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# Nonlocal cosmology II. Cosmic acceleration without fine tuning or dark energy

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**Abstract.** We present an improved version of our original cosmological model to explain the current phase of cosmological acceleration without resorting to a cosmological constant or any other mass scale. Like the original, this phenomenological approach is based on an effective quantum gravitational action, but now depends on the original nonlocal dimensionless scalar  $X = \square^{-1}R$  only through  $Y = \square^{-1}g^{\mu\nu}X_{,\mu}X_{,\nu}$ . Both  $X$  and  $Y$  are quiescent during the radiation-dominated ( $R = 0$ ) era, both only grow logarithmically during matter dominance, and neither affects the propagation of gravitational radiation. However, while  $X$  has the same sign for gravitationally bound systems as for cosmology, we show that the sign of  $Y$  differs for the two cases: it is positive for cosmology and negative for strongly gravitationally bound systems. We can therefore enforce the  $\Lambda$ CDM expansion history by making a suitable choice of the nonlocal distortion function  $f(Y)$  for  $Y > 0$ , while ensuring that there is no change in the heavily constrained solar system phenomenology simply by making  $f$  vanish for  $Y < 0$  without discontinuity. The required  $f(Y > 0)$  is determined numerically to have a strikingly simple exponential form.

**Keywords:** dark energy theory, modified gravity, quantum cosmology

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## 1 Introduction

This work is a continuation, and shares the basis and philosophy, of our original cosmological model [1–3]. The present cosmological acceleration phase of the universe [4] is a major, if originally unexpected, feature of late time expansion. An explanation not invoking new physics or fine tuning is clearly to be preferred; ours was a nonlocal one, based on a function of the dimensionless scalar  $X[g] = \square^{-1}R$ . The argument was that it represents current effects of the necessarily abundant infrared gravitons in the early universe [5, 6].

Extensive studies have been made of the theory’s cosmological perturbations [7–12]. There have also been studies of future cosmological evolution [13], solar system constraints [14], and the generation of gravitational radiation [15].<sup>1</sup>

Our original model assumed that  $X[g]$  had opposite signs in the cosmological (–) and the (smaller scale) gravitationally bound (+) contexts. That would prevent — unwanted — effects in the latter. However, it was recently pointed out that  $X[g]$  is negative definite [24]. We overcome this difficulty by a simple modification: replacing  $X[g]$  by the (equally nonlocal)  $Y[g] \equiv \square^{-1}[g^{\mu\nu}\partial_\mu X\partial_\nu X]$ , removes the problem without losing the explanation of accelerated expansion: while both  $X[g]$  and  $Y[g]$  vanish during radiation domination ( $R = 0$ ), and only grow slowly thereafter,  $Y[g]$  does have opposite signs in strongly bound matter ( $Y < 0$ ) and in the large ( $Y > 0$ ); so we merely define the nonlocal distortion function  $f(Y)$  to vanish for  $Y < 0$ , and have the proper details for  $Y > 0$ , thus restoring the desired behavior throughout.<sup>2</sup> An additional, highly desirable property of both the original and the new theories is that there is no change in the constrained propagation of gravitational radiation [25].

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<sup>1</sup>A similar model has also been proposed which is based on the dimensionful nonlocal scalar  $\square^{-2}R$  [16, 17]. Many studies have been made of the phenomenology of this model [18–23].

<sup>2</sup>It turns out that enforcing the  $\Lambda$ CDM requires  $f(Y) \rightarrow Y^2 \ln(Y)$  as  $Y$  approaches zero from above. This means that making  $f(Y < 0) = 0$  leads to no discontinuity in either  $f(Y)$  or  $f'(Y)$  at  $Y = 0$ .

Section 2 defines our model and discusses how it might emerge from fundamental considerations. It also explains why the new nonlocal scalar  $Y[g]$  changes sign from cosmological to gravitationally bound systems. Section 3 gives an explicit numerical determination of the nonlocal distortion function  $f(Y)$  to reproduce the  $\Lambda$ CDM expansion history without a cosmological constant. It also derives an amazingly simple exponential fit to  $f(Y)$ . Section 4 presents conclusions.

## 2 The new model

In this section, we define and discuss the improved model. The original problem and its remedy are explained. We close with comments on its possible origin in a more fundamental setting.

### 2.1 Defining the new model

Our two nonlocal scalars are

$$X[g] \equiv \frac{1}{\square} R, \quad Y[g] \equiv \frac{1}{\square} \left( g^{\mu\nu} \partial_\mu X[g] \partial_\nu X[g] \right); \quad \square \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right), \quad (2.1)$$

where  $\square^{-1}$  is defined by retarded boundary conditions which require that  $X[g](x)$ ,  $Y[g](x)$  and their first derivatives all vanish on the initial value surface. Our nonlocal modification is defined by the distortion function  $f(Y)$ ,

$$\mathcal{L}_{\text{nonlocal}} \equiv \frac{1}{16\pi G} R \left[ 1 + f(Y[g]) \right] \sqrt{-g}. \quad (2.2)$$

Just as the original model could be localized through the introduction of two auxiliary scalar fields [26], the new model requires four auxiliaries,<sup>3</sup>

$$\mathcal{L}_{\text{local}} \equiv \frac{\sqrt{-g}}{16\pi G} \left[ R \left( 1 + U + f(Y) \right) + \left( \partial_\mu X \partial_\nu U + \partial_\mu Y \partial_\nu V + V \partial_\mu X \partial_\nu X \right) g^{\mu\nu} \right]. \quad (2.3)$$

It is important to bear in mind that the auxiliary scalars do *not* have arbitrary initial value data, which would result in new degrees of freedom, half being ghosts [3, 27]. Instead, all obey retarded boundary conditions, hence introduce no excitations.

The  $U$ ,  $V$  Lagrange multipliers, whose variations lead to the equations which, with retarded boundary conditions, define  $X[g]$  and  $Y[g]$ ,

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta S}{\delta U} = -\square X + R = 0 \quad \implies \quad X[g] = \frac{1}{\square} R, \quad (2.4)$$

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta S}{\delta V} = -\square Y + g^{\mu\nu} X_{,\mu} X_{,\nu} = 0 \quad \implies \quad Y[g] = \frac{1}{\square} \left[ g^{\mu\nu} X_{,\mu} X_{,\nu} \right]. \quad (2.5)$$

Variation with respect to  $X$  and  $Y$  leads to similar equations for  $U$  and  $V$ , which are also solved with retarded boundary conditions,

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta S}{\delta X} = -\square U - 2D_\mu (V D^\mu X) = 0 \quad \implies \quad U[g] = -\frac{2}{\square} D_\mu (V D^\mu X), \quad (2.6)$$

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta S}{\delta Y} = -\square V + R f'(Y) = 0 \quad \implies \quad V[g] = \frac{1}{\square} R f'(Y). \quad (2.7)$$

<sup>3</sup>We shall abuse the notation slightly by using the same symbols  $X$  and  $Y$  for auxiliary scalars in the localized model (2.3) as for their retarded solutions (2.1).

