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STABILITY OF REGIME-SWITCHING DIFFUSION SYSTEMS WITH DISCRETE STATES BELONGING TO A COUNTABLE SET*

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Abstract. This work focuses on the stability of regime-switching diffusions consisting of continuous and discrete components, in which the discrete component switches in a countably infinite set and its switching rates at the current time depend on the continuous component. In contrast to the existing approach, this work provides a more practically viable approach with more feasible conditions for stability. A classical approach for asymptotic stability using Lyapunov function techniques shows that the Lyapunov function evaluated at the solution process goes to 0 as time $t \to \infty$. A distinctive feature of this paper is the precise estimate of pathwise rates of convergence, which pinpoint how fast the aforementioned convergence takes place. In addition, some examples are given to illustrate our findings.

Key words. switching diffusion, countable discrete-state space, stability, rate of convergence

AMS subject classifications. 93D05, 93D20, 34D20, 60H10, 60J60

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1. Introduction.

In the new era, because of the pressing needs in networked systems (including physical, biological, ecological, and social dynamic systems), large-scale optimization, and wired and wireless communications, many new sophisticated control systems have come into being. Hybrid systems in which discrete and continuous states coexist and interact are such a representative. In particular, taking random disturbances into consideration, the so-called regime-switching diffusion systems have drawn resurgent and increasing attention. A regime-switching diffusion is a two-component process $(X(t), \alpha(t))$, a continuous component and a discrete component taking values in a set consisting of isolated points. When the discrete component takes a value i (i.e., $\alpha(t) = i$), the continuous component $X(t)$ evolves according to the diffusion process whose drift and diffusion coefficients depend on i. Asymptotic properties of such systems such as stability have been studied intensively because of numerous applications. For example, many issues such as permanence, extinction, persistence, etc., of species in population dynamics and ecology are all linked to the stability issues.

Because many systems are in operation for a long period of time, an important problem of great interest is the stability of such systems. Many results on different types of stability have been given for switching diffusions when the state space of $\alpha(t)$ is finite (see, e.g., [9, 12, 21, 22, 24, 25]). For $\alpha(t)$ taking values in a countable state space, the stability of the underlying systems is more difficult to analyze. To the best of our knowledge, very few papers have considered the stability of switching diffusion with countable switching states. In [18], some conditions for the stability of those systems were given by approximating the generator of a continuous state-dependent

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switching process by that of a Markov chain with finite state space.

To find sufficient conditions for stability, it is desirable to find some common threads that are shared by many specific systems. Our motivation is based on the following thoughts. First, although the dynamics of $X(t)$ depend on the residence of the state of $\alpha(t)$, the structures of equations for different states of $\alpha(t)$ are not drastically different but rather similar in a certain sense. This observation suggests finding a Lyapunov function that has a similar form in different states of $\alpha(t)$. For instance, suppose that there is a Lyapunov function $V(x)$ such that in each discrete state i we have $\mathcal{L}_iV(x) \le c_iV(x)$, where $\mathcal{L}_i$ is the generator of the diffusion in regime i (more conditions and explanations for this inequality and related issues will be given in the next sections). In this case, there is a common Lyapunov function shared by all the discrete states (or the Lyapunov function is independent of the discrete states). It is well known that the sign of c_i determines the stability of the diffusion in each state i. For the regime-switching diffusion, one can expect that the stability of the system depends not only on $\{c_i\}$ but also on the generator $Q(x)$ of the switching part. A natural question is, "Under what relation between $\{c_i\}$ and $Q(x)$ is the regime-switching diffusion stable?" When the number of regimes is finite, this question has been answered relatively completely; see [9, 18]. However, it is not straightforward to answer this question for the case of the discrete states belonging to a countable state space. We aim to take the challenges here. Moreover, this paper also considers a generalization when the condition $\mathcal{L}_iV(x) \le c_iV(x)$ is replaced with a condition of the type $\mathcal{L}_iV(x) \le c_ig(V(x))$ with g being an appropriate function.

To date, much work has been devoted to the asymptotic stability of diffusions and switching diffusions. A commonly used technique is based on the Lyapunov stability argument. For example, treating asymptotic stability, much effort has been devoted to obtaining sufficient conditions under which the Lyapunov function evaluated at the solutions of the processes goes to 0 as $t \to \infty$. However, the question of how fast the Lyapunov function goes to 0 is unknown to date to the best of our knowledge. The current paper settles this issue. We estimate the convergence rate of the solution to the equilibrium point by the use of properties of the function $g(\cdot)$.

Treating switching diffusions as Markov processes, one may obtain sufficient conditions for stability by using a Lyapunov function satisfying certain properties. However, the conditions are often not directly related to the given system coefficients (such as the drifts and diffusion matrices). To obtain conditions that are based on coefficients of the systems, we examine the associated linearized (about the point of equilibrium) systems. The idea originated from the topological equivalence of the linearized systems and original nonlinear systems due to the well-known Hartman–Grobman theorem in differential equations. Here, in addition to linearizing the systems about the equilibrium point, we also replace $Q(x)$ by $Q(0)$, resulting in replacing the state-dependent switching by a continuous-time Markov chain.

The rest of this paper is organized as follows. In section 2, we formulate the equation for a regime-switching diffusion and pose appropriate conditions for the existence and uniqueness of solutions. We then provide the definitions of certain types of stability as well as give general conditions for the stability of switching. In section 3, conditions for the stability and instability of regime-switching diffusions are given. Applications of these conditions to linearizable systems are given in section 4, and examples are provided in section 5 to illustrate our findings. Section 6 is devoted to several remarks. Finally, we provide the proofs of a number of technical results in Appendix A.

2. Formulation and auxiliary results. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition; i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Throughout the paper, we work with $W(t)$, an $\mathcal{F}_t$-adapted and $\mathbb{R}^d$-valued standard Brownian motion, and $\mathfrak{p}(dt, dz)$, an $\mathcal{F}_t$-adapted Poisson measure independent of the Brownian motion $W(t)$ (see (2.3)). Suppose $b(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^{n\times d}$. Consider the two-component process $(X(t), \alpha(t))$, where $\alpha(t)$ is a pure jump process taking value in $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$, the set of positive integers, and $X(t) \in \mathbb{R}^n$ satisfies

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t). \tag{2.1}$$

We assume that the jump intensity of $\alpha(t)$ depends on the current state of $X(t)$, that is, there are functions $q_{ij}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for $i, j \in \mathbb{Z}_+$ satisfying

$$\begin{aligned} &\mathbb{P}\{\alpha(t+\Delta) = j | \alpha(t) = i, X(s), \alpha(s), s \leq t\} = q_{ij}(X(t))\Delta + o(\Delta) \quad \text{if } i \neq j,\\ &\mathbb{P}\{\alpha(t+\Delta) = i | \alpha(t) = i, X(s), \alpha(s), s \leq t\} = 1 - q_i(X(t))\Delta + o(\Delta). \end{aligned} \tag{2.2}$$

Throughout this paper, $q_{ij}(x) \geq 0$ for each $i \neq j$ and $\sum_{j\in\mathbb{Z}_+} q_{ij}(x) = 0$ for each i and all $x \in \mathbb{R}^n$. Denote $q_i(x) = \sum_{j=1,j\neq i}^{\infty} q_{ij}(x)$ (so $q_{ii}(x) = -q_i(x)$) and $Q(x) = (q_{ij}(x))_{\mathbb{Z}_+\times\mathbb{Z}_+}$. The process $\alpha(t)$ can be defined rigorously as the solution to a stochastic differential equation with respect to a Poisson random measure. For each function $x \in \mathbb{R}^n$, $i \in \mathbb{Z}_+$, let $\Delta_{ij}(x)$, $j \neq i$, be the consecutive left-closed, right-open intervals of the real line, each having length $q_{ij}(x)$. That is,

$$\begin{aligned} &\Delta_{i1}(x) = [0, q_{i1}(x)),\\ &\Delta_{ij}(x) = \left[\sum_{k=1,k\neq i}^{j-1} q_{ik}(x), \sum_{k=1,k\neq i}^{j} q_{ik}(x)\right), \quad j > 1, \quad j \neq i. \end{aligned}$$

Define $h : \mathbb{R}^n \times \mathbb{Z}_+ \times \mathbb{R} \mapsto \mathbb{R}$ by $h(x, i, z) = \sum_{j=1,j\neq i}^{\infty} (j - i)\mathbf{1}_{\{z\in\Delta_{ij}(x)\}}$. Recall that in our case, both the $\mathbb{R}^d$-valued Brownian motion $W(\cdot)$ and the Poisson random measure $\mathfrak{p}(dt, dz)$ being independent of the Brownian motion are defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and are $\mathcal{F}_t$-adapted. Then the process $\alpha(t)$ can be defined as the solution to

$$d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t-), z)\mathfrak{p}(dt, dz), \tag{2.3}$$

where $a(t-) = \lim_{s\to t^-} \alpha(s)$ and $\mathfrak{p}(dt, dz)$ is a Poisson random measure with intensity $dt \times \mathfrak{m}(dz)$ and $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{R}$. The pair $(X(t), \alpha(t))$ is therefore a solution to

$$\begin{cases} dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t),\\ d\alpha(t) = \displaystyle\int_{\mathbb{R}} h(X(t), \alpha(t-), z)\mathfrak{p}(dt, dz). \end{cases} \tag{2.4}$$

A strong solution to (2.4) on $[0, T]$ with initial data $(x, i) \in \mathbb{R}^n \times \mathbb{Z}_+$ is an $\mathcal{F}_t$-adapted process $(X(t), \alpha(t))$ such that the following hold:

- $X(t)$ is continuous and $\alpha(t)$ is cadlag (right continuous with left limits) with probability 1 (w.p.1).
- $X(0) = x$ and $\alpha(0) = i_0$.
- $(X(t), \alpha(t))$ satisfies (2.4) for all $t \in [0, T]$ w.p.1.

