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On twisted group frames [☆]

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A B S T R A C T

In this paper, we study (G, α)-frames for finite dimensional Hilbert spaces, where G is a finite group and α is a unitary Schur multiplier of G. We apply the characterizations for tight (G, α)-frames and central (G, α)-frames to investigation of (G, α)-frames that have the maximal spanning property. We apply the theory of twisted group frames to obtain a criterion for maximal spanning vectors that generalizes the main result of [16, Theorem 1.7]. Moreover, we prove that a projective representation does not admit any maximal spanning vector if it has an at least 2-dimensional subrepresentation that is equivalent to a central projection induced subrepresentation of the α-regular representation.

1. Introduction

Historically, frame theory was originated in the study of signal decomposition. But it has ever-increasing applications today to problems from pure and applied mathematics, physics, computer science etc. (cf. [5, Section 1]). In particular, frames with symmetries,

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i.e. groups frames, have important applications in signal analysis and in quantum information theory (cf. [22]). In this paper, we examine some phase-retrieval related problems for *twisted group frames*, which are also referred to as projective representation frames or group-like systems in the literature (cf. [8–11,16]). We apply the classification results for tight and central tight twisted group frames to the existence problem ([16]) for maximal spanning vectors. In particular, we provide two proofs for a special case of [16, Conjecture], one is based on a similar calculation as in [16] and one is an application of the theory we developed for (G, α)-frames (cf. Sections 3.2 and 3.4). As a special consequence, we obtain that central frames do not have the maximal span property unless the representation is one-dimensional. In particular, this is true for (non-twisted) abelian group frames.

First we give the precise definition of a twisted group frame. Let G be a finite group. A *Schur multiplier* (or a *factor set* or a *2-cocycle*) on G is a map $\alpha : G \times G \to \mathbb{C}^{\times}$ such that

(1) $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$;
(2) $\alpha(x, 1) = \alpha(1, x) = 1$ for all $x \in G$.

We always assume that α is *unitary*, i.e., $\operatorname{Im}\alpha \subset \mathbb{C}^{\times}_{|\cdot|=1}$. A *projective representation* of G over a finite dimensional $\mathbb{C}$-vector space V is a map $\pi : G \to \mathrm{GL}(V)$ such that $\pi(x)\pi(y) = \alpha(x, y)\pi(xy)$ for all $x, y \in G$, where α is the associated multiplier. Since every projective representation V admits an inner product that is invariant under π, we may and will assume that the projective representation is given by $\pi : G \to \mathbf{U}(V)$. Here $\mathbf{U}(V)$ denotes the set of unitary operators on a Hilbert space V.

Definition 1.1. Let V be a finite dimensional Hilbert space. An *α-twisted G-frame* (or a *(G, α)-frame*) for V is a frame $\Phi = \{\phi_g \mid g \in G\}$ for V, for which there exists a unitary α-projective representation $\pi : G \to \mathbf{U}(V)$ with

$$g\phi_h (:= \pi(g)\phi_h) = \alpha(g, h)\phi_{gh}.$$

If $\Phi = \{\phi_g \mid g \in G\}$ is a (G, α)-frame for V, we call ϕ_1 a *frame vector* for V, where 1 is the group unit.

Let $\Phi = \{\phi_g \mid g \in G\}$ be a (G, α)-frame. The symmetry on Φ gives special structures on the objects associated with Φ. An obvious property is that Φ is an equal norm frame. (Indeed, in the definition take $h = 1$, then we have $g\phi_1 = \alpha(g, 1)\phi_g = \phi_g$.) In Section 2.1, we show that a frame is a (G, α)-frame if and only if its Gramian is a (G, α)-matrix (Definition 2.1). We then show that the (G, α)-matrices form a sub-algebra of the matrix algebra $M_{n\times n}(\mathbb{C})$ isomorphic to the twisted group algebra $\mathbb{C}[G]_\alpha$. Here $n = |G|$.

In Section 2.2 and Section 2.3, like group frames we examine some basic properties about (G, α)-frames with respect to the irreducible decomposition of the projective representation. We particularly examine a special type of (G, α)-frames, namely, central frames. The characterizations of tight (G, α)-frames and tight central frames are obtained in Theorem 2.11 and Theorem 2.22. We also give explicit examples with non-trivial α, which show that (G, α)-frames are different from G-frames.

Applying Theorem 2.11, it is easy to prove that $\pi \otimes \pi^* : G \to \mathbf{U}(V \otimes V^*)$, which is a projective representation of G with multiplier $\alpha\alpha^{-1} = 1$, admits tight G-frames. Moreover, [16, Conjecture] conjectures that, if V is irreducible, there always exists a G-frame of the form $\{gw \mid g \in G\}$ with $w = v \otimes v^*$ for some $v \in V$. Such a vector v is called a *maximal spanning vector* of V, which is automatically *phase-retrievable*. The case for G abelian of [16, Conjecture] has been verified (cf. [16, Theorem 1.7]). In Section 3.2, we give a slightly different proof of this result. Both our argument and argument in [16] are based on the special structure of projective representations of abelian groups. Starting with the abelian case, in Section 3.3, we verify the conjecture for certain projective representations of some solvable groups. Section 3.4 is devoted to applying the theory of twisted group frames to obtain a criterion for maximal spanning vectors. In particular, Proposition 3.11 or [16, Theorem 1.7] can be considered as a special case of the criteria. For reducible representations, we prove that a central frame vector cannot be a maximal spanning vector, and in particular we obtain that an α-projective representation does not admit any maximal spanning vector if it has a subrepresentation that has dimension at least 2 and is equivalent to a central projection induced subrepresentation of the α-regular representation (Corollary 3.5).

1.1. Notation

In this paper, G is a finite group with $|G| = n$, $\alpha \in Z^2(G, \mathbb{C}^\times)$ is a unitary multiplier. A subgroup H of G is called *α-symmetric* if $\alpha(x, y) = \alpha(y, x)$ for any $x, y \in H$.