Note that all four auxiliary scalars propagate along the characteristic curves of the scalar d'Alembertian  $\square$ , so the sound speed should agree with the speed of light, alleviating the problems which can occur in some modified gravity theories [28]. The gravitational field equations are,

$$\begin{aligned} & \left( G_{\mu\nu} + g_{\mu\nu} \square - D_\mu D_\nu \right) \left( 1 + U + f(Y) \right) + \partial_{(\mu} X \partial_{\nu)} U + \partial_{(\mu} Y \partial_{\nu)} V \\ & + V \partial_\mu X \partial_\nu X - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \left( \partial_\rho X \partial_\sigma U + \partial_\rho Y \partial_\sigma V + V \partial_\rho X \partial_\sigma X \right) = 8\pi G T_{\mu\nu} ; \end{aligned} \quad (2.8)$$

here parenthesized indices are symmetrized and  $T_{\mu\nu}$  is the matter stress-energy tensor *without* dark energy.

## 2.2 The signs of $Y$

To determine the sign of  $Y$  in regions of bound matter, we assume the metric there to be (quasi-) static, i.e., time-independent and diagonal; it could more generally be stationary,  $g_{0i} \neq 0$ , but the same considerations should still hold with a bit more matrix detail. Then  $g^{\mu\nu} X_{,\mu} X_{,\nu} \rightarrow g^{ij} X_{,i} X_{,j}$  is positive in our  $(-+++)$  convention. We argue next that, instead,  $\square^{-1}$  is negative so that there  $Y < 0$  also. Recall that in flat space,  $\square^{-1}$  acting on a time independent source reduces, upon time-integration, to  $\nabla^{-2}$ . But our  $\square$  has the flat-space  $-\partial_t^2$  form, namely  $-g^{00}[-g_{00}\nabla^2 - \partial_t^2]$ , except for the overall  $-g^{00}$  and the (irrelevant) metric dependence of our  $\nabla^2$ ; there is also a (strictly positive) factor  $\sqrt{-g} = \sqrt{3}g\sqrt{-g_{00}}$  upstairs. Thus, after time integration, our net inverse Laplacian is  $\frac{1}{\nabla^2}$  (since  $g^{00}g_{00} = 1$ ), a negative quantity when operating on the positive  $g^{ij} X_{,i} X_{,j}$ . In section 3 we show, by explicit computation, that  $Y$  is positive in the purely time dependent cosmological region.

Another, global, way of understanding the signs of both  $X$  and  $Y$  is by taking the flat space limit. The retarded Green's function  $G_{\text{ret}}[g](x; x')$  which implements  $\square^{-1}$  reduces, when  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ , to the usual flat

$$G_{\text{ret}}[\eta](x; x') = -\frac{\delta(t-t' - \frac{1}{c}\|\vec{x}-\vec{x}'\|)}{4\pi\|\vec{x}-\vec{x}'\|}. \quad (2.9)$$

This simple form makes it easy to derive explicit expressions for  $X$  and  $Y$ . For example, if the Ricci scalar is a positive constant and the initial value surface is at  $t = 0$ ,

$$R(x) = \frac{1}{\ell^2} \implies X(x) = -\frac{c^2 t^2}{2\ell^2}, \quad Y(x) = +\frac{c^4 t^4}{12\ell^4}. \quad (2.10)$$

This situation in which the time dependence of  $X$  dominates is relevant to cosmology. On the other hand, suppose the Ricci scalar is a positive constant within a sphere of radius  $\ell$ , and we consider some time  $t$  much larger than either  $\ell$  or  $\|\vec{x}\| \equiv r$ ,

$$R(x) = \frac{\theta(\ell - \|\vec{x}\|)}{\ell^2} \implies X(x) = -\frac{1}{3} \left[ \frac{3}{2} - \frac{r^2}{2\ell^2} \right] \theta(\ell - r) - \frac{\ell}{3r} \theta(r - \ell). \quad (2.11)$$

The result for  $X$  is still negative definite, but the space derivatives dominate,

$$R(x) = \frac{\theta(\ell - \|\vec{x}\|)}{\ell^2} \implies g^{\mu\nu} \partial_\mu X \partial_\nu X = \left( \frac{\partial X}{\partial r} \right)^2 = \frac{r^2}{9\ell^4} \theta(\ell - r) + \frac{\ell^2}{9r^4} \theta(r - \ell). \quad (2.12)$$

That reverses the sign of  $Y$  from the cosmological case (2.10),

$$R(x) = \frac{\theta(\ell - \|\vec{x}\|)}{\ell^2} \implies Y(x) = -\frac{1}{18} \left[ \frac{3}{2} - \frac{r^4}{10\ell^4} \right] \theta(\ell - r) - \frac{1}{18} \left[ \frac{12\ell}{5r} - \frac{\ell^2}{r^2} \right] \theta(r - \ell). \quad (2.13)$$

The arguments we have given for  $Y < 0$  are based on ignoring time derivatives of  $X[g]$ , and should apply to systems which are strongly gravitationally bound. This should certainly be valid for the solar system, and even for the formation of large scale structure because peculiar velocities are expected to be small. However, around denser and/or warmer objects it may not be valid to ignore time derivatives of  $X$ , and this may modify the gravitational forces in these systems. This may perhaps turn out to be a critical test of our model.

### 2.3 Unwanted homogeneous solutions

For us the key insight of [24] was that  $X[g]$  is negative definite. However, it is worth commenting on the *additional* argument by those authors that matching with the cosmological solution results in a small time-dependent homogeneous contribution to  $X[g]$  inside gravitationally bound systems, extending over Megaparsec distances and even into strongly bound regions such as the solar system. This would induce a small time dependence in the effective Newton constant, which violates the constraints from lunar laser ranging.

The effective Newton constant in our model is  $G \times [1 + U + f(Y)]$  and it is obvious that the same argument does not necessarily apply to either  $Y[g]$  or  $U[g]$  because they are sourced differently from  $X[g]$ . On a deeper level, we question the plausibility of time-dependent homogeneous solutions carrying cosmological time dependence deep inside gravitationally bound systems even for  $X[g]$ . In general relativity the various gravitational fields also possess time-dependent homogeneous solutions, yet there is no leakage of cosmological time dependence inside strongly gravitationally bound systems. The authors of [24] point out that they solved the scalar problem in the background of precisely such gravitational fields and still found time dependent solutions, but that is not realistic. The *mechanism* through which the gravitational fields of general relativity manage to avoid exciting time dependent homogeneous solutions inside bound systems is feedback at the time the structure forms. Structures will form *differently* in our model than in general relativity, and this will lead to feedback which involves the auxiliary scalars as well as the gravitational fields. We think it likely that this feedback will prevent the excitation of unwanted homogeneous solutions for our model in the same way as it does in general relativity. However, this potential problem clearly deserves further study.