Let $f(\cdot,\cdot):\mathbb{R}^n\times\mathbb{Z}_+\mapsto\mathbb{R}$ be twice continuously differentiable in x. We define the operator $\mathcal{L}f(\cdot,\cdot):\mathbb{R}^n\times\mathbb{Z}_+\mapsto\mathbb{R}$ by

(2.5)
$$\begin{aligned}\mathcal{L}f(x,i)&=[\nabla f(x,i)]^\top b(x,i)+\frac{1}{2}\operatorname{tr}\Big(\nabla^2 f(x,i)A(x,i)\Big)+\sum_{j=1,j\neq i}^{\infty}q_{ij}(x)\big[f(x,j)-f(x,i)\big]\\&=\sum_{k=1}^{n}b_k(x,i)f_k(x,i)+\frac{1}{2}\sum_{k,l=1}^{n}a_{kl}(x,i)f_{kl}(x,i)+\sum_{j=1,j\neq i}^{\infty}q_{ij}(x)\big[f(x,j)-f(x,i)\big],\end{aligned}$$

where $\nabla f(x,i)=(f_1(x,i),\dots,f_n(x,i))\in\mathbb{R}^{1\times n}$ and $\nabla^2 f(x,i)=(f_{ij}(x,i))_{n\times n}$ are the gradient and Hessian of $f(x,i)$ with respect to x, respectively, with

$$\begin{aligned}&f_k(x,i)=(\partial/\partial x_k)f(x,i),\ \ f_{kl}(x,i)=(\partial^2/\partial x_k\partial x_l)f(x,i),\\&A(x,i)=(a_{ij}(x,i))_{n\times n}=\sigma(x,i)\sigma^\top(x,i),\end{aligned}$$

where $z^\top$ denotes the transpose of z. If $(X(t),\alpha(t))$ satisfies (2.4), then by modifying the proof of [19, Lemma 3, p. 104], we have the generalized Itô formula

$$f(X(t),\alpha(t))-f(X(0),\alpha(0))=\int_0^t\mathcal{L}f(X(s),\alpha(s-))ds+M_1(t)+M_2(t),$$

where $M_1(\cdot)$ and $M_2(\cdot)$ are two local martingales defined by

(2.6)
$$\begin{aligned}M_1(t)&=\int_0^t\nabla f(X(s),\alpha(s-))\sigma(X(s),\alpha(s-))dW(s),\\M_2(t)&=\int_0^t\int_{\mathbb{R}}\big[f\big(X(s),\alpha(s-)+h(X(s),\alpha(s-),z)\big)-f(X(s),\alpha(s-))\big]\mu(ds,dz),\end{aligned}$$

and $\mu(ds,dz)$ is the compensated Poisson random measure given by

$$\mu(ds,dz)=\mathfrak{p}(ds,dz)-m(dz)ds.$$

For discussion on martingales, see [10] and references therein. Throughout this paper, we assume that either one of the following assumptions is satisfied. Under either of the conditions, it is proved in [13] that (2.4) has a unique solution with given initial data. Moreover, the solution is a Markov–Feller process.

Assumption 2.1. We assume the following conditions:

1. For each $i\in\mathbb{Z}_+$, $H>0$, there is a positive constant $L_{i,H}$ such that
$$|b(x,i)-b(y,i)|+|\sigma(y,i)-\sigma(x,i)|\le L_{i,H}|x-y|$$
if $x,y\in\mathbb{R}^n$ and $|x|$, $|y|\le H$.
2. For each $i\in\mathbb{Z}_+$, there is a positive constant $\widetilde{L}_i$ such that
$$|b(x,i)|+|\sigma(x,i)|\le\widetilde{L}_i(|x|+1).$$
3. The $q_{ij}(x)$ is continuous in $x\in\mathbb{R}^n$ for each $(i,j)\in\mathbb{Z}_+^2$. Moreover,
$$M:=\sup_{x\in\mathbb{R}^n,i\in\mathbb{Z}_+}\{|q_i(x)|\}<\infty.$$

Assumption 2.2. We assume the following conditions hold:

1. For each $i \in \mathbb{Z}_+$, $H > 0$, there is a positive constant $L_{i,H}$ such that
$$|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \le L_{i,H}|x - y|$$
if $x, y \in \mathbb{R}^n$ and $|x|, |y| \le H$.
2. There is a positive constant $\widetilde{L}$ such that
$$|b(x,i)| + |\sigma(x,i)| \le \widetilde{L}(|x| + 1).$$
3. The $q_{ij}(x)$ is continuous in $x \in \mathbb{R}^n$ for each $(i,j) \in \mathbb{Z}_+^2$. Moreover, for any $H > 0$,
$$M_H := \sup_{x\in\mathbb{R}^n, |x|\le H, i\in\mathbb{Z}_+} \{|q_i(x)|\} < \infty.$$

We suppose throughout this paper that $b(0,i) = 0$ and $\sigma(0,i) = 0$ for $i \in \mathbb{Z}_+$ and give the following definitions of stability.

Definition 2.3. *The trivial solution $X(t) \equiv 0$ is said to be*

- *stable in probability if for any $h > 0$,*
$$\lim_{x\to 0} \inf_{i\in\mathbb{Z}_+} \mathbb{P}_{x,i}\big\{X(t) \le h \,\forall\, t \ge 0\big\} = 1 \textit{ and}$$
- *asymptotic stable in probability if it is stable in probability and*
$$\lim_{x\to 0} \inf_{i\in\mathbb{Z}_+} \mathbb{P}_{x,i}\left\{\lim_{t\to\infty} X(t) = 0\right\} = 1.$$

We state a general result that can be proved by well-known arguments; see [25, section 7.2].

Theorem 2.4. *Let D be a neighborhood of $0 \in \mathbb{R}^n$. Suppose there exist three functions $V(x,i) : D \times \mathbb{Z} \mapsto \mathbb{R}_+$, $\mu_1(x) : D \mapsto \mathbb{R}_+$, $\mu_2(x) : D \mapsto \mathbb{R}_+$ such that the following hold:*

- *$\mu_1(x), \mu_2(x)$ are continuous on D, and $\mu_k(x) = 0$ if and only if $x = 0$ for $k = 1, 2$.*
- *$V(x,i)$ is continuous on D and twice continuously differentiable in $\mathcal{D} \setminus \{0\}$ for each $i \in \mathbb{Z}_+$.*
- *$\mu_1(x) \le V(x,i)$ for any $(x,i) \in D \times \mathbb{Z}_+$.*

Then the following conclusions hold:

- *If $\mathcal{L}V(x,i) \le 0$ for any $(x,i) \in D \times \mathbb{Z}_+$, the trivial solution is stable in probability.*
- *If $\mathcal{L}V(x,i) \le -\mu_2(x)$ for any $(x,i) \in D \times \mathbb{Z}_+$ the trivial solution is asymptotically stable in probability.*

Definition 2.5. *Let $\widehat{\alpha}(t)$ be the Markov chain with bounded generator $Q(0)$ and transition probability $\widehat{p}_{ij}(t)$. The Markov chain $\widehat{\alpha}(t)$ is said to be*

- *ergodic if it has an invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ and*
$$\lim_{t\to\infty} \widehat{p}_{ij}(t) = \nu_j \quad \textit{for any } i, j \in \mathbb{Z}_+$$
or, equivalently,
$$\lim_{t\to\infty} \sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \nu_j| = 0 \quad \textit{for any } i \in \mathbb{Z}_+,$$

- *strongly ergodic if*

$$\lim_{t\to\infty}\sup_{i\in\mathbb{Z}_+}\left\{\sum_{j\in\mathbb{Z}_+}|\widehat{p}_{ij}(t)-\nu_j|\right\}=0,\ \textit{and}$$

- *strongly exponentially ergodic if there exist $C>0$ and $\lambda>0$ such that*

$$\sum_{j\in\mathbb{Z}_+}|\widehat{p}_{ij}(t)-\nu_j|\le Ce^{-\lambda t}\quad \textit{for any } i\in\mathbb{Z}_+,\ t\ge 0. \tag{2.7}$$

We refer the reader to [1] for some properties and sufficient conditions for the aforementioned ergodicity.

3. Certain practical conditions for stability and instability. For each $h>0$, denote by $B_h\subset\mathbb{R}^n$ the open ball centered at 0 with radius h. Throughout this section, let D be a neighborhood of 0 satisfying $D\subset B_1$. We also denote by $\widehat{\alpha}(t)$ the continuous-time Markov chain with generator $Q(0)$. Denote by $\mathcal{L}_i$ the generator of the diffusion when the discrete component is in state i, that is,

$$\mathcal{L}_iV(x)=\nabla V(x)b(x,i)+\frac{1}{2}\operatorname{tr}\Big(\nabla^2V(x)A(x,i)\Big).$$

We first state a theorem, which generalizes [9, Theorem 4.3], a result for switching diffusions when the switching takes values in a finite set.

THEOREM 3.1. *Suppose that the Markov chain $\widehat{\alpha}(t)$ is strongly exponentially ergodic with invariant probability measure $\boldsymbol{\nu}=(\nu_1,\nu_2,\dots)$ and that*

$$\sup_{i\in\mathbb{Z}_+}\sum_{j\neq i}|q_{ij}(x)-q_{ij}(0)|\to 0\ \ \textit{as}\ \ x\to 0. \tag{3.1}$$

Let D be a neighborhood of 0 and $V:D\mapsto\mathbb{R}_+$ which satisfies that $V(x)=0$ if and only if $x=0$ and that $V(x)$ is continuous on D, twice continuously differentiable in $D\setminus\{0\}$. Suppose that there is a bounded sequence of real numbers $\{c_i: i\in\mathbb{Z}_+\}$ such that

$$\mathcal{L}_iV(x)\le c_iV(x)\quad\forall\, x\in D\setminus\{0\}. \tag{3.2}$$

Then, if $\sum_{i\in\mathbb{Z}_+}c_i\nu_i<0$, the trivial solution is asymptotically stable in probability.

Proof. Let $\lambda=-\sum_{i\in\mathbb{Z}_+}c_i\nu_i$. Since $\sum_{i\in\mathbb{Z}_+}\nu_i=1$, we have $\sum_{i\in\mathbb{Z}_+}(c_i+\lambda)\nu_i=0$. Since $\widehat{\alpha}(t)$ is strongly exponentially ergodic, it follows from Lemma A.1 that there exists a bounded sequence of real numbers $\{\gamma_i: i\in\mathbb{Z}_+\}$ such that

$$\sum_{j\in\mathbb{Z}_+}q_{ij}(0)\gamma_j=\lambda+c_i\quad\text{for any } i\in\mathbb{Z}_+. \tag{3.3}$$

Since $\sum_{j\in\mathbb{Z}_+}q_{ij}(0)=0$ for any $i\in\mathbb{Z}_+$ it follows from (3.3) that

$$\sum_{j\in\mathbb{Z}_+}q_{ij}(0)\gamma_j=\sum_{j\in\mathbb{Z}_+}q_{ij}(0)(1-p\gamma_j)=-p(\lambda+c_i)\quad\text{for any } i\in\mathbb{Z}_+. \tag{3.4}$$

Since $\{\gamma_i\}$ is bounded, we can choose $p\in(0,1)$ such that

$$p|\gamma_i|\le\min\{0.25\lambda,0.5\}. \tag{3.5}$$

In view of (3.1) and (3.5), there is an $h > 0$ sufficiently small such that

$$\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)| < \frac{p\lambda}{4} \quad \forall\, x\in B_h. \tag{3.6}$$

Define the function $U(x,i): B_h\times\mathbb{Z}_+\mapsto\mathbb{R}_+$ by $U(x,i)=(1-p\gamma_i)V^p(x)$. By Itô's formula, (3.1), (3.4), and (3.6), we have

(3.7)

$$\begin{aligned}
&\mathcal{L}U(x,i)\\
&= p(1-p\gamma_i)V^{p-1}\mathcal{L}_iV(x) - \frac{p(1-p)}{2}V^{p-2}\left|V_x(x)\sigma(x,i)\right|^2 + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)q_{ij}(x)\\
&\le c_ip(1-p\gamma_i)V^{p-1} + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)q_{ij}(0) + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)|\\
&\le c_ip(1-p\gamma_i)V^{p-1} - p(\lambda+c_i)V^p(x) + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)|\\
&\le p(-\lambda-p\gamma_i)V^p(x) + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)|\\
&\le -0.75p\lambda V^p(x) + 0.25p\lambda V^p(x) = -0.5p\lambda V^p(x) \quad \text{for } (x,i)\in B_h\times\mathbb{Z}_+.
\end{aligned}$$

Using Theorem 2.4, it follows from (3.7) that the trivial solution is asymptotically stable. ∎

The hypothesis of this theorem seems to be restrictive. It requires the strongly exponential ergodicity of $Q(0)$ and the uniform convergence to 0 of $\sum_{j\ne i}|q_{ij}(x)-q_{ij}(0)|$. To treat cases in which $Q(0)$ is strongly ergodic (not exponentially ergodic) or even only ergodic, as well as to relax the condition (3.1), we need a more complicated method. Our method, which is inspired by the idea in [4], utilizes the ergodicity of $Q(0)$ and the analysis of the Laplace transform. Similar techniques that use the Laplace transform can also be seen in the large deviations theory and related applications [3, 26]. We also take a step further by estimating the pathwise rate of convergence of solutions.

