $\{\phi(x, a_{i,f(i)}) : i \in \omega\}$ is consistent.
 (2) A theory T is NTP_2 if no formula has TP_2 relatively to it.

Remark 5.2. [4, Lemma 3.2] If T is not NTP_2 , one can find a formula as in Definition 5.1(1) with $k = 2$.

We will use the following formula-free characterization of NTP_2 from [4, Section 1].

Fact 5.3. *Let T be a theory and $\mathbb{M} \models T$ a monster model. Let $\kappa := |T|^+$. The following are equivalent:*

(1) T is NTP_2 .

- (2) For any array $(a_{i,j} : i \in \kappa, j \in \omega)$ of finite tuples with mutually indiscernible rows (i.e. for each $i \in \kappa$, the sequence $\bar{a}_i := (a_{i,j} : j \in \omega)$ is indiscernible over $\{a_{i',j} : i' \in \kappa \setminus \{i\}, j \in \omega\}$) and a finite tuple b , there is some $\alpha \in \kappa$ satisfying the following: for any $i > \alpha$ there is some b' such that $\text{tp}(b/a_{i,0}) = \text{tp}(b'/a_{i,0})$ and \bar{a}_i is indiscernible over b' .

The following can be proved using finitary Ramsey theorem and compactness, see [4, Section 1] for the details.

Fact 5.4. Let $(a_{\alpha,i} : \alpha, i \in \kappa)$ be an array of tuples from $\mathbb{M} \models T$. Then there is an array $(b_{\alpha,i} : \alpha, i \in \kappa)$ with mutually indiscernible rows based on $(a_{\alpha,i} : \alpha, i \in \kappa)$, i.e. such that for every finite set of formulas Δ , any $\alpha_0, \dots, \alpha_{n-1} \in \kappa$ and any strictly increasing finite tuples $\bar{j}_0, \dots, \bar{j}_{n-1}$ from κ , there are some strictly increasing tuples $\bar{i}_0, \dots, \bar{i}_{n-1}$ from κ such that

$$\begin{aligned} & \models \Delta((b_{\alpha_0,i} : i \in \bar{j}_0), \dots, (b_{\alpha_{n-1},i} : i \in \bar{j}_{n-1})) \iff \\ & \models \Delta((a_{\alpha_0,i} : i \in \bar{i}_0), \dots, (a_{\alpha_{n-1},i} : i \in \bar{i}_{n-1})). \end{aligned}$$

Remark 5.5. If $\phi(x, y)$ and $(a_{\alpha,i} : \alpha, i \in \kappa)$ satisfy the condition in Definition 5.1(1) and $(b_{\alpha,i} : \alpha, i \in \kappa)$ is based on it, then $\phi(x, y)$ and $(b_{\alpha,i} : \alpha, i \in \kappa)$ also satisfy the condition in Definition 5.1(1).

Theorem 5.6. For any nice graph C , we have that $\text{Th}(G(C))$ is NTP_2 if and only if $\text{Th}(C)$ is NTP_2 .

Proof. As before, let $G \models \text{Th}(G(C))$ be a monster model, let X be a transversal, and let H be a set in $Z(G)$ which is linearly independent over G' such that $G = \langle X \rangle \times \langle H \rangle$. Moreover, fix κ to be \aleph_0^+ . If $\text{Th}(G(C))$ is NTP_2 then $\text{Th}(C)$ is also NTP_2 as C is interpretable in $G(C)$.

Now suppose that $\text{Th}(C)$ is NTP_2 , but $\text{Th}(G(C))$ has TP_2 . By compactness and Remark 5.2 we can find some formula $\phi(x, y)$ and an array $(\bar{a}_{i,j} : i, j \in \kappa)$ of tuples in G witnessing TP_2 as in Definition 5.1(1) with $k = 2$. Then for all $i, j \in \kappa$ we have $\bar{a}_{i,j} = t_{i,j}(\bar{x}_{i,j}, \bar{h}_{i,j})$ for some terms $t_{i,j} \in L_G$ and some finite tuples $\bar{x}_{i,j}$ from X and $\bar{h}_{i,j}$ from H .