### 2.4 Connection to fundamentals

We do not believe that nonlocality is fundamental; it is rather a conjecture for the most cosmologically significant part of the quantum gravitational effective action. The underlying idea [29] is that the problem of the cosmological constant [30, 31] may have no resolution: general relativity really does have a large, positive cosmological constant, and this is what started primordial inflation. However, accelerated expansion led to the production of a vast ensemble of infrared gravitons [32], and the self-gravitation between these gravitons grew without bound as more and more came into causal contact. This self-gravitation provides a sort of quantum gravitational friction which slows inflation by an amount that eventually becomes nonperturbatively large. No one has yet devised a way of passing beyond perturbation theory to derive the result but the natural supposition is that this quantum gravitational effect eventually screened the large bare cosmological constant and brought inflation to a close.

Because what is being cancelled is a *constant*, whereas the screening mechanism is *dynamical*, depending on how many gravitons can see one another as the past light-cone opens up, it is obvious that the persistence of perfect screening can only occur in one geometry. We believe this “perfect” geometry is radiation domination, and that the transition to matter

domination disrupts perfect screening, after which a small fraction of the original large bare cosmological constant peeks out from under the blanket of infrared gravitons which had previously completely screened it.

The key nonlocal ingredient in our model is the inverse scalar d'Alembertian  $\square^{-1}$  which can be roughly motivated [33, 34] by the secular growth factors that arise in explicit loop corrections to gravitational radiation [35, 36] and to gravitational forces [37, 38] on de Sitter background. For the rest, the Ricci scalar is the simplest curvature scalar upon which it might act, and the combination in  $Y[g](x)$  seems to be the simplest form which both matches the perturbative secular growth on de Sitter and also changes sign inside strongly gravitationally bound systems. At this stage there is of course no way to derive the nonlocal distortion function  $f(Y)$ , but simply accepting the model as a residual effect from the gravitational screening of inflationary gravitons does motivate two of its features which would otherwise seem parachuted in:

- There is an initial value surface upon which the initial conditions of  $\square^{-1}$  can be defined; and
- There are modifications of gravity on large, but not small, distances without fine tuning or an explicit  $\Lambda$ .

### 3 Enforcing the $\Lambda$ CDM expansion history

In this section, we solve for the distortion function  $f(Y)$  which supports the  $\Lambda$ CDM expansion history without dark energy. We begin by specializing the model to cosmology, then describe the procedure for numerically determining the required  $f(Y)$ . The section closes with a very simple and accurate exponential fit to this function.

#### 3.1 The cosmological sector

Cosmology's geometry is well described by a scale factor  $a(t)$ ,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x}, \quad (3.1)$$

whose expansion is quantified by the Hubble and first slow roll parameters,

$$H \equiv \frac{\dot{a}}{a}, \quad \epsilon \equiv -\frac{\dot{H}}{H^2}. \quad (3.2)$$

In this geometry the nonzero covariant derivative operators become,

$$\square \longrightarrow -\left(\frac{d}{dt}\right)^2 - 3H\frac{d}{dt}, \quad D_0 D_0 \longrightarrow \left(\frac{d}{dt}\right)^2, \quad D_i D_j \longrightarrow g_{ij} H \frac{d}{dt}. \quad (3.3)$$

The time-time component of the gravitational field equations (2.8) is,

$$3H\left(\frac{d}{dt} + H\right)(1 + U + f(Y)) + \frac{1}{2}(\dot{X}\dot{U} + \dot{Y}\dot{V} + \dot{X}^2) = 8\pi G\rho, \quad (3.4)$$

where  $\rho$  is the energy density without dark energy. The space-space component is  $g_{ij}$  times,

$$-\left(\frac{d^2}{dt^2} + 2H\frac{d}{dt} + 2\dot{H} + 3H^2\right)(1 + U + f(Y)) + \frac{1}{2}(\dot{X}\dot{U} + \dot{Y}\dot{V} + \dot{X}^2) = 8\pi Gp, \quad (3.5)$$

where  $p$  is the pressure, also without dark energy.

The best time variable is  $N \equiv \ln(\frac{a_0}{a})$ , the number of e-foldings until the present. The various differentials and derivatives then simplify,

$$dN = -Hdt, \quad \frac{d}{dt} = -H \frac{d}{dN}, \quad \frac{d^2}{dt^2} = H^2 \left( \frac{d^2}{dN^2} + \epsilon \frac{d}{dN} \right). \quad (3.6)$$

We seek to determine the function  $f(Y)$  to enforce the  $\Lambda$ CDM expansion history without a cosmological constant. This means the Hubble parameter, energy density and pressure take the forms,

$$H^2 = H_0^2 (\Omega_r e^{4N} + \Omega_m e^{3N} + \Omega_\Lambda) \equiv H_0^2 \times \tilde{H}^2, \quad (3.7)$$

$$8\pi G\rho = 3H_0^2 (\Omega_r e^{4N} + \Omega_m e^{3N}) \equiv H_0^2 \times \tilde{\rho}, \quad (3.8)$$

$$8\pi Gp = 3H_0^2 \times \frac{1}{3} \Omega_r e^{4N} \equiv H_0^2 \times \tilde{p}, \quad (3.9)$$

where  $H_0$  is the current value of the Hubble parameter and  $\Omega_r$ ,  $\Omega_m$  and  $\Omega_\Lambda = 1 - \Omega_r - \Omega_m$  are the  $\Lambda$ CDM fractions of the energy density in radiation, matter and vacuum energy.<sup>4</sup> In this notation the scalar equations are,

$$\left[ \tilde{H} e^{-3N} X' \right]' = -12 \left( 1 - \frac{1}{2} \epsilon \right) \tilde{H} e^{-3N}, \quad \left[ \tilde{H} e^{-3N} Y' \right]' = \tilde{H} e^{-3N} X'^2, \quad (3.10)$$

$$\left[ \tilde{H} e^{-3N} V' \right]' = -12 \left( 1 - \frac{1}{2} \epsilon \right) \tilde{H} e^{-3N} f'(Y), \quad U' = -2X'V, \quad (3.11)$$

where a prime denotes differentiation with respect to the natural argument —  $Y$  for  $f(Y)$  and  $N$  for  $\tilde{H}(N)$ ,  $X(N)$ ,  $Y(N)$ ,  $U(N)$  and  $V(N)$ . Note that equations (3.10) give explicit integral expressions for  $X'(N)$  and  $Y'(N)$ ,

$$X'(N) = \frac{e^{3N}}{\tilde{H}(N)} \int_N^\infty dN' \frac{e^{-3N'}}{\tilde{H}(N')} \left[ 3\Omega_m e^{3N'} + 12\Omega_\Lambda \right], \quad (3.12)$$

$$Y'(N) = -\frac{e^{3N}}{\tilde{H}(N)} \int_N^\infty dN' e^{-3N'} \tilde{H}(N') \left[ X'(N') \right]^2. \quad (3.13)$$

Figure 1 shows  $X(N)$  and  $Y(N)$  and their derivatives. Since  $X'(N)$  and  $Y'(N)$  have definite signs here, both  $X$  and  $Y$  are monotonic, hence invertible.