Denote by Rep_G^α the category of projective representations of G with multiplier α. It is well-known that, up to isomorphism, there are finitely many irreducible projective representations in Rep_G^α. Denote those representations by $\rho_i : G \to \mathbf{U}(V_i)$, where V_i is a $\mathbb{C}$-vector space with dimension d_i $(1 \le i \le r)$. Denote by χ_i the character of ρ_i $(1 \le i \le r)$. We refer to [15] for details on Schur multipliers and refer to [6,7] for details on projective representations.

For the terminology on frames and group frames, we follow the book *Finite Frames. Theory and Applications* edited by Peter G. Casazza and Gitta Kutyniok, especially the articles [5,17,22].

2. Twisted G-frames

In this section, we study (G, α)-frames with the same strategy in the study of group frames (cf. [17,19,20]). Some of the basic properties about (G, α)-frames maybe known in the literature. However, we still include the proof details for self-completeness and for the purpose of discussing their connections with the phase-retrievable problem.

2.1. Group matrices and the Gramian of a (G,α)-frame

Let $\Phi = \{g\phi_1 \mid g \in G\}$ be a (G,α)-frame for V. Then

$$\langle \phi_g, \phi_h \rangle = \langle g\phi_1, h\phi_1 \rangle = \langle \phi_1, \pi(g)^{-1}\pi(h)\phi_1 \rangle = \langle \phi_1, \frac{\alpha(g^{-1},h)}{\alpha(g,g^{-1})}\pi(g^{-1}h)\phi_1 \rangle.$$

This motivates the following definition.

Definition 2.1. A (G,α)-*matrix* is a matrix A with entries indexed by elements of a group G and of the form

$$A = (f(g,h))_{g,h\in G},$$

where

$$f(g,h) = \frac{\alpha(g,g^{-1})}{\alpha(g^{-1},h)}\eta(g^{-1}h) \text{ for some } \eta : G \to \mathbb{C}.$$

We denote the matrix A by $M(\eta)$.

Certainly, the Gramian of a (G,α)-frame is a (G,α)-matrix. Conversely, we have the following result, which is a projective version of [20, Theorem 4.1]. See also [22, Theorem 5.2] and [13, Theorem 3.2].

Theorem 2.2. *Let G be a finite group. Assume that $\Phi = \{\phi_g \mid g \in G\}$ is a frame of V indexed by elements of G. Then Φ is a (G,α)-frame if and only if its Gramian* $\mathrm{Gram}(\Phi)$ *is a (G,α)-matrix.*

Proof. We only need to prove the *if* part. Let $\eta : G \to \mathbb{C}$ be the function such that $\mathrm{Gram}(\Phi) = (\frac{\alpha(g,g^{-1})}{\alpha(g^{-1},h)}\eta(g^{-1}h))_{g,h\in G}$. It suffices to construct a projective representation $U : G \to \mathbf{U}(V)$ with multiplier α such that $U_g\phi_h = \alpha(g,h)\phi_{gh}$.

Let $\tilde{\Phi} = \{\tilde{\phi}_g := S^{-1}\phi_g \mid g \in G\}$ be the canonical dual frame of Φ (cf. [5, Definition 1.19, Proposition 1.13]). Here S is the frame operator of Φ. For any $v \in V$, we have

$$v = \sum_{h_1\in G}\langle v, \tilde{\phi}_{h_1}\rangle \phi_{h_1} = \sum_{h_1\in G}\langle v, \phi_{h_1}\rangle \tilde{\phi}_{h_1}. \tag{2.1}$$

Define $U : G \to \mathrm{GL}(V)$ by

$$U_g(v) = \sum_{h_1\in G}\langle v, \phi_{h_1}\rangle \alpha(g,h_1)\tilde{\phi}_{gh_1}.$$

This U satisfies the following properties.

(1) For any $g, h, h_1 \in G$, we have

$$\langle \phi_h, \phi_{h_1} \rangle \alpha(g, h_1) = \langle \phi_{gh}, \phi_{gh_1} \rangle \alpha(g, h). \tag{2.2}$$

Indeed, in the first equation of the definition of Schur multipliers, take $y = x^{-1}$ and $z = x$, we have $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$ for all $x \in G$. Equality (2.2) then follows from the following computation

$$\begin{aligned}\alpha(g, h_1)\alpha(h^{-1}g^{-1}, gh_1)\alpha(h, h^{-1}) &= \alpha(h^{-1}g^{-1}, g)\alpha(h^{-1}, h_1)\alpha(h, h^{-1}) \\ &= [\alpha(h^{-1}g^{-1}, g)\alpha(h^{-1}, h)]\alpha(h^{-1}, h_1) \\ &= \alpha(h^{-1}g^{-1}, gh)\alpha(g, h)\alpha(h^{-1}, h_1) \\ &= \alpha(gh, h^{-1}g^{-1})\alpha(g, h)\alpha(h^{-1}, h_1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}U_g(\phi_h) &= \sum_{h_1 \in G} \langle \phi_h, \phi_{h_1} \rangle \alpha(g, h_1)\tilde{\phi}_{gh_1} \\ &= \sum_{h_1 \in G} \langle \phi_{gh}, \phi_{gh_1} \rangle \alpha(g, h)\tilde{\phi}_{gh_1} = \alpha(g, h)\phi_{gh}.\end{aligned}$$

(2) For any $\phi_h \in \Phi$,

$$\begin{aligned}U_{g_1}(U_{g_2}\phi_h) &= U_{g_1}(\alpha(g_2, h)\phi_{g_2h}) \\ &= \alpha(g_2, h)\alpha(g_1, g_2h)\phi_{g_1g_2h} \\ &= \alpha(g_1, g_2)(\alpha(g_1g_2, h)\phi_{g_1g_2h}) = \alpha(g_1, g_2)U_{g_1g_2}\phi_h.\end{aligned}$$

Hence U defines a projective representation with multiplier α.