As $\kappa > |L_G| + \aleph_0$, passing to a subsequence of each row, and then to a subsequence of the rows, we may assume that $t_{i,j} = t \in L_G$ and $\bar{x}_{i,j} = \bar{x}_{i,j}^\nu \widehat{\bar{x}_{i,j}^p} \bar{x}_{i,j}^t$ with $|\bar{x}_{i,j}^\nu|, |\bar{x}_{i,j}^p|, |\bar{x}_{i,j}^t|, |\bar{h}_{i,j}|$ constant for all $i, j \in \kappa$. Again as in the NIP case, we add the handles of the elements in the tuple $\bar{x}_{i,\alpha}^p$ to the beginning of $\bar{x}_{i,\alpha}^\nu$ for all $i, j \in \kappa$. Taking $\psi(x, y') := \phi(x, t(y'))$ with $|y'| = |\bar{x}_{i,j}^\nu \widehat{\bar{x}_{i,j}^p} \bar{x}_{i,j}^t|$ and $\bar{b}_{i,j} := \bar{x}_{i,j}^\nu \widehat{\bar{x}_{i,j}^p} \bar{x}_{i,j}^t$, we have that $\psi(x, y') \in L_G$ and the array $(\bar{b}_{i,j} : i, j \in \kappa)$ still satisfy the condition in Definition 5.1(1) with $k = 2$.

By Fact 5.4, let $(\bar{c}_{i,j} : i, j \in \kappa)$ with $\bar{c}_{i,j} = \bar{y}_{i,j} \widehat{\bar{m}_{i,j}}$ be an array with mutually indiscernible rows based on $(\bar{b}_{i,j} : i, j \in \kappa)$. Then, arguing as in the proofs of Theorems 3.2 and 4.7 using type-definability of the relevant properties from Proposition 2.18 and Remark 5.5, we have:

- (1) $\psi(x, y')$ and the array $(\bar{c}_{i,j} : i, j \in \kappa)$ satisfy the condition in Definition 5.1(1) with $k = 2$;
- (2) For $\bar{y}_{i,j} = \bar{y}_{i,j}^\nu \widehat{\bar{y}_{i,j}^p} \bar{y}_{i,j}^t$ we have that:
 - all of these tuples are of fixed length and list elements of the corresponding type,
 - the handle of the n -th element of $\bar{y}_{i,j}^p$ is the n -th element of $\bar{y}_{i,j}^\nu$;

- (3) the set of all elements of G appearing in $(\bar{y}_{i,j} : i, j \in \kappa)$ is a partial transversal of G and can be extended to a transversal Y of G ;
- (4) the set of all elements of G appearing in $(\bar{m}_{i,j} : i, j \in \kappa)$ is a set of elements in $Z(G)$ linearly independent over G' , hence can be extended to a set of generators M such that $G = \langle Y \rangle \times \langle M \rangle$.

Let now \bar{b} be a tuple in G such that $G \models \{\psi(\bar{b}, \bar{c}_{i,0}) : i \in \kappa\}$. We have that $\bar{b} = s(\bar{y}, \bar{m})$ for some term $s \in L_G$ and some finite tuples \bar{y} in Y and \bar{m} in M . Let $\bar{y} = \bar{y}^\nu \frown \bar{y}^p \frown \bar{y}^t$, each listing the elements of the corresponding type. In the same way as for each of the $\bar{y}_{i,j}$'s, we add the handles of the elements in the tuple \bar{y}^p to the beginning of \bar{y}^ν so that the handle of the n -th element of \bar{y}^p is the n -th element of \bar{y}^ν . Taking $\theta(x', y') := \psi(s(x'), y')$, we still have that $\bar{y} \frown \bar{m} \models \{\theta(x', \bar{c}_{i,0}) : i \in \kappa\}$ and the set of formulas $\{\theta(x', \bar{c}_{i,j}) : j \in \kappa\}$ is 2-inconsistent for each $i \in \kappa$. Moreover, after possibly throwing away finitely many rows, we may assume that the rows are mutually indiscernible over $\bar{y} \frown \bar{m} \cap \bigcup \{\bar{c}_{i,0} : i \in \kappa\}$ (if an element of $\bar{y} \frown \bar{m}$ appears in $\bar{c}_{i,0}$, then the rows of the array $(\bar{c}_{i',j} : i' \in \kappa, i' \neq i, j \in \kappa)$ are mutually indiscernible over it). This implies that if $z \in \bar{y} \cap \bar{y}_{i,0}$ for some i and z is the n -th element in the tuple $\bar{y}_{i,0}$, then it is the n -th element in any tuple $\bar{y}_{j,0}$ with $j \in \kappa$.