Let Γ be a family of increasing and differentiable functions $g:\mathbb{R}_+\mapsto\mathbb{R}_+$ such that $g(y)=0$ if and only if $y=0$ and $\frac{dy}{dy}$ is bounded on $[0,1]$. Since $\frac{dg}{dy}(y)$ is bounded on $[0,1]$ and $g(0)=0$, it is easy to show that the function

$$G(y) := -\int_y^1 \frac{dz}{g(z)} \quad \text{on } [0,1] \tag{3.8}$$

is nonpositive and strictly decreasing and $\lim_{y\to0}G(y)=-\infty$. Its inverse $G^{-1}:(-\infty,0]\mapsto(0,1]$ satisfies

$$\lim_{t\to\infty}G^{-1}(-t)=0.$$

We state some assumptions to be used in what follows; we will also provide some lemmas whose proofs are relegated to Appendix A.

Assumption 3.2. There are functions $g\in\Gamma$, $V:D\mapsto\mathbb{R}_+$ such that the following hold:

- $V(x)=0$ if and only if $x=0$.
- $V(x)$ is continuous on D and twice continuously differentiable in $D\setminus\{0\}$.
- There is a bounded sequence of real numbers $\{c_i: i\in\mathbb{Z}_+\}$ such that

$$\mathcal{L}_iV(x)\le c_ig(V(x)) \quad \forall\, x\in D\setminus\{0\}. \tag{3.9}$$

LEMMA 3.3. *Under Assumption 3.2, for any $\varepsilon, T, h > 0$, there exists an $\tilde{h} = \tilde{h}(\varepsilon, T, h)$ such that*

$$\mathbb{P}_{x,i}\{\tau_h \le T\} < \varepsilon \quad \forall (x,i) \in B_{\tilde{h}} \times \mathbb{Z}_+,$$

where $\tau_h = \inf\{t \ge 0 : |X(t)| \ge h\}$.

LEMMA 3.4. *Let Y be a random variable, let $\theta_0 > 0$ be a constant, and suppose*

$$\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y) \le K_1.$$

Then the log-Laplace transform $\phi(\theta) = \ln \mathbb{E}\exp(\theta Y)$ is twice differentiable on $\left[0, \frac{\theta_0}{2}\right)$ and

$$\frac{d\phi}{d\theta}(0) = \mathbb{E}Y \quad \text{and} \quad 0 \le \frac{d^2\phi}{d\theta^2}(\theta) \le K_2\,, \theta \in \left[0, \frac{\theta_0}{2}\right)$$

for some $K_2 > 0$. As a result of Taylor's expansion, we have

$$\phi(\theta) \le \theta \mathbb{E}Y + \theta^2 K_2 \quad \text{for } \theta \in [0, 0.5\theta_0).$$

LEMMA 3.5. *Under the assumption $b(0,i) = 0, \sigma(0,i) = 0$, $i \in \mathbb{Z}_+$, we have*

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \ge 0\} = 0 \quad \text{for any } x \ne 0,\, i \in \mathbb{Z}_+.$$

With the auxiliary results above, we can prove our main results.

THEOREM 3.6. *Suppose that the Markov chain $\widehat{\alpha}(t)$ is ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ and Assumption 3.2 is satisfied with additional conditions*

$$\limsup_{i \to \infty} c_i < 0 \tag{3.10}$$

and

$$M_g := \sup_{x \in D, i \in \mathbb{Z}_+} \left\{ \left| \frac{V_x(x)\sigma(x,i)}{g(V(x))} \right| \right\} < \infty. \tag{3.11}$$

Then, if $\sum_{i \in \mathbb{Z}_+} c_i \nu_i < 0$, the trivial solution is asymptotically stable in probability. That is, for any $h > 0$ with $B_h \subset D$, and $\varepsilon > 0$, there exists $\delta = \delta(h, \varepsilon) > 0$ such that

$$\mathbb{P}_{x,i}\left\{X(t) < h \;\forall\, t \ge 0 \quad \text{and} \quad \lim_{t \to \infty} X(t) = 0\right\} > 1 - \varepsilon \quad \text{for any } (x,i) \in B_\delta \times \mathbb{Z}_+.$$

Moreover, there is a $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\lim_{t \to \infty} \frac{V(X(t))}{G^{-1}(-\lambda t)} \le 1\right\} > 1 - \varepsilon \quad \text{for any } (x,i) \in B_\delta \times \mathbb{Z}_+. \tag{3.12}$$

Remark 3.7. Before proceeding to the proof of the theorem, let us make a brief comment. In addition to providing sufficient conditions for asymptotic stability, a significant new element here is the rate of convergence given in (3.12). Although there are numerous treatments of stochastic stability by a host of authors for diffusions and switching diffusions, the rate result in Theorem 3.6 appears to be the first one of its kind.

Proof. The proof is divided into two steps. We first show that the trivial solution is stable in probability, and then we prove asymptotic stability and estimate the pathwise convergence rate.

Step 1: Stability. Shrinking D if necessary, we can assume without loss of generality that $V(x) \le 1$ in D. Let $h > 0$ such that $B_h \subset D$. Since $\{c_i\}$ is bounded,

$$\lim_{k\to\infty} \sum_{i\le k} c_i\nu_i = \sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0. \tag{3.13}$$

This and (3.10) show that there exists $k_0 \in \mathbb{Z}_+$ such that

$$-\lambda_1 := \sum_{i\le k_0} c_i\nu_i < 0$$

and

$$-2\lambda_2 := \sup_{i>k_0} c_i < 0.$$

Let $\overline{c} = \sup_{i\in\mathbb{Z}_+} |c_i|$, and let m_0 be a positive integer satisfying $m_0\lambda_2 > \overline{c} + M_g + 1$. Define $G(y) = -\int_y^1 g^{-1}(z)dz$. In view of Lemma 3.5, if $X(0) \ne 0$, then $X(t) \ne 0$ a.s, which leads to $g(V(X(t))) \ne 0$ a.s. Thus, we have from Itô's formula and the increasing property of $g(\cdot)$ that

$$\begin{aligned} G\big(V(X(\tau_h\wedge t))\big) &= G(V(x)) + \int_0^{\tau_h\wedge t} \frac{\mathcal{L}_{\alpha(s)}V(X(s))}{g(V(X(s)))}ds \\ &\quad - \int_0^{\tau_h\wedge t} \frac{\frac{dg}{dy}(V(X(s)))\Big|V_x(X(s))\sigma(X(s),\alpha(s))\Big|^2}{2g^2(V(X(s)))}ds \\ &\quad + \int_0^{\tau_h\wedge t} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s) \le G(V(x)) + H(t), \end{aligned} \tag{3.14}$$

where

$$H(t) = \int_0^{\tau_h\wedge t} c_{\alpha(s)}ds + \int_0^{\tau_h\wedge t} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s).$$

By Itô's formula,

$$\begin{aligned} e^{\theta H(t)} &= 1 + \int_0^{t\wedge\tau_h} e^{\theta H(s)} \left[\theta c_{\alpha(s)} + \frac{\theta^2}{2}\frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}\right] ds \\ &\quad + \theta\int_0^{t\wedge\tau_h} e^{\theta H(s)} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s). \end{aligned} \tag{3.15}$$

Let $\varsigma_k = \inf\{t \ge 0 : |H(t)| \ge k\}$. It follows from (3.15) that

$$\begin{aligned} \mathbb{E}_{x,i}e^{\theta H(t\wedge\varsigma_k)} &= 1 + \mathbb{E}_{x,i}\int_0^{t\wedge\varsigma_k\wedge\tau_h} e^{\theta H(s)} \left[\theta c_{\alpha(s)} + \frac{\theta^2}{2}\frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}\right] ds \\ &\le 1 + [\overline{c} + M_g]\mathbb{E}_{x,i}\int_0^{t\wedge\varsigma_k\wedge\tau_h} e^{\theta H(s)}ds \\ &\le 1 + [\overline{c} + M_g]\int_0^t \mathbb{E}_{x,i}e^{\theta H(s\wedge\varsigma_k)}ds. \end{aligned}$$

In view of Gronwall's inequality, for any $t \geq 0$ and $(x, i) \in B_h \times \mathbb{Z}_+$, we have

$$\mathbb{E}_{x,i} e^{\theta H(t\wedge\varsigma_k)} \leq e^{\theta[\bar{c}+M_g]t}, \quad \theta \in [-1, 1]. \tag{3.16}$$

Letting $k \to \infty$ and applying the Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}_{x,i} e^{\theta H(t)} \leq e^{\theta[\bar{c}+M_g]t}, \quad \theta \in [-1, 1]. \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} \mathbb{E}_{x,i} H(t) &\leq \mathbb{E}_{x,i} \int_0^{\tau_h \wedge t} c_{\alpha(s)} ds \\ &\leq \mathbb{E}_{x,i} \int_0^t c_{\alpha(s)} ds - \mathbb{E}_{x,i} \int_{\tau_h \wedge t}^t c_{\alpha(s)} ds \\ &\leq \mathbb{E}_{x,i} \int_0^t c_{\alpha(s)} ds + t\bar{c}\mathbb{P}_{x,i}\{\tau_h < t\}. \end{aligned} \tag{3.18}$$

Because of the ergodicity of $\widehat{\alpha}(t)$, there exists a $T > 0$ depending on k_0 such that

$$\mathbb{E}_{0,i} \int_0^t c_{\alpha(s)} ds = \mathbb{E}_i \int_0^t c_{\widehat{\alpha}(s)} ds \leq -\frac{3\lambda_1}{4} t \quad \forall\, t \geq T, i \leq k_0. \tag{3.19}$$