(3) For any $\phi_{h_1}, \phi_{h_2} \in \Phi$,

$$\langle U_g\phi_{h_1}, U_g\phi_{h_2} \rangle = \frac{\alpha(g, h_1)}{\alpha(g, h_2)} \frac{\alpha(gh_1, (gh_1)^{-1})}{\alpha((gh_1)^{-1}, gh_2)} \eta(h_1^{-1}h_2).$$

We claim that $U_g \in \mathbf{U}(V)$, i.e. $\langle U_g\phi_{h_1}, U_g\phi_{h_2} \rangle = \langle \phi_{h_1}, \phi_{h_2} \rangle$. To prove this, it suffices to check that

$$\frac{\alpha(g, h_1)}{\alpha(g, h_2)} \frac{\alpha(gh_1, (gh_1)^{-1})}{\alpha((gh_1)^{-1}, gh_2)} = \frac{\alpha(h_1, h_1^{-1})}{\alpha(h_1^{-1}, h_2)}.$$

Note that

$$\alpha(g, h_1)\alpha(gh_1, (gh_1)^{-1}) = \alpha(g, g^{-1})\alpha(h_1, h_1^{-1}g^{-1})$$

and

$$\alpha((gh_1)^{-1}, gh_2)\alpha(g, h_2) = \alpha(h_1^{-1}g^{-1}, g)\alpha(h_1^{-1}, h_2),$$

it suffices to check that

$$\alpha(g, g^{-1})\alpha(h_1, h_1^{-1}g^{-1}) = \alpha(h_1^{-1}g^{-1}, g)\alpha(h_1, h_1^{-1}). \tag{2.3}$$

Multiply both sides with $\alpha(h_1^{-1}, g^{-1})$, it is easy to see that equation (2.3) holds, hence the claim follows.

Therefore $U : G \to \mathbf{U}(V)$ is the desired projective representation and the theorem follows. □

To understand more on (G, α)-matrices, we start with the following observation.

Lemma 2.3. *Let G be a finite group. The sum and product of (G, α)-matrices are (G, α)-matrices.*

Proof. Let $\nu : G \to \mathbb{C}$ and $\mu : G \to \mathbb{C}$ be two functions. Then $M(\mu) + M(\nu) = M(\mu + \nu)$. Next we compute $M(\nu)M(\mu)$. The (g, h)-entry of $M(\nu)M(\mu)$ is

$$\begin{aligned}
&\sum_{f\in G} \frac{\alpha(g, g^{-1})}{\alpha(g^{-1}, f)}\nu(g^{-1}f)\frac{\alpha(f, f^{-1})}{\alpha(f^{-1}, h)}\mu(f^{-1}h) \\
=&\sum_{t\in G} \frac{\alpha(g, g^{-1})}{\alpha(g^{-1}, ht)}\nu(g^{-1}ht)\frac{\alpha(ht, t^{-1}h^{-1})}{\alpha(t^{-1}h^{-1}, h)}\mu(t^{-1}) \text{ (here } f = ht) \\
=&\frac{\alpha(g, g^{-1})}{\alpha(g^{-1}, h)} \sum_{t\in G} \frac{\alpha(t, t^{-1})}{\alpha(g^{-1}h, t)}\nu(g^{-1}ht)\mu(t^{-1}).
\end{aligned}$$

To verify the last equality, as $\alpha(g^{-1}, h)\alpha(g^{-1}h, t) = \alpha(g^{-1}, ht)\alpha(h, t)$, it suffices to check that

$$\alpha(h, t)\alpha(ht, t^{-1}h^{-1}) = \alpha(t, t^{-1})\alpha(t^{-1}h^{-1}, h).$$

This is clear since both sides are equal to $\alpha(h, h^{-1})\alpha(t, t^{-1}h^{-1})$. Define $\eta : G \to \mathbb{C}$ by $\eta(g) = \sum_{t\in G} \frac{\alpha(t,t^{-1})}{\alpha(g,t)}\nu(gt)\mu(t^{-1})$. Then $M(\nu)M(\mu) = M(\eta)$. The lemma follows. □

Let $\mathbb{C}[G]_\alpha$ be the twisted group algebra over $\mathbb{C}$ attached to (G, α), i.e., it is a $\mathbb{C}$-algebra with a basis $\{a_g \mid g \in G\}$ indexed by elements of G. Each element $f \in \mathbb{C}[G]_\alpha$ can be uniquely written as

$$f = \sum_{g\in G} c_g a_g, \quad c_g \in \mathbb{C}$$

and the multiplication is given by

$$a_g a_h = \alpha(g,h) a_{gh}.$$

In the following, we write $\sum_{g\in G} c_g g$ for $\sum_{g\in G} c_g a_g \in \mathbb{C}[G]_\alpha$. By Lemma 2.3, the set $\mathcal{M}_\alpha$ of (G,α)-matrices is a sub-algebra of the matrix algebra $M_{n\times n}(\mathbb{C})$. Define

$$\begin{aligned} \Pi : \mathcal{M}_\alpha &\to \mathbb{C}[G]_\alpha \\ M(\nu) &\mapsto \sum_{g\in G} \nu(g) g. \end{aligned} \tag{2.4}$$

We have the following result (cf. [20, Proposition 4.1]).

Proposition 2.4. *The map* Π *is an isomorphism of* $\mathbb{C}$*-algebras.*

Proof. Let $L^2(G)$ be the set of functions from G to $\mathbb{C}$. Let e_g be the characteristic function of $\{g\}$. Then define

$$\begin{aligned} \Pi' : \mathbb{C}[G]_\alpha &\to \mathcal{M}_\alpha \\ g &\mapsto M(e_g) \end{aligned}$$

It is easy to check that Π' is the inverse of Π. To prove the proposition, it suffices to check that Π is a morphism of $\mathbb{C}$-algebras. But this follows from $M(e_g)M(e_h) = \alpha(g,h)M(e_{gh})$. $\square$

2.2. Tight (G,α)-frames

Let $\Phi = \{\phi_g \mid g \in G\}$ be a (G,α)-frame for V. Let $S : V \to V$ be the frame operator, i.e.,

$$S(v) = \sum_{g\in G} \langle v, \phi_g\rangle \phi_g.$$

Then

$$\begin{aligned} S(hv) &= \sum_{g\in G} \langle hv, \phi_g\rangle \phi_g = \sum_{g\in G} \langle h^{-1}(hv), h^{-1}\phi_g\rangle \phi_g \\ &= \sum_{g\in G} \langle \alpha(h^{-1},h)v, \alpha(h^{-1},g)\phi_{h^{-1}g}\rangle \phi_g \\ &= \sum_{g\in G} \frac{\alpha(h^{-1},h)}{\alpha(h^{-1},g)\alpha(h,h^{-1}g)} \langle v, \phi_{h^{-1}g}\rangle h\phi_{h^{-1}g} = hS(v). \end{aligned}$$

