AN EFFICIENT ENSEMBLE ALGORITHM FOR NUMERICAL APPROXIMATION OF STOCHASTIC STOKES-DARCY EQUATIONS

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Abstract. We propose and analyze an efficient ensemble algorithm for fast computation of multiple realizations of the stochastic Stokes-Darcy model with a random hydraulic conductivity tensor. The algorithm results in a common coefficient matrix for all realizations at each time step making solving the linear systems much less expensive while maintaining comparable accuracy to traditional methods that compute each realization separately. Moreover, it decouples the Stokes-Darcy system into two smaller sub-physics problems, which reduces the size of the linear systems and allows parallel computation of the two sub-physics problems. We prove the ensemble method is long time stable and first-order in time convergent under a time-step condition and two parameter conditions. Numerical examples are presented to support the theoretical results and illustrate the application of the algorithm.

1. Introduction. Many engineering and geological applications require effective simulations of the coupling of groundwater flows (in porous media) and surface flows. Accurate simulations are usually not feasible due to the fact it is physically impossible to know the exact parameter values, e.g., the hydraulic conductivity tensor, at every point in the domain as the realistic domains are of large scale and natural randomness occur at small scales. Consequently, these uncertainties must be taken into account to obtain meaningful results. The usual way is to model the parameter of interest as a stochastic function that is determined by an underlying random field with an prescribed (usually experimentally determined) covariance structure, and then recast the original deterministic system as a stochastic system. As a result, numerical approximations that involve repeated sampling and simulations pose great challenges on the computer resources and capability. A recently developed ensemble algorithm was devoted to address this issue. Jiang and Layton [26] studied an efficient ensemble algorithm for solving multiple realizations of evolutionary Navier-Stokes equations. The algorithm results in a common coefficient matrix for all realizations corresponding to different initial conditions or body forces, and thus efficient direct or iterative solvers can be used to reduce both required storage and computational time. This algorithm has been extensively tested and shown to be able to significantly reduce the computational cost, [15, 27, 28, 31, 37]. Herein we follow the same idea and develop an efficient ensemble algorithm for simulating the coupling of groundwater flows and surface flows.

In this report, we consider a linear Stokes-Darcy model for the coupling of the surface and porous media flows, where the Stokes equations describe the incompressible surface fuild flow and the Darcy model describes the groundwater flow in porous media. For derivation and more detailed discussions of the Stokes-Darcy model, see [3], [9], [10], [38], [35], [12], [24]. Let D_f denote the surface fluid flow region and D_p the porous media flow region, where $D_f, D_p \subset R^d (d = 2, 3)$ are both open, bounded domains. These two domains lie across an interface, I, from each other, and $D_f \cap D_p = \emptyset, \bar{D}_f \cap \bar{D}_p = I$, see Figure 1.1.

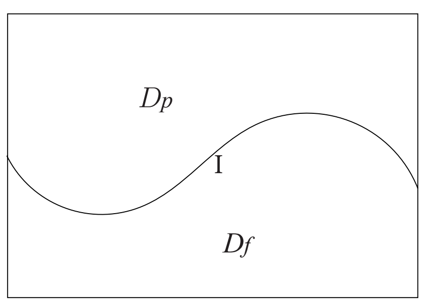

Fig. 1.1: A sketch of the porous median domain D_p, fluid domain D_f, and the interface I.

The Stokes-Darcy model is: Find fluid velocity $u(x,t)$, fluid pressure $p(x,t)$, and hydraulic head $\phi(x,t)$

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that satisfy

$$
\begin{aligned}
u_t - \nu\Delta u + \nabla p = f_f(x,t), \nabla \cdot u = 0, \text{ in } D_f,\\
S_0\phi_t - \nabla\cdot(\mathcal{K}(x)\nabla\phi) = f_p(x,t), \text{ in } D_p,\\
\phi(x,0) = \phi_0(x), \text{ in } D_p \text{ and } u(x,0) = u_0(x), \text{ in } D_f,\\
\phi(x,t) = 0, \text{ in } \partial D_p\backslash I \text{ and } u(x,t) = 0, \text{ in } \partial D_f\backslash I.
\end{aligned} \tag{1.1}
$$

Let $\widehat{n}_{f/p}$ denote the outward unit normal vector on I associated with $D_{f/p}$, where $\widehat{n}_f = -\widehat{n}_p$. The coupling conditions across I are conservation of mass, balance of forces and the Beavers-Joseph-Saffman condition on the tangential velocity:

$$
\begin{aligned}
u\cdot\widehat{n}_f - \mathcal{K}\nabla\phi\cdot\widehat{n}_p = 0 \text{ and } p - \nu\ \widehat{n}_f\cdot\nabla u\cdot\widehat{n}_f = g\phi \text{ on } I,\\
-\nu\ \nabla u\cdot\widehat{n}_f = \tfrac{\alpha_{\text{BJS}}}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}\widehat{\tau}_i}} u\cdot\widehat{\tau}_i \text{ on } I, \text{ for any tangential vector } \widehat{\tau}_i \text{ on } I.
\end{aligned}
$$

see [4], [41], [25]. Here, g, $\mathcal{K}$, ν and S_0 are the gravitational acceleration constant, hydraulic conductivity tensor, kinematic viscosity and specific mass storativity coefficient, respectively, which are all positive. $\mathcal{K}$ is assumed to be symmetric positive definite (SPD).

In simulations of porous media flows, the major difficulty is the determination of the hydraulic conductivity tensor $\mathcal{K}$. In the simplest case of isotropic homogeneous media, the hydraulic conductivity tensor is diagonal and constant. But in most geophysical and engineering applications, the media are usually randomly heterogeneous, and each component $k_{ij}(x,w)$ of the hydraulic conductivity tensor is a random function that depends on spatial coordinates. Then the problem becomes solving a stochastic PDE system instead of a deterministic PDE system and the goal of mathematical analysis and computer simulations is the prediction of statistical moments of the solution, such as the mean and variance. The most popular approach in solving a PDE system with random inputs is the Monte Carlo method, which is easy to implement and allows the use of existing deterministic codes. The main disadvantage of the Monte Carlo method is its very slow convergence rate $1/\sqrt{J}$, which inevitably requires computation of a large number of realizations to obtain useful statistical information from the solutions. Other ensemble-based methods have been devised to produce faster convergence rates and reduce numerical efforts including multilevel Monte Carlo method [2], quasi-Monte Carlo sequences [32], Latin hypercube sampling [23], centroidal Voronoi tessellations [40], and more recently developed stochastic collocation methods [1, 45] and non-intrusive polynomial chaos methods [22, 39]. All these methods are non-intrusive in the sense that the stochastic and spatial degrees of freedom are decoupled and deterministic codes can be used directly without any modification. However, repetitive runs of an existing deterministic solver can be prohibitively costly when the governing equations take complicated forms.

A recent ensemble algorithm aiming at significantly reducing the computational cost of the ensemble simulations and consequently improving the performance of the aforementioned ensemble-based stochastic approaches was proposed in [26]. This ensemble algorithm solves all realizations simultaneously instead of solving them individually. It utilizes the mean of the solutions at each time step to form a coefficient matrix that is independent of the realization index j, that is, all realizations have the same coefficient matrix at each time step. Then the problem reduces to solving one linear system with multiple right hand sides, for which the computational cost can be significantly reduced. This ensemble algorithm has been extensively studied and tested for ensemble simulations to account for uncertainties in initial conditions and forcing terms [26, 29, 27, 30, 31, 37, 44]. Some recent work include incorporating model reduction techniques to further reduce computational cost [15, 16], and devising ensemble algorithms to account for various model parameters of Navier-Stokes equations [17, 18], Boussinesq equations [14] and a simple elliptic equation [36]. In this paper, we will further develop the ensemble algorithm for computing an ensemble of the Stokes-Darcy systems to account for uncertainties in initial conditions, forcing terms and the hydraulic conductivity tensor. Herein we consider computing an ensemble of J Stokes-Darcy systems corresponding to J different parameter sets $(u_j^0, \phi_j^0, f_{fj}, f_{pj}, \mathcal{K}_j)$, $j = 1, ..., J$,

$$
\begin{aligned}
u_{j,t} - \nu_j\Delta u_j + \nabla p_j = f_{f,j}(x,t), \ \nabla\cdot u_j = 0, \text{ in } D_f,\\
S_0\phi_{j,t} - \nabla\cdot(\mathcal{K}_j(x)\nabla\phi_j) = f_{p,j}(x,t), \text{ in } D_p,
\end{aligned} \tag{1.2}
$$

$$\phi_j(x,t)=0, \text{ in } \partial D_p\backslash I \text{ and } u_j(x,t)=0, \text{ in } \partial D_f\backslash I.$$

Here we assume there are uncertainties in initial conditions $u^0(x), \phi^0(x)$, forcing terms $f_f(x,t), f_p(x,t)$ and the hydraulic conductivity tensor $\mathcal{K}(x)$, and $(u_j^0, \phi_j^0, f_{fj}, f_{pj}, \mathcal{K}_j)$ is one of the samples drawn from the respective probabilistic distributions. J is the number of total samples.

The Stokes-Darcy equations have intrinsic difficulties with the coupling of equations. There is a vast literature on numerical methods for solving evolutionary Stokes-Darcy problem, including monolithic methods that use implicit time discretization followed by domain decomposition iterations, [5, 6, 9, 11], and partitioned timestepping methods that decouple the original problem into two subregion problems reducing the size of the linear systems to be solved and allowing parallel computation of the two subregion problems, [8, 33, 35, 38, 42, 43]. In this paper we study a partitioned, ensemble timestepping method to compute the Stokes-Darcy models with different parameter sets. Specifically, the algorithm reads

Algorithm 1.1. *Find* $(u_j^{n+1}, p_j^{n+1}, \phi_j^{n+1}) \in X_f \times Q_f \times X_p$ *satisfying* $\forall\ (v,q,\psi) \in X_f \times Q_f \times X_p$,

$$\begin{aligned}
&\left(\frac{u_j^{n+1}-u_j^n}{\Delta t}, v\right)_f + \nu(\nabla u_j^{n+1}, \nabla v)_f + \sum_i \int_I \bar{\eta}_i (u_j^{n+1}\cdot \widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds \\
&+\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)(u_j^n\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds - \left(p_j^{n+1}, \nabla\cdot v\right)_f + c_I(v,\phi_j^n) = (f_{f,j}^{n+1}, v)_f, \\
&(q, \nabla\cdot u_j^{n+1})_f = 0, \\
&gS_0\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}, \psi\right)_p + g(\bar{\mathcal{K}}\nabla\phi_j^{n+1}, \nabla\psi)_p + g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_j^n, \nabla\psi)_p \\
&\qquad\qquad - c_I(u_j^n, \psi) = g(f_{p,j}^{n+1}, \psi)_p.
\end{aligned} \tag{1.3}$$

where

$$\bar{\mathcal{K}} = \frac{1}{J}\sum_{j=1}^{J}\mathcal{K}_j, \quad \eta_{i,j} = \frac{\alpha_{\text{BJS}}}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}_j\widehat{\tau}_i}} \quad \textit{and} \quad \bar{\eta}_i = \frac{1}{J}\sum_{j=1}^{J}\eta_{i,j}.$$

This algorithm decouples the original problem into two sub-physics problems, which can be run in parallel. Moreover, at each time step, all realizations share the same coefficient matrix, which allows the use of efficient block solvers, e.g, block CG [13], block GMRES [21], or direct solvers such as LU factorization, to reduce both storage and computation time.

This paper is organized as follows. Section 2 gives mathematical preliminaries and defines notation. In Section 3 we prove the long time stability of the proposed method under a time-step condition and two parameter conditions. In Section 4, we discuss an alternative approach for the case that $\mathcal{K}$ has simpler structures. We prove this method is long time stable under a similar time-step condition, *without* any parameter conditions. In section 5, we study the convergence and error estimates for the proposed method and prove that it is first order convergent in time. Section 6 gives a brief introduction on how to combine our ensemble algorithm with the Monte Carlo method to approximate the stochastic Stoke-Darcy system and derives an error estimate on the expectation of the L^2 norm of the error for approximating $E[u(x,t_n,\omega)]$. Section 7 numerically tests the proposed ensemble method and illustrates our theoretical results. Final conclusions and future directions are discussed in Section 8.

2. Notation and Preliminaries.

We denote the $L^2(I)$ norm by $\|\cdot\|_I$ and the $L^2(D_{f/p})$ norms by $\|\cdot\|_{f/p}$; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. Further, we denote the $H^k(D_{f/p})$ norm by $\|\cdot\|_{H^k(D_{f/p})}$. The following inequalities will be used in the proofs, [35].

$$\|\phi\|_I \le C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}, \tag{2.1}$$

$$\|u\|_I \le C(D_f)\sqrt{\|u\|_f\|\nabla u\|_f}, \tag{2.2}$$

where $C(D_{f/p}) = \mathcal{O}(\sqrt{L_{f/p}}), L_{f/p} = diameter(D_{f/p})$.

Define the function spaces:

$$\begin{aligned}
\text{Velocity: } & X_f := \{v \in \left(H^1(D_f)\right)^d : v = 0 \text{ on } \partial D_f \backslash I\}, \\
\text{Pressure: } & Q_f := \left\{q \in L^2(D_f) : \int_\Omega q\, dx = 0\right\}, \\
\text{Hydraulic Head: } & X_p := \{\psi \in H^1(D_p) : \psi = 0 \text{ on } \partial D_p \backslash I\}.
\end{aligned}$$

To discretize the Stokes-Darcy problem in space by the finite element method, we choose conforming velocity, pressure, hydraulic head finite element spaces based on an edge to edge triangulation ($d = 2$) or tetrahedralization ($d = 3$) of the domain $D_{f/p}$ with maximum element diameter h:

$$X_f^h \subset X_f\ ,\ Q_f^h \subset Q_f,\ X_p^h \subset X_p.$$

The continuity across the interface I between the finite element meshes in the two subdomains is not assumed. The finite element spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup $/LBB^h$ condition for stability of the discrete pressure, see [19] for more on this condition. Taylor-Hood elements, [19], are one such choice used in the numerical tests in Section 7.