By the Feller property of $(X(t), \alpha(t))$, there exists an $h_1 \in (0, h)$ such that

$$\mathbb{E}_{x,i} \int_0^t c_{\alpha(s)} ds \leq -\frac{\lambda_1}{2} t \quad \forall\, t \in [T, T_2], |x| \leq h_1, i \leq k_0, \tag{3.20}$$

where $T_2 = (m_0 + 1)T$. In view of Lemma 3.3, there exists an $h_2 \in (0, h_1)$ such that

$$\bar{c}\mathbb{P}_{x,i}\{\tau_h < m_0 T + T\} \leq \frac{\lambda_1}{4}, \text{ provided } |x| \leq h_2,\ i \in \mathbb{Z}_+. \tag{3.21}$$

Applying (3.20) and (3.21) to (3.18), we obtain

$$\mathbb{E}_{x,i} H(t) \leq -\frac{\lambda_1}{4} t \quad \text{if } 0 < |x| \leq h_2,\ i \leq k_0,\ t \in [T, T_2]. \tag{3.22}$$

By Lemma 3.4, it follows from (3.17) and (3.22) that for $\theta \in [0, 0.5]$, $0 < |x| < h_2$, $i \leq k_0$, $t \in [T, T_2]$, we have

$$\begin{aligned} \ln \mathbb{E}_{x,i} e^{\theta H(t)} &\leq \theta \mathbb{E}_{x,i} H(t) + \theta^2 K \\ &\leq -\theta \frac{\lambda_1 t}{4} + \theta^2 K \end{aligned} \tag{3.23}$$

for some $K > 0$ depending on T_2, $\bar{c}$, and M_g. Letting $\theta \in (0, 0.5]$ such that

$$\theta K < \frac{\lambda_1 T}{8} \quad \text{and} \quad \theta M_g < \lambda_2, \tag{3.24}$$

we have

$$\ln \mathbb{E}_{x,i} e^{\theta H(t)} \leq -\frac{\theta \lambda_1 t}{8} \quad \text{for } 0 < |x| < h_2,\ i \leq k_0,\ t \in [T, T_2]$$

or, equivalently,

$$\mathbb{E}_{x,i} e^{\theta H(t)} \leq \exp\left\{-\frac{\theta \lambda_1 t}{8}\right\} \quad \text{for } 0 < |x| < h_2,\ i \leq k_0,\ t \in [T, T_2]. \tag{3.25}$$

In what follows, we fix a $\theta \in (0, 0.5]$ satisfying (3.24). Exponentiating both sides of the inequality $G(V(X(\tau_h \wedge t))) \le G(V(x)) + H(t)$, we have for $0 < |x| < h_2$, $i \le k_0$, $t \in [T, T_2]$ that

$$\mathbb{E}_{x,i} U(X(\tau_h \wedge t)) \le U(x) \mathbb{E}_{x,i} e^{\theta H(t)} \le U(x) \exp\left\{-\frac{\theta \lambda_1 t}{8}\right\}, \tag{3.26}$$

where $U(x) = \exp(\theta G(V(x)))$. Since $\lim_{x \to 0} G(V(x)) = -\infty$, then

$$\lim_{x \to 0} U(x) = 0. \tag{3.27}$$

Using the inequality $G(V(X(\tau_h \wedge t))) \le G(V(x)) + H(t)$ and (3.17), we have

$$\mathbb{E}_{x,i} U(X(\tau_h \wedge t)) \le U(x) \exp\{\theta[\overline{c} + M_g]t\} \quad \forall (x, i) \in B_h \times \mathbb{Z}_+,\ t \ge 0. \tag{3.28}$$

Now, let $\Delta = \inf\{U(x) : h_2 \le |x| \le h\} > 0$. Define stopping times

$$\xi = \inf\{t \ge 0 : \alpha(t) \le k_0\} \quad \text{and} \quad \zeta = \inf\{t \ge 0 : U(X(t)) \ge \Delta\}.$$

Clearly, if $X(0) \in B_h$, then $\zeta \le \tau_h$, and if $t < \zeta$, then $|X(t)| < h_2$. By computation and (3.24), we have

$$\begin{aligned} \mathcal{L}_i U(x) &\le \theta U(x) \left[c_i + [\theta - \dot{g}(V(x))] \frac{|V_x(x)\sigma(x,i)|^2}{g(V(x))} \right] \\ &\le \theta(-2\lambda_2 + \theta M_g) U(x) \\ &\le -\theta \lambda_2 U(x) \quad \text{for } 0 < |x| < h,\ i > k_0. \end{aligned}$$

It follows from Itô's formula that

$$\begin{aligned} &\mathbb{E}_{x,i} e^{\theta \lambda_2 (t \wedge \xi \wedge \zeta)} U(X(t \wedge \xi)) \\ &\quad = U(x) + \mathbb{E}_{x,i} \int_0^{t \wedge \xi \wedge \zeta} e^{\lambda_2 s} \left[\theta \lambda_2 U(X(s)) + \mathcal{L}_{\alpha(t)} U(X(s))\right] ds \\ &\quad \le U(x) \quad \text{for } 0 < |x| < h,\ i \in \mathbb{Z}_+. \end{aligned} \tag{3.29}$$

We have the following estimate for $0 < |x| < h$, $i > k_0$:

$$\begin{aligned} &\mathbb{E}_{x,i} e^{\theta \lambda_2 (T_2 \wedge \xi \wedge \zeta)} U(X(T_2 \wedge \xi \wedge \zeta)) \\ &\quad = \mathbb{E}_{x,i} \mathbf{1}_{\{\xi \wedge \zeta < m_0 T\}} e^{\theta \lambda_2 (T_2 \wedge \xi \wedge \zeta)} U(X(T_2 \wedge \xi \wedge \zeta)) \\ &\qquad + \mathbb{E}_{x,i} \mathbf{1}_{\{m_0 T \le \xi \wedge \zeta < T_2\}} e^{\theta \lambda_2 (T_2 \wedge \xi \wedge \zeta)} U(X(T_2 \wedge \xi \wedge \zeta)) \\ &\qquad + \mathbb{E}_{x,i} \mathbf{1}_{\{\xi \wedge \zeta \ge T_2\}} e^{\theta \lambda_2 (T_2 \wedge \xi \wedge \zeta)} U(X(T_2 \wedge \xi \wedge \zeta)) \\ &\quad \ge \mathbb{E}_{x,i} \mathbf{1}_{\{\xi \wedge \zeta \le m_0 T\}} U(X(\xi \wedge \zeta)) \\ &\qquad + e^{\theta \lambda_2 m_0 T} \mathbb{E}_{x,i} \mathbf{1}_{\{m_0 T \le \xi \wedge \zeta < T_2\}} U(X(\xi \wedge \zeta)) \\ &\qquad + e^{\theta \lambda_2 T_2} \mathbb{E}_{x,i} \mathbf{1}_{\{\xi \ge T_2\}} U(X(T_2)). \end{aligned} \tag{3.30}$$

Since $\mathbb{P}_{x,i}\{\zeta = 0\} = 1$ if $i \le k_0$, (3.30) holds for $0 < |x| < h$, $i \in \mathbb{Z}_+$. Noting that $U(x) \wedge \Delta \le \Delta$ for any $x \in B_h$, we have

$$\mathbb{E}\Big[U(X(T_2 \wedge \tau_h)) \wedge \Delta \Big| \zeta < m_0 T, \zeta \le \xi\Big] \le \Delta \le U(X(\zeta)) = U(X(\xi \wedge \zeta)).$$

If $\xi < \zeta$, then $U(X(\xi)) < \Delta$. By the strong Markov property of $(X(t), \alpha(t))$, (3.26), and (3.17), we have

$$\mathbb{E}\Big[U(X(T_2 \wedge \tau_h)) \wedge \Delta \Big| \xi < m_0 T \wedge \zeta\Big] \le U(X(\xi)) = U(X(\xi \wedge \zeta))$$

and

$$\mathbb{E}\Big[U(X(T_2\wedge\tau_h))\wedge\Delta\Big| m_0 T \le \xi < T_2\wedge\zeta\Big] \le U(X(\xi))e^{\theta(\overline{c}+M_g)T} = U(X(\xi\wedge\zeta))e^{\theta(\overline{c}+M_g)T}.$$

From the three estimates above, we have

$$\begin{aligned}\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\le m_0T\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] &= \mathbb{E}_{x,i}\mathbf{1}_{\{\zeta<m_0T,\zeta\le\xi\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big]\\ &\quad + \mathbb{E}_{x,i}\mathbf{1}_{\{\xi<m_0T\wedge\zeta\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big]\\ &\le \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta<m_0T\}}U(X(\xi\wedge\zeta))\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] &\le e^{\theta(\overline{c}+M_g)T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}}U(X(\xi\wedge\zeta))\\ &\le e^{\theta\lambda_2 m_0T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}}U(X(\xi\wedge\zeta)),\end{aligned} \tag{3.32}$$

where the last line follows from $m_0\lambda_2 > \overline{c} + M_g + 1$. Applying (3.31) and (3.32) to (3.30), we obtain

$$\mathbb{E}_{x,i}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le U(x) \quad \text{for any } (x, i) \in B_h \times \mathbb{Z}.$$

Since $\mathbb{E}_{x,i}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le \Delta$, we have

$$\mathbb{E}_{x,i}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le U(x) \wedge \Delta \quad \text{for any } (x, i) \in B_h \times \mathbb{Z}. \tag{3.33}$$

This together with the Markov property of $(X(t), \alpha(t))$ implies that

$$\{M(k) := \big[U(X(kT_2 \wedge \tau_h)) \wedge \Delta\big], k \in \mathbb{Z}_+\} \text{ is a supermartingale.}$$

Let $\eta = \inf\{k \in \mathbb{Z}_+ : M(k) = \Delta\}$. Clearly, $\{\eta < \infty\} \supset \{\tau_h < \infty\}$. For any $\varepsilon > 0$, if $U(x) < \varepsilon\Delta$, we have that

$$\mathbb{P}_{x,i}\{\eta < k\} \le \frac{\mathbb{E}_{x,i}M(\eta\wedge k)}{\Delta} \le \frac{U(x)}{\Delta} \le \varepsilon.$$

Letting $k \to \infty$ yields

$$\mathbb{P}_{x,i}\{\tau_h < \infty\} \le \mathbb{P}_{x,i}\{\eta < \infty\} \le \varepsilon \quad \text{if } U(x) < \varepsilon\Delta. \tag{3.34}$$

We complete the proof of this step by noting that $\{x : U(x) < \varepsilon\Delta\}$ is a neighborhood of x due to the fact that $\lim_{x\to0} U(x) = 0$.