### 3.2 The reconstruction procedure

The two gravitational field equations are,

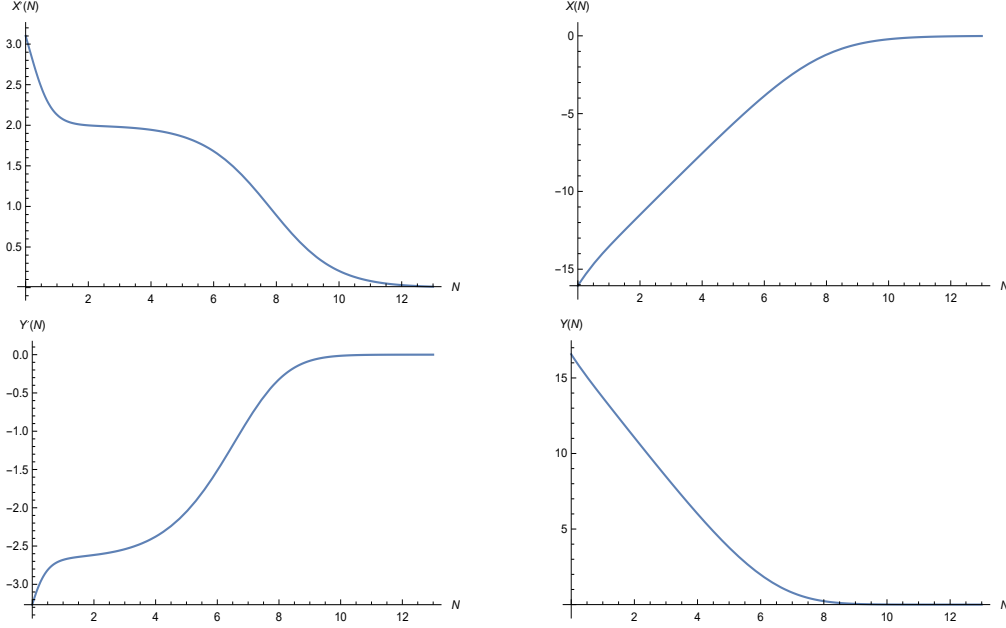
$$-3(\partial_N - 1) \left[ U + f(Y) \right] + \frac{1}{2} \left[ X'U' + Y'V' + X'^2 \right] = -\frac{3\Omega_\Lambda}{\tilde{H}^2}, \quad (3.14)$$

$$-\left( \partial_N^2 - (2 - \epsilon)\partial_N + 3 - 2\epsilon \right) \left[ U + f(Y) \right] + \frac{1}{2} \left[ X'U' + Y'V' + X'^2 \right] = \frac{3\Omega_\Lambda}{\tilde{H}^2}. \quad (3.15)$$

As for the original model [2], the first step in constructing a nonlocal distortion function which supports the  $\Lambda$ CDM expansion history is to take the difference of (3.14) and (3.15),

$$(\partial_N - 3 + \epsilon)(\partial_N - 2) \left[ U + f(Y) \right] = -\frac{6\Omega_\Lambda}{\tilde{H}^2}. \quad (3.16)$$

<sup>4</sup>We use  $\Omega_m/\Omega_r \equiv 1 + z_{eq} \simeq 3403$ ,  $\Omega_m \simeq 0.3153$  and  $\Omega_\Lambda \simeq 0.6847$  [39].



**Figure 1.** The left hand graphs show numerical simulations of  $X'(N)$  and  $Y'(N)$  as defined by (3.12)–(3.13). The right hand graphs give their integrals.

This can be integrated to give exactly the same result as for the original model [2],

$$U + f(Y) = -6\Omega_\Lambda e^{2N} \int_N^\infty dN' \frac{e^{N'}}{\tilde{H}(N')} \int_{N'}^\infty dN'' \frac{e^{-3N''}}{\tilde{H}(N'')} \equiv g(N). \quad (3.17)$$

The next step is to derive a differential equation for the function,

$$G(N) \equiv \frac{Y'(N)}{X'(N)} f'(Y(N)) - \frac{g'(N)}{X'(N)}. \quad (3.18)$$

Differentiating relation (3.17), using (3.11) and dividing by  $X'(N)$  gives,

$$-2V(N) + G(N) = 0. \quad (3.19)$$

Acting  $\partial_N^2 - (3 - \epsilon)\partial_N$  on (3.19) and using relation (3.11) produces,

$$(\partial_N - 3 + \epsilon)\partial_N G + 24\left(1 - \frac{1}{2}\epsilon\right)\frac{X'}{Y'}G = -24\left(1 - \frac{1}{2}\epsilon\right)\frac{g'}{Y'}. \quad (3.20)$$

The procedure from this point is clear: we numerically solve (3.20) for  $G(N)$ , extract  $\partial_N f(Y) = Y' \times f'(Y)$  using relation (3.18), numerically integrate to recover  $f(Y)$  as a function of  $N$ , and finally exploit the one-to-one relation between  $Y$  and  $N$  to numerically express  $f(Y)$  as a function of  $Y$ .

### 3.3 Solution for $f(Y)$

The initial conditions at large  $N$  follow from exact results, derived in appendix A, by retaining only the leading dependence on  $\Omega_\Lambda$ . Because  $\Omega_\Lambda$  is irrelevant until late times, expressions (A.1)–(A.9) are accurate to three digits for  $N > 2$ . The functions we need for

equation (3.20) can be usefully expanded in powers of the variable  $y \equiv (1 + z_{eq})e^{-N}$ ,

$$\epsilon \longrightarrow +2 - \frac{1}{2}y + \frac{1}{2}y^2 - \frac{1}{2}y^3 + O(y^4), \quad (3.21)$$

$$X' \longrightarrow +\frac{3}{2}y - \frac{5}{4}y^2 + \frac{35}{32}y^3 - \frac{63}{64}y^4 + O(y^5), \quad (3.22)$$

$$Y' \longrightarrow -\frac{3}{4}y^2 + \frac{33}{32}y^3 - \frac{367}{320}y^4 + \frac{4577}{3840}y^5 + O(y^6), \quad (3.23)$$

$$g' \longrightarrow \frac{\Omega_\Lambda \Omega_r^3}{\Omega_m^4} \left\{ \frac{4}{5}y^4 - \frac{11}{14}y^5 + \frac{429}{560}y^6 - \frac{142}{192}y^7 + O(y^8) \right\}. \quad (3.24)$$