We will also consider the discretely divergence-free space:

$$V_f^h := \{v_h \in X_f^h : (q_h, \nabla \cdot v_h)_f = 0,\ \forall q_h \in Q_f^h\}.$$

Define

$$c_I(u, \phi) = g \int_I \phi u \cdot \widehat{n}_f\, ds.$$

Let $C_{P,f}$ and $C_{P,p}$ be the Poincaré constants of the indicated domains and $\bar{k}_{min}(x)$ be the minimum eigenvalue of the mean hydraulic conductivity tensor $\bar{\mathcal{K}}(x)$. Define $\bar{k}_{min} = \min_{x\in\Omega_p} \bar{k}_{min}(x)$ and two parameter-dependent constants

$$C_1 = \frac{C_{P,f}^2[gC(D_f)C(D_p)]^4}{4\nu^2}, \qquad C_2 = \frac{C_{P,p}^2 g^2[C(D_f)C(D_p)]^4}{4\bar{k}_{min}^2}.$$

Then we have the following estimates for the coupling term $c_I(u, \phi)$.

Lemma 2.1. *For any $(u, \phi) \in X_f \times X_p$ and any $\epsilon_1, \epsilon_2, \alpha_1, \beta_1 > 0$,*

$$|c_I(u, \phi)| \le \frac{1}{4\epsilon_1}\|\phi\|_p^2 + \frac{\epsilon_1}{\alpha_1^2} C_1 \|\nabla\phi\|_p^2 + \alpha_1 \nu \|\nabla u\|_f^2, \tag{2.3}$$

$$|c_I(u, \phi)| \le \frac{1}{4\epsilon_2}\|u\|_f^2 + \frac{\epsilon_2}{\beta_1^2} C_2 \|\nabla u\|_f^2 + \beta_1 g\bar{k}_{min} \|\nabla\phi\|_p^2. \tag{2.4}$$

Proof. The proof is similar to that in [35]. Using inequalities (2.1) and (2.2), as well as the inequality $abc \le \frac{1}{4}a^4 + \frac{1}{4}b^4 + c^2$, we have

$$\begin{aligned}
c_I(u, \phi) &= g \int_I \phi u \cdot \widehat{n}_f\, ds \le gC(D_f)C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}\sqrt{\|u\|_f\|\nabla u\|_f} \\
&\le \left(\frac{1}{\epsilon_1^{1/4}}\|\phi\|_p^{1/2}\right)\left(gC(D_f)C(D_p)\epsilon_1^{1/4}\frac{1}{\alpha_1^{1/2}\nu^{1/2}}C_{P,f}^{1/2}\|\nabla\phi\|_p^{1/2}\right)\left(\alpha_1^{1/2}\nu^{1/2}\|\nabla u\|_f\right) \\
&\le \frac{1}{4\epsilon_1}\|\phi\|_p^2 + \frac{\epsilon_1}{\alpha_1^2}C_1\|\nabla\phi\|_p^2 + \alpha_1\nu\|\nabla u\|_f^2,
\end{aligned}$$

and

$$c_I(u, \phi) = g \int_I \phi u \cdot \widehat{n}_f\, ds \le gC(D_f)C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}\sqrt{\|u\|_f\|\nabla u\|_f}$$

$$\le \left(\frac{1}{\epsilon_2^{1/4}}\|u\|_f^{1/2}\right)\left(gC(D_f)C(D_p)\epsilon_2^{1/4}\frac{1}{\beta_1^{1/2}(g\bar{k}_{min})^{1/2}}C_{P,p}^{1/2}\|\nabla u\|_f^{1/2}\right)\left(\beta_1^{1/2}(g\bar{k}_{min})^{1/2}\|\nabla\phi\|_p\right)$$
$$\le \frac{1}{4\epsilon_2}\|u\|_f^2+\frac{\epsilon_2}{\beta_1^2}C_2\|\nabla u\|_f^2+\beta_1 g\bar{k}_{min}\|\nabla\phi\|_p^2.$$

□

The fully discrete approximation of (1.2) is:
ALGORITHM 2.2. *Find* $(u_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ *satisfying* $\forall\ (v_h, q_h, \psi_h) \in X_f^h \times Q_f^h \times X_p^h$,

$$\begin{aligned}
&\left(\frac{u_{j,h}^{n+1}-u_{j,h}^n}{\Delta t}, v_h\right)_f + \nu(\nabla u_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds \\
&+\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)(u_{j,h}^n\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds - \left(p_{j,h}^{n+1}, \nabla\cdot v_h\right)_f + c_I(v_h, \phi_{j,h}^n) = (f_{f,j}^{n+1}, v_h)_f, \\
&(q_h, \nabla\cdot u_{j,h}^{n+1})_f = 0, \qquad\qquad (2.5)\\
&gS_0\left(\frac{\phi_{j,h}^{n+1}-\phi_{j,h}^n}{\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1}, \nabla\psi_h)_p + g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_{j,h}^n, \nabla\psi_h)_p \\
&\qquad - c_I(u_{j,h}^n, \psi_h) = g(f_{p,j}^{n+1}, \psi_h)_p.
\end{aligned}$$

Moving all the known quantities to the right hand side, the algorithm is as follows.

$$\begin{aligned}
&\left(\frac{u_{j,h}^{n+1}-u_{j,h}^n}{\Delta t}, v_h\right)_f + \nu(\nabla u_{j,h}^{n+1}, \nabla v)_f + \sum_i \int_I \bar{\eta}_i (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds \\
&-\left(p_{j,h}^{n+1}, \nabla\cdot v_h\right)_f = (f_{f,j}^{n+1}, v_h)_f - \sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)(u_{j,h}^n\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds - c_I(v_h, \phi_{j,h}^n), \\
&(q_h, \nabla\cdot u_{j,h}^{n+1})_f = 0, \qquad\qquad (2.6)\\
&gS_0\left(\frac{\phi_{j,h}^{n+1}-\phi_{j,h}^n}{\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1}, \nabla\psi)_p \\
&\qquad = g(f_{p,j}^{n+1}, \psi_h)_p - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_{j,h}^n, \nabla\psi)_p + c_I(u_h^n, \psi_h).
\end{aligned}$$

Each time step requires the solution of two sets of linear systems (for (u_j, p_j) and ϕ_j respectively), where each set has the same, shared coefficient matrix:

$$A\left[\begin{array}{c|c|c} u_1 & \cdots & u_J \\ p_1 & \cdots & p_J \end{array}\right] = \left[RHS_1\,|\cdots|\,RHS_J\right], \tag{2.7}$$

$$B\left[\begin{array}{c|c|c} \phi_1 & \cdots & \phi_J \end{array}\right] = \left[RHS_1^*\,|\cdots|\,RHS_J^*\right]. \tag{2.8}$$

This structure of the linear systems allows the use of efficient iterative solvers or direct solvers such as LU factorization for fast calculation. The two sets of linear systems (2.7) and (2.8) can also be run in parallel to reduce the computation time.

3. Stability Analysis. Let $|\cdot|_2$ denote the 2-norm of either vectors or matrices. Let $k_{j,min}(x)$, $\bar{k}_{min}(x)$ be the minimum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, $\bar{\mathcal{K}}(x)$ respectively, and $\rho_j'(x)$ be the spectral radius of the fluctuation of hydraulic conductivity tensor $\mathcal{K}_j(x)-\bar{\mathcal{K}}(x)$. Since both $\mathcal{K}_j(x)$ and $\bar{\mathcal{K}}(x)$ are symmetric, $|\mathcal{K}_j(x)-\bar{\mathcal{K}}(x)|_2 = \rho_j'(x)$. We then define the following quantities that will be used in our proof.

$$\eta_{i,j}'^{max} = \max_{x\in I}|\eta_{i,j}(x)-\bar{\eta}_i(x)|, \qquad \eta_i'^{max} = \max_j \eta_{i,j}'^{max}, \qquad \bar{\eta}_i^{min} = \min_{x\in I}\bar{\eta}_i(x),$$

$$k_{j,min} = \min_{x\in D_p} k_{j,min}(x), \qquad k_{min} = \min_j k_{j,min}, \qquad \bar{k}_{min} = \min_{x\in D_p} \bar{k}_{min}(x),$$
$$\rho'_{j,max} = \max_{x\in D_p} \rho'_{j,max}(x), \qquad \rho'_{max} = \max_j \rho'_{j,max}.$$

We prove long time stability of Algorithm 2.2 under a time-step condition and two parameter conditions

$$\Delta t \le \min\left\{ \frac{2(1-\alpha_1-\alpha_2)\beta_1^2}{[C(D_f)C(D_p)]^4 C_{P,p}^2}\frac{\nu\bar{k}_{min}^2}{g^2}, \frac{2(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2}{[C(D_f)C(D_p)]^4 C_{P,f}^2}\frac{\nu^2\bar{k}_{min}S_0}{g^2} \right\}, \tag{3.1}$$
$$\eta_i'^{max} \le \bar{\eta}_i^{min}, \tag{3.2}$$
$$\rho'_{max} < \bar{k}_{min}. \tag{3.3}$$

Remark 3.1. *The two parameter conditions* (3.2) *and* (3.3) *relate to the probability distribution of the random hydraulic conductivity tensor. They require the magnitude of the fluctuations be smaller than the magnitude of the mean. In many applications, this can be easily achieved by dividing the ensemble of samples into smaller ensembles.*

Theorem 3.2 (Long time stability of Algorithm 2.2). *If the two parameter conditions* (3.2), (3.3) *hold, and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $(0,1)$ such that the time-step condition* (3.1) *also holds, then the Algorithm 2.2 is long time stable: for any $N > 0$,*

$$\begin{aligned}
&\frac{1}{2}\|u_{j,h}^N\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^N\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla u_{j,h}^N\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_I (u_{j,h}^N\cdot\widehat{\tau}_i)^2\, ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_{j,h}^N\|_p^2 \\
&\le \frac{1}{2}\|u_{j,h}^0\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^0\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla u_{j,h}^0\|_f^2 + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_I (u_{j,h}^0\cdot\widehat{\tau}_i)^2\, ds \\
&+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi_{j,h}^0\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned} \tag{3.4}$$

Proof. Setting $v_h = u_{j,h}^{n+1}$, $\psi_h = \phi_{j,h}^{n+1}$ in Algorithm 2.2 and adding all three equations yields

$$\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^n\|_f^2 + \frac{1}{2\Delta t}\|u_{j,h}^{n+1}-u_{j,h}^n\|_f^2 + \nu\|\nabla u_{j,h}^{n+1}\|_f^2 \\
&+\sum_i\int_I \bar{\eta}_i(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^n\|_p^2 \\
&+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^n\|_p^2 + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1},\nabla\phi_{j,h}^{n+1})_p + c_I(u_{j,h}^{n+1},\phi_{j,h}^n) - c_I(u_{j,h}^n,\phi_{j,h}^{n+1}) \\
&= (f_{f,j}^{n+1},u_{j,h}^{n+1})_f + g(f_{p,j}^{n+1},\phi_{j,h}^{n+1})_p - \sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)(u_{j,h}^n\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds \\
&\quad - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_{j,h}^n,\nabla\phi_{j,h}^{n+1})_p.
\end{aligned} \tag{3.5}$$

Note that

$$\begin{aligned}
&c_I(u_{j,h}^{n+1},\phi_{j,h}^n) - c_I(u_{j,h}^n,\phi_{j,h}^{n+1}) \\
&\quad = \left[c_I(u_{j,h}^{n+1},\phi_{j,h}^n) - c_I(u_{j,h}^{n+1},\phi_{j,h}^{n+1})\right] - \left[c_I(u_{j,h}^n,\phi_{j,h}^{n+1}) - c_I(u_{j,h}^{n+1},\phi_{j,h}^{n+1})\right] \\
&\quad = -c_I(u_{j,h}^{n+1},\phi_{j,h}^{n+1}-\phi_{j,h}^n) + c_I(u_{j,h}^{n+1}-u_{j,h}^n,\phi_{j,h}^{n+1}).
\end{aligned} \tag{3.6}$$

Applying estimates (2.3) and (2.4) with $\epsilon_1 = \frac{\Delta t}{2gS_0}$, $\epsilon_2 = \frac{\Delta t}{2}$, and if the time-step condition (3.1) holds, we have

$$c_I(u_{j,h}^{n+1}-u_{j,h}^n,\phi_{j,h}^{n+1}) - c_I(u_{j,h}^{n+1},\phi_{j,h}^{n+1}-\phi_{j,h}^n) \tag{3.7}$$

$$
\begin{aligned}
&\geq -\frac{1}{2\Delta t}\|u_{j,h}^{n+1}-u_{j,h}^{n}\|_f^2-\frac{\Delta t}{2}\frac{C_2}{\beta_1^2}\|\nabla(u_{j,h}^{n+1}-u_{j,h}^{n})\|_f^2-\beta_1 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2\\
&\quad-\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^{n}\|_p^2-\frac{\Delta t}{2gS_0}\frac{C_1}{\alpha_1^2}\|\nabla(\phi_{j,h}^{n+1}-\phi_{j,h}^{n})\|_p^2-\alpha_1\nu\|\nabla u_{j,h}^{n+1}\|_f^2\\
&\geq -\frac{1}{2\Delta t}\|u_{j,h}^{n+1}-u_{j,h}^{n}\|_f^2-\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^{n+1}\|_f^2+\|\nabla u_{j,h}^{n}\|_f^2\right)-\beta_1 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2\\
&\quad-\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^{n}\|_p^2-\frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2+\|\nabla\phi_{j,h}^{n}\|_p^2\right)-\alpha_1\nu\|\nabla u_{j,h}^{n+1}\|_f^2.
\end{aligned}
$$

Applying Cauchy-Schwarz and Young's inequalities to the source terms, for any $\alpha_2>0,\beta_2>0$ we have

$$
\begin{aligned}
&(f_{f,j}^{n+1},u_{j,h}^{n+1})_f+g(f_{p,j}^{n+1},\phi_{j,h}^{n+1})_p\\
&\quad\leq\|f_{f,j}^{n+1}\|_f\|u_{j,h}^{n+1}\|_f+g\|f_{p,j}^{n+1}\|_p\|\phi_{j,h}^{n+1}\|_p\\
&\quad\leq C_{P,f}\|f_{f,j}^{n+1}\|_f\|\nabla u_{j,h}^{n+1}\|_f+gC_{P,p}\|f_{p,j}^{n+1}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p\\
&\quad\leq\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2+\alpha_2\nu\|\nabla u_{j,h}^{n+1}\|_f^2+\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2+\beta_2 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2.
\end{aligned}
\tag{3.8}
$$