Step 2: Asymptotic stability and pathwise convergence rate. To prove the asymptotic stability in probability, fix $h > 0$ and define $U(x)$, T_2, m_0, and Δ depending on h as in the first step. By virtue of (3.30), we have

$$\begin{aligned}\mathbb{E}_{x,i}e^{\theta\lambda_2(T_2\wedge\xi\wedge\zeta)}U(X(T_2\wedge\xi\wedge\zeta)) &\ge \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta<m_0T\}}U(X(\xi\wedge\zeta))\\ &\quad + e^{\theta\lambda_2 m_0T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}}U(X(\xi\wedge\zeta))\\ &\quad + e^{\theta\lambda_2 T_2}\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}}U(X(T_2))\\ &\ge \mathbb{E}_{x,i}\mathbf{1}_{\{\xi<m_0T,\zeta>\xi\}}U(X(\xi))\\ &\quad + e^{\theta\lambda_2 m_0T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi<T_2,\zeta>\xi\}}U(X(\xi))\\ &\quad + e^{\theta\lambda_2 T_2}\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}}U(X(T_2)).\end{aligned} \tag{3.35}$$

Recalling that $\zeta \le \tau_h$ and $X(t) < h_2$ if $t < \zeta$, we have from (3.25) and (3.28) that

$$
\begin{aligned}
\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(T_2)) &= \mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(T_2\wedge\tau_h))\\
&\le \mathbb{E}_{x,i}\mathbf{1}_{\{\xi<\zeta\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(T_2\wedge\tau_h))\\
&\le \mathbb{E}_{x,i}\left[\mathbf{1}_{\{\xi<m_0T\wedge\zeta\}}U(X(\xi))\exp\left\{-\theta\frac{\lambda}{8}(T_2-\xi)\right\}\right]\\
&\le \exp\left\{-\frac{\theta\lambda T}{8}\right\}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{\zeta\ge\xi\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(\xi))\right]
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
&\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi<T_2,\zeta\ge T_2\}}U(X(T_2))\\
&\quad\le \mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(T_2\wedge\tau_h))\\
&\quad\le \mathbb{E}_{x,i}\left[\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(\xi))\exp\left\{\theta(\overline{c}+M_g)(T_2-\xi)\right\}\right]\\
&\quad\le \exp\left\{\theta(\overline{c}+M_g)T\right\}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(\xi))\right]\\
&\quad\le \exp\{-\theta T\}\exp\left\{\theta\lambda_2m_0T\right\}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(\xi))\right].
\end{aligned}
\tag{3.37}
$$

On the other hand, we can write

$$
\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}}U(X(T_2)) = e^{-\theta\lambda_2T_2}e^{\theta\lambda_2T_2}\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}}U(X(T_2)).
\tag{3.38}
$$

Letting $p = \max\left\{\exp\left\{-\frac{\theta\lambda T}{8}\right\},\exp\{-\theta T\},\exp\{-\theta\lambda_2T_2\}\right\} < 1$ and adding (3.36), (3.37), and (3.38) side by side and then using (3.35), we have

$$
\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}U(X(T_2)) \le pU(x) \quad \text{for } (x,i)\in B_h\times\mathbb{Z}_+.
$$

By the strong Markov property of the process $(X(t),\alpha(t))$ (see, e.g., [13, 17]),

$$
\begin{aligned}
\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge 2T_2\}}U(X(2T_2)) &= \mathbb{E}_{x,i}\left[\mathbf{1}_{\{\zeta\ge T_2\}}\mathbb{E}_{X(T_2),\alpha(T_2)}\mathbf{1}_{\{\zeta\ge T_2\}}U(X(T_2))\right]\\
&\le p\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}U(X(T_2))\\
&\le p^2U(x) \quad \text{for } (x,i)\in B_h\times\mathbb{Z}_+.
\end{aligned}
$$

Continuing this way, we have

$$
\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge kT_2\}}U(X(kT_2)) \le p^kU(x) \quad \text{for } (x,i)\in B_h\times\mathbb{Z}_+.
$$

Since $2\theta < 1$, we have from (3.17) that $\mathbb{E}_{x,i}e^{2\theta H(s)} \le e^{2\theta[\overline{c}+M_g]s}$. This and the Burkholder–Davis–Gundy inequality imply that

$$
\begin{aligned}
&\mathbb{E}_{x,i}\sup_{t\le T_2}\left|\int_0^{t\wedge\tau_h} e^{\theta H(s)}\frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s)\right|\\
&\qquad\le \left[\mathbb{E}_{x,i}\int_0^{T_2\wedge\tau_h} e^{2\theta H(s)}\frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{g^2(V(X(s)))}ds\right]^{\frac{1}{2}}\\
&\qquad\le \left[M_g^2\mathbb{E}_{x,i}\int_0^{T_2} e^{2\theta H(s)}ds\right]^{\frac{1}{2}}\\
&\qquad\le \left[M_g^2\int_0^{T_2} e^{2\theta[\overline{c}+M_g]s}ds\right]^{\frac{1}{2}} := \widetilde{K}_1.
\end{aligned}
\tag{3.39}
$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}_{x,i}\sup_{t\le T_2}\left|\int_0^{t\wedge\tau_h} e^{\theta H(s)}\left[\theta c_{\alpha(s)}+\frac{\theta^2}{2}\frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}\right]ds\right|\\
&\quad\le(\overline{c}+M_g)\mathbb{E}_{x,i}\int_0^{T_2\wedge\tau_h}e^{\theta H(s)}ds\\
&\quad\le(\overline{c}+M_g)\int_0^{T_2}e^{\theta[\overline{c}+M_g]}ds:=\widetilde{K}_2.
\end{aligned}\tag{3.40}$$

It follows from (3.39) and (3.40) that

$$\begin{aligned}
\mathbb{E}_{x,i}\sup_{t\le T_2}U(X(t\wedge\tau_h))&=U(x)\mathbb{E}_{x,i}\sup_{t\le T_2}e^{\theta H_t}\\
&\le U(x)[1+\widetilde{K}_1+\widetilde{K}_2]:=U(x)\widetilde{K}_3.
\end{aligned}\tag{3.41}$$

By the strong Markov property of $(X(t),\alpha(t))$, we derive from (3.41) that

$$\begin{aligned}
&\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))\\
&\quad\le\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge kT_2\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))\\
&\quad\le\widetilde{K}_3\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge kT_2\}}U(X(kT_2))\\
&\quad\le\widetilde{K}_3U(x)\rho^k,
\end{aligned}\tag{3.42}$$

which combined with Markov's inequality leads to

$$\begin{aligned}
&\mathbb{P}_{x,i}\left\{\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))>(\rho+\widetilde{\varepsilon})^k\right\}\\
&\quad\le\frac{1}{(\rho+\widetilde{\varepsilon})^k}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))\right]\\
&\quad\le\widetilde{K}_3U(x)\frac{\rho^k}{(\rho+\widetilde{\varepsilon})^k},\quad k\in\mathbb{Z}_+,
\end{aligned}\tag{3.43}$$

where $\widetilde{\varepsilon}$ is any number in $(0,1-\rho)$. In view of the Borel–Cantelli lemma, for almost all $\omega\in\Omega$, there exists random integer $k_1=k_1(\omega)$ such that

$$\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t))<(\rho+\widetilde{\varepsilon})^k\quad\text{for any }k\ge k_1.$$

Thus, for almost all $\omega\in\{\zeta=\infty\}$, we have

$$G(V(X(t)))\le[t/T_2]\ln(\rho+\widetilde{\varepsilon})\le-\lambda t\quad\text{for }t\ge k_1T_2,\tag{3.44}$$

where $[t/T_2]$ is the integer part of t/T_2 and $\lambda=-\frac{\ln(\rho+\widetilde{\varepsilon})}{2T_2}>0$. Since $G(y)$ is decreasing and maps $(0,h]$ onto $(-\infty,0]$, (3.12) follows from (3.34) and (3.44). The proof is complete. ∎

In Theorem 3.6, under the condition that $\alpha(t)$ is merely ergodic, we need an additional condition (3.10) to obtain the stability in probability of the system. If $\alpha(t)$ is strongly ergodic, the condition (3.10) can be removed.

THEOREM 3.8. *Suppose that the following hod:*

- *For any $T > 0$ and a bounded function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$, we have*

$$\lim_{x\to 0}\sup_{i\in\mathbb{Z}_+}\left\{\left|\mathbb{E}_{x,i}\int_0^T f(\alpha(s))ds - \mathbb{E}_i\int_0^T f(\widehat{\alpha}(s))ds\right|\right\} = 0. \tag{3.45}$$

- *Assumption 3.2 is satisfied.*
- *The Markov chain $\widehat{\alpha}(t)$ is strongly ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$.*

Suppose further that (3.11) is satisfied and $\sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0$. Then the conclusion of Theorem 3.6 holds.

Remark 3.9. In Appendix A, we will prove that (3.45) holds if Assumption 3.2 and (3.1) hold.

Proof of Theorem 3.8. Let $\lambda = -\sum_{i\in\mathbb{Z}_+} c_i\nu_i$. Because of the uniform ergodicity of $\widehat{\alpha}(t)$, there exists a $T > 0$ such that

$$\mathbb{E}_{0,i}\int_0^t c_{\alpha(s)}ds = \mathbb{E}_i\int_0^t c(\widehat{\alpha}(s))ds \le -\frac{3\lambda}{4}t \quad \forall\, t \ge T,\, i \in \mathbb{Z}_+. \tag{3.46}$$

By (3.45), there exists an $h_1 \in (0, h)$ such that

$$\mathbb{E}_{x,i}\int_0^T c_{\alpha(s)}ds \le -\frac{\lambda}{2}T \quad \forall\, |x| \le h_1,\, i \in \mathbb{Z}_+. \tag{3.47}$$

In view of Lemma 3.3, there exists an $h_2 \in (0, h_1)$ such that

$$\overline{c}\mathbb{P}_{x,i}\{\tau_h < T\} \le \frac{\lambda}{4}, \text{ provided } |x| \le h_2,\, i \in \mathbb{Z}_+. \tag{3.48}$$

Applying (3.47) and (3.48) to (3.18), we have

$$\mathbb{E}_{x,i}H(T) \le -\frac{\lambda}{4}T \quad \text{ if } 0 < |x| \le h_2,\, i \in \mathbb{Z}_+. \tag{3.49}$$

Using (3.49), we can use arguments in the proof Theorem 3.6 to show that

$$\mathbb{E}_{x,i}e^{\theta H(T)} \le \exp\left\{-\frac{\theta\lambda T}{8}\right\} \qquad \text{for } 0 < |x| < h_2 \tag{3.50}$$

for a sufficiently small $\theta > 0$. This implies that

$$\mathbb{E}_{x,i}U\big(X(T\wedge\tau_h)\big) \le \exp\left\{-\frac{\theta\lambda T}{8}\right\}U(x), \tag{3.51}$$

where $U(x) = \exp(\theta G(V(x)))$. Thus, $\{M_k := U\big(X\big((kT)\wedge\tau_h\big)\big),\, k = 0, 1, \dots\}$ is a bounded supermartingale. Then we can easily obtain the stability in probability of the trivial solution. Moreover, proceeding as in Step 2 of the proof of Theorem 3.6, we can obtain the asymptotic stability as well as the rate of convergence. The arguments are actually simpler because (3.51) holds uniformly in $i \in \mathbb{Z}_+$, rather than $i \in \{1, \dots, k_0\}$ in the proof of Theorem 3.6. ∎

Remark 3.10. Consider the special case $g(y) \equiv y$. With this function, $U(X(t)) = V(X(t))$. Thus, if Assumption 3.2 holds with $g(y) \equiv y$, then the conclusions on stability in Theorems 3.6 and 3.8 are still true without the condition (3.11) because we still have $\mathbb{E}V(X(t\wedge\tau_h)) \le V(x)e^{\overline{c}t}$, which can be used in place of (3.28). However, in order to obtain asymptotic stability and rate of convergence, (3.11) is needed. In that case, if the initial value is sufficiently closed to 0, then $V(X(t))$ will converge exponentially fast to 0 with a large probability.