Because  $1 + z_{eq} \simeq 3403 \simeq \exp[+8.132]$  is so large, these expansions are only accurate for  $N > 10$ . Employing the expansions (3.21)–(3.24) allows us to factorize the large  $N$  limiting form of the differential operator in equation (3.20),

$$F_1(y) \frac{d}{dN} \left\{ F_2(y) \frac{d}{dN} [F_3(y)G] \right\} = F_4(y), \quad (3.25)$$

where the four factors are,

$$F_1(y) = \frac{1}{y^4} \left[ 1 + \frac{3}{8}y - \frac{13}{960}y^2 - \frac{13}{4608}y^3 + \dots \right], \quad (3.26)$$

$$F_2(y) = y^7 \left[ 1 - \frac{5}{4}y + \frac{1151}{960}y^2 - \frac{6071}{5760}y^3 + \dots \right], \quad (3.27)$$

$$F_3(y) = \frac{1}{y^3} \left[ 1 + \frac{7}{8}y + \frac{47}{960}y^2 + \frac{137}{23040}y^3 + \dots \right], \quad (3.28)$$

$$F_4(y) = \frac{\Omega_\Lambda \Omega_r^3}{\Omega_m^4} \left\{ \frac{32}{5}y^3 - \frac{136}{35}y^4 + \frac{3869}{1050}y^5 - \frac{2587}{720}y^6 + \dots \right\}. \quad (3.29)$$

Expression (3.25) is a second order differential equation and possesses two homogeneous solutions. However, only one of these falls off for large  $N$ ,

$$G_h(N) = \frac{1}{F_3(y)} = y^3 - \frac{7}{8}y^4 + \frac{43}{60}y^5 - \frac{85}{144}y^6 + O(y^7). \quad (3.30)$$

The large  $N$  limiting form of  $G(N)$  can be inferred from (3.25),

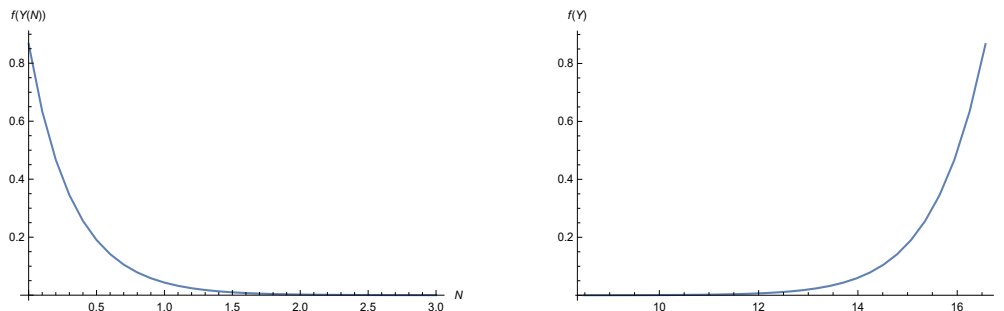
$$G(N) \longrightarrow \frac{\Omega_\Lambda \Omega_r^3}{\Omega_m^4} \left\{ \frac{32}{35}G_h(N) \ln(y) + \frac{5}{14}y^4 - \frac{1247}{4200}y^5 + \frac{71117}{302400}y^6 + O(y^7) \right\}. \quad (3.31)$$

This provides the initial conditions to evolve (3.20) from finite  $N$ .

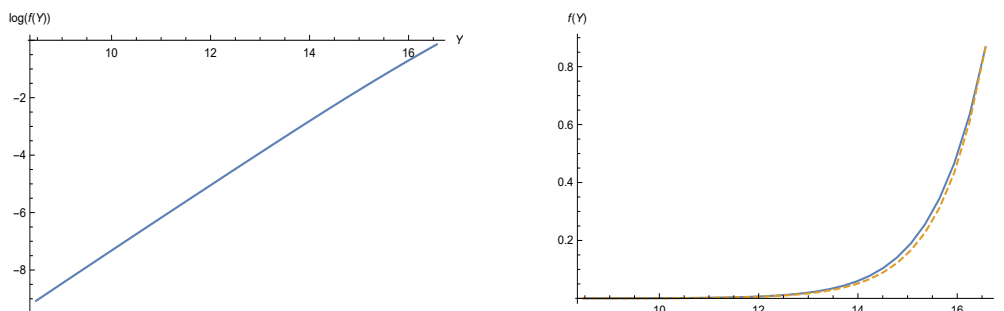
The expansion (3.31) fixes the small  $Y$  behavior of  $f(Y)$ ,

$$f(Y) = \frac{\Omega_\Lambda \Omega_r^3}{\Omega_m^4} \left\{ -\frac{128}{105}Y^2 \ln(Y) + O(Y^2) \right\}. \quad (3.32)$$

This means that making  $f(Y)$  vanish for all  $Y < 0$  leads to no discontinuity in either  $f(Y)$  or  $f'(Y)$  at  $Y = 0$ . Numerical evolution gives the result for general  $Y > 0$ , which is depicted



**Figure 2.** The left hand graph shows a numerical simulation of  $f(Y)$  as a function of the evolution variable  $N$ . The right hand graph also gives  $f(Y)$ , but now as a function of its natural argument  $Y$ .



**Figure 3.** The left hand graph shows that  $\ln[f(Y)]$  is nearly a straight line. The right hand graph compares the full numerical determination of  $f(Y)$  (in solid, blue) to the resulting exponential fit (3.33) (in dashed, yellow).

in figure 2. Figure 3 shows that  $f(Y)$  is well fit by the strikingly simple form, duly matched to (3.32) at small  $Y$ ,<sup>5</sup>

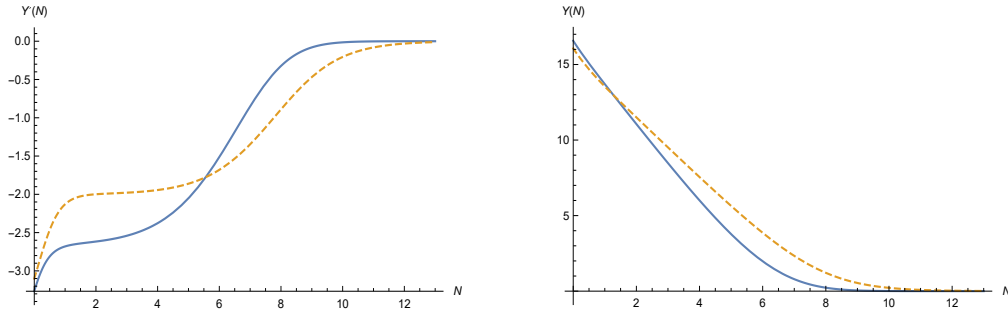
$$f(Y) \simeq \exp\left[1.1\left(Y - 16.7\right)\right]. \quad (3.33)$$

## 4 Discussion

We have presented a simple variant of our original model [1] to explain the current phase of cosmic acceleration without dark energy. Like its ancestor, the new model is based on augmenting the Hilbert action by the addition of  $R$  times a function of a dimensionless, nonlocal scalar; only the scalar has changed from  $X[g] = \square^{-1}R$  to  $Y[g] = \square^{-1}g^{\mu\nu}\partial_\mu X\partial_\nu X$ . Both  $X[g]$  and  $Y[g]$  are quiescent during radiation domination, and thereafter only grow logarithmically, which provides a natural explanation for why the onset of acceleration is delayed to late in cosmic history. Both scalars also vanish for gravitational radiation which means that they do not affect the — tightly constrained — propagation velocity [25].