The other two terms on the right hand side of (3.5) can be bounded as follows.

$$
\begin{aligned}
&-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\;ds\\
&\qquad\leq\sum_i\int_I|\eta_{i,j}-\bar{\eta}_i|\left|(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\right|ds\\
&\qquad\leq\sum_i\eta_{i,j}'^{max}\int_I\left|(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\right|\;ds,\\
&\qquad\leq\sum_i\left[\frac{\eta_i'^{max}}{2}\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\;ds+\frac{\eta_i'^{max}}{2}\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right],
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
-g\left((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_{j,h}^{n},\nabla\phi_{j,h}^{n+1}\right)_p&\leq g\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\mathcal{K}_j-\bar{\mathcal{K}}|_2|\nabla\phi_{j,h}^{n}|_2\,dx\\
&\leq g\int_{D_p}\rho_j'(x)|\nabla\phi_{j,h}^{n+1}|_2|\nabla\phi_{j,h}^{n}|_2\,dx\\
&\leq g\rho_{j,max}'\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\nabla\phi_{j,h}^{n}|_2\,dx\\
&\leq g\rho_{j,max}'\|\nabla\phi_{j,h}^{n}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p\\
&\leq\frac{g\rho_{max}'}{2}\|\nabla\phi_{j,h}^{n}\|_p^2+\frac{g\rho_{max}'}{2}\|\nabla\phi_{j,h}^{n+1}\|_p^2.
\end{aligned}
$$

Using above estimates, equation (3.5) becomes

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2-\frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2+\left(1-\alpha_1-\alpha_2-\Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla u_{j,h}^{n+1}\|_f^2\\
&+\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^{n+1}\|_f^2-\|\nabla u_{j,h}^{n}\|_f^2\right)+\sum_i\left[\frac{\bar{\eta}_i^{min}}{2}-\frac{\eta_i'^{max}}{2}\right]\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\\
&+\sum_i\frac{\bar{\eta}_i^{min}}{2}\left[\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\;ds-\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\;ds\right]
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&+\sum_i \left[\frac{\bar{\eta}_i^{min}}{2} - \frac{\eta_i'^{max}}{2}\right] \int_I (u_{j,h}^n \cdot \widehat{\tau}_i)^2 \; ds + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^n\|_p^2 \\
&+ (1 - \beta_1 - \beta_2 - \Delta t \frac{1}{g^2 S_0 \bar{k}_{min}} \frac{2C_1}{\alpha_1^2} - \frac{\rho'_{max}}{\bar{k}_{min}}) g\bar{k}_{min} \|\nabla \phi_{j,h}^{n+1}\|_p^2 \\
&+ \left(\Delta t \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g\rho'_{max}}{2}\right)\left(\|\nabla \phi_{j,h}^{n+1}\|_p^2 - \|\nabla \phi_{j,h}^n\|_p^2\right) \\
&\leq \frac{C_{P,f}^2}{4\alpha_2 \nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2 \bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
$$

To obtain stability, we need

$$1 - \alpha_1 - \alpha_2 - \Delta t \frac{2C_2}{\beta_1^2 \nu} \geq 0, \tag{3.11}$$

$$\frac{\bar{\eta}_i^{min}}{2} - \frac{\eta_i'^{max}}{2} \geq 0, \tag{3.12}$$

$$1 - \beta_1 - \beta_2 - \Delta t \frac{1}{g^2 S_0 \bar{k}_{min}} \frac{2C_1}{\alpha_1^2} - \frac{\rho'_{max}}{\bar{k}_{min}} \geq 0. \tag{3.13}$$

Recall that $\alpha_1, \alpha_2, \beta_1, \beta_2, \Delta t, \eta_i'^{max}, \rho'_{max}$ are all positive, we then have the following constraints on these parameters.

$$0 < \alpha_1 < 1, \quad 0 < \alpha_2 < 1, \qquad 0 < \beta_1 < 1, \quad 0 < \beta_2 < 1, \tag{3.14}$$

$$\frac{\rho'_{max}}{\bar{k}_{min}} < 1, \quad \eta_i'^{max} \leq \bar{\eta}_i^{min}, \tag{3.15}$$

$$
\begin{aligned}
\Delta t &\leq \min\left\{\frac{(1-\alpha_1-\alpha_2)\beta_1^2\nu}{2C_2}, \frac{(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 g^2 S_0 \bar{k}_{min}}{2C_1}\right\} \\
&\leq \min\left\{\frac{2(1-\alpha_1-\alpha_2)\beta_1^2}{[C(D_f)C(D_p)]^4 C_{P,p}^2}\frac{\nu \bar{k}_{min}^2}{g^2}, \frac{2(1-\beta_1-\beta_2-\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2}{[C(D_f)C(D_p)]^4 C_{P,f}^2}\frac{\nu^2 \bar{k}_{min} S_0}{g^2}\right\}.
\end{aligned}
\tag{3.16}
$$

(3.15) leads to the two parameter conditions (3.2) and (3.3), and (3.16) leads to the time-step condition (3.1) required for stability.

Now if the time-step condition (3.1) and the two parameter conditions (3.2) and (3.3) all hold, (3.10) reduces to

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^n\|_f^2 + \Delta t \frac{C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^{n+1}\|_f^2 - \|\nabla u_{j,h}^n\|_f^2\right) \\
&+ \sum_i \frac{\bar{\eta}_i^{min}}{2}\left[\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2 \; ds - \int_I (u_{j,h}^n \cdot \widehat{\tau}_i)^2 \; ds\right] + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^n\|_p^2 \\
&+ \left(\Delta t \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g\rho'_{max}}{2}\right)\left(\|\nabla \phi_{j,h}^{n+1}\|_p^2 - \|\nabla \phi_{j,h}^n\|_p^2\right) \\
&\leq \frac{C_{P,f}^2}{4\alpha_2 \nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2 \bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
\tag{3.17}
$$

Sum (3.17) from $n = 0$ to $N - 1$ and multiply through by Δt to get

$$
\begin{aligned}
&\frac{1}{2}\|u_{j,h}^N\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^N\|_p^2 + \Delta t^2 \frac{C_2}{\beta_1^2}\|\nabla u_{j,h}^N\|_f^2 + \Delta t \sum_i \frac{\bar{\eta}_i^{min}}{2}\int_I (u_{j,h}^N \cdot \widehat{\tau}_i)^2 \; ds \\
&+ \left(\Delta t^2 \frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t \frac{g\rho'_{max}}{2}\right)\|\nabla \phi_{j,h}^N\|_p^2
\end{aligned}
\tag{3.18}
$$

$$\le \frac{1}{2}\|u^0_{j,h}\|^2_f + \frac{gS_0}{2}\|\phi^0_{j,h}\|^2_p + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla u^0_{j,h}\|^2_f + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{2}\int_I (u^0_{j,h}\cdot\widehat{\tau}_i)^2\ ds$$
$$+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g\rho'_{max}}{2}\right)\|\nabla\phi^0_{j,h}\|^2_p + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f^{n+1}_{f,j}\|^2_f + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f^{n+1}_{p,j}\|^2_p.$$

□

4. An alternative approach. Let $k_{j,max}(x)$ be the maximum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, and we define

$$\eta_{i,j}^{max} = \max_{x\in I}\eta_{i,j}(x), \qquad \eta_i^{max} = \max_j \eta_{i,j}^{max}, \qquad k_{j,max} = \max_{x\in D_p} k_{j,max}(x), \qquad k_{max} = \max_j k_{j,max}.$$

If it is easy to identify the minimum and maximum eigenvalues of the hydraulic conductivity tensor $\mathcal{K}_j(x)$ (e.g., $\mathcal{K}_j(x)$ is a diagonal matrix function), then the following algorithm can be used, which removes one of the parameter conditions for stability.

Algorithm 4.1. *Find* $(u^{n+1}_{j,h}, p^{n+1}_{j,h}, \phi^{n+1}_{j,h}) \in X_f^h\times Q_f^h\times X_p^h$ *satisfying* $\forall\ (v_h, q_h, \psi_h) \in X_f^h\times Q_f^h\times X_p^h$,

$$\begin{aligned}
&\left(\frac{u^{n+1}_{j,h}-u^n_{j,h}}{\Delta t}, v_h\right)_f + \nu(\nabla u^{n+1}_{j,h},\nabla v)_f + \sum_i\int_I \eta_i^{max}(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds\\
&+\sum_i\int_I(\eta_{i,j}-\eta_i^{max})(u^n_{j,h}\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds - \left(p^{n+1}_{j,h},\nabla\cdot v_h\right)_f + c_I(v_h,\phi^n_{j,h}) = (f^{n+1}_{f,j}, v_h)_f,\\
&(q_h,\nabla\cdot u^{n+1}_{j,h})_f = 0, \qquad (4.1)\\
&gS_0\left(\frac{\phi^{n+1}_{j,h}-\phi^n_{j,h}}{\Delta t},\psi_h\right)_p + k_{max}g(\nabla\phi^{n+1}_{j,h},\nabla\psi)_p + g((\mathcal{K}_j - k_{max}\mathcal{I})\nabla\phi^n_{j,h},\nabla\psi)_p\\
&\qquad - c_I(u^n_{j,h},\psi_h) = g(f^{n+1}_{p,j},\psi_h)_p.
\end{aligned}$$

For this approach, since $\mathcal{K}_j(x)$ and $k_{max}\mathcal{I}$ are both symmetric, we have $|\mathcal{K}_j(x) - k_{max}\mathcal{I}|_2 \le k_{max} - k_{min}$. We then prove long time stability of Algorithm 4.1 under a similar time-step condition, *without* any parameter conditions.

$$\Delta t \le \min\left\{\frac{2(1-\alpha_1-\alpha_2)\beta_1^2}{[C(D_f)C(D_p)]^4C_{P,p}^2}\frac{\nu k_{max}^2}{g^2}, \frac{2(1-\beta_1-\beta_2-\frac{k_{max}-k_{min}}{k_{max}})\alpha_1^2}{[C(D_f)C(D_p)]^4C_{P,f}^2}\frac{\nu^2k_{max}S_0}{g^2}\right\}. \qquad (4.2)$$

Theorem 4.2 (Long time stability of Algorithm 4.1). *If there exist* $\alpha_1,\alpha_2,\beta_1,\beta_2$ *in* $(0,1)$ *such that the time-step condition* (4.2) *holds, then the Algorithm 4.1 is long time stable: for any* $N>0$,

$$\begin{aligned}
&\frac{1}{2}\|u^N_{j,h}\|^2_f + \frac{gS_0}{2}\|\phi^N_{j,h}\|^2_p + \frac{\Delta t\nu}{8}\|\nabla u^N_{j,h}\|^2_f + \Delta t\sum_i\frac{\eta_i^{max}}{2}\int_I(u^N_{j,h}\cdot\widehat{\tau}_i)^2\ ds \qquad (4.3)\\
&+\frac{\Delta t}{8}gk_{max}\|\nabla\phi^N_{j,h}\|^2_p + \Delta t\sum_{n=0}^{N-1}\frac{\nu}{4}\|\nabla u^{n+1}_{j,h}\|^2_f\\
&\le \frac{1}{2}\|u^0_{j,h}\|^2_f + \frac{gS_0}{2}\|\phi^0_{j,h}\|^2_p + \frac{\Delta t\nu}{8}\|\nabla u^0_{j,h}\|^2_f + \Delta t\sum_i\frac{\eta_i^{max}}{2}\int_I(u^0_{j,h}\cdot\widehat{\tau}_i)^2\ ds\\
&+\frac{\Delta t}{8}gk_{max}\|\nabla\phi^0_{j,h}\|^2_p + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{\nu}\|f^{n+1}_{f,j}\|^2_f + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{k_{max}}\|f^{n+1}_{p,j}\|^2_p.
\end{aligned}$$

Proof. Setting $v_h = u^{n+1}_{j,h}$, $\psi_h = \phi^{n+1}_{j,h}$ in Algorithm 4.1 and adding all three equations yields

$$\frac{1}{2\Delta t}\|u^{n+1}_{j,h}\|^2_f - \frac{1}{2\Delta t}\|u^n_{j,h}\|^2_f + \frac{1}{2\Delta t}\|u^{n+1}_{j,h}-u^n_{j,h}\|^2_f + \nu\|\nabla u^{n+1}_{j,h}\|^2_f \qquad (4.4)$$

$$
\begin{aligned}
&+\sum_i \int_I \eta_i^{max}(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2 \\
&+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^{n}\|_p^2 + gk_{max}(\nabla\phi_{j,h}^{n+1},\nabla\phi_{j,h}^{n+1})_p + c_I(u_{j,h}^{n+1},\phi_{j,h}^{n}) - c_I(u_{j,h}^{n},\phi_{j,h}^{n+1}) \\
&=(f_{f,j}^{n+1},u_{j,h}^{n+1})_f + g(f_{p,j}^{n+1},\phi_{j,h}^{n+1})_p - \sum_i\int_I(\eta_{i,j}-\eta_i^{max})(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds \\
&\quad -g((\mathcal{K}_j-k_{max}\mathcal{I})\nabla\phi_{j,h}^{n},\nabla\phi_{j,h}^{n+1})_p.
\end{aligned}
$$

The main difference from the proof of Theorem (3.2) is on the estimates of the following two terms.

$$
\begin{aligned}
&-\sum_i\int_I(\eta_{i,j}-\eta_i^{max})(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds \\
&\qquad\le \sum_i\int_I|\eta_{i,j}-\eta_i^{max}|\,(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds \\
&\qquad\le \sum_i \eta_i^{max}\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds \\
&\qquad\le \sum_i\left[\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds + \frac{\eta_i^{max}}{2}\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds\right],
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
-g\left((\mathcal{K}_j-k_{max}\mathcal{I})\nabla\phi_{j,h}^{n},\nabla\phi_{j,h}^{n+1}\right)_p &\le g\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\mathcal{K}_j-k_{max}\mathcal{I}|_2|\nabla\phi_{j,h}^{n}|_2\,dx \\
&\le g(k_{max}-k_{min})\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\nabla\phi_{j,h}^{n}|_2\,dx \\
&\le g(k_{max}-k_{min})\|\nabla\phi_{j,h}^{n}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p \\
&\le \frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^{n}\|_p^2 + \frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^{n+1}\|_p^2.
\end{aligned}
$$

Then we have the following inequality

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2 + \left(1-\alpha_1-\alpha_2-\Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla u_{j,h}^{n+1}\|_f^2 \\
&+\Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^{n+1}\|_f^2-\|\nabla u_{j,h}^{n}\|_f^2\right) + \sum_i\frac{\eta_i^{max}}{2}\left[\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds - \int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right] \\
&+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2 + (1-\beta_1-\beta_2-\Delta t\frac{1}{g_0^2k_{max}}\frac{2C_1}{\alpha_1^2}-\frac{k_{max}-k_{min}}{k_{max}})gk_{max}\|\nabla\phi_{j,h}^{n+1}\|_p^2 \\
&+\left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2}+\frac{g(k_{max}-k_{min})}{2}\right)\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2-\|\nabla\phi_{j,h}^{n}\|_p^2\right) \\
&\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2k_{max}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
\tag{4.6}
$$

Since we assume $\mathcal{K}_j$ is SPD, and any two ensemble members have different hydraulic conductivity tensor $\mathcal{K}$, we have $k_{max} > k_{min} > 0$ and thus $0 < \frac{k_{max}-k_{min}}{k_{max}} < 1$. So we do not need any constraints on these parameters.