Here the last equality follows as $\frac{\alpha(h^{-1},h)}{\alpha(h^{-1},g)\alpha(h,h^{-1}g)} = 1$. Therefore, the frame operator $S \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V,V)$. It is easy to see that the canonical dual frame of a (G,α)-frame is also a (G, α)-frame (cf. [5, Definition 1.19]). Moreover, by Schur's Lemma (cf. [6, Lemma 2.1]), S is a scalar multiplication if $\pi : G \to \mathbf{U}(V)$ is irreducible. Assume now this is the case. Let $v \in V$ be a nonzero vector. Then $V = \mathrm{Span}\{gv \mid g \in G\}$. Hence $Gv = \{gv \mid g \in G\}$ is a frame for V. Even better, we have the following result.

Proposition 2.5. *With the notation as above, for $v \neq 0$, Gv is a tight (G, α)-frame for V.*

Proof. Let $S_v : V \to V$ be the frame operator associated with Gv. By the above discussion, we must have $S_v = \lambda_v \,\mathrm{id}_V$ for a constant $\lambda_v \in \mathbb{C}$. As S_v is self-adjoint and positive, $\lambda_v \in \mathbb{R}_{>0}$ as $v \neq 0$. The proposition follows. $\square$

Remark 2.6. We may obtain λ_v by the following computation (cf. [5, Proposition 1.11]). Let $\{w_i \mid 1 \leq i \leq \dim V\}$ be an orthonormal basis of V. Then for any j,

$$S_v(w_j) = \sum_{g \in G} \langle w_j, gv\rangle gv = \lambda_v w_j.$$

Hence $\lambda_v = \sum_{g \in G} |\langle w_j, gv\rangle|^2$. So

$$\lambda_v \dim V = \sum_j \sum_{g \in G} |\langle w_j, gv\rangle|^2 = \sum_{g \in G} ||gv||^2 = n \cdot ||v||^2.$$

Remark 2.7 *(The Gabor frame).* Let G be a finite group. Suppose that $G = H \times F$ is a direct product of two groups. Let $F \times H \to \mathbb{C}^\times$ be a bi-homomorphism and denote it by $(f, h) \mapsto f(h) \in \mathbb{C}^\times$. Define $\alpha : G \times G \to \mathbb{C}^\times$ by

$$\alpha((h, f), (h', f')) = f(h') \text{ for all } h, h' \in H \text{ and } f, f' \in F.$$

It is easy to check that $\alpha \in Z^2(G, \mathbb{C}^\times)$. Let V be the (right) regular representation of H. Fix a basis $\{e_h \mid h \in H\}$. Define a map $\pi : G \to \mathrm{GL}(V)$ by

$$\pi((h, f))e_i = f(i)e_{hi} \text{ for all } h, i \in H \text{ and } f \in F.$$

Then $\pi : G \to \mathrm{GL}(V)$ is a projective representation of G with multiplier α. As an H-representation, V contains all irreducible linear representations of H. Yet the projective representation π may be irreducible. Let us consider the following special case (cf. [4,17]).

Let H be an abelian group. Let $F = \hat{H} := \mathrm{Hom}(H, \mathbb{C}^\times)$ be the dual group of H. Take the natural pairing between $\hat{H}$ and H. Note that $\alpha((h, f), (h', f')) = \alpha((h', f'), (h, f))$ if and only if $f(h') = f'(h)$. Thus the maximal α-symmetric subgroup of G has order $|H|$. Indeed, every maximal α-symmetric subgroup of G has the form $K \times K^\perp$, where K is a subgroup of H, and $K^\perp = \{f \in \hat{H} \mid f(K) = 1\}$ (cf. [6, Section 2.3]). By [6, Proposition 2.14], this π is the only irreducible element (up to isomorphism) in Rep^α_G. Hence for any nonzero $v \in V$, Gv is a tight (G, α)-frame for V.

Next, we do not assume the irreducibility of V. In this case, write $V = V_1 \oplus \cdots \oplus V_k$ as an orthogonal direct sum of irreducible projective representations with multiplier α. Let $v \in V$ be a nonzero vector. Write $v = v_1 + \cdots + v_k$ with $v_i \in V_i$ for $1 \leq i \leq k$. Then we have the following result.

Proposition 2.8. *With the notation as above, $Gv = \{gv \mid g \in G\}$ is a tight (G, α)-frame for V if and only if the following two conditions hold:*

(1) $\frac{||v_i||^2}{||v_j||^2} = \frac{\dim V_i}{\dim V_j}$ *for all* $1 \leq i, j \leq k$;
(2) $\sum_{g \in G} \langle v_i, gv_i \rangle gv_j = 0$ *for all* $i \neq j$.

Proof. Note that Gv is tight if and only if there exists $\lambda \in \mathbb{R}_{>0}$ with $\sum_{g \in G} \langle f, gv \rangle gv = \lambda f$ for all $f \in V$. Assume that Gv is tight. Take $f_i \in V_i$, then

$$\begin{aligned}\sum_{g \in G} \langle f_i, gv \rangle gv &= \sum_{g \in G} \langle f_i, gv_i \rangle gv \\ &= \sum_{g \in G} \langle f_i, gv_i \rangle gv_i + \sum_{g \in G} \sum_{j \neq i} \langle f_i, gv_i \rangle gv_j.\end{aligned}$$

We must have

$$\begin{cases} \sum_{g \in G} \langle f_i, gv_i \rangle gv_i = \lambda_i f_i, \\ \sum_{g \in G} \langle f_i, gv_i \rangle gv_j = 0 \text{ for any } j \neq i. \end{cases}$$

The second condition holds by taking $f_i = v_i$. Since V_i is irreducible and $v_i \neq 0$, by Remark 2.6, we have $\lambda_i = \frac{n||v_i||^2}{\dim V_i}$. Moreover, $\lambda_i = \lambda = \lambda_j$. Hence $\frac{||v_i||^2}{||v_j||^2} = \frac{\dim V_i}{\dim V_j}$ for all $1 \leq i, j \leq k$. We obtain the first condition as well.