Because section 3 employed the parameter  $\Omega_\Lambda$  to make our model reproduce the  $\Lambda$ CDM expansion history, one might question our claim of no fine tuning. However, it is best to view fine tuning from the perspective of how precisely the parameters of the Lagrangian must be adjusted to explain late time acceleration. Our model (2.2) amounts to replacing the standard

<sup>5</sup>Note that this form only pertains for positive  $Y$  somewhat greater than zero. We still require  $f(Y)$  to vanish as in (3.32) as  $Y \rightarrow 0^+$ , and to vanish for all  $Y \leq 0$ .



**Figure 4.** The left hand graph compares  $Y'(N)$  (solid blue) with  $-X'(N)$  (dashed yellow). The right hand graph compares  $Y(N)$  (solid blue) with  $-X(N)$  (dashed yellow).

$R$  term of general relativity by  $R + R \times f(Y)$ , where  $Y[g]$  obeys (2.1). In this model nothing needs to be done to delay the onset of acceleration until very late in cosmological history; that happens naturally because  $Y[g]$  is sourced by  $R$ , which vanishes during radiation domination, and because  $Y[g]$  only grows logarithmically after the transition to matter domination. Nor are there any new scales; the function  $f(Y)$  is dimensionless, as is  $Y[g]$  itself. As long as  $f(Y)$  grows with  $Y$  there will be a phase of late time acceleration. Just how little the old model [1] changes with variations of the nonlocal distortion function has already been explored [9] and would not differ in the new model. Contrast this with the two local alternatives of general relativity with a cosmological constant or a scalar quintessence model,

$$R \longrightarrow R - 2\Lambda, \quad (4.1)$$

$$R \longrightarrow R - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi g^{\mu\nu} - V(\varphi). \quad (4.2)$$

In the first case (4.1) the dimensionful parameter  $\Lambda$  must be fine-tuned to make the dimensionless product  $G\Lambda \simeq 10^{-122}$  vanish to 122 decimal places! A similar amount of fine tuning must be imposed on the potential  $V(\varphi)$  of quintessence models (4.2). From this perspective, our model is indeed a non-fine-tuned one!

Figure 4 shows that  $Y$  is close to  $-X$  for cosmology. This made it simple to determine the nonlocal distortion function  $f(Y)$  numerically in order to reproduce exactly the  $\Lambda$ CDM expansion history without dark energy. That was done in section 3, with results shown on figure 2. An unexpected consequence was the simple exponential approximation (3.33) for  $f(Y)$ , whose accuracy can be seen from figure 3.

The new model differs from the original one in that  $Y$  (unlike  $X$ ) changes sign from cosmology (with  $Y > 0$ ) to strongly gravitationally bound systems (with  $Y < 0$ ). Because cosmology only fixes  $f(Y)$  for  $Y \geq 0$ , with  $f(0) = 0$ , simply assuming  $f(Y) = 0$  for  $Y < 0$  protects the model from changing the heavily constrained physics of the solar system. The huge advantages of this model can be seen by comparison with  $F(R)$  theories of gravity, which must invoke ever more exotic physics such as the chameleon mechanism [40] to evade solar system constraints. Note also that the only stable choice of  $F(R)$  which exactly reproduces the  $\Lambda$ CDM expansion history is  $F(R) = R - 2\Lambda$  [41].

Now that the nonlocal distortion function  $f(Y)$  has been fixed the model is complete. Because  $f(Y)$  has been chosen to exactly reproduce the  $\Lambda$ CDM expansion history, with no changes inside gravitationally bound systems, tests of the model must come from its predictions for cosmological perturbations and the growth of structures. Stability is another important constraint to study.

Finally, we return to the presumed local sources of our model, the gravitons of primordial inflation. Their loop effects can grow non-perturbatively strongly during the primordial inflation era [36], and  $\square^{-1}$  does correctly capture this growth on de Sitter background, but it is clearly a major unsolved problem to follow their temporal effects in any detail. We have no explanation, other than simplicity and dimensionlessness, for the combination  $X = \square^{-1}R$ , nor can we justify the appearance of  $Y$ . It might, however, be worth noting that nonlocal realizations of MODified Newtonian Dynamics (MOND) [42–44] involve a similar nonlocal scalar [45, 46]. This raises the hope that there is a master effective action describing the full range of cosmic history from the build-up of gravitational back-reaction during inflation, and giving rise to both the present model and to MOND as residual effects.

For now we can strictly only offer our phenomenological (but dimensionless) construction. Nevertheless, the presumed inflationary origin does provide two vital answers that otherwise seem unnatural: the existence of an initial value surface from which one may launch the initial conditions defining our inverse differential operators, i.e., the propagators, *and* why the corrections only modify classical general relativity on cosmological, rather than on the smaller (bound matter) scales, where no “improvement” is needed!

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## A Exact expressions to leading order in $\Omega_\Lambda$

Expressions for  $X(N)$ ,  $Y(N)$  and  $g(N)$  simplify dramatically for large  $N$  when  $\Omega_\Lambda$  becomes negligible relative to  $\Omega_m e^{3N}$  and  $\Omega_r e^{4N}$ . Setting  $\Omega_\Lambda = 0$  and making the change of variable  $y \equiv \frac{\Omega_m}{\Omega_r} e^{-N}$  in expression (3.12) reduces  $X'(N)$  to an elementary function,

$$X'(N) \longrightarrow \frac{3}{y\sqrt{1+y}} \int_0^y \frac{y' dy'}{\sqrt{1+y'}} = \frac{2(z-1)(z+2)}{z(z+1)}, \quad (\text{A.1})$$

where  $z \equiv \sqrt{1+y}$ . Integrating expression (A.1) gives,

$$X(N) \longrightarrow -2 \left( \frac{z-1}{z+1} \right) - 4 \ln \left( \frac{1}{2} z + \frac{1}{2} \right). \quad (\text{A.2})$$