Now if the time-step condition (4.2) holds, (4.6) reduces to

$$
\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2 + \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^{n+1}\|_f^2-\|\nabla u_{j,h}^{n}\|_f^2\right)
\tag{4.7}
$$

$$+\sum_i \frac{\eta_i^{max}}{2}\left[\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds - \int_I (u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right] + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2$$
$$+\left(\Delta t\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \frac{g(k_{max}-k_{min})}{2}\right)\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2 - \|\nabla\phi_{j,h}^{n}\|_p^2\right)$$
$$\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f_{p,j}^{n+1}\|_p^2.$$

Sum (4.7) from $n=0$ to $N-1$ and multiply through by Δt to get

$$\frac{1}{2}\|u_{j,h}^N\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^N\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla u_{j,h}^N\|_f^2 + \Delta t\sum_i \frac{\eta_i^{max}}{2}\int_I (u_{j,h}^N\cdot\widehat{\tau}_i)^2\ ds \tag{4.8}$$
$$+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g(k_{max}-k_{min})}{2}\right)\|\nabla\phi_{j,h}^N\|_p^2$$
$$\le \frac{1}{2}\|u_{j,h}^0\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^0\|_p^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla u_{j,h}^0\|_f^2 + \Delta t\sum_i \frac{\eta_i^{max}}{2}\int_I (u_{j,h}^0\cdot\widehat{\tau}_i)^2\ ds$$
$$+\left(\Delta t^2\frac{1}{gS_0}\frac{C_1}{\alpha_1^2} + \Delta t\frac{g(k_{max}-k_{min})}{2}\right)\|\nabla\phi_{j,h}^0\|_p^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{4\beta_2 k_{max}}\|f_{p,j}^{n+1}\|_p^2.$$

□

5. Error Analysis. In this section, we analyze the error of Algorithm 2.2. The error analysis for Algorithm 4.1 can be done similarly with minor modification. We assume the finite element spaces satisfy the approximation properties of piecewise polynomials on quasiuniform meshes

$$\inf_{v_h\in X_f^h}\|v-v_h\|_f \le Ch^{k+1}\|u\|_{H^{k+1}(D_f)} \qquad \forall v\in[H^{k+1}(D_f)]^d, \tag{5.1}$$
$$\inf_{v_h\in X_f^h}\|\nabla(v-v_h)\|_f \le Ch^{k}\|v\|_{H^{k+1}(D_f)} \qquad \forall v\in[H^{k+1}(D_f)]^d, \tag{5.2}$$
$$\inf_{q_h\in Q_f^h}\|q-q_h\|_f \le Ch^{s+1}\|q\|_{H^{s+1}(D_f)} \qquad \forall q\in H^{s+1}(D_f), \tag{5.3}$$
$$\inf_{\psi_h\in X_p^h}\|\psi-\psi_h\|_p \le Ch^{m+1}\|\psi\|_{H^{m+1}(D_p)} \qquad \forall \psi\in H^{m+1}(D_p), \tag{5.4}$$
$$\inf_{\psi_h\in X_p^h}\|\nabla(\psi-\psi_h)\|_p \le Ch^{m}\|\psi\|_{H^{m+1}(D_p)} \qquad \forall \psi\in H^{m+1}(D_p), \tag{5.5}$$

where the generic constant $C>0$ is independent of the mesh size h. An example for which both the LBB^h stability condition and the approximation properties are satisfied is the finite elements (P_{l+1}–P_l–P_{l+1}), $l\ge 1$, see [19, 20, 34] for more details.

We also assume the following regularity on the true solution of the Stokes-Darcy equations.

$$u_j\in L^\infty(0,T;H^{k+1}(D_f)), u_{j,t}\in L^2(0,T;H^{k+1}(D_f)), u_{j,tt}\in L^2(0,T;L^2(D_f)),$$
$$\phi_j\in L^\infty(0,T;H^{m+1}(D_p)), \phi_{j,t}\in L^2(0,T;H^{m+1}(D_p)), \phi_{j,tt}\in L^2(0,T;L^2(D_p)),$$
$$p_j\in L^2(0,T;H^{s+1}(D_f)).$$

For functions $v(x,t)$ defined on $(0,T)$, we define the continuous norm

$$\|v\|_{m,k,r} := \|v\|_{L^m(0,T;H^k(D_r))},\ r\in\{f,p\}.$$

Given a time step Δt, let $t_n=n\Delta t, T=N\Delta t$, $v^n=v(x,t_n)$ and define the discrete norms

$$|||v|||_{\infty,k,r} = \max_{0\le n\le N}\|v^n\|_{H^k(D_r)} \qquad \text{and}$$
$$|||v|||_{m,k,r} := \left(\sum_{n=0}^{N}\|v^n\|_{H^k(D_r)}^m\Delta t\right)^{1/m},\ \ r\in\{f,p\}.$$

Let $u_j^n = u_j(x,t_n), p_j^n = p_j(x,t_n), \phi_j^n = \phi_j(x,t_n)$. Denote the errors by $e_{j,u}^n := u_j^n - u_{j,h}^n, e_{j,\phi}^n := \phi_j^n - \phi_{j,h}^n$. We prove the convergence of Algorithm 2.2 under a time-step condition and two parameter conditions.

$$\Delta t \le \min\left\{ \frac{(1-\alpha_1-\alpha_2)\beta_1^2 \bar{k}_{min}}{C_{P,p}^2}, \frac{(1-\sigma_4-\beta_1-\beta_2-(1+\sigma_3)\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 S_0 \nu}{C_{P,f}^2} \right\} \frac{2\nu \bar{k}_{min}}{g^2[C(D_f)C(D_p)]^4}, \tag{5.6}$$

$$\eta_i'^{max} \le \bar{\eta}_i^{min}, \tag{5.7}$$

$$\rho'_{max} < \bar{k}_{min}. \tag{5.8}$$

Note that the two parameter conditions are the same as those for stability. The time-step condition is slightly different with two extra constants $\sigma_3 > 0$ and $\sigma_4 \in (0,1)$.

THEOREM 5.1 (Error Estimate). *For any $j = 1, \dots, J$, if the two parameter conditions* (5.7) *and* (5.8) *hold, and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma_4 \in (0,1)$ and $\sigma_3 > 0$ such that the time-step condition* (5.6) *also holds, then there is a positive constant C independent of the time step Δt and mesh size h such that*

$$\begin{aligned}
&\frac{1}{2}\|e_{j,u}^N\|_f^2 + \Delta t^2 \frac{C_2}{\beta_1^2}\|\nabla e_{j,u}^N\|_f^2 + \frac{gS_0}{2}\|e_{j,\phi}^N\|_p^2 + \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^N\|_p^2 \\
&\le \frac{1}{2}\|e_{j,u}^0\|_f^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla e_{j,u}^0\|_f^2 + \Delta t\sum_i \frac{1}{2}\eta_i'^{max}\int_I (e_{j,u}^0\cdot\widehat{\tau}_i)^2\, ds + \frac{gS_0}{2}\|e_{j,\phi}^0\|_p^2 \\
&+ \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^0\|_p^2 + C\Delta t^2\|u_{j,t}\|_{2,1,f}^2 + Ch^{2k}|||u_j|||_{2,k+1,f}^2 + C\Delta t^2\|u_{j,tt}\|_{2,0,f}^2 \\
&+ C\Delta t^2\|\phi_{j,t}\|_{2,1,p}^2 + C\Delta t^2\|\phi_{j,tt}\|_{2,0,p}^2 + Ch^{2k+2}\|u_{j,t}\|_{2,k+1,f} + Ch^{2k+2}\|\phi_{j,t}\|_{2,m+1,p} \\
&+ Ch^{2k}|||\phi_j|||_{2,m+1,p}^2 + Ch^{2s+2}|||p_j|||_{2,s+1,f}^2 + Ch^{2k+2}|||u_j|||_{\infty,k+1,f} + C\Delta t^2 h^{2k}|||u_j|||_{\infty,k+1,f} \\
&+ Ch^{2k+2}|||\phi_j|||_{\infty,m+1,p} + C\Delta t h^{2k}|||\phi_j|||_{\infty,m+1,p}.
\end{aligned} \tag{5.9}$$

In particular, if Taylor-Hood elements ($k = 2$, $s = 1$) are used for approximating (u_j, p_j), i.e., the C^0 piecewise-quadratic velocity space X_f^h and the C^0 piecewise-linear pressure space Q_f^h, and P_2 element ($m = 2$) is used for X_p^h, we have the following estimate.

COROLLARY 5.2. *Assume that $\|e_{j,u}^0\|$, $\|\nabla e_{j,u}^0\|$, $\|e_{j,\phi}^0\|$ and $\|\nabla e_{j,\phi}^0\|$ are all $O(h^2)$ accurate or better. Then, if (X_f^h, Q_f^h, X_p^h) are chosen as the (P_2, P_1, P_2) elements, we have*

$$\frac{1}{2}\|e_{j,u}^N\|_f^2 + \Delta t^2\frac{C_2}{\beta_1^2}\|\nabla e_{j,u}^N\|_f^2 + \frac{gS_0}{2}\|e_{j,\phi}^N\|_p^2 + \Delta t\left(\frac{1}{2}g\rho'_{max} + \frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^N\|_p^2 \le C(h^4 + \Delta t^2)\ .$$

Proof. (of Theorem 5.1) For $\forall v_h \in V_f^h, \forall \psi_h \in X_p^h, \forall \lambda_h^{n+1} \in Q_f^h$, the true solution (u_j, p_j, ϕ_j) satisfies

$$\begin{aligned}
&\left(\frac{u_j^{n+1}-u_j^n}{\Delta t}, v_h\right)_f + \nu(\nabla u_j^{n+1}, \nabla v_h)_f + \sum_i \int_I \eta_{i,j}(u_j^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds \\
&- \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot v_h\right)_f + c_I(v_h, \phi_j^n) = (f_{f,j}^{n+1}, v_h)_f + \epsilon_{j,f}^{n+1}(v_h), \\
&gS_0\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}, \psi_h\right)_p + g(\mathcal{K}_j\nabla\phi_j^{n+1}, \nabla\psi_h)_p - c_I(u_j^n, \psi_h) \\
&= g(f_{p,j}^{n+1}, \psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h).
\end{aligned} \tag{5.10}$$

The consistency errors $\epsilon_{j,f}^{n+1}(v_h), \epsilon_{j,p}^{n+1}(\psi_h)$ are defined by

$$\epsilon_{j,f}^{n+1}(v_h) := \left(\frac{u_j^{n+1}-u_j^n}{\Delta t} - u_{j,t}^{n+1}, v_h\right)_f - c_I(v_h, \phi_j^{n+1} - \phi_j^n),$$

$$\epsilon_{j,p}^{n+1}(\psi_h) := gS_0\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}-\phi_{j,t}^{n+1},\psi_h\right)_p + c_I(u_j^{n+1}-u_j^n,\psi_h).$$

Subtracting (2.5) from (5.10) gives, for $\forall v_h \in V_f^h, \forall \psi_h \in X_p^h, \forall \lambda_h^{n+1} \in Q_f^h$,

$$\begin{aligned}
&\left(\frac{e_{j,u}^{n+1}-e_{j,u}^n}{\Delta t},v_h\right)_f + \nu(\nabla e_{j,u}^{n+1},\nabla v_h)_f + \sum_i \int_I \bar{\eta}_i (e_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds \\
&+\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,(e_{j,u}^n\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds - \left(p_j^{n+1}-\lambda_h^{n+1},\nabla\cdot v_h\right)_f + c_I(v_h,e_{j,\phi}^n) \\
&= -\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds + \epsilon_{j,f}^{n+1}(v_h), \\
&gS_0\left(\frac{e_{j,\phi}^{n+1}-e_{j,\phi}^n}{\Delta t},\psi_h\right)_p + g(\bar{\mathcal{K}}\nabla e_{j,\phi}^{n+1},\nabla\psi_h)_p + g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla e_{j,\phi}^n,\nabla\psi_h)_p \\
&- c_I(e_{j,u}^n,\psi_h) = -g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi_j^{n+1}-\phi_j^n),\nabla\psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h).
\end{aligned} \tag{5.11}$$

Let U_j^{n+1}, Φ_j^{n+1} be an interpolation of u_j^{n+1} and ϕ_j^{n+1} in V_f^h and X_p^h correspondingly. Denote

$$\begin{aligned}
e_{j,u}^{n+1} &= \left(u_j^{n+1}-U_j^{n+1}\right) + \left(U_j^{n+1}-u_{j,h}^{n+1}\right) =: \mu_{j,u}^{n+1}+\xi_{j,u}^{n+1}, \\
e_{j,\phi}^{n+1} &= \left(\phi_j^{n+1}-\Phi_j^{n+1}\right) + \left(\Phi_j^{n+1}-\phi_{j,h}^{n+1}\right) =: \mu_{j,\phi}^{n+1}+\xi_{j,\phi}^{n+1}.
\end{aligned}$$