Consider all of the elements in \bar{y}^ν and $(\bar{y}_{i,j}^\nu : i, j \in \kappa)$ as elements in $\Gamma(G)$, a saturated model of $\text{Th}(C)$, and note that as $\Gamma(G)$ is interpretable in G we have that the array $(\bar{y}_{i,j}^\nu : i, j \in \kappa)$ has mutually indiscernible rows in $\Gamma(G)$. As $\text{Th}(\Gamma(G))$ is NTP₂, it follows by Fact 5.3 that there is some $\alpha \in \kappa$ such that for each $i > \alpha$ there is some tuple \bar{y}''^ν such that $\text{tp}_\Gamma(\bar{y}''^\nu / \bar{y}_{i,0}^\nu) = \text{tp}_\Gamma(\bar{y}^\nu / \bar{y}_{i,0}^\nu)$ and the sequence $(\bar{y}_{i,j}^\nu : j \in \kappa)$ is L_Γ -indiscernible over \bar{y}''^ν , i. e. $\text{tp}_\Gamma(\bar{y}''^\nu, \bar{y}_{i,0}^\nu) = \text{tp}_\Gamma(\bar{y}''^\nu, \bar{y}_{i,0}^\nu)$. Let σ_0 be the bijection which maps $\bar{y}^\nu \frown \bar{y}_{i,0}$ to $\bar{y}''^\nu \frown \bar{y}_{i,0}$. Now we want to extend this bijection σ_0 to a bijection σ defined on each element of $\bar{y} \frown \bar{y}_{i,0}$ in a type and handle preserving way. To do so, we have to choose an image for each element in $\bar{y}^p \frown \bar{y}^t$. Let z be the n -th element of \bar{y}^p and let u be the n -th element of \bar{y}^ν (i. e. the handle of z).

- If $z \notin \bar{y}_{i,0}^p$, then choose $\sigma(z)$ to be any element in Y^p which has handle $\sigma_0(u)$ and is not contained in $\bar{y}_{i,0}^p$ (as Y is a $|G|$ -cover and $\text{tp}_\Gamma(u) = \text{tp}_\Gamma(\sigma_0(u))$, using Fact 2.11(4) there must be infinitely many elements in Y^p for which $\sigma_0(u)$ is a handle, so we can choose one of them which is not contained in the finite tuple $\bar{y}_{i,0}^p$).
- If $z \in \bar{y}_{i,0}^p$, then we have that σ_0 fixes z as well as the handle u of z . In this case let $\sigma(z)$ be equal to z .

Finally, we define σ on each element of \bar{y}^t as the identity map. Let $\bar{y}' = \bar{y}''^\nu \frown \sigma(\bar{y}^p) \frown \bar{y}^t$. Then we have that for all $y \in \bar{y} \frown \bar{y}_{i,0}$:

- (1) σ is well defined;
- (2) σ fixes all elements in $\bar{y}_{i,0}$;
- (3) σ respects types and handles by construction;
- (4) $\text{tp}_\Gamma(\bar{y}''^\nu, \bar{y}_{i,0}^\nu) = \text{tp}_\Gamma(\sigma(\bar{y}''^\nu, \bar{y}_{i,0}^\nu))$ as $\sigma(y) = \sigma_0(y)$ for all $y \in \bar{y}''^\nu \frown \bar{y}_{i,0}^\nu$.