Conversely, assume that conditions (1) and (2) hold. From the above discussion and Proposition 2.5, to show that $Gv = \{gv \mid g \in G\}$ is a tight (G, α)-frame for V, it suffices to show that $\sum_{g \in G} \langle f_i, gv_i \rangle gv_j = 0$ for any $f_i \in V_i$ and $j \neq i$. First take $f_i = hv_i$ for $h \in G$, we have

$$\begin{aligned}\sum_{g \in G} \langle f_i, gv_i \rangle gv_j &= \sum_{g \in G} \langle hv_i, gv_i \rangle gv_j \\ &= \sum_{g \in G} \langle v_i, h^{-1}(gv_i) \rangle gv_j = \sum_{g \in G} \langle v_i, (h^{-1}g)v_i) \rangle \frac{1}{\alpha(h^{-1}, g)} gv_j\end{aligned}$$

Applying h^{-1} on both sides of the above equality, we have

$$h^{-1}(\sum_{g \in G} \langle f_i, gv_i \rangle gv_j) = h^{-1}(\sum_{g \in G} \langle v_i, (h^{-1}g)v_i) \rangle \frac{1}{\alpha(h^{-1}, g)} gv_j)$$

$$= \sum_{g\in G} \langle v_i, (h^{-1}g)v_i\rangle \frac{1}{\alpha(h^{-1}, g)} h^{-1}(gv_j)$$
$$= \sum_{g\in G} \langle v_i, (h^{-1}g)v_i\rangle (h^{-1}g)v_j = 0.$$

The proposition then follows since $\sum_{g\in G}\langle f_i, gv_i\rangle gv_j$ is linear as a map on f_i. □

Corollary 2.9. *With the notation as in Proposition 2.8, Gv is a frame if and only if $v_i \neq 0$ for all i and $\sum_{g\in G}\langle v_i, gv_i\rangle gv_j = 0$ for any $j \neq i$.*

Proof. From the same argument in Proposition 2.8, the tightness gives us the first condition. The corollary follows as Gv is a tight (G, α)-frame if V is irreducible. □

In order to simplify the second condition in Proposition 2.8, we first prove a frame version of Schur's Lemma.

Lemma 2.10. *Let $\pi_i : G \to \mathbf{U}(V_i)$ $i = 1, 2$ be irreducible projective representations of G with multiplier α. Fix $v_i \in V_i$ nonzero vector for $i = 1, 2$. Define*

$$S := S_{v_1,v_2} : V_1 \to V_2$$
$$f \mapsto \sum_{g\in G} \langle f, gv_1\rangle gv_2.$$

Then $S = 0$ if V_1 and V_2 are not isomorphic in Rep_G^α. If $\sigma : V_1 \to V_2$ is an isomorphism in Rep_G^α, then

$$S(f) = \frac{n||v_1||^2}{(\dim V_1)||\sigma v_1||^2}\langle v_2, \sigma v_1\rangle \sigma(f) \text{ for all } f \in V_1.$$

Proof. A simple computation shows that $S \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_1, V_2)$. If V_1 and V_2 are irreducible and non-isomorphic, Schur's Lemma tells us $S = 0$. On the other hand, if $V_1 \cong V_2$, S must be σ multiplying with a constant number λ. Take $f = v_1$, we have

$$\langle Sv_1, \sigma v_1\rangle = \langle \sum_{g\in G}\langle v_1, gv_1\rangle gv_2, \sigma v_1\rangle = \sum_{g\in G}\langle v_1, gv_1\rangle\langle gv_2, \sigma v_1\rangle$$
$$= \sum_{g\in G}\langle v_1, gv_1\rangle \alpha(g^{-1}, g)\langle v_2, g^{-1}(\sigma v_1)\rangle$$
$$= \sum_{g\in G}\langle g^{-1}v_1, v_1\rangle\langle v_2, \sigma(g^{-1}v_1)\rangle$$
$$= \langle v_2, \sigma(\sum_{g\in G}\langle v_1, gv_1\rangle gv_1)\rangle$$

By Remark 2.6, we have $\sum_{g\in G}\langle v_1, gv_1\rangle gv_1 = \frac{n}{\dim V_1}||v_1||^2 v_1$. The lemma follows. □

Combining Proposition 2.8 and Lemma 2.10, we have the following result.

Theorem 2.11. *With the notation as in Proposition 2.8, $Gv = \{gv \mid g \in G\}$ is a tight (G, α)-frame for V if and only if the following two conditions hold*

(1) $\frac{||v_i||^2}{||v_j||^2} = \frac{\dim V_i}{\dim V_j}$ *for all* $1 \le i, j \le k$;
(2) $\langle \sigma v_i, v_j\rangle = 0$ *for any* $\sigma \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_i, V_j)$ *and* $i \ne j$.

Similarly, the following result holds.

Corollary 2.12. *With the notation as in Proposition 2.8, Gv is a frame if and only if $v_i \ne 0$ for all i and $\langle \sigma v_i, v_j\rangle = 0$ for any $\sigma \in \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_i, V_j)$ and $i \ne j$.*

Remark 2.13. Let $\Phi = \{gv \mid g \in G\}$ be a (G, α)-frame for V. Write $v = v_1 + \cdots + v_k$ as above. Then the *canonical dual frame* of Φ is given by $\tilde{\Phi} = \{gv' \mid g \in G\}$, with $v' = \frac{1}{\lambda_1}v_1 + \cdots + \frac{1}{\lambda_k}v_k$.

The results in this section are projective version of the corresponding results in [19, Section 6]. See also [14, Theorem 6.1] and [21, Theorem 2.8].

2.3. *Central tight (G, α)-frames*

Definition 2.14. A function $f : G \to \mathbb{C}$ is called an *α-class function* if for all $g, h \in G$,

$$f(hgh^{-1}) = \frac{\alpha(h, h^{-1})}{\alpha(h, gh^{-1})\alpha(g, h^{-1})} f(g) = \frac{\alpha(h, h^{-1})}{\alpha(h, g)\alpha(hg, h^{-1})} f(g).$$

Let $\mathbb{H}_\alpha$ denote the space of α-class functions on G. The characters of projective representations belong to $\mathbb{H}_\alpha$ and $\{\chi_i\}$ forms a basis of $\mathbb{H}_\alpha$.