Then (5.11) can be rewritten as

$$\begin{aligned}
&\left(\frac{\xi_{j,u}^{n+1}-\xi_{j,u}^n}{\Delta t},v_h\right)_f + \nu(\nabla \xi_{j,u}^{n+1},\nabla v_h)_f + \sum_i \int_I \bar{\eta}_i (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds \\
&\quad+\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,(\xi_{j,u}^n\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds - \left(p_j^{n+1}-\lambda_h^{n+1},\nabla\cdot v_h\right)_f + c_I(v_h,\xi_{j,\phi}^n) \\
&= -\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds + \epsilon_{j,f}^{n+1}(v_h), \\
&\quad-\left(\frac{\mu_{j,u}^{n+1}-\mu_{j,u}^n}{\Delta t},v_h\right)_f - \nu(\nabla \mu_{j,u}^{n+1},\nabla v_h)_f - \sum_i \int_I \bar{\eta}_i (\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds \\
&\quad-\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,(\mu_{j,u}^n\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds - c_I(v_h,\mu_{j,\phi}^n), \\
&gS_0\left(\frac{\xi_{j,\phi}^{n+1}-\xi_{j,\phi}^n}{\Delta t},\psi_h\right)_p + g(\bar{\mathcal{K}}\nabla \xi_{j,\phi}^{n+1},\nabla\psi_h)_p + g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla \xi_{j,\phi}^n,\nabla\psi_h)_p - c_I(\xi_{j,u}^n,\psi_h) \\
&= -g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi_j^{n+1}-\phi_j^n),\nabla\psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h) - gS_0\left(\frac{\mu_{j,\phi}^{n+1}-\mu_{j,\phi}^n}{\Delta t},\psi_h\right)_p \\
&\quad- g(\bar{\mathcal{K}}\nabla \mu_{j,\phi}^{n+1},\nabla\psi_h)_p - g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla \mu_{j,\phi}^n,\nabla\psi_h)_p + c_I(\mu_{j,u}^n,\psi_h).
\end{aligned} \tag{5.12}$$

Letting $v_h = \xi_{j,u}^{n+1}, \psi_h = \xi_{j,\phi}^{n+1}$ in (5.12) and adding the two equations yields

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,u}^n\|_f^2 + \frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}-\xi_{j,u}^n\|_f^2 + \nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \sum_i\int_I \bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\, ds \\
&+\frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^n\|_p^2 + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}-\xi_{j,\phi}^n\|_p^2 + g(\bar{\mathcal{K}}\nabla\xi_{j,\phi}^{n+1},\nabla\xi_{j,\phi}^{n+1})_p + c_I(\xi_{j,u}^{n+1},\xi_{j,\phi}^n) - c_I(\xi_{j,u}^n,\xi_{j,\phi}^{n+1})
\end{aligned} \tag{5.13}$$

$$= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\xi^n_{j,u} \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,u} \cdot \widehat{\tau}_i)\; ds - \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,((u^{n+1}_j - u^n_j) \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,u} \cdot \widehat{\tau}_i)\; ds$$

$$+ \epsilon^{n+1}_{j,f}(\xi^{n+1}_{j,u}) - \left(\frac{\mu^{n+1}_{j,u} - \mu^n_{j,u}}{\Delta t}, \xi^{n+1}_{j,u}\right)_f - \nu(\nabla \mu^{n+1}_{j,u}, \nabla \xi^{n+1}_{j,u})_f - \sum_i \int_I \bar{\eta}_i (\mu^{n+1}_{j,u} \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,u} \cdot \widehat{\tau}_i)\; ds$$

$$- \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\mu^n_{j,u} \cdot \widehat{\tau}_i)(\xi^{n+1}_{j,u} \cdot \widehat{\tau}_i)\; ds - c_I(\xi^{n+1}_{j,u}, \mu^n_{j,\phi}) + \left(p^{n+1}_j - \lambda^{n+1}_h, \nabla \cdot \xi^{n+1}_{j,u}\right)_f$$

$$- g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi^{n+1}_j - \phi^n_j), \nabla \xi^{n+1}_{j,\phi})_p + \epsilon^{n+1}_{j,p}(\xi^{n+1}_{j,\phi}) - gS_0 \left(\frac{\mu^{n+1}_{j,\phi} - \mu^n_{j,\phi}}{\Delta t}, \xi^{n+1}_{j,\phi}\right)_p$$

$$- g(\bar{\mathcal{K}}\nabla \mu^{n+1}_{j,\phi}, \nabla \xi^{n+1}_{j,\phi})_p - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla \mu^n_{j,\phi}, \nabla \xi^{n+1}_{j,\phi})_p + c_I(\mu^n_{j,u}, \xi^{n+1}_{j,\phi}) - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla \xi^n_{j,\phi}, \nabla \xi^{n+1}_{j,\phi})_p.$$

Using the same techniques in the stability proof (see (3.6) and (3.7)), we have

$$\begin{aligned} c_I(\xi^{n+1}_{j,u}, &\xi^n_{j,\phi}) - c_I(\xi^n_{j,u}, \xi^{n+1}_{j,\phi}) \\ &= c_I(\xi^{n+1}_{j,u} - \xi^n_{j,u}, \xi^{n+1}_{j,\phi}) - c_I(\xi^{n+1}_{j,u}, \xi^{n+1}_{j,\phi} - \xi^n_{j,\phi}) \\ &\geq -\frac{1}{2\Delta t}\|\xi^{n+1}_{j,u} - \xi^n_{j,u}\|_f^2 - \Delta t \frac{C_2}{\beta_1^2}\left(\|\nabla \xi^{n+1}_{j,u}\|_f^2 + \|\nabla \xi^n_{j,u}\|_f^2\right) - \beta_1 g\bar{k}_{min}\|\nabla \xi^{n+1}_{j,\phi}\|_p^2 \\ &\quad - \frac{gS_0}{2\Delta t}\|\xi^{n+1}_{j,\phi} - \xi^n_{j,\phi}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\left(\|\nabla \xi^{n+1}_{j,\phi}\|_p^2 + \|\nabla \xi^n_{j,\phi}\|_p^2\right) - \alpha_1 \nu \|\nabla \xi^{n+1}_{j,u}\|_f^2. \end{aligned} \tag{5.14}$$

Next we bound the terms on the right hand side of (5.13).

$$\begin{aligned} &- \left(\frac{\mu^{n+1}_{j,u} - \mu^n_{j,u}}{\Delta t}, \xi^{n+1}_{j,u}\right)_f - gS_0 \left(\frac{\mu^{n+1}_{j,\phi} - \mu^n_{j,\phi}}{\Delta t}, \xi^{n+1}_{j,\phi}\right)_p \\ &\leq \frac{5C_{P,f}^2}{4\alpha_2 \nu}\|\frac{\mu^{n+1}_{j,u} - \mu^n_{j,u}}{\Delta t}\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2 + \frac{C_{P,p}^2 g S_0^2}{\beta_2 \bar{k}_{min}}\|\frac{\mu^{n+1}_{j,\phi} - \mu^n_{j,\phi}}{\Delta t}\|_p^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla \xi^{n+1}_{j,\phi}\|_p^2 \\ &\leq \frac{5C_{P,f}^2}{4\alpha_2 \nu}\left\|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mu_{j,u,t}\; dt\right\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2 + \frac{C_{P,p}^2 g S_0^2}{\beta_2 \bar{k}_{min}}\left\|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mu_{j,\phi,t}\; dt\right\|_p^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla \xi^{n+1}_{j,\phi}\|_p^2 \\ &\leq \frac{5C_{P,f}^2}{4\alpha_2 \nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \|\mu_{j,u,t}\|_f^2\; dt + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2 + \frac{C_{P,p}^2 g S_0^2}{\beta_2 \bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \|\mu_{j,\phi,t}\|_p^2\; dt + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla \xi^{n+1}_{j,\phi}\|_p^2. \end{aligned} \tag{5.15}$$

$$\begin{aligned} &- \nu(\nabla \mu^{n+1}_{j,u}, \nabla \xi^{n+1}_{j,u})_f - g(\bar{\mathcal{K}}\nabla \mu^{n+1}_{j,\phi}, \nabla \xi^{n+1}_{j,\phi})_p \\ &\quad \leq C\left(\|\nabla \mu^{n+1}_{j,u}\|_f^2 + \|\nabla \mu^{n+1}_{j,\phi}\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla \xi^{n+1}_{j,\phi}\|_p^2. \end{aligned} \tag{5.16}$$

By trace theorem, we have the following estimates

$$\begin{aligned} &- c_I(\xi^{n+1}_{j,u}, \mu^n_{j,\phi}) + c_I(\mu^n_{j,u}, \xi^{n+1}_{j,\phi}) \\ &\quad \leq C\left(\|\nabla \mu^n_{j,u}\|_f^2 + \|\nabla \mu^n_{j,\phi}\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla \xi^{n+1}_{j,\phi}\|_p^2. \end{aligned} \tag{5.17}$$

The pressure term can be bounded as follows.

$$\left(p^{n+1}_j - \lambda^{n+1}_h, \nabla \cdot \xi^{n+1}_{j,u}\right)_f \leq C\|p^{n+1}_j - \lambda^{n+1}_h\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2. \tag{5.18}$$

Next we bound the consistency errors.

$$\epsilon^{n+1}_{j,f}(\xi^{n+1}_{j,u}) \leq C\left\|\frac{u^{n+1}_j - u^n_j}{\Delta t} - u^{n+1}_{j,t}\right\|_f^2 + C\|\nabla(\phi^{n+1}_j - \phi^n_j)\|_p^2 + \frac{\alpha_2}{5}\nu\|\nabla \xi^{n+1}_{j,u}\|_f^2 \tag{5.19}$$

$$\leq C\Delta t\int_{t^n}^{t^{n+1}} \|u_{j,tt}\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla\phi_{j,t}\|_p^2\,dt + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2.$$

$$\epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) \leq C\Big\|\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t} - \phi_{j,t}^{n+1}\Big\|_p^2 + C\|\nabla(u_j^{n+1}-u_j^n)\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 \tag{5.20}$$
$$\leq C\Delta t\int_{t^n}^{t^{n+1}} \|\phi_{j,tt}\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|_f^2\,dt + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.$$

The rest of the terms on the right hand side of (5.13) can be bounded as follows.

$$-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\,(\xi_{j,u}^n\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\;ds \tag{5.21}$$
$$\leq \sum_i \eta_{i,j}'^{max}\int_I|(\xi_{j,u}^n\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\;ds$$
$$\leq \sum_i\left[\frac{\eta_i'^{max}}{2}\int_I(\xi_{j,u}^n\cdot\widehat{\tau}_i)^2\;ds + \frac{\eta_i'^{max}}{2}\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right].$$

By (2.2) and Poincaré inequality, we have, for any $\sigma_1 > 0$

$$-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\,((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\;ds \tag{5.22}$$
$$\leq \sum_i\eta_{i,j}'^{max}\int_I|((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\;ds$$
$$\leq \sum_i\eta_i'^{max}\left[\frac{1}{2\sigma_1}\int_I\left((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i\right)^2\;ds + \frac{\sigma_1}{2}\int_I\left(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i\right)^2\;ds\right]$$
$$\leq \sum_i\left[\frac{\eta_i'^{max}}{2\sigma_1}\|u_j^{n+1}-u_j^n\|_I^2 + \frac{\sigma_1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right]$$
$$\leq \sum_i\left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i'^{max}\|\nabla(u_j^{n+1}-u_j^n)\|_f^2 + \frac{\sigma_1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right]$$
$$\leq \sum_i\left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\;dt + \frac{\sigma_1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right].$$

Similarly, for any $\sigma_2 > 0$

$$-\sum_i\int_I\bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\;ds \tag{5.23}$$
$$\leq \sum_i\left[\frac{1}{4\sigma_2}\int_I\bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds + \sigma_2\int_I\bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right]$$
$$\leq \sum_i\left[\frac{1}{4\sigma_2}\bar{\eta}_i^{max}\|\mu_{j,u}^{n+1}\|_I^2 + \sigma_2\int_I\bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right]$$
$$\leq \sum_i\left[\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \sigma_2\int_I\bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\;ds\right].$$

$$-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\,(\mu_{j,u}^n\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\;ds \tag{5.24}$$
$$\leq \sum_i\eta_{i,j}'^{max}\int_I|(\mu_{j,u}^n\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\;ds$$

$$\le \sum_i \eta_i'^{max} \left[\frac{1}{2\sigma_1} \int_I (\mu_{j,u}^n \cdot \widehat{\tau}_i)^2 \ ds + \frac{\sigma_1}{2} \int_I (\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)^2 \right] \ ds$$
$$\le \sum_i \left[\frac{1}{2\sigma_1} \eta_i'^{max} \|\mu_{j,u}^n\|_I^2 + \frac{\sigma_1}{2} \eta_i'^{max} \int_I (\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)^2 \ ds \right]$$
$$\le \sum_i \left[\frac{C^2(D_f) C_{P,f}}{2\sigma_1} \eta_{i,j}'^{max} \|\nabla \mu_{j,u}^n\|_f^2 + \frac{\sigma_1}{2} \eta_i'^{max} \int_I (\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)^2 \ ds \right].$$

The hydraulic conductivity tensor terms are estimated as follows.

$$\begin{aligned} -g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \xi_{j,\phi}^n, \nabla \xi_{j,\phi}^{n+1})_p & \\ &\le g \int_{D_p} |\nabla \xi_{j,\phi}^n|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \int_{D_p} \rho_j'(x) |\nabla \xi_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \rho_{j,max}' \int_{D_p} |\nabla \xi_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \rho_{max}' \|\nabla \xi_{j,\phi}^n\|_p \|\nabla \xi_{j,\phi}^{n+1}\|_p \\ &\le \frac{g \rho_{max}'}{2} \|\nabla \xi_{j,\phi}^n\|_p^2 + \frac{g \rho_{max}'}{2} \|\nabla \xi_{j,\phi}^{n+1}\|_p^2. \end{aligned} \tag{5.25}$$