Now consider \bar{m} and $(\bar{m}_{i,j} : i, j \in \kappa)$ as tuples of elements in $\langle M \rangle$, which is a model of the stable theory $\text{Th}(\langle M \rangle)$. Moreover, as $(\bar{m}_{i,j} : i, j \in \kappa)$ has L_G -mutually indiscernible rows and $\text{Th}(\langle M \rangle)$ eliminates quantifiers, $(\bar{m}_{i,j} : i, j \in \kappa)$ has mutually indiscernible rows in the sense of $\text{Th}(\langle M \rangle)$. Hence, by Fact 5.3 again, there is some $\beta \in \kappa$ such that for each $i > \beta$ there is some $\tau \in \text{Aut}(\langle M \rangle)$ fixing $\bar{m}_{i,0}$ and such that $(\bar{m}_{i,j} : j \in \kappa)$ is indiscernible over $\bar{m}' := \tau(\bar{m})$.

Fix some $i > \max\{\alpha, \beta\}$ and let \bar{y}' and \bar{m}' be chosen as above. Then by Lemma 2.14 we find an automorphism of G which maps $\bar{y}\bar{m} \frown \bar{y}_{i,0}\bar{m}_{i,0}$ to $\bar{y}'(\bar{m}') \frown \bar{y}_{i,0}\bar{m}_{i,0}$, hence

$$\text{tp}_G(\bar{y}'\bar{m}'/\bar{y}_{i,0}\bar{m}_{i,0}) = \text{tp}_G(\bar{y}\bar{m}/\bar{y}_{i,0}\bar{m}_{i,0}).$$

In particular, $G \models \theta(\bar{y}'\bar{m}', \bar{y}_{i,0}\bar{m}_{i,0})$. We will show that

$$\text{tp}_G(\bar{y}_{i,0}\bar{m}_{i,0}/\bar{y}'\bar{m}') = \text{tp}_G(\bar{y}_{i,1}\bar{m}_{i,1}/\bar{y}'\bar{m}'),$$

which would then contradict the assumption that $\{\theta(x', \bar{y}_{i,j}\bar{m}_{i,j}) : j \in \kappa\}$ is 2-inconsistent.

We show that sending $\bar{y}'\bar{y}_{i,0}$ to $\bar{y}'\bar{y}_{i,1}$ is a well-defined bijection f_0 . The only thing to check is that if the n -th element z of $\bar{y}_{i,0}$ is an element of \bar{y}' , then the n -th element of $\bar{y}_{i,1}$ is equal to z . This is true as by construction we have that the sequence $(\bar{y}_{i,j} : j \in \kappa)$ is indiscernible over $\bar{y}' \cap \bar{y}_{i,0}$ (as $\bar{y}' \cap \bar{y}_{i,0} = \bar{y} \cap \bar{y}_{i,0}$ by construction, and $(\bar{y}_{i,j} : j \in \kappa)$ is indiscernible over $\bar{y} \cap \bar{y}_{i,0}$ by assumption). Moreover, we have the following properties for f_0 :

- (1) f_0 fixes all elements in \bar{y}' (by construction);
- (2) f_0 respects types and handles (by construction);
- (3) $\text{tp}_\Gamma(\bar{y}^\nu, \bar{y}_{i,0}^\nu) = \text{tp}_\Gamma(f_0(\bar{y}^\nu, \bar{y}_{i,0}^\nu))$ (since by the choice of \bar{y}^ν above, we have that $(\bar{y}_{i,j}^\nu : j \in \kappa)$ is indiscernible over \bar{y}^ν in $\Gamma(G)$).

Similarly, by the choice of \bar{m}' above, the sequence $(\bar{m}_{i,j} : j \in \kappa)$ is indiscernible over \bar{m}' , so $\text{tp}_{\langle M \rangle}(\bar{m}_{i,0}, \bar{m}') = \text{tp}_{\langle M \rangle}(\bar{m}_{i,1}, \bar{m}')$

Again, Lemma 2.14 gives us an automorphism of G sending $\bar{y}_{i,0}\bar{m}_{i,0}$ to $\bar{y}_{i,1}\bar{m}_{i,1}$ and fixing $\bar{y}'\bar{m}'$, as wanted. \square

Remark 5.7. A slight modification of the same proof shows that $\text{Th}(G(C))$ is *strong* if and only if $\text{Th}(C)$ is strong (see [4, Sections 2 and 3] for the relevant definitions).