Definition 2.15. Let $\Phi = \{\phi_g \mid g \in G\}$ be a (G, α)-frame. We say that Φ is *central* if $\nu : G \to \mathbb{C}$ defined by

$$\nu(g) = \langle \phi_1, \phi_g\rangle$$

is an α^{-1}-class function.

Remark 2.16. Let $\pi : G \to \mathbf{U}(V)$ be a projective representation with multiplier α. We have

$$\langle \pi(h)v, \pi(g)\pi(h)v\rangle = \frac{\alpha(h^{-1}, h)}{\alpha(h^{-1}, g)\alpha(h^{-1}g, h)} \langle v, \pi(h^{-1}gh)v\rangle.$$

Hence v is a central (G, α)-frame vector for V if and only if

$$\langle \pi(h)v, \pi(g)\pi(h)v\rangle = \langle v, \pi(g)v\rangle \text{ for all } g, h \in G.$$

In order to classify all the central tight (G,α)-frames, we first review the structure of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$, the center of the twisted group algebra $\mathbb{C}[G]_\alpha$. We use the notation introduced in Section 1.1. Define $\omega_i = \frac{d_i}{n}\omega_{\chi_i} := \frac{d_i}{n}\sum_{g\in G}\bar{\chi}_i(g)g \in \mathbb{C}[G]_\alpha$. Here $\bar{\chi}_i$ is the complex conjugation of χ_i.

Lemma 2.17. *With the notation as above, $\{\omega_i \mid 1 \le i \le r\}$ is a basis for* $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$. *Moreover $\omega_i\omega_i = \omega_i$ and $\omega_i\omega_j = 0$ if $i \ne j$.*

Proof. Let $\chi : G \to \mathbb{C}$ be an α-class function. Let $\tau = \sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})g$. Then τ is an element of $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$. Indeed, it suffices to show that $h(\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})g)$ $= (\sum_{g\in G}\alpha(g,g^{-1})^{-1}\chi(g^{-1})g)h$ for any $h \in G$. This is equivalent to

$$\begin{aligned}&\alpha(hgh^{-1},hg^{-1}h^{-1})^{-1}\chi(hg^{-1}h^{-1})\alpha(hgh^{-1},h) = \alpha(g,g^{-1})^{-1}\chi(g^{-1})\alpha(h,g)\\ \Leftrightarrow&\alpha(hgh^{-1},hg^{-1}h^{-1}) = \frac{\alpha(hgh^{-1},h)\alpha(h,h^{-1})\alpha(g,g^{-1})}{\alpha(h,g^{-1}h^{-1})\alpha(g^{-1},h^{-1})\alpha(h,g)}.\end{aligned} \tag{2.5}$$

This identity follows from

$$\begin{aligned}\alpha(xgx^{-1},xhx^{-1}) &= \frac{\alpha(x,ghx^{-1})\alpha(gx^{-1},xhx^{-1})}{\alpha(x,gx^{-1})}\\ &= \frac{\alpha(x,ghx^{-1})}{\alpha(x,gx^{-1})}\frac{\alpha(gx^{-1},x)\alpha(g,hx^{-1})}{\alpha(x,hx^{-1})}\\ &= \frac{\alpha(x,ghx^{-1})\alpha(gx^{-1},x)}{\alpha(x,gx^{-1})\alpha(x,hx^{-1})}\frac{\alpha(g,h)\alpha(gh,x^{-1})}{\alpha(h,x^{-1})}\\ &= \frac{\alpha(x,ghx^{-1})\alpha(g,h)\alpha(gh,x^{-1})\alpha(x,x^{-1})}{\alpha(x,gx^{-1})\alpha(x,hx^{-1})\alpha(h,x^{-1})\alpha(g,x^{-1})}.\end{aligned} \tag{2.6}$$

If χ is a character, then $\bar{\chi}(g) = \alpha(g,g^{-1})^{-1}\chi(g^{-1})$. Therefore, $\{\omega_i \mid 1 \le i \le r\}$ is a basis for $\mathrm{Cent}\,.\mathbb{C}[G]_\alpha$. We check that $\omega_i\omega_i = \omega_i$. The other identities follow by similar argument. For simplicity, we omit the subscript i in the computation. Write $\rho : G \to \mathrm{GL}(V)$ in the matrix form as $\rho(g) = (r_{ij}(g))$. Then as a consequence of Schur's Lemma, we have

$$\frac{1}{n}\sum_{g\in G}\alpha(g,g^{-1})^{-1}r_{i_2j_2}(g^{-1})r_{j_1i_1}(g) = \frac{1}{\dim V}\delta_{i_2i_1}\delta_{j_2j_1}.$$

Note that

$$\omega^2 = \sum_{g\in G}(\sum_{h\in G}\alpha(gh^{-1},h)\bar{\chi}(gh^{-1})\bar{\chi}(h))g.$$

The identity $\omega^2 = \omega$ follows from

$$
\begin{aligned}
&(\sum_{h\in G}\alpha(gh^{-1},h)\bar{\chi}(gh^{-1})\bar{\chi}(h))^{-} \\
=&\sum_{h\in G}\alpha(gh^{-1},h)^{-1}\chi(gh^{-1})\chi(h) \\
=&\sum_{h\in G}\alpha(gh^{-1},h)^{-1}\alpha(g,h^{-1})^{-1}\cdot\mathrm{Tr}[(r_{ij}(g))(r_{ij}(h^{-1}))]\cdot\mathrm{Tr}(r_{ij}(h)) \\
=&\sum_{h\in G}\sum_{i,j,k}\alpha(h,h^{-1})^{-1}r_{ij}(g)r_{ji}(h^{-1})r_{kk}(h) \\
=&\frac{n}{d}\sum_{k}r_{kk}(g)=\frac{n}{d}\chi(g).
\end{aligned}
$$

This completes the proof of the lemma. □

We have the following result on central (G,α)-frames, which is a projective version of [20, Theorem 5.1].