For any $\sigma_3 > 0$, we have

$$\begin{aligned} -g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla (\phi_j^{n+1} - \phi_j^n), \nabla \xi_{j,\phi}^{n+1})_p & \\ &\le g \int_{D_p} |\nabla (\phi_j^{n+1} - \phi_j^n)|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \int_{D_p} \rho_j'(x) |\nabla (\phi_j^{n+1} - \phi_j^n)|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \rho_{j,max}' \int_{D_p} |\nabla (\phi_j^{n+1} - \phi_j^n)|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \rho_{max}' \|\nabla (\phi_j^{n+1} - \phi_j^n)\|_p \|\nabla \xi_{j,\phi}^{n+1}\|_p \\ &\le \frac{g \rho_{max}'}{2\sigma_3} \|\nabla (\phi_j^{n+1} - \phi_j^n)\|_p^2 + \frac{\sigma_3}{2} g \rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \\ &\le \frac{g \rho_{max}'}{2\sigma_3} \| \int_{t^n}^{t^{n+1}} \nabla \phi_{j,t} \, dt\|_p^2 + \frac{\sigma_3}{2} g \rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \\ &\le \frac{g \rho_{max}'}{2\sigma_3} \Delta t \int_{t^n}^{t^{n+1}} \|\nabla \phi_{j,t}\|_p^2 \, dt + \frac{\sigma_3}{2} g \rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2. \end{aligned} \tag{5.26}$$

Similarly,

$$\begin{aligned} -g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \mu_{j,\phi}^n, \nabla \xi_{j,\phi}^{n+1})_p &\le g \int_{D_p} |\nabla \mu_{j,\phi}^n|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \int_{D_p} \rho_j'(x) |\nabla \mu_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \rho_{j,max}' \int_{D_p} |\nabla \mu_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\ &\le g \rho_{max}' \|\nabla \mu_{j,\phi}^n\|_p \|\nabla \xi_{j,\phi}^{n+1}\|_p \\ &\le \frac{g \rho_{max}'}{2\sigma_3} \|\nabla \mu_{j,\phi}^n\|_p^2 + \frac{\sigma_3}{2} g \rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2. \end{aligned} \tag{5.27}$$

Since $\bar{\mathcal{K}}$ is SPD, for any $\sigma_4 > 0$

$$\begin{aligned} -g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1}, \nabla\xi_{j,\phi}^{n+1})_p &= -g(\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\mu_{j,\phi}^{n+1}, \bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1})_p \\ &\le g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\mu_{j,\phi}^{n+1}\|_p\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1}\|_p \\ &\le \frac{1}{4\sigma_4}g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \sigma_4 g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1}\|_p^2 \\ &\le \frac{1}{4\sigma_4}g\bar{k}^{max}\|\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \sigma_4 g\|\bar{\mathcal{K}}^{\frac{1}{2}}\nabla\xi_{j,\phi}^{n+1}\|_p^2. \end{aligned} \tag{5.28}$$

Combining all these estimates, we have the following inequality

$$\begin{aligned} &\frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,u}^{n}\|_f^2 + \left(1 - \alpha_1 - \alpha_2 - \Delta t\frac{2C_2}{\beta_1^2\nu}\right)\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 \\ &+ \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\xi_{j,u}^{n+1}\|_f^2 - \|\nabla\xi_{j,u}^{n}\|_f^2\right) + \sum_i\left((1-\sigma_2)\bar{\eta}_i^{min} - (1+\sigma_1)\eta_i'^{max}\right)\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\ ds \\ &+ \sum_i\frac{1}{2}\eta_i'^{max}\left(\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\ ds - \int_I(\xi_{j,u}^{n}\cdot\widehat{\tau}_i)^2\ ds\right) + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n}\|_p^2 \\ &+ \left((1 - \sigma_4 - \beta_1 - \beta_2 - \Delta t\frac{2C_1}{g^2S_0\bar{k}_{min}\alpha_1^2}) - (1+\sigma_3)\frac{\rho'_{max}}{\bar{k}_{min}}\right)g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 \\ &+ \left(\frac{1}{2}g\rho'_{max} + \frac{\Delta tC_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 - \|\nabla\xi_{j,\phi}^{n}\|_p^2\right) \\ &\le \sum_i\frac{C_{P,f}C^2(D_f)}{4\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\ dt + \sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\bar{\eta}_i^{max}\|\nabla\mu_{j,u}^{n+1}\|_f^2 \\ &+ \sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\eta_{i,j}'^{max}\|\nabla\mu_{j,u}^{n}\|_f^2 + C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt \\ &+ C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt + \frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,u,t}\|_f^2\ dt \\ &+ \frac{C_{P.g}^2gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\ dt + C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right) + C\left(\|\nabla\mu_{j,u}^{n}\|_f^2 + \|\nabla\mu_{j,\phi}^{n}\|_p^2\right) \\ &+ C\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2 + \frac{g\rho'_{max}}{4\sigma_2}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt + \frac{1}{4\sigma_2}g\bar{k}^{max}\|\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \frac{g\rho'_{max}}{4\sigma_2}\|\nabla\mu_{j,\phi}^{n}\|_p^2. \end{aligned} \tag{5.29}$$

To make sure the third, fifth and ninth term on the left hand side are non-negative, we need $0 < \alpha_1, \alpha_2, \sigma_2, \sigma_4, \beta_1, \beta_2 < 1$, and

$$\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}} \le \frac{1-\sigma_2}{1+\sigma_1}, \qquad \frac{\rho'_{max}}{\bar{k}_{min}} < \frac{1}{1+\sigma_3}. \tag{5.30}$$

For $\forall\sigma_2 \in (0,1), \forall\sigma_1 > 0, \forall\sigma_3 > 0$, we can derive that $\frac{1-\sigma_2}{1+\sigma_1}, \frac{1}{1+\sigma_3} \in (0,1)$. Now if the two parameter conditions (3.2) and (3.3) are satisfied, we have $\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}}, \frac{\rho'_{max}}{\bar{k}_{min}} \in (0,1)$. Then we can easily find $\sigma_2 \in (0,1), \sigma_1 > 0$ such that $\frac{\eta_i'^{max}}{\bar{\eta}_i^{min}} = \frac{1-\sigma_2}{1+\sigma_1}$, and $\sigma_3 > 0$ such that $\frac{\rho'_{max}}{\bar{k}_{min}} < \frac{1}{1+\sigma_3}$.

Then under two the parameter conditions (5.7) and (5.8), and the time-step condition (5.6), (5.29) reduces to

$$\begin{aligned} &\frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,u}^{n}\|_f^2 + \Delta t\frac{C_2}{\beta_1^2}\left(\|\nabla\xi_{j,u}^{n+1}\|_f^2 - \|\nabla\xi_{j,u}^{n}\|_f^2\right) \\ &+ \sum_i\frac{1}{2}\eta_i'^{max}\left(\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\ ds - \int_I(\xi_{j,u}^{n}\cdot\widehat{\tau}_i)^2\ ds\right) + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n}\|_p^2 \end{aligned} \tag{5.31}$$

$$+\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta t C_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi_{j,\phi}^{n+1}\|_p^2-\|\nabla\xi_{j,\phi}^{n}\|_p^2\right)$$
$$\le\sum_i\frac{C_{P,f}C^2(D_f)}{4\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu_{j,u}^{n+1}\|_f^2$$
$$+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\eta_{i,j}'^{max}\|\nabla\mu_{j,u}^{n}\|_f^2+C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt$$
$$+C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt+\frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,u,t}\|_f^2\,dt$$
$$+\frac{C_{P.g}^2gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt+C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2+\|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right)$$
$$+C\left(\|\nabla\mu_{j,u}^{n}\|_f^2+\|\nabla\mu_{j,\phi}^{n}\|_p^2\right)+C\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2+\frac{g\rho'_{max}}{4\sigma_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt$$
$$+\frac{1}{4\sigma_3}g\bar{k}^{max}\|\nabla\mu_{j,\phi}^{n+1}\|_p^2+\frac{g\rho'_{max}}{4\sigma_4}\|\nabla\mu_{j,\phi}^{n}\|_p^2.$$

Summing up from $n=0$ to $n=N-1$ and multiplying through by Δt yields

$$\frac{1}{2}\|\xi_{j,u}^N\|_f^2+\Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi_{j,u}^N\|_f^2+\Delta t\sum_i\frac{1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^N\cdot\widehat{\tau}_i)^2\,ds \tag{5.32}$$
$$+\frac{gS_0}{2}\|\xi_{j,\phi}^N\|_p^2+\Delta t\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta tC_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^N\|_p^2$$
$$\le\frac{1}{2}\|\xi_{j,u}^0\|_f^2+\Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi_{j,u}^0\|_f^2+\Delta t\sum_i\frac{1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^0\cdot\widehat{\tau}_i)^2\,ds+\frac{gS_0}{2}\|\xi_{j,\phi}^0\|_p^2$$
$$+\Delta t\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta tC_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^0\|_p^2+\Delta t\sum_{n=1}^{N-1}\Big\{\sum_i\frac{C_{P,f}C^2(D_f)}{4\sigma_1}\eta_i'^{max}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt$$
$$+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu_{j,u}^{n+1}\|_f^2+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_1}\eta_{i,j}'^{max}\|\nabla\mu_{j,u}^{n}\|_f^2+C\Delta t\int_{t^n}^{t^{n+1}}\|u_{j,tt}\|_f^2\,dt$$
$$+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\phi_{j,tt}\|_p^2\,dt+C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla u_{j,t}\|_f^2\,dt$$
$$+\frac{5C_{P,f}^2}{4\alpha_2\nu}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,u,t}\|_f^2\,dt+\frac{C_{P.g}^2gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt+C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2+\|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right)$$
$$+C\left(\|\nabla\mu_{j,u}^{n}\|_f^2+\|\nabla\mu_{j,\phi}^{n}\|_p^2\right)+C\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2+\frac{g\rho'_{max}}{4\sigma_3}\Delta t\int_{t^n}^{t^{n+1}}\|\nabla\phi_{j,t}\|_p^2\,dt$$
$$+\frac{1}{4\sigma_3}g\bar{k}^{max}\|\nabla\mu_{j,\phi}^{n+1}\|_p^2+\frac{g\rho'_{max}}{4\sigma_4}\|\nabla\mu_{j,\phi}^{n}\|_p^2\Big\}.$$

Using interpolation inequalities, we obtain

$$\frac{1}{2}\|\xi_{j,u}^N\|_f^2+\Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi_{j,u}^N\|_f^2+\Delta t\sum_i\frac{1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^N\cdot\widehat{\tau}_i)^2\,ds \tag{5.33}$$
$$+\frac{gS_0}{2}\|\xi_{j,\phi}^N\|_p^2+\Delta t\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta tC_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^N\|_p^2$$
$$\le\frac{1}{2}\|\xi_{j,u}^0\|_f^2+\Delta t^2\frac{C_2}{\beta_1^2}\|\nabla\xi_{j,u}^0\|_f^2+\Delta t\sum_i\frac{1}{2}\eta_i'^{max}\int_I(\xi_{j,u}^0\cdot\widehat{\tau}_i)^2\,ds+\frac{gS_0}{2}\|\xi_{j,\phi}^0\|_p^2$$
$$+\Delta t\left(\frac{1}{2}g\rho'_{max}+\frac{\Delta tC_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^0\|_p^2+C\Delta t^2\|u_{j,t}\|_{2,1,f}^2+Ch^{2k}|||u_j|||_{2,k+1,f}^2+C\Delta t^2\|u_{j,tt}\|_{2,0,f}^2$$

$$+ C\Delta t^2\|\phi_{j,t}\|^2_{2,1,p} + C\Delta t^2\|\phi_{j,tt}\|^2_{2,0,p} + Ch^{2k+2}\|u_{j,t}\|_{2,k+1,f} + Ch^{2k+2}\|\phi_{j,t}\|_{2,m+1,p}$$
$$+ Ch^{2k}|||\phi_j|||^2_{2,m+1,p} + Ch^{2s+2}|||p_j|||^2_{2,s+1,f}.$$

Using triangle inequality on the error equations yields (5.9) and completes the proof. ☐

6. Stochastic Stokes-Darcy equations. In this section, we consider using the presented ensemble algorithm for solving stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(x,w)$. Note that the ensemble algorithm can also deal with uncertainties in initial conditions and the forcing terms. Here for simplicity of presentation, we only consider the example that has a random hydraulic conductivity tensor. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. Here Ω is the set of outcomes, $\mathcal{F} \in 2^{\Omega}$ is the $\sigma-$algebra of events, and $\mathcal{P} : \mathcal{F} \to [0,1]$ is a probability measure. The stochastic Stokes-Darcy system considered reads: Find the functions $u : D_f \times [0,T] \times \Omega \to \mathbb{R}^d$ $(d = 2,3)$, $p : D_f \times [0,T] \times \Omega \to \mathbb{R}$, and $\phi : D_p \times [0,T] \times \Omega \to \mathbb{R}$, such that it holds $\mathcal{P} - a.e.$ in Ω, or in other words, almost surely

$$\begin{gathered}
u_t(x,t,\omega) - \nu\Delta u(x,t,\omega) + \nabla p(x,t,\omega) = f_f(x,t), \quad \nabla\cdot u(x,t,\omega) = 0, \text{ in } D_f \times \Omega, \\
S_0\phi_t(x,t,\omega) - \nabla\cdot(\mathcal{K}(x,\omega)\nabla\phi(x,t,\omega)) = f_p(x,t), \text{ in } D_p \times \Omega, \\
\phi(x,0) = \phi_0(x), \text{ in } D_p, \text{ and } u(x,0) = u_0(x), \text{ in } D_f, \\
\phi(x,t,\omega) = 0, \text{ in } \partial D_p \backslash I \text{ and } u(x,t,\omega) = 0, \text{ in } \partial D_f \backslash I,
\end{gathered} \tag{6.1}$$

where $f_f(x,t) \in L^2(D_f), f_p(x,t) \in L^2(D_p)$. The hydraulic conductivity $\mathcal{K}(x,\omega)$ is a stochastic function, which is assumed to have continuous and bounded correlation function.