THEOREM 3.11. *Consider the case that the state space of $\alpha(t)$ is finite, say $\mathcal{M} = \{1,\dots,m_0\}$ for some positive integer m_0, rather than $\mathbb{Z}_+$. Suppose that $Q(0)$ is irreducible and that $\boldsymbol{\nu}$ is the invariant probability measure of the Markov chain with generator $Q(0)$. If $\sum_{i\in\mathcal{M}} c_i\nu_i < 0$, then the trivial solution is asymptotically stable in probability, and for any $\varepsilon > 0$, there are $\lambda > 0$ and $\delta > 0$ such that*

$$\mathbb{P}_{x,i}\left\{\lim_{t\to\infty}\frac{V(X(t))}{G^{-1}(-\lambda_3 t)} \le 1\right\} > 1-\varepsilon \quad \text{for any } (x,i)\in B_\delta\times\mathcal{M}.$$

We now provide some conditions for instability in probability.

THEOREM 3.12. *Suppose that the Markov chain $\widehat{\alpha}(t)$ is ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1,\nu_2,\dots)$ and that there are functions $g\in\Gamma$, $V : D\mapsto\mathbb{R}_+$ such that the following hold:*

- *$V(x) = 0$ if and only if $x = 0$.*
- *$V(x)$ is continuous on D and twice continuously differentiable in $D\setminus\{0\}$.*
- *There is a bounded sequence of real numbers $\{c_i : i\in\mathbb{Z}_+\}$ such that*

$$\mathcal{L}_i V(x) \ge c_i g(V(x)) \quad \forall\, x\in D\setminus\{0\}. \tag{3.52}$$

If (3.11) is satisfied and if $\sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0$ and $\limsup_{i\to\infty} c_i < 0$, then the trivial solution is unstable in probability.

Proof. Define $G(y) = -\int_y^1 g^{-1}(z)dz$ as in Theorem 3.6. We have from Itô's formula that

$$\begin{aligned}-G\big(V(X(\tau_h\wedge t))\big) = &- G(V(x)) - \int_0^{\tau_h\wedge t}\frac{\mathcal{L}_{\alpha(s)}V(X(s))}{g(V(X(s)))}ds\\ &+\int_0^{\tau_h\wedge t}\frac{\dot g(V(X(s)))|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}ds\\ &-\int_0^{\tau_h\wedge t}\frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s) \le -G(V(x)) + \widetilde{H}(t),\end{aligned}\tag{3.53}$$

where

$$\widetilde{H}(t) = -\int_0^{\tau_h\wedge t} c_{\alpha(s)}ds - \int_0^{\tau_h\wedge t}\frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s).$$

Then, using (3.53) and proceeding in the same manner as in the proof of Theorem 3.6 with $H(t)$ replaced with $\widetilde{H}(t)$, we can find a sufficiently small $\widetilde{\theta},\widetilde{\Delta} > 0$ and a sufficiently large $T_3 > 0$ such that

$$\mathbb{E}_{x,i}\mathbf{1}_{\{\widetilde{\zeta}\ge kT_3\}}\widetilde{U}(X(kT_2)) \le p^k\widetilde{U}(x) \quad \text{for } (x,i)\in B_h\times\mathbb{Z}_+,$$

where $\widetilde{U}(x) = \exp\{-\widetilde{\theta}G(V(x))\}$, and $\widetilde{\zeta} = \inf\{k\ge 0 : U(X(kT_3)) \le \widetilde{\Delta}^{-1}\}$. Note that, unlike $U(x)$, we have $\lim_{x\to 0}\widetilde{U}(x) = \infty$. Since $U(X(kT_3)) \ge \widetilde{\Delta}^{-1}$ if $\widetilde{\zeta}\ge k$, we have that

$$\mathbb{P}_{x,i}\{\widetilde{\zeta} = \infty\} = \lim_{k\to\infty} \mathbb{P}_{x,i}\{\widetilde{\zeta} \geq k\} = 0. \qquad \square$$

Similarly, we can obtain a counterpart of Theorem 3.8 for instability.

THEOREM 3.13. *Suppose that the Markov chain $\widehat{\alpha}(t)$ is strongly ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ and that there are functions $g \in \Gamma$ and $V : D \mapsto \mathbb{R}_+$ such that the following hold:*

- *$V(x) = 0$ if and only if $x = 0$.*
- *$V(x)$ is continuous on D and twice continuously differentiable in $D \setminus \{0\}$.*
- *There is a bounded sequence of real numbers $\{c_i : i \in \mathbb{Z}_+\}$ such that*

$$\mathcal{L}_i V(x) \geq c_i g(V(x)) \quad \forall\, x \in D \setminus \{0\}. \tag{3.54}$$

If (3.11) *and* (3.1) *are satisfied and if $\sum_{i\in\mathbb{Z}_+} c_i\nu_i > 0$, then the trivial solution is unstable in probability.*

4. Stability and instability of linearized systems. Suppose that (3.45) is satisfied and that $\widehat{\alpha}(t)$ is a strongly ergodic Markov chain.

Assumption 4.1. Suppose that for $i \in \mathbb{Z}_+$, there exist $b(i)$ and $\sigma_k(i) \in \mathbb{R}^{n\times n}$ bounded uniformly for $i \in \mathbb{Z}_+$ such that

$$\xi_i(x) := b(x,i) - b(i)x, \quad \zeta_i(x) := \sigma(x,i) - (\sigma_1(i)x, \dots, \sigma_d(i)x)$$

satisfying

$$\lim_{x\to 0} \sup_{i\in\mathbb{Z}_+} \left\{ \frac{|\xi_i(x)| \vee |\zeta_i(x)|}{|x|} \right\} = 0. \tag{4.1}$$

For $i \in \mathbb{Z}_+$, $k \in \{1, \dots, n\}$, let $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$ be the maximum eigenvalues of $\frac{b(i)+b^\top(i)}{2}$ and $\sigma_k(i)\sigma_k^\top(i)$, respectively. Similarly, denote by $\lambda_{1,i}$ and $\lambda_{2,i,k}$ the minimum eigenvalues of $\frac{b(i)+b^\top(i)}{2}$ and $\sigma_k(i)\sigma_k^\top(i)$, respectively.

Suppose that $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+$; then we claim that if

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left(\Lambda_{1,i} + \frac{1}{2}\sum_{k=1}^{n} \Lambda_{2,i,k} \right) < 0,$$

then the trivial solution is asymptotically stable.

To show that, let $\varepsilon > 0$ be sufficiently small such that

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left(\varepsilon + \Lambda_{1,i} + \frac{1}{2}\sum_{k=1}^{n} \Lambda_{2,i,k} \right) < 0. \tag{4.2}$$

Defining $V(x) = |x|^p$, carrying out the calculation, and obtaining the estimates as in those of [9, Theorem 4.3], we can find a sufficiently small $p > 0$ and $\hbar > 0$ such that

$$\mathcal{L}_i V(x) \leq p \left(\varepsilon + \Lambda_{1,i} + \frac{1}{2}\sum_{k=1}^{n} \Lambda_{2,i,k} \right) V(x) \quad \text{for } 0 < |x| < \hbar. \tag{4.3}$$

(Note that the existence of such p and $\hbar$ satisfying (4.3) uniformly for $i \in \mathbb{Z}_+$ is due to (4.3) and the boundedness of $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$.)

By (4.2) and (4.3), it follows from Theorem 3.8 that the trivial solution is asymptotic stable and that for any $\varepsilon > 0$, there exist $\delta > 0$, $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\lim_{t\to\infty} e^{\lambda t}|X(t)| \le 1\right\} \ge 1-\varepsilon \quad \text{for } (x,i) \in B_\delta \times \mathbb{Z}_+.$$

Similarly, if $\sum_{i\in\mathbb{Z}_+} \nu_i(\lambda_{1,i} + \frac{1}{2}\sum_{k=1}^n \lambda_{2,i,k}) > 0$ and if $\lambda_{1,i}$ and $\lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+$, $k = 1,\dots,n$, we have that the trivial solution is unstable. To sum up, we have the following result.

Proposition 4.2. *Let Assumption* 4.1 *be satisfied. Then the following hold:*

- *If $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+$ and*

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left(\Lambda_{1,i} + \frac{1}{2}\sum_{k=1}^n \Lambda_{2,i,k}\right) < 0,$$

  then the trivial solution is asymptotically stable in probability.
- *If $\lambda_{1,i}$ and $\lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+$, $k = 1,\dots,n$, and*

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left(\lambda_{1,i} + \frac{1}{2}\sum_{k=1}^n \lambda_{2,i,k}\right) > 0,$$

  then the trivial solution is unstable in probability.

5. Examples.

This section provides several examples.