Proposition 2.18. *Let G be a finite group and Φ be a (G,α)-frame. Φ is a central (G,α)-frame if and only if its Gramian is given by*

$$\mathrm{Gram}(\Phi)=M(\sum_{i\in I\subset\{1\le i\le r\}}a_i\frac{d_i}{n}\bar{\chi}_i).$$

Here $a_i\in\mathbb{C}^{\times}$.

Proof. By definition, Φ is central if and only if $\mathrm{Gram}(\phi)=M(\eta)$ with η being an α^{-1}-class function. The proposition follows as $\{\bar{\chi}_i \mid 1\le i\le r\}$ is a basis of the space $\mathbb{H}_{\alpha^{-1}}$. □

Remark 2.19. We may realize central frames explicitly as follows. Let $\rho_i : G\to\mathbf{U}(V_i)$ be the irreducible projective representation of G with dimension d_i and character χ_i. Choose an orthogonal basis $\{v_{i1},\dots,v_{id_i}\}$ of V_i with the same length. Consider $v_i = v_{i1}\oplus\cdots\oplus v_{id_i}\in V_i^{\oplus d_i}$. Then v_i is a central (G,α)-frame vector for $\pi : G\to\mathbf{U}(V_i^{\oplus d_i})$. The corresponding Gramian is $M(||v_{i1}||^2\bar{\chi}_i)$. Other central (G,α)-frames could be built by direct sum (cf. [20, Corollary 2.1]).

Theorem 2.20. *Let $\pi : G\to\mathbf{U}(V)$ be a projective representation with multiplier α. Then the following are equivalent.*

(1) *π admits a central (G,α)-frame vector.*
(2) *Every tight (G,α)-frame vector is central.*
(3) *π is equivalent to a central projection of the α-regular representation.*
(4) *$\langle\pi(h)v,\pi(g)\pi(h)v\rangle=\langle v,\pi(g)v\rangle$ for all $g,h\in G$.*

Proof. Only the part (1) $\Rightarrow$ (2) needs explanation. Write $V = V_1^{\oplus d_1} \oplus \cdots \oplus V_s^{\oplus d_s}$ as a direct sum of irreducible projective representations, where $d_i = \dim V_i$ and $V_i \ncong V_j$ if $i \neq j$. Let $v = v_{11} + \cdots v_{1d_1} + \cdots + v_{s1} + \cdots + v_{sd_s}$, where v_{ij} is the component of v in the j-th copy of V_i ($1 \leq i \leq s$ and $1 \leq j \leq d_i$).

Assume that v is a tight (G, α)-frame vector for V, by Theorem 2.11, for any $1 \leq i \leq s$, $\{v_{i1}, \dots, v_{id_i}\}$ is an orthogonal basis of V_i with the same length and $\frac{||v_{i1}||^2}{||v_{j1}||^2} = \frac{d_i}{d_j}$. Write $a = \frac{n||v_{i1}||^2}{d_i} \in \mathbb{C}^\times$, which is a constant independent of the choice of i. Then the Gramian of $\{gv \mid g \in G\}$ is $M(\sum_{i=1}^s \bar{\chi}_i ||v_{i1}||^2) = M(a \sum_{i=1}^s \frac{d_i}{n} \bar{\chi}_i)$. Hence v is central. $\square$

Corollary 2.21. *Let $\pi : G \to \mathbf{U}(V)$ be a projective representation that admits central (G, α)-frames. Then every central (G, α)-frame vector for V is similar to a Parseval frame vector.*

We now classify all central tight (G, α)-frames by describing their Gramian matrices. Suppose that $\Phi = \{\phi_g \mid g \in G\}$ is a central normalized tight (G, α)-frame. Then its Gramian is a (G, α)-matrix

$$P := \mathrm{Gram}(\Phi) = M(\nu), \text{ with } \nu(g) = \langle \phi_1, g\phi_1 \rangle.$$

Since Φ is central, ν is an α^{-1}-class function. Moreover, Φ is a normalized tight frame, by [20, Theorem 2.1], P is an orthogonal projection, i.e. $P^2 = P = \bar{P}^t$. Under the map $\Pi : \mathcal{M}_\alpha \to \mathbb{C}[G]_\alpha$, the image $\Pi(P)$ is an idempotent. Write $p = \Pi(P) = \sum_{g \in G} \nu(g) g$. Since $\nu(g)$ is an α^{-1}-class function, $p \in \mathrm{Cent}\, .\mathbb{C}[G]_\alpha$. By Lemma 2.17, we may write

$$p = \sum_{i=1}^r a_i \omega_i.$$

Since $p^2 = p$, it is easy to see that $a_i \in \{0, 1\}$. Let $I = \{1 \leq i \leq r \mid a_i = 1\}$. Then $p = \sum_{i \in I} \omega_i$.

Conversely, assume that $P = \mathrm{Gram}(\Phi) = M(\nu)$ with $\nu = \sum_{i \in I} \frac{d_i}{n} \bar{\chi}_i$ for some $I \subset \{1 \leq i \leq r\}$. Then $P^2 = P = \bar{P}^t$ as they have the same image under Π. By [20, Theorem 2.1] again, Φ is a central normalized tight (G, α)-frame. Summing up the discussion, we have proved the first part of the following result.

Theorem 2.22. *Let G be a finite group and Φ be a (G, α)-frame.*

(1) *Φ is a central normalized tight (G, α)-frame if and only if its Gramian is given by*

$$\mathrm{Gram}(\Phi) = M(\sum_{i \in I \subset \{1 \leq i \leq r\}} \frac{d_i}{n} \bar{\chi}_i).$$

(2) *Choose normalized tight (G, α)-frames Φ_i for W_i, with*

$$\mathrm{Gram}(\Phi_i) = M(\frac{d_i}{n}\bar{\chi}_i), \quad \dim(W_i) = d_i^2.$$

Then the unique (up to isomorphism in Rep_G^α) central normalized tight (G, α)-frame with Gramian $M(\sum_{i\in I} \frac{d_i}{n}\bar{\chi}_i)$ is given by the direct sum

$$\Phi = \oplus_{i\in I}\Phi_i \subset W := \oplus_{i\in I} W_i.$$

Proof. Note that for $i \neq j$, $\mathrm{Gram}(\Phi_i)\,\mathrm{Gram}(\Phi_j) = O$ since $\Pi(\mathrm{Gram}(\Phi_i)\,\mathrm{Gram}(\Phi_j)) = \omega_i\omega_j = 0$. Therefore the second part of the theorem follows from [20, Corollary 2.1]. □