The Monte Carlo method is one of the most classical approaches for solving stochastic PDEs. It consists of repeated sampling of the input parameter and solving the corresponding deterministic PDEs using standard numerical methods, which generates identically distributed approximations of the solution. Then the approximate solutions are further analyzed to yield statistical moments or distributions. The Monte Carlo method is known to be computationally expensive as it usually requires a large number of sample points at a high resolution level. Herein we investigate incorporating the proposed ensemble algorithm with the Monte Carlo method to solve the stochastic Stokes-Darcy equations at reduced computational cost. The computation procedure is as follows.

(1) Generate a number of independently, identically distributed (i.i.d.) samples for the random hydraulic conductivity $\mathcal{K}(x,\omega_j)$, $j = 1, \cdots, J$;

(2) Apply a numerical method to solve for approximate solutions $u^{n+1}_{j,h}(x), p^{n+1}_{j,h}(x)$, $\phi^{n+1}_{j,h}(x)$, $j = 1, \cdots, J$;

(3) Output required statistical information such as the expectation of $u(x,t_n,\omega)$: $E[u(x,t_n,\omega)] \approx \frac{1}{J}\sum_{j=1}^{J} u^n_{j,h}(x)$.

Remark 6.1. *Similar procedures can also be carried out for other ensemble-based UQ methods, such as sparse grid collocation methods and non-intrusive polynomial chaos methods. The Monte Carlo method is chosen here for a simple demonstration of the effectiveness and efficiency of our ensemble algorithm.*

Let $u^n(x,\omega) = u(x,t_n,\omega)$. The error for approximating $E[u(x,t_n,\omega)]$ is then

$$\begin{aligned}
E[u^n] - \frac{1}{J}\sum_{j=1}^{J} u^n_{j,h} &= \left(E[u^n] - \frac{1}{J}\sum_{j=1}^{J} u^n_j\right) + \left(\frac{1}{J}\sum_{j=1}^{J} u^n_j - \frac{1}{J}\sum_{j=1}^{J} u^n_{j,h}\right) \\
&= \mathcal{E}^n_{MC,u} + \mathcal{E}^n_{EN,u},
\end{aligned} \tag{6.2}$$

where $\mathcal{E}^n_{MC,u}$ represents the numerical error from using the Monte Carlo method while $\mathcal{E}^n_{EN,u}$ is the error due to using the ensemble algorithm for numerical solution.

Theorem 6.2. *If the time step condition* (5.6) *holds, and the two parameter conditions* (5.7), (5.8) *all hold, then for any* $N > 0$, *there holds*

$$E\Big[\|E[u^n] - \frac{1}{J}\sum_{j=1}^{J} u^n_{j,h}\|^2\Big] \le \frac{1}{J}E[\|u^n\|^2_f] + C(h^4 + \Delta t^2). \tag{6.3}$$

Proof.

$$E[\|\mathcal{E}^n_{MC,u}\|^2] = E\left[\left(E[u^n] - \frac{1}{J}\sum_{j=1}^{J} u_j^n, E[u^n] - \frac{1}{J}\sum_{j=1}^{J} u_j^n\right)_f\right] \tag{6.4}$$

$$= E\left[\left(\frac{1}{J}\sum_{j=1}^{J}(E[u^n] - u_j^n), \frac{1}{J}\sum_{j=1}^{J}(E[u^n] - u_j^n)\right)_f\right]$$

$$= \frac{1}{J^2}\sum_{i=1}^{J}\sum_{j=1}^{J} E\left[\left(E[u^n] - u_i^n, E[u^n] - u_j^n\right)_f\right].$$

Since $u_1^n(x), u_2^n(x), \cdots, u_J^n(x)$ are i.i.d., we have

$$E\left[\left(E[u^n] - u_i^n, E[u^n] - u_j^n\right)_f\right] = 0, \qquad \text{if } i \neq j. \tag{6.5}$$

Therefore,

$$E[\|\mathcal{E}^n_{MC,u}\|^2] = \frac{1}{J^2}\sum_{j=1}^{J} E\left[\left(E[u^n] - u_j^n, E[u^n] - u_j^n\right)_f\right] \tag{6.6}$$

$$= \frac{1}{J^2}\sum_{j=1}^{J} E\left[\|E[u^n]\|_f^2 - 2(u_j^n, E[u^n])_f + \|u_j^n\|_f^2\right]$$

$$= \frac{1}{J^2}\sum_{j=1}^{J}\|E[u^n]\|_f^2 - \frac{2}{J^2}\sum_{j=1}^{J}\|E[u^n]\|_f^2 + \frac{1}{J^2}\sum_{j=1}^{J}E[\|u_j^n\|_f^2]$$

$$= \frac{1}{J}E[\|u^n\|_f^2] - \frac{1}{J}\|E[u^n]\|_f^2$$

$$\leq \frac{1}{J}E[\|u^n\|_f^2].$$

$$\|\mathcal{E}^n_{EN_u}\|_f^2 = \left\|\frac{1}{J}\sum_{j=1}^{J}u_j^n - \frac{1}{J}\sum_{j=1}^{J}u_{j,h}^n\right\|_f^2 = \left\|\frac{1}{J}\sum_{j=1}^{J}\left(u_j^n - u_{j,h}^n\right)\right\|_f^2 = \left\|\frac{1}{J}\sum_{j=1}^{J}e_{j,u}^n\right\|_f^2 \tag{6.7}$$

$$\leq \frac{1}{J}\sum_{j=1}^{J}\left\|e_{j,u}^n\right\|_f^2 \leq C(h^4 + \Delta t^2).$$

Then we have the following estimate on the expectation of the L^2 norm of the error for approximating $E[u(x, t_n, \omega)]$.

$$E\Big[\|E[u^n] - \frac{1}{J}\sum_{j=1}^{J}u_{j,h}^n\|^2\Big] = E\Big[\|\mathcal{E}^n_{MC,u} + \mathcal{E}^n_{EN,u}\|^2\Big] \tag{6.8}$$

$$\leq E\Big[\|\mathcal{E}^n_{MC,u}\|^2\Big] + E\Big[\|\mathcal{E}^n_{EN,u}\|^2\Big]$$

$$\leq \frac{1}{J}E[\|u^n\|_f^2] + C(h^4 + \Delta t^2).$$

□

7. Numerical Illustrations. In this section, we use two numerical examples to illustrate the features of the proposed ensemble scheme for the Stokes-Darcy system. The first example is to test the convergence of the ensemble algorithm with a known exact solution. The second example is to illustrate how to combine our ensemble algorithm with the Monte Carlo method to efficiently simulate the Stokes-Darcy system with a random hydraulic conductivity tensor. We show the efficiency and effectiveness of the ensemble algorithm by comparing the numerical results and computation time with those of independent, individual simulations. For both tests, we effect spatial discretization using the Taylor-Hood element pair for the Stokes equations, i.e., continuous piecewise linear and quadratic finite element spaces for the approximation of p and $\overrightarrow{u}$, respectively, and for the Darcy equation, continuous piecewise quadratic finite element space for the approximation of ϕ.

7.1. Convergence test. In this section, we test the convergence rate of our ensemble algorithm by computing the numerical error between the numerical approximation and a known exact solution. Specifically we consider the model problem on $D = [0, \pi] \times [-1, 1]$, where $D_p = [0, \pi] \times [-1, 0]$, and $D_f = [0, \pi] \times [0, 1]$. We take $\alpha_{BJS} = 1$, $\nu = 1$, $g = 1$, $S_0 = 1$, and

$$\mathcal{K} = \mathcal{K}_j = \begin{bmatrix} k_{11}^j & 0 \\ 0 & k_{22}^j \end{bmatrix}, \quad j = 1, \ldots, J,$$

where $\mathcal{K}$ is the random hydraulic conductivity tensor and $\mathcal{K}_j$ is one of the samples of $\mathcal{K}$. In this simple test, we only consider the case that k_{11}, k_{22} are random variables that are independent of spatial coordinates. The boundary condition functions and the source terms are chosen such that the following functions are the exact solutions.

$$\begin{aligned} &\phi_D = (e^y - e^{-y}) sin(x) e^t, \\ &\overrightarrow{u}_S = [\frac{k_{11}^j}{\pi} sin(2\pi y) cos(x), (-2k_{22}^j + \frac{k_{22}^j}{\pi^2} sin^2(\pi y)) sin(x)]^T e^t, \\ &p_S = 0. \end{aligned}$$

All the numerical results below are for $t = T = 1$.

We consider a group of simulations with $J = 3$ members. The three members are corresponding to different hydraulic conductivity tensors, i.e. $k_{11}^1 = k_{22}^1 = 2.21, k_{11}^2 = k_{22}^2 = 4.11, k_{11}^3 = k_{22}^3 = 6.21$. As $\mathcal{K}$ is diagonal, we use Algorithm 4.1 for computation, and thus there are no parameter conditions for both stability and convergence. In order to check the convergence order in time, we uniformly refine the mesh size h and time step size Δt from the initial mesh size 1/4 and time step size $\Delta t = h^3$. The approximation errors of the ensemble method are listed in Table 7.1, Table 7.2 and Table 7.3, for the velocity $\overrightarrow{u}$, the hydraulic head ϕ and the pressure p respectively. From these tables, we can find the rate of convergence is $O(h^3 + \triangle t) = O(h^3) = O(\Delta t)$ with respect to L^2 norms for $\overrightarrow{u}$ and ϕ, which confirms that our ensemble algorithm is first order in time convergent in both fluid velocity and hydraulic head. In this test, the time step seems to be small enough as we did not observe any instabilies. The convergence rate for the pressure p is somehow better than expected, which may be because the exact solution for the pressure vanishes, [7].

Table 7.1: Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h^3$.

h	$\Vert\overrightarrow{u}_h - \overrightarrow{u}\Vert_0^{E,1}$	rate	$\Vert\overrightarrow{u}_h - \overrightarrow{u}\Vert_0^{E,2}$	rate	$\Vert\overrightarrow{u}_h - \overrightarrow{u}\Vert_0^{E,3}$	rate
1/4	6.0818×10^{-2}	–	1.1996×10^{-1}	–	1.7971×10^{-1}	–
1/8	7.5907×10^{-3}	3.00	1.4960×10^{-2}	3.00	2.2409×10^{-2}	3.00
1/16	9.3433×10^{-4}	3.02	1.8431×10^{-3}	3.02	2.7611×10^{-3}	3.02
1/32	1.1534×10^{-4}	3.01	2.3009×10^{-4}	3.00	3.4513×10^{-4}	3.00
h	$\vert\overrightarrow{u}_h - \overrightarrow{u}\vert_1^{E,1}$	rate	$\vert\overrightarrow{u}_h - \overrightarrow{u}\vert_1^{E,2}$	rate	$\vert\overrightarrow{u}_h - \overrightarrow{u}\vert_1^{E,3}$	rate
1/4	1.2578×10^{0}	–	2.5143×10^{0}	–	3.7713×10^{0}	–
1/8	3.3416×10^{-1}	1.91	6.6823×10^{-1}	1.91	1.0023×10^{0}	1.91
1/16	8.5725×10^{-2}	1.96	1.7144×10^{-1}	1.96	2.5717×10^{-1}	1.96
1/32	2.1431×10^{-2}	2.00	4.2861×10^{-2}	2.00	6.4292×10^{-2}	2.00

Table 7.2: Errors and convergence rates of the ensemble algorithm ($J=3$) for $\triangle t = h^3$.

h	$\|\phi_h-\phi\|_0^{E,1}$	rate	$\|\phi_h-\phi\|_0^{E,2}$	rate	$\|\phi_h-\phi\|_0^{E,3}$	rate
1/4	1.1563×10^{-1}	–	4.5348×10^{-2}	–	2.2165×10^{-2}	–
1/8	1.4786×10^{-2}	2.97	5.6293×10^{-3}	3.01	2.6717×10^{-3}	3.05
1/16	1.8504×10^{-3}	2.99	6.9932×10^{-4}	3.00	3.3003×10^{-4}	3.01
1/32	2.3132×10^{-4}	3.00	8.7305×10^{-5}	3.00	3.7074×10^{-5}	3.1
h	$\|\phi_h-\phi\|_1^{E,1}$	rate	$\|\phi_h-\phi\|_1^{E,2}$	rate	$\|\phi_h-\phi\|_1^{E,3}$	rate
1/4	3.5679×10^{-1}	–	2.7501×10^{-1}	–	2.6241×10^{-1}	–
1/8	7.3695×10^{-2}	2.08	6.7565×10^{-2}	2.03	6.6760×10^{-2}	1.99
1/16	1.7274×10^{-2}	2.09	1.6874×10^{-2}	2.00	1.6824×10^{-2}	2.00
1/32	4.1129×10^{-3}	2.07	4.1156×10^{-3}	2.03	4.1061×10^{-3}	2.03

Table 7.3: Errors and convergence rates of the ensemble algorithm ($J=3$) for $\triangle t = h^3$.

h	$\|p_h-p\|_0^{E,1}$	rate	$\|p_h-p\|_0^{E,2}$	rate	$\|p_h-p\|_0^{E,3}$	rate
1/4	4.4572×10^{-1}	–	7.2784×10^{-1}	–	1.0725×10^{0}	–
1/8	5.5340×10^{-2}	3.00	9.0644×10^{-2}	3.00	1.3392×10^{-1}	3.00
1/16	6.2909×10^{-3}	3.03	9.7592×10^{-3}	3.12	1.4333×10^{-2}	3.13
1/32	7.7665×10^{-4}	3.01	1.2048×10^{-3}	3.02	1.7479×10^{-3}	3.03

7.2. Random hydraulic conductivity tensor. Next, we consider approximating the stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(\vec{x},\omega)$ that depends on spatial coordinates, using the Monte Carlo method for sampling and our ensemble algorithm for numerical simulations, as described in Section 6.