Setting  $\Omega_\Lambda = 0$  in expression (3.13) similarly reduces  $Y'(N)$  to an elementary function,

$$Y'(N) \longrightarrow \frac{-1}{y\sqrt{1+y}} \int_0^y dy' \sqrt{1+y'} \left[ X'(N') \right]^2, \quad (\text{A.3})$$

$$= \frac{-8}{z(z+1)} \left[ \frac{1}{3} (z^2 + z - 11) + \frac{2}{z+1} + \frac{4 \ln(\frac{1}{2} z + \frac{1}{2})}{z-1} \right]. \quad (\text{A.4})$$

Integrating (A.4) to get  $Y(N)$  produces a dilogarithm in addition to elementary functions,

$$Y(N) \longrightarrow \frac{8}{(z+1)^2} - \frac{112}{3(z+1)} + \left[ \frac{16}{3} - \frac{32z}{z^2-1} \right] \ln \left( \frac{1}{2} z + \frac{1}{2} \right) + \frac{37}{24} + 8 \ln^2 \left( \frac{1}{2} z + \frac{1}{2} \right) + 16 \text{Li}_2 \left( \frac{1}{2} - \frac{1}{2} z \right), \quad (\text{A.5})$$

where

$$\text{Li}_2(x) \equiv - \int_0^x dt \frac{\ln(1-t)}{t}. \quad (\text{A.6})$$

The function  $g(N)$  actually vanishes with  $\Omega_\Lambda$  so its large  $N$  limit derives from preserving the initial factor in (3.17),

$$g(N) \longrightarrow - \frac{6\Omega_\Lambda\Omega_r^3}{\Omega_m^4 y^2} \int_0^y \frac{dy'}{\sqrt{1+y'}} \int_0^{y'} dy'' \frac{y''^4}{\sqrt{1+y''}}, \quad (\text{A.7})$$

$$= - \frac{\Omega_\Lambda\Omega_r^3}{\Omega_m^4} \frac{(z-1)^4}{105} \left[ 28z^2 + 112z + 156 + \frac{64}{z+1} + \frac{32}{(z+1)^2} \right]. \quad (\text{A.8})$$

Differentiating this gives,

$$g'(N) \longrightarrow \frac{\Omega_\Lambda\Omega_r^3}{\Omega_m^4} \frac{(z-1)^4}{105} \left[ 84z^2 + 336z + 508 + \frac{352}{z} + \frac{96}{z(z+1)} + \frac{96}{z(z+1)^2} \right]. \quad (\text{A.9})$$

All of these expressions are accurate to three digits for  $N > 2$ .

## References

- [1] S. Deser and R.P. Woodard, *Nonlocal Cosmology*, *Phys. Rev. Lett.* **99** (2007) 111301 [[arXiv:0706.2151](#)] [[INSPIRE](#)].
- [2] C. Deffayet and R.P. Woodard, *Reconstructing the Distortion Function for Nonlocal Cosmology*, *JCAP* **08** (2009) 023 [[arXiv:0904.0961](#)] [[INSPIRE](#)].
- [3] S. Deser and R.P. Woodard, *Observational Viability and Stability of Nonlocal Cosmology*, *JCAP* **11** (2013) 036 [[arXiv:1307.6639](#)] [[INSPIRE](#)].
- [4] DES collaboration, *Dark Energy Survey Year 1 Results: Constraints on Extended Cosmological Models from Galaxy Clustering and Weak Lensing*, *Phys. Rev. D* **99** (2019) 123505 [[arXiv:1810.02499](#)] [[INSPIRE](#)].
- [5] R.P. Woodard, *Nonlocal Models of Cosmic Acceleration*, *Found. Phys.* **44** (2014) 213 [[arXiv:1401.0254](#)] [[INSPIRE](#)].
- [6] R.P. Woodard, *The Case for Nonlocal Modifications of Gravity*, *Universe* **4** (2018) 88 [[arXiv:1807.01791](#)] [[INSPIRE](#)].
- [7] S. Park and S. Dodelson, *Structure formation in a nonlocally modified gravity model*, *Phys. Rev. D* **87** (2013) 024003 [[arXiv:1209.0836](#)] [[INSPIRE](#)].
- [8] S. Dodelson and S. Park, *Nonlocal Gravity and Structure in the Universe*, *Phys. Rev. D* **90** (2014) 043535 [Erratum *ibid.* **98** (2018) 029904] [[arXiv:1310.4329](#)] [[INSPIRE](#)].
- [9] S. Park and A. Shafieloo, *Growth of perturbations in nonlocal gravity with non- $\Lambda$ CDM background*, *Phys. Rev. D* **95** (2017) 064061 [[arXiv:1608.02541](#)] [[INSPIRE](#)].
- [10] H. Nersisyan, A.F. Cid and L. Amendola, *Structure formation in the Deser-Woodard nonlocal gravity model: a reappraisal*, *JCAP* **04** (2017) 046 [[arXiv:1701.00434](#)] [[INSPIRE](#)].
- [11] S. Park, *Revival of the Deser-Woodard nonlocal gravity model: Comparison of the original nonlocal form and a localized formulation*, *Phys. Rev. D* **97** (2018) 044006 [[arXiv:1711.08759](#)] [[INSPIRE](#)].
- [12] L. Amendola, Y. Dirian, H. Nersisyan and S. Park, *Observational Constraints in Nonlocal Gravity: the Deser-Woodard Case*, *JCAP* **03** (2019) 045 [[arXiv:1901.07832](#)] [[INSPIRE](#)].