Example 5.1. Consider a real-valued switching diffusion

$$dX(t) = b(\alpha(t))X(t)[|X(t)|^\gamma \vee 1]dt + \sigma(\alpha(t))\sin^2 X(t)dW(t), \quad 0<\gamma<1, \tag{5.1}$$

where $a \vee b = \max(a,b)$ for two real numbers a and b, and $Q(x) = \big(q_{ij}(x)\big)_{\mathbb{Z}_+\times\mathbb{Z}_+}$ with

$$q_{ij}(x) = \begin{cases} -\check{p}_1(x) & \text{if } i=j=1,\\ \check{p}_1(x) & \text{if } i=1,\ j=2,\\ -\widehat{p}_i(x) - \check{p}_i(x) & \text{if } i=j\ge 2,\\ \widehat{p}_i(x) & \text{if } i\ge 2,\ j=i-1,\\ \check{p}_i(x) & \text{if } i\ge 2,\ j=i+1. \end{cases}$$

Note that the drift grows faster than linear and the diffusion coefficient is locally like x^2 near the origin for the continuous state. Suppose that $b(i), \sigma(i), \check{p}_i(x), \widehat{p}_i(x)$ are bounded for $(x,i) \in \mathbb{R}\times\mathbb{Z}_+$ and $\check{p}_i(x), \widehat{p}_i(x)$ are continuous in $\mathbb{R}^n$ for each $i \in \mathbb{Z}_+$. It is well known (see [1, Chapter 8]) that if

$$\nu^* := \sum_{k=2}^\infty \prod_{\ell=2}^k \frac{\check{p}_{\ell-1}(0)}{\widehat{p}_\ell(0)} < \infty,$$

then $\widehat{\alpha}(t)$ is ergodic with the invariant measure $\boldsymbol{\nu}$ given by

$$\nu_1 = \frac{1}{\nu^*}, \quad \nu_k = \frac{1}{\nu^*}\prod_{\ell=2}^k \frac{\check{p}_{\ell-1}(0)}{\widehat{p}_\ell(0)}, \quad k\ge 2.$$

We suppose that

$$\sum_i b(i)\nu_i < 0 \quad \text{and} \quad \limsup_{i\to\infty} b(i) < 0.$$

We will show that the trivial solution is stable. Let $0 < \varepsilon < -\sum_i b(i)\nu_i$. Then $\sum_i [b(i)+\varepsilon]\nu_i < 0$. Let

$$V(x) = x^2.$$

We have

$$\mathcal{L}_i V(x) = 2b(i)|x|^{2+2\gamma} + \sigma^2(i)\sin^4(x).$$

Since $\gamma < 1$ and $\sigma(i)$ is bounded, there exists an $\hbar > 0$ such that $\sigma^2(i)\sin^4(x) \le \varepsilon|x|^{2+2\gamma}$, given that $|x| \le \hbar$. Then

$$\mathcal{L}_i V(x) \le [2b(i)+\varepsilon]|x|^{2+2\gamma} = [2b(i)+\varepsilon]V^{1+\gamma}(x) \quad \text{ in } [-\hbar,\hbar]\times\mathbb{Z}_+.$$

By Theorem 3.6, the trivial solution is asymptotically stable in probability. Moreover, for the function $g(y) = y^{1+\gamma}$,

$$G(y) := -\int_y^1 \frac{1}{g(z)}ds = 1 - y^{-\gamma}, \quad y \in (0,1],$$

has the inverse

$$G^{-1}(-t) = \frac{1}{[t+1]^{1/\gamma}} \quad \text{ for } t \ge 0.$$

Thus, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $(x,i) \in [0,\delta]\times\mathbb{Z}_+$, then there exists a $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\limsup_{t\to\infty} t^{1/\gamma}X^2(t) \le \lambda\right\} > 1-\varepsilon.$$

Example 5.2. This example considers a random-switching linear system of differential equations:

$$dX(t) = A(\alpha(t))X(t)dt, \tag{5.2}$$

where $A(i) \in \mathbb{R}^{n\times n}$ satisfies $\sup_{i\in\mathbb{Z}_+}\{|\lambda_i| \vee |\Lambda_i|\} < \infty$ with λ_i, Λ_i being the minimum and maximum eigenvalues of $A(i)$, respectively. Let

$$Q(x) = \begin{pmatrix} -1-\sin|x| & 1+\sin|x| & 0 & 0 & 0 & \cdots \\ 1+\sin|x| & -2-2\sin|x| & 1+\sin|x| & 0 & 0 & \cdots \\ 1+\sin|x| & 0 & -2-2\sin|x| & 1+\sin|x| & 0 & \cdots \\ 1+\sin|x| & 0 & 0 & -2-2\sin|x| & 1+\sin|x| & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By [1, Proposition 3.3], it is easy to verify that the Markov chain $\widehat{\alpha}(t)$ with generators $Q(0)$ is strongly ergodic. Solving the system of equations

$$\boldsymbol{\nu}Q(0) = 0, \quad \sum \boldsymbol{\nu}_i = 1,$$

we obtain that the invariant measure of $\widehat{\alpha}(t)$ is $(\nu_i)_{i=1}^\infty = (2^{-i})_{i=1}^\infty$. Thus, if $\sum_i \lambda_i 2^{-i} > 0$, the trivial solution to (5.2) is unstable. In case $\sum_i \Lambda_i 2^{-i} < 0$, the trivial solution to (5.2) is asymptotically stable in probability. In particular, suppose that $n = 2$ and $A(i)$ are upper triangle matrices, that is,

$$A(i) = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix}.$$

If a_i and c_i are positive for $i \geq 2$, then the system $dX(t) = A(i)X(t)dt$ is unstable. However, if $a_1, c_1 < -\sup_{i\geq 2}\{a_i, c_i\}$, then $\sum_i (a_i \vee c_i)2^{-i} < 0$. Thus, the switching differential system is asymptotically stable. The stability of the system at state 1 and the switching process become a stabilizing factor.

On the other hand, if $a_i \wedge c_i$ is negative for $i \geq 2$, then the system $dX(t) = A(i)X(t)dt$ is asymptotically stable. Suppose further that $a_1, c_1 > \sup_{i\geq 2}\{-(a_i \wedge c_i)\}$; then $\sum_i (a_i \vee c_i)2^{-i} > 0$. Under this condition, the switching differential system is unstable.

6. Further remarks. We developed a new method to provide sufficient conditions for the stability and instability in probability of a class of regime-switching diffusion systems with switching states belonging to a countable set. The conditions are based on the relation of a "switching-independent" Lyapunov function and the generator of the switching part.

Although the systems under consideration are memoryless, the main results of this paper hold if we assume that the switching intensities q_{ij} depend on the history of $\{X(t)\}$ rather than the current state of $X(t)$; see [13, 14, 15, 16] for the main properties of such processes. The problem can be formulated as follows. Let r be a fixed positive number. Denote by $\mathcal{C}$ the set of $\mathbb{R}^n$-valued continuous functions defined on $[-r, 0]$. For $\phi \in \mathcal{C}$, use the norm $\|\phi\| = \sup\{|\phi(t)| : t \in [-r, 0]\}$, and for $t \geq 0$, denote by y_t the so-called segment function (or memory segment function) $y_t = \{y(t+s) : -r \leq s \leq 0\}$. We assume that the jump intensity of $\alpha(t)$ depends on the trajectory of $X(t)$ in the interval $[t-r, t]$. That is, there are functions $q_{ij}(\cdot) : \mathcal{C} \to \mathbb{R}$ for $i, j \in \mathbb{Z}_+$ satisfying that $q_i(\phi) := \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)$ is uniformly bounded in $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$ and that $q_i(\cdot)$ and $q_{ij}(\cdot)$ are continuous such that

$$\begin{aligned} &\mathbb{P}\{\alpha(t+\Delta) = j | \alpha(t) = i, X_s, \alpha(s), s \leq t\} = q_{ij}(X_t)\Delta + o(\Delta) \quad \text{ if } i \neq j, \\ &\mathbb{P}\{\alpha(t+\Delta) = i | \alpha(t) = i, X_s, \alpha(s), s \leq t\} = 1 - q_i(X_t)\Delta + o(\Delta). \end{aligned} \tag{6.1}$$

It was proved in [13] that if either Assumption 2.1 or 2.2 is satisfied with $x, y \in \mathbb{R}^n$ replaced by $\phi, \psi \in \mathcal{C}$, then there is a unique solution to the switching diffusion (2.1) and (6.1) with a given initial value. Moreover, the process $(X_t, \alpha(t))$ has the Markov–Feller property. With slight modifications in the proofs, the theorems in section 3 still hold for system (2.1) and (6.1).

Our method can also be applied to regime-switching jump diffusion processes. Thus our method generalizes the existing results (see, e.g., [20, 23, 26]) to regime-switching jump diffusions with the random switching taking values in countable state space.

Note that there is a gap between sufficient conditions for stability and instability in Proposition 4.2. To overcome the difficulty, we need to make a polar coordinate transformation to decompose of $X(t)$ into the radial part $r(t) = |X(t)|$ and the angular part $Y(t) = X(t)/r(t)$. Then the Lyapunov exponents with respect to invariant measures of the linearized process of $(Y(t), \alpha(t))$ will determine whether or not the system is stable. This approach has been used to treat many linear and linearized stochastic systems (see, e.g., [2, 3, 5, 8]). In our setting, the switching process $\alpha(t)$

takes values in a noncompact space; thus, it is more difficult to examine invariant measures. We will address this problem together with necessary conditions of stability in a subsequent paper.

Appendix A.

Proof of Lemma 3.3. Since $V(0) = 0$ and $V(x)$ is continuous on D, we can find $h_* > 0$ such that $B_{h_*} \subset D$ and $V(x) \le 1$ for any $x \in B_{h_*}$. Because $\tau_{h_1} \le \tau_{h_2}$ if $h_1 \le h_2$, it suffices to prove the lemma for any $h \le h_*$.

Since g is continuously differentiable and $g(0) = 0$, there exists a $K_g > 0$ such that $g(z) \le K_g|z|$ for $|z| \le 1$. Thus, we have

$$\mathcal{L}_i V(x) \le K_g \sup_{i\in\mathbb{Z}_+}\{|c_i|\}V(x), \quad (x,i) \in B_{h_*} \times \mathbb{Z}_+.$$

Letting $\tilde{K} = K_g \sup_{i\in\mathbb{Z}_+}\{|c_i|\}$, by Itô's formula,

$$\begin{aligned}\mathbb{E}_{x,i}V(X(t\wedge\tau_h)) &\le V(x) + \tilde{K}\mathbb{E}_{x,i}\int_0^{t\wedge\tau_h} V(X(s))ds\\ &\le V(x) + \tilde{K}\mathbb{E}_{x,i}\int_0^t \mathbb{E}_{x,i}V(X(s\wedge\tau_h))ds.\end{aligned}$$

By the Gronwall inequality, we obtain

$$\mathbb{E}_{x,i}V(X(T\wedge\tau_h)) \le V(x)e^{KT}.$$

Letting $v_h = \inf\{V(x) : |x| = h\} > 0$, an application of Markov's inequality yields that

$$\mathbb{P}_{x,i}\{\tau_h \le T\} \le \frac{V(x)e^{KT}}{v_h}.$$

Since $V(0) = 0$ and V is continuous on D, there is an $\tilde{h} > 0$ such that

$$\mathbb{P}_{x,i}\{\tau_h \le T\} \le \frac{V(x)e^{KT}}{v_h} \le \varepsilon$$

for any $x \in B_{\tilde{h}}$ as desired. ☐

Proof of Lemma 3.4. The proof was already given in [7]. To be self-contained, we present the proof here. It is easy to show that there exists some $K_2 > 0$ such that

$$|y|^k \exp(\theta y) \le K_2(\exp(\theta_0 y) + \exp(-\theta_0 y)), \quad k = 1, 2,$$

for $\theta \in \left[0, \frac{\theta_0}{2}\right]$, $y \in \mathbb{R}$. For any $y \in \mathbb{R}$, let $\xi(y)$ be a number lying between y and 0 such that $\exp(\xi(y)) = \frac{e^y-1}{y}$. Pick out a $\theta \in \left[0, \frac{\theta_0}{2}\right]$ and let $h \in \mathbb{R}$ such that $0 \le \theta + h \le \frac{\theta_0}{2}$. Then

$$\lim_{h\to 0}\frac{\exp((\theta+h)Y) - \exp(\theta Y)}{h} = Y\exp(\theta Y) \text{ a.s.},$$

where Y is as defined in Lemma 3.4, and

$$\left|\frac{\exp((\theta+h)Y) - \exp(\theta Y)}{h}\right| = |Y|\exp(\theta Y + \xi(hY)) \le 2K_3[\exp(\theta_0 Y) + \exp(-\theta_0 Y)].$$