7.2.1. Diagonal hydraulic conductivity tensor. We first consider the case that the hydraulic conductivity tensor is diagonal and Algorithm 4.1 will be used for computation. We construct the random hydraulic conductivity tensor that varies in the vertical direction as follows

$$\mathcal{K}(\vec{x},\omega) = \begin{bmatrix} k_{11}(\vec{x},\omega) & 0 \\ 0 & k_{22}(\vec{x},\omega) \end{bmatrix}, \quad \text{and}$$

$$k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) = k(\vec{x},\omega) = a_0 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f}\sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)],$$

where $\vec{x}=(x,y)^T, \lambda_0 = \frac{\sqrt{\pi}L_c}{2}$, $\lambda_i = \sqrt{\pi}L_c e^{-\frac{(i\pi L_c)^2}{4}}$ for $i=1,\ldots,n_f$ and $Y_0,\ldots,Y_{2n_f}$ are uncorrelated random variables with zero mean and unit variance. In the following numerical test, we take the desired physical correlation length $L_c = 0.25$ for the random field and $a_0=1$, $\sigma=0.15$, $n_f=3$. We assume the random variables $Y_0,\ldots,Y_{2n_f}$ are independent and uniformly distributed in the interval $[-\sqrt{3},\sqrt{3}]$. Note that in this setting, the random functions $k_{11}(\vec{x},w), k_{22}(\vec{x},w)$ are guaranteed to be positive, and the corresponding $\mathcal{K}(\vec{x},\omega)$ is SPD.

The domain and parameters are the same as those in the first test. But in this test, the problem is associated with the forcing terms as follows:

$$f_p = (e^y - e^{-y})sin(x)e^t,$$
$$f_{f_1} = [(1+\nu+4\nu\pi^2)\frac{k(\vec{x},\omega)}{\pi}]sin(2\pi y)cos(x)e^t,$$
$$f_{f_2} = -2\nu k(\vec{x},\omega)cos(2\pi y)sin(x)e^t + (1+\nu)[-2k(\vec{x},\omega) + \frac{k(\vec{x},\omega)}{\pi^2}sin^2(\pi y)]sin(x)e^t.$$

The Dirichlet boundary condition

$$\phi = (e^y - e^{-y})sin(x)e^t,$$
$$\overrightarrow{u} = [\frac{k(\vec{x},\omega)}{\pi}sin(2\pi y)cos(x), (-2k(\vec{x},\omega) + \frac{k(\vec{x},\omega)}{\pi^2}sin^2(\pi y))sin(x)]^T e^t,$$

will be used on the boundary of the domain, and the initial conditions are chosen by

$$\phi = (e^y - e^{-y})sin(x),$$
$$\overrightarrow{u} = [\frac{k(\vec{x},\omega)}{\pi}sin(2\pi y)cos(x), (-2k(\vec{x},\omega) + \frac{k(\vec{x},\omega)}{\pi^2}sin^2(\pi y))sin(x)]^T.$$

We simulate the system over the time interval $[0, 0.5]$, and the uniform triangulation with mesh size $h = 1/32$ and uniform time partition with time step size $\triangle t = h^3$ are used.

We generate a set of J random samples of $\mathcal{K}$ by the Monte Carlo sampling, and run our code for simulating the ensemble of the system associated with the J realizations. We use Algorithm 4.1 for ensemble computation since $\mathcal{K}$ is diagonal and multifrontal LU factorization as the linear solver. We first check the rate of convergence with respect to the number of samples, J. As the exact solution to the stochastic Stokes-Darcy system is unknown, we take the ensemble mean of numerical solutions of $J_0 = 1000$ realizations as our exact solution (expectation) and then evaluate the approximation errors based on this. The numerical results with $J = 10, 20, 40, 80, 160$ realizations are listed in Table 7.4. Using linear regression, the errors in Table 7.4 satisfy

$$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0 \approx 0.0291J^{-0.5074}, \ |\overrightarrow{u}_h - \overrightarrow{u}|_1 \approx 0.2534J^{-0.4870},$$
$$\|p_h - p\|_0 \approx 0.0267J^{-0.5199}, \ \|\phi_h - \phi\|_0 \approx 0.0540J^{-0.4996}.$$

The values of $\|\cdot\|_0$ and $|\cdot|_1$ together with their linear regression models are plotted in Figure 7.1. It is seen that the rate of convergence with respect to J is close to -0.5, which coincides with our theoretical results.

Table 7.4: Errors of ensemble simulations.

J	10	20	40	80	100
$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^E$	9.0319×10^{-3}	6.2865×10^{-3}	4.5452×10^{-3}	3.2131×10^{-3}	2.1762×10^{-3}
$\lvert\overrightarrow{u}_h - \overrightarrow{u}\rvert_1^E$	8.2725×10^{-2}	5.7495×10^{-2}	4.3362×10^{-2}	3.0247×10^{-2}	2.1095×10^{-2}
$\|\phi_h - \phi\|_0^E$	8.0585×10^{-3}	5.5982×10^{-3}	3.9585×10^{-3}	2.8010×10^{-3}	1.8792×10^{-3}
$\lvert\phi_h - \phi\rvert_1^E$	1.7074×10^{-2}	1.2073×10^{-2}	8.5392×10^{-3}	6.1365×10^{-3}	4.2391×10^{-3}

We next test the efficiency of our ensemble algorithm. We first run the ensemble simulations with $J = 1, 10, 20, 40, 80$ realizations using our ensemble algorithm and record the respective elapsed CPU times. Then we run the simulations again with the same parameter samples using the traditional approach, i.e., solving each realization individually. A comparison of the elapsed CPU time is presented in Table 7.5, from which one can clearly see that our ensemble algorithm is much faster than the traditional approach. For example, the elapsed CPU time for the ensemble simulation using our ensemble algorithm is 473 seconds, while running simulations individually takes total 3.998×10^3 seconds when $J = 80$. The ensemble algorithm saves about 88% of the computation time.

Table 7.5: CPU elapsed time of ensemble simulations.

J	1	10	20	40	80
individual	50	498	1001	2000	3998
ensemble	56	160	281	365	473

We also plot numerical results of our ensemble algorithm and those of individual runs for comparison. The speed contours and velocity streamlines of the ensemble mean computed from both approaches at $T = 0.5$ with $J = 80$ realizations are presented in Figure 7.2. It is observed that both approaches capture the same general behavior of the flow, while our ensemble algorithm saves 88% of computation time.

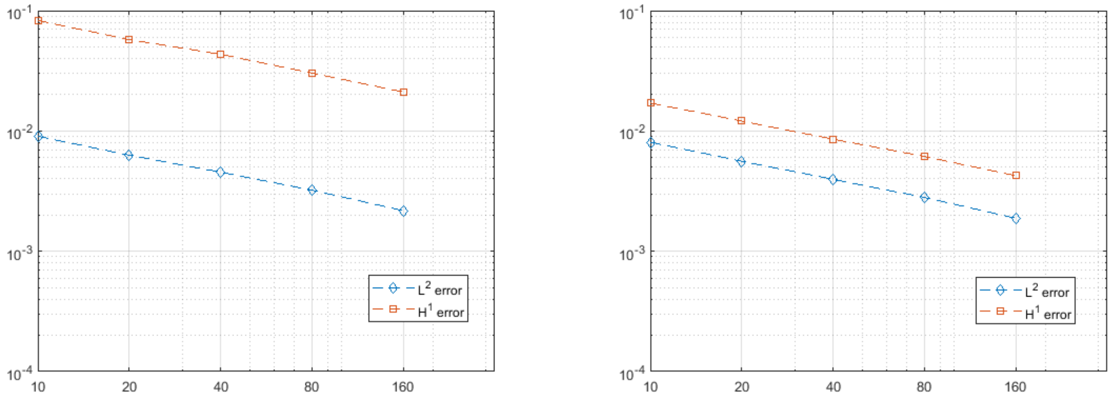

Fig. 7.1: the rate of Ensemble simulations errors is $O(1/\sqrt{J})$ for $\overrightarrow{u}$ (left) and ϕ (right).

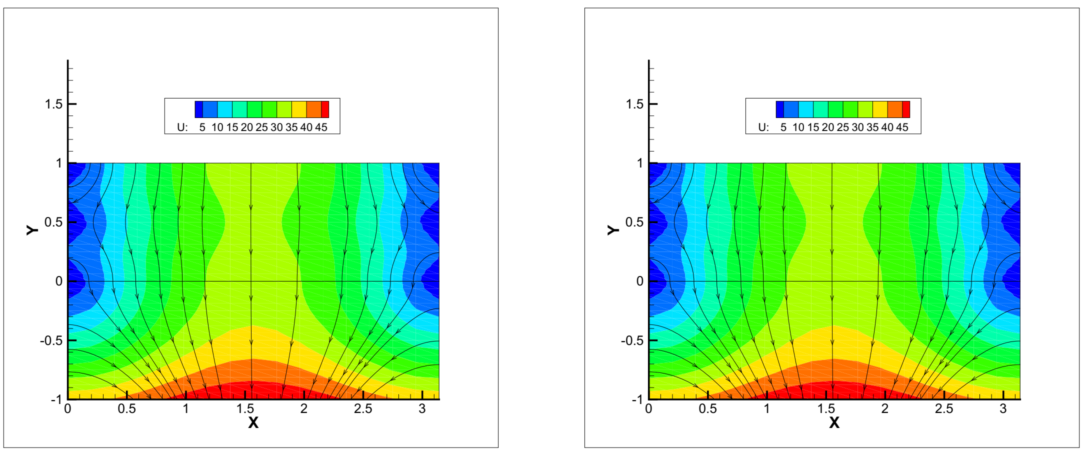

Fig. 7.2: Speed contours and velocity streamlines of the ensemble mean obtained from individual runs (left) and computation using our ensemble algorithm (right) with $J = 80$ at $T = 0.5$.

7.2.2. Non-diagonal hydraulic conductivity tensor. Here we consider the more realistic case where the hydraulic conductivity tensor is non-diagonal, for which we need to use Algorithm 2.2 for ensemble computation. Let

$$\mathcal{K}(\vec{x},\omega) = \begin{bmatrix} k_{11}(\vec{x},\omega) & k_{12}(\vec{x},\omega) \\ k_{21}(\vec{x},\omega) & k_{22}(\vec{x},\omega) \end{bmatrix},$$

where $k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) \neq 0$ and $k_{21}(\vec{x},\omega) = k_{12}(\vec{x},\omega) \neq 0$, i.e. $\mathcal{K}(\vec{x},\omega)$ is not diagonal but symmetric and

$$k_{11}(\vec{x},\omega) = k_{22}(\vec{x},\omega) = a_1 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f}\sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)],$$

$$k_{21}(\vec{x},\omega) = k_{12}(\vec{x},\omega) = a_2 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f}\sigma\sqrt{\lambda_i}[Y_i(\omega)cos(i\pi y) + Y_{n_f+i}(\omega)sin(i\pi y)].$$

The corresponding forcing term for the Darcy equation is $f_p = (1+k_{11}(\vec{x},\omega)-k_{22}(\vec{x},\omega))(e^y - e^{-y})sin(x)e^t - (k_{12}(\vec{x},\omega) + k_{21}(\vec{x},\omega))(e^y - e^{-y})cos(x)e^t$; for the Stokes equations, f_{f_1} and f_{f_2} are the same as those in Section 7.2.1. The boundary conditions and initial conditions are also the same as those in Section 7.2.1.

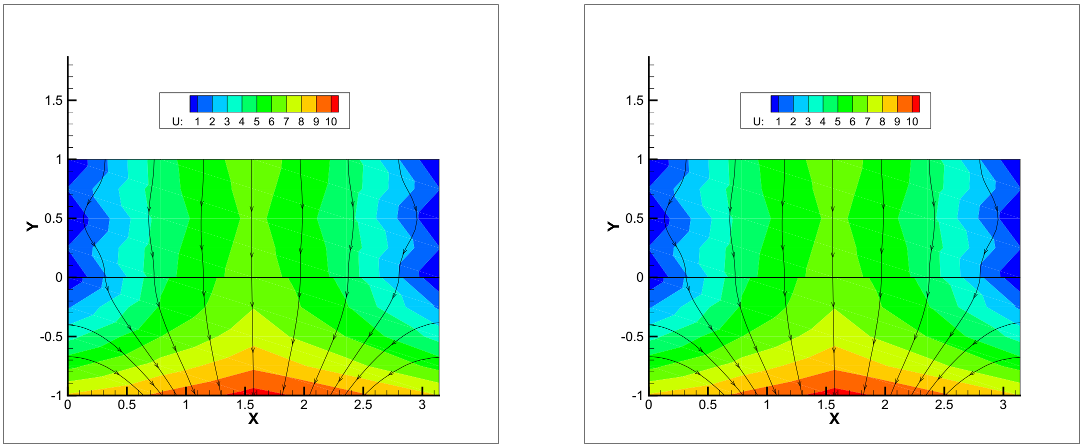

Fig. 7.3: Speed contours and velocity streamlines of the ensemble mean obtained from individual runs (left) and computation using our ensemble algorithm (right) with $J = 100$ at $T = 0.5$.

We take $a_1 = 10$ and $a_2 = 1$ so that the random hydraulic conductivity tensor $\mathcal{K}(\vec{x}, \omega)$ is SPD. We consider a group of simulations with $J = 100$ using the Monte Carlo method for sampling. We plot the numerical results of our ensemble algorithm (Algorithm 2.2) and those of individual runs for comparison. The speed contours and velocity streamlines of the ensemble mean computed from both approaches at $T = 0.5$ with $J = 100$ realizations are presented in Figure 7.3. It can be seen that both approaches capture the same general behavior of the flow while our ensemble algorithm is much faster.

8. Conclusions. We proposed an efficient, decoupled ensemble algorithm for fast computation of the Stokes-Darcy systems with different parameters sets. The proposed algorithm results in one common coefficient matrix for all realizations at each time step, which allows the use of efficient iterative or direct methods for solving the linear systems at greatly reduced computational cost. Moreover, it also decouples the original coupled problem into two sub-physics problems, which reduces the size of the linear systems to be solved and allows parallel computation of the two sub-physics problems. We proved the algorithm is long time stable and first order in time convergent under a time-step condition and two parameter conditions. We also presented an alternative algorithm for the problems in which it is easy to identify the eigenvalues of the hydraulic conductivity tensor, e.g. $\mathcal{K}$ is diagonal. We proved this algorithm is long time stable under a time-step condition, *without* any parameter conditions. Several numerical experiments were presented to show the algorithm is first-order in time convergent, illustrate how to incorporate the ensemble algorithm with the Monte Carlo method and demonstrate the efficiency and effectiveness of the ensemble algorithm. This is only the first step to study efficient ensemble algorithms in the application of uncertainty quantification (UQ) for surface-groundwater flows. The natural next step is to study higher order ensemble methods and their applications in UQ.

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