However, since in the proof we have to throw away a finite, but unknown number of rows, this leaves the following problem.

Problem 5.8. Assume that $\text{Th}(C)$ is of finite burden. Does $\text{Th}(G(C))$ also have to be of finite burden?

Finally, it would be interesting to investigate what other properties from generalized stability are preserved by Mekler's construction. For example:

Conjecture 1. If $\text{Th}(C)$ is NSOP₁ then $\text{Th}(G(C))$ is also NSOP₁.

We expect that this could be verified using the methods of this paper and the criterion from [9] and [13].

REFERENCES

- [1] Hans Adler, *An introduction to theories without the independence property*, Archive for Mathematical Logic **5** (2008).
- [2] Andreas Baudisch, *Mekler's construction preserves CM-triviality*, Annals of Pure and Applied Logic **115** (2002), no. 1-3, 115–173.
- [3] Itai Ben Yaacov and Artem Chernikov, *An independence theorem for NTP₂ theories*, The Journal of Symbolic Logic **79** (2014), no. 1, 135–153.
- [4] Artem Chernikov, *Theories without the tree property of the second kind*, Annals of Pure and Applied Logic **165** (2014), no. 2, 695–723.

- [5] Artem Chernikov and Itay Kaplan, *Forking and dividing in NTP_2 theories*, The Journal of Symbolic Logic **77** (2012), no. 1, 1–20.
- [6] Artem Chernikov, Itay Kaplan, and Saharon Shelah, *On non-forking spectra*, Journal of the European Mathematical Society **18** (2016), no. 12, 2821–2848.
- [7] Artem Chernikov, Itay Kaplan, and Pierre Simon, *Groups and fields with NTP_2* , Proceedings of the American Mathematical Society **143** (2015), no. 1, 395–406.
- [8] Artem Chernikov, Daniel Palacin, and Kota Takeuchi, *On n -dependence*, Notre Dame Journal of Formal Logic, accepted (arXiv:1411.0120).
- [9] Artem Chernikov and Nicholas Ramsey, *On model-theoretic tree properties*, Journal of Mathematical Logic **16** (2016), no. 02, 1650009.
- [10] Artem Chernikov and Saharon Shelah, *On the number of Dedekind cuts and two-cardinal models of dependent theories*, Journal of the Institute of Mathematics of Jussieu **15** (2016), no. 4, 771–784.
- [11] Nadja Hempel, *On n -dependent groups and fields*, Mathematical Logic Quarterly **62** (2016), no. 3, 215–224.
- [12] Wilfrid Hodges, *Model theory*, Vol. 42, Cambridge University Press, 1993.
- [13] Itay Kaplan and Nicholas Ramsey, *On Kim-independence*, Journal of the European Mathematical Society, accepted (arXiv:1702.03894) (2017).
- [14] Dugald Macpherson and Charles Steinhorn, *One-dimensional asymptotic classes of finite structures*, Transactions of the American Mathematical Society **360** (2008), no. 1, 411–448.
- [15] Alan H Mekler, *Stability of nilpotent groups of class 2 and prime exponent*, Journal of Symbolic Logic (1981), 781–788.
- [16] Cédric Milliet, *Definable envelopes in groups having a simple theory*, J. Algebra **492** (2017), 298–323.
- [17] William Mitchell, *Aronszajn trees and the independence of the transfer property*, Annals of Mathematical Logic **5** (1972), no. 1, 21–46.
- [18] Saharon Shelah, *Simple unstable theories*, Annals of Mathematical Logic **19** (1980), no. 3, 177–203.
- [19] ———, *Classification theory: and the number of non-isomorphic models*, Vol. 92, Elsevier, 1990.
- [20] ———, *Strongly dependent theories*, Israel Journal of Mathematics **204** (2014), no. 1, 1–83.
- [21] ———, *Definable groups for dependent and 2-dependent theories*, Sarajevo J. Math. **13(25)** (2017), no. 1, 3–25.
- [22] Frank O. Wagner, *Simple theories*, Mathematics and its Applications, vol. 503, Kluwer Academic Publishers, Dordrecht, 2000.