By the Lebesgue dominated convergence theorem,

$$\frac{d\mathbb{E}\exp(\theta Y)}{d\theta} = \lim_{h\to 0}\mathbb{E}\frac{\exp((\theta+h)Y) - \exp(\theta Y)}{h} = \mathbb{E}Y\exp(\theta Y).$$

Similarly,

$$\frac{d^2\mathbb{E}\exp(\theta Y)}{d\theta^2} = \mathbb{E}Y^2\exp(\theta Y).$$

As a result, we obtain

$$\frac{d\phi}{d\theta} = \frac{\mathbb{E}Y\exp(\theta Y)}{\mathbb{E}\exp(\theta Y)},$$

which implies that

$$\frac{d\phi}{d\theta}(0) = \mathbb{E}Y$$

and

$$\frac{d^2\phi}{d\theta^2} = \frac{\mathbb{E}Y^2\exp(\theta Y)\mathbb{E}\exp(\theta Y) - [\mathbb{E}Y\exp(\theta Y)]^2}{[\mathbb{E}\exp(\theta Y)]^2}.$$

By Hölder's inequality, we have $\mathbb{E}Y^2\exp(\theta Y)\mathbb{E}\exp(\theta Y) \geq [\mathbb{E}Y\exp(\theta Y)]^2$, and therefore

$$\frac{d^2\phi}{d\theta^2} \geq 0 \quad \forall\, \theta \in \left[0, \frac{\theta_0}{2}\right].$$

Moreover,

$$\begin{aligned}\frac{d^2\phi}{d\theta^2} &\leq \frac{\mathbb{E}Y^2\exp(\theta Y)}{\mathbb{E}\exp(\theta Y)}\\ &\leq \frac{K_3(\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y))}{\exp(\theta\mathbb{E}Y)}\\ &\leq \frac{K_3(\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y))}{\exp(-\theta_0|\mathbb{E}Y|)} := K_2,\end{aligned}$$

which concludes the proof. ∎

Proof of Lemma 3.5. Let $\overline{\tau}_n$ be the nth jump moment of $\alpha(t)$. Letting $T > 0$, in view of [11, Lemma 4.3.2], we have

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [0, T\wedge\overline{\tau}_1]\} = 0 \quad \text{ for any } x \neq 0,\, i \in \mathbb{Z}_+.$$

Since $X(T\wedge\overline{\tau}_1) \neq 0$ a.s., applying [11, Lemma 4.3.2] again yields

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [T\wedge\overline{\tau}_1, T\wedge\overline{\tau}_2]\} = 0 \quad \text{ for any } x \neq 0,\, i \in \mathbb{Z}_+.$$

Continuing this way, we have

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [0, T\wedge\overline{\tau}_n]\} = 0 \quad \text{ for any } x \neq 0,\, i \in \mathbb{Z}_+,\, n \in \mathbb{Z}_+. \tag{A.1}$$

In [13, Theorems 3.1 and 3.3], we have that $\lim_{n\to\infty}\overline{\tau}_n = \infty$. This and (A.1) imply that

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [0, T]\} = 0 \quad \text{ for any } x \neq 0,\; i \in \mathbb{Z}_+,\; n \in \mathbb{Z}_+.$$

Since T is taken arbitrarily, we obtain the desired result. ∎

Lemma A.1. *If the Markov chain $\widehat{\alpha}(t)$ is strongly exponentially ergodic with generator $\widehat{Q}$ and invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)^\top$, then if $\mathbf{b} = (b_1, b_2, \dots)^\top$ is bounded satisfying $\sum_i \nu_i b_i = 0$, then there exists a bounded vector $\mathbf{c} = (c_1, c_2, \dots)^\top$ such that $b_i = \sum_j \widehat{q}_{ji} c_j$.*

Proof. Let $\widehat{P}(t) = \widehat{p}_{ij}(t)$, where $\widehat{p}_{ij}(t) = \mathbb{P}\{\widehat{\alpha}(t) = j|\alpha(0) = i\}$, the transition matrix of $\widehat{\alpha}(t)$. Let $\mathbf{c} = (c_1, c_2, \dots)^\top$, where $c_i = \int_0^\infty [\nu_j b_j - \widehat{P}_{ij}(t) b_i] dt$. In view of (2.7), it is easy to see that $\mathbf{c}$ is bounded. Let $\mathbb{1} = (1, 1, \dots)$. We have

$$\begin{aligned}\widehat{Q}\mathbf{c} &= \int_0^\infty \left[\widehat{Q}\boldsymbol{\nu}\mathbb{1}\mathbf{b} - \widehat{Q}\widehat{P}(t)\mathbf{b}\right] dt \\ &= -\int_0^\infty \widehat{Q}\widehat{P}(t)\mathbf{b}dt \\ &= -\int_0^\infty \widehat{P}(t)\mathbf{b}dt = -\widehat{P}(t)\mathbf{b}\Big|_0^\infty \\ &= -\mathbb{1}\boldsymbol{\nu}\mathbf{b} + \mathbf{b} = \mathbf{b}.\end{aligned}$$

∎

LEMMA A.2. *Suppose that Assumption* 3.2 *and* (3.1) *hold. Then, for any* $T > 0$ *and a bounded function* $f : \mathbb{Z}_+ \mapsto \mathbb{R}$*, we have*

$$\lim_{x\to 0} \sup_{i\in\mathbb{Z}_+, t\in[0,T]} \{|\mathbb{E}_{x,i} f(\alpha(t)) - \mathbb{E}_i f(\widehat{\alpha}(t))|\} = 0. \tag{A.2}$$

Proof. By the basic coupling method (see, e.g., [6, p. 11]), we can consider the joint process $(X(t), \alpha(t), \widehat{\alpha}(t))$ as a switching diffusion where the diffusion $X(t) \in \mathbb{R}^n$ satisfies

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t) \tag{A.3}$$

and the switching part $(\alpha(t), \widehat{\alpha}(t)) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ has the generator $\widetilde{Q}(X(t))$ which is defined by

$$\begin{aligned}\widetilde{Q}(x)\widetilde{f}(k,l) &= \sum_{j,i\in\mathbb{Z}_+} \widetilde{q}_{(k,l)(j,i)}(x) \left(\widetilde{f}(j,i) - \widetilde{f}(k,l)\right) \\ &= \sum_{j\in\mathbb{Z}_+} [q_{kj}(x) - q_{lj}(0)]^+ (\widetilde{f}(j,l) - \widetilde{f}(k,l)) \\ &\quad + \sum_{j\in\mathbb{Z}_+} [q_{lj}(0) - q_{kj}(x)]^+ (\widetilde{f}(k,j) - \widetilde{f}(k,l)) \\ &\quad + \sum_{j\in\mathbb{Z}_+} [q_{kj}(x) \wedge q_{lj}(0)](\widetilde{f}(j,j) - \widetilde{f}(k,l)).\end{aligned} \tag{A.4}$$

In what follows, we use the notation $\mathbb{E}_{x,i,j}$ and $\mathbb{P}_{x,i,j}$ to denote the corresponding conditional expectation and probability for the coupled process $(X(t), \alpha(t), \widehat{\alpha}(t))$ conditioned on $(X(0), \alpha(0), \widehat{\alpha}(0)) = (x, i, j)$. Let $\vartheta = \inf\{t \geq 0 : \alpha(t) \neq \widehat{\alpha}(t)\}$. Define $\tilde{g} : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{R}$ by $\widetilde{g}(k,l) = \mathbf{1}_{\{k=l\}}$. By the definition of the function $\widetilde{g}$, we have

$$\begin{aligned}\widetilde{Q}(x)\widetilde{g}(k,k) &= \sum_{j\in\mathbb{Z}_+, j\neq k} [q_{kj}(x) - q_{kj}(0)]^+ + \sum_{j\in\mathbb{Z}_+, j\neq k} [q_{kj}(0) - q_{kj}(x)]^+ \\ &= \sum_{j\in\mathbb{Z}_+, j\neq k} |q_{kj}(x) - q_{kj}(0)| =: \Xi(x,k).\end{aligned} \tag{A.5}$$

For any $\varepsilon > 0$, let $h > 0$ such that $B_h \in D$ and $\sup_{(x,k)\in B_h\times\mathbb{Z}_+} \Xi(x,k) < \frac{\varepsilon}{2T}$. Applying Itô's formula and noting that $\alpha(t) = \widehat{\alpha}(t)$, $t \leq \vartheta$, we obtain that

$$\begin{aligned}\mathbb{P}_{x,i,i}\{\vartheta \le T\wedge\tau_h\} &= \mathbb{E}_{x,i,i}\widetilde{g}\left(\alpha(\vartheta\wedge T\wedge\tau_h),\widehat{\alpha}(\vartheta\wedge T\wedge\tau_h)\right)\\ &= \mathbb{E}_{x,i,i}\int_0^{\vartheta\wedge T\wedge\tau_h}\widetilde{Q}(X(t))\widetilde{g}(\alpha(t),\widehat{\alpha}(t))dt\\ &= \mathbb{E}_{x,i,i}\int_0^{\vartheta\wedge T\wedge\tau_h}\Xi(X(t),\alpha(t))dt\\ &\le T\sup_{(x,i)\in B_h\times\mathbb{Z}_+}\Xi(x,k)\le\frac{\varepsilon}{2}.\end{aligned}\tag{A.6}$$

Thus, in view of Lemma 3.3, there is a $\delta > 0$ such that $\mathbb{P}_{x,i,i}\{\tau_h \le T\} \le \frac{\varepsilon}{2}$. This and (A.6) lead to

$$\mathbb{P}_{x,i,i}\{\vartheta\wedge\tau_h \le T\} \le \mathbb{P}_{x,i,i}\{\vartheta \le T\wedge\tau_h\} + \mathbb{P}_{x,i,i}\{\tau_h \le T\} \le \varepsilon.$$

We have that

$$\begin{aligned}|\mathbb{E}_{x,i}f(\alpha(t)) - \mathbb{E}_{0,i}f(\alpha(t))| &= |\mathbb{E}_{x,i,i}\left[f(\alpha(t)) - f(\widehat{\alpha}(t))\right]|\\ &= \left|\mathbb{E}_{x,i,i}\mathbf{1}_{\{\vartheta\wedge\tau_h\le t\}}\left[f(\alpha(t)) - f(\widehat{\alpha}(t))\right]\right|\\ &\le 2M_f\mathbb{P}_{x,i,i}\{\vartheta\wedge\tau_h\le t\} \le 2M_f\varepsilon \quad \text{for } t\in[0,T],\end{aligned}$$

where $M_f = \sup_{i\in\mathbb{Z}_+}|f(i)|$. The lemma is proved. □

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