- [13] T. Koivisto, *Dynamics of Nonlocal Cosmology*, *Phys. Rev. D* **77** (2008) 123513 [[arXiv:0803.3399](#)] [[INSPIRE](#)].
- [14] T.S. Koivisto, *Newtonian limit of nonlocal cosmology*, *Phys. Rev. D* **78** (2008) 123505 [[arXiv:0807.3778](#)] [[INSPIRE](#)].
- [15] Y.-Z. Chu and S. Park, *Does nonlocal gravity yield divergent gravitational energy-momentum fluxes?*, *Phys. Rev. D* **99** (2019) 044052 [[arXiv:1811.04647](#)] [[INSPIRE](#)].
- [16] M. Maggiore and M. Mancarella, *Nonlocal gravity and dark energy*, *Phys. Rev. D* **90** (2014) 023005 [[arXiv:1402.0448](#)] [[INSPIRE](#)].
- [17] M. Maggiore, *Nonlocal Infrared Modifications of Gravity. A Review*, *Fundam. Theor. Phys.* **187** (2017) 221 [[arXiv:1606.08784](#)] [[INSPIRE](#)].
- [18] Y. Dirian, S. Foffa, N. Khosravi, M. Kunz and M. Maggiore, *Cosmological perturbations and structure formation in nonlocal infrared modifications of general relativity*, *JCAP* **06** (2014) 033 [[arXiv:1403.6068](#)] [[INSPIRE](#)].
- [19] A. Barreira, B. Li, W.A. Hellwing, C.M. Baugh and S. Pascoli, *Nonlinear structure formation in Nonlocal Gravity*, *JCAP* **09** (2014) 031 [[arXiv:1408.1084](#)] [[INSPIRE](#)].
- [20] Y. Dirian and E. Mitsou, *Stability analysis and future singularity of the  $m^2 R \Box^{-2} R$  model of non-local gravity*, *JCAP* **10** (2014) 065 [[arXiv:1408.5058](#)] [[INSPIRE](#)].
- [21] Y. Dirian, S. Foffa, M. Kunz, M. Maggiore and V. Pettorino, *Non-local gravity and comparison with observational datasets*, *JCAP* **04** (2015) 044 [[arXiv:1411.7692](#)] [[INSPIRE](#)].
- [22] Y. Dirian, S. Foffa, M. Kunz, M. Maggiore and V. Pettorino, *Non-local gravity and comparison with observational datasets. II. Updated results and Bayesian model comparison with  $\Lambda$ CDM*, *JCAP* **05** (2016) 068 [[arXiv:1602.03558](#)] [[INSPIRE](#)].
- [23] H. Nersisyan, Y. Akrami, L. Amendola, T.S. Koivisto and J. Rubio, *Dynamical analysis of  $R \frac{1}{\Box^2} R$  cosmology: Impact of initial conditions and constraints from supernovae*, *Phys. Rev. D* **94** (2016) 043531 [[arXiv:1606.04349](#)] [[INSPIRE](#)].
- [24] E. Belgacem, A. Finke, A. Frassino and M. Maggiore, *Testing nonlocal gravity with Lunar Laser Ranging*, *JCAP* **02** (2019) 035 [[arXiv:1812.11181](#)] [[INSPIRE](#)].
- [25] S. Boran, S. Desai, E.O. Kahya and R.P. Woodard, *GW170817 Falsifies Dark Matter Emulators*, *Phys. Rev. D* **97** (2018) 041501 [[arXiv:1710.06168](#)] [[INSPIRE](#)].
- [26] S. Nojiri and S.D. Odintsov, *Modified non-local- $F(R)$  gravity as the key for the inflation and dark energy*, *Phys. Lett. B* **659** (2008) 821 [[arXiv:0708.0924](#)] [[INSPIRE](#)].
- [27] S. Park and R.P. Woodard, *Exciting the scalar ghost mode through time evolution*, *Phys. Rev. D* **99** (2019) 024014 [[arXiv:1809.06841](#)] [[INSPIRE](#)].
- [28] I. Sawicki and E. Bellini, *Limits of quasistatic approximation in modified-gravity cosmologies*, *Phys. Rev. D* **92** (2015) 084061 [[arXiv:1503.06831](#)] [[INSPIRE](#)].
- [29] N.C. Tsamis and R.P. Woodard, *Quantum gravity slows inflation*, *Nucl. Phys. B* **474** (1996) 235 [[hep-ph/9602315](#)] [[INSPIRE](#)].
- [30] S. Weinberg, *The Cosmological Constant Problem*, *Rev. Mod. Phys.* **61** (1989) 1 [[INSPIRE](#)].
- [31] S.M. Carroll, *The Cosmological constant*, *Living Rev. Rel.* **4** (2001) 1 [[astro-ph/0004075](#)] [[INSPIRE](#)].
- [32] L.P. Grishchuk, *Amplification of gravitational waves in an isotropic universe*, *Sov. Phys. JETP* **40** (1975) 409 [[INSPIRE](#)].
- [33] N.C. Tsamis and R.P. Woodard, *Nonperturbative models for the quantum gravitational back reaction on inflation*, *Annals Phys.* **267** (1998) 145 [[hep-ph/9712331](#)] [[INSPIRE](#)].

- [34] M.G. Romania, N.C. Tsamis and R.P. Woodard, *Quantum Gravity and Inflation*, *Lect. Notes Phys.* **863** (2013) 375 [[arXiv:1204.6558](#)] [[INSPIRE](#)].
- [35] N.C. Tsamis and R.P. Woodard, *One loop graviton selfenergy in a locally de Sitter background*, *Phys. Rev. D* **54** (1996) 2621 [[hep-ph/9602317](#)] [[INSPIRE](#)].
- [36] P.J. Mora, N.C. Tsamis and R.P. Woodard, *Hartree approximation to the one loop quantum gravitational correction to the graviton mode function on de Sitter*, *JCAP* **10** (2013) 018 [[arXiv:1307.1422](#)] [[INSPIRE](#)].
- [37] S. Park and R.P. Woodard, *Scalar Contribution to the Graviton Self-Energy during Inflation*, *Phys. Rev. D* **83** (2011) 084049 [[arXiv:1101.5804](#)] [[INSPIRE](#)].
- [38] S. Park, T. Prokopec and R.P. Woodard, *Quantum Scalar Corrections to the Gravitational Potentials on de Sitter Background*, *JHEP* **01** (2016) 074 [[arXiv:1510.03352](#)] [[INSPIRE](#)].
- [39] PLANCK collaboration, *Planck 2018 results. VI. Cosmological parameters*, [arXiv:1807.06209](#) [[INSPIRE](#)].
- [40] J. Khoury and A. Weltman, *Chameleon fields: Awaiting surprises for tests of gravity in space*, *Phys. Rev. Lett.* **93** (2004) 171104 [[astro-ph/0309300](#)] [[INSPIRE](#)].
- [41] P.K.S. Dunsby, E. Elizalde, R. Goswami, S. Odintsov and D.S. Gomez, *On the LCDM Universe in  $f(R)$  gravity*, *Phys. Rev. D* **82** (2010) 023519 [[arXiv:1005.2205](#)] [[INSPIRE](#)].
- [42] M. Milgrom, *A Modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis*, *Astrophys. J.* **270** (1983) 365 [[INSPIRE](#)].
- [43] M. Milgrom, *A Modification of the Newtonian dynamics: Implications for galaxies*, *Astrophys. J.* **270** (1983) 371 [[INSPIRE](#)].
- [44] M. Milgrom, *A modification of the Newtonian dynamics: implications for galaxy systems*, *Astrophys. J.* **270** (1983) 384 [[INSPIRE](#)].
- [45] C. Deffayet, G. Esposito-Farese and R.P. Woodard, *Nonlocal metric formulations of MOND with sufficient lensing*, *Phys. Rev. D* **84** (2011) 124054 [[arXiv:1106.4984](#)] [[INSPIRE](#)].
- [46] C. Deffayet, G. Esposito-Farese and R.P. Woodard, *Field equations and cosmology for a class of nonlocal metric models of MOND*, *Phys. Rev. D* **90** (2014) 064038 [[arXiv:1405.0393](#)] [[INSPIRE](#)].