Remark 2.23. We give an explicit realization of W_i. Let $\rho_i : G \to \mathbf{U}(\mathbb{C}^{d_i})$ be the projective representation of G with character χ_i. Define

$$\phi_g^i := (d_i/n)^{1/2}\rho_i(g) \in \mathbf{U}(\mathbb{C}^{d_i}) \subset \mathbb{C}^{d_i\times d_i} \cong \mathbb{C}^{d_i^2} =: W_i.$$

Define the inner product on matrices by $\langle A, B\rangle = \mathrm{Tr}\,\bar{B}^t A$. Then $\Phi_i = \{\phi_g^i \mid g \in G\}$ is a central tight (G, α)-frame for W_i with Gramian $M(\frac{d_i}{n}\bar{\chi}_i)$.

Indeed, since ρ_i is irreducible, $\mathrm{Span}\{\rho_i(g) \mid g \in G\} = \mathbb{C}^{d_i\times d_i}$. Define $\pi : G \to \mathrm{GL}(\mathbb{C}^{d_i\times d_i})$ by

$$\pi(h)\rho_i(g) = \alpha(h, g)\rho_i(hg) \text{ for all } h, g \in G.$$

It is easy to check that π defines a unitary projective representation with multiplier α. To prove the claim, it suffices to check that this frame has the desired Gramian. This follows from

$$\langle\phi_g^i, \phi_h^i\rangle = \frac{d_i}{n}\,\mathrm{Tr}(\overline{\rho_i(h)}^t\rho_i(g)) = \frac{\alpha(g, g^{-1})}{\alpha(g^{-1}, h)}\frac{d_i}{n}\bar{\chi}_i(g^{-1}h).$$

Note that the second equality follows from the fact that $\rho_i(h)$ is unitary and the fact $\bar{\chi}_i(g) = \alpha(g, g^{-1})^{-1}\chi_i(g^{-1})$ for any $g \in G$. Putting all the W_i together, the Gramian $M(\frac{1}{n}\sum_{1\le i\le r} d_i\bar{\chi}_i)$ is realized by the α-regular representation of G (cf. [6, Section 2.1], [12, Theorem 6], [14, Corollary 5.5]).

2.4. An example

Let $G = D_{4m}$ be the dihedral group with $4m$ elements. Fix a presentation of G

$$G = \langle a, b \mid a^{2m} = 1, b^2 = 1, bab = a^{-1}\rangle.$$

We know that $H^2(G, \mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$ (see for example [15, 2.11.3 Theorem]). Let ζ be a primitive $2m$-th root of unity. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be the cocycle defined by

$$\alpha(a^i b^j, a^{i'} b^{j'}) = \zeta^{ji'}.$$

Here $0 \leq i \leq 2m-1$ and $b \in \{0,1\}$. Then it is a unitary cocycle that represents the non-trivial element in $H^2(G, \mathbb{C}^\times)$ (cf. [15, 2.11.1 Lemma and 2.11.3 Theorem]). We compute the tight (G,α)-frames.

2.4.1. The explicit projective representations

By [6, Theorem 3.9], we know that there are m non-isomorphic irreducible elements in Rep_G^α and all of them have dimension 2. They are given by

$$\begin{aligned} \pi_k : G &\to \mathrm{GL}(V_k) = \mathrm{GL}_2(\mathbb{C}) \\ a^i b^j &\mapsto C_k^i B^j, \end{aligned} \tag{2.7}$$

where $1 \leq k \leq m$, $C_k = \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{1-k} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2.4.2. The tight (G,α)-frames

Let $\pi : G \to \mathbf{U}(W)$ be an element in Rep_G^α. Then W admits a tight (G,α)-frame if and only if $\dim \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_k, W) \leq 2$ for all $1 \leq k \leq m$. On the other hand, if $\dim \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_k, W) \leq 2$ for all $1 \leq k \leq m$, it is easy to write down a tight (G,α)-frame by Theorem 2.11.

2.4.3. The central tight (G,α)-frames

Let $\pi : G \to \mathbf{U}(W)$ be an element in Rep_G^α. Then W admits a central tight (G,α)-frame if and only if $\dim \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(V_k, W) = 2$ or 0. Those frames could be written down explicitly via Remark 2.23 and they are different from the G-frames constructed from linear representations.

For example, let $G = D_8$ and $\zeta = \sqrt{-1}$. Arrange the elements of G in the order $1, a, a^2, a^3, b, ab, a^2b, a^3b$, the central tight (G,α)-frames for $V_k^{\oplus 2}$ $(k = 1, 2)$ are given by

$$\Phi_1 = \{\frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}\zeta\\0\\0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}\zeta^2\\0\\0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}\zeta^3\\0\\0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\\zeta\\1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\\zeta^2\\1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\\zeta^3\\1\\0\end{pmatrix}\}$$

and

$$\Phi_2 = \{\frac{1}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}-1\\0\\0\\-\zeta\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\0\\0\\\zeta^2\end{pmatrix}, \frac{1}{2}\begin{pmatrix}-1\\0\\0\\-\zeta^3\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$
$$\frac{1}{2}\begin{pmatrix}0\\-1\\-\zeta\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\1\\\zeta^2\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0\\-1\\-\zeta^3\\0\end{pmatrix}\}.$$

3. The maximal spanning vectors

3.1. Definition and the general discussion

Let $\pi : G \to \mathbf{U}(V)$ be a projective representation of G with multiplier α over a complex vector space V. Denote by d the dimension of V. Let (V^*, π^*) be the dual projective representation of V, i.e. $V^* = \mathrm{Hom}(V, \mathbb{C})$ and $\pi^*(g) = \alpha(g, g^{-1})^{-1}{}^t\pi(g^{-1})$. By definition,

$$\begin{aligned}\pi^*(gh) &= \alpha(gh, (gh)^{-1})^{-1}{}^t\pi((gh)^{-1}) = \alpha(gh, (gh)^{-1})^{-1}{}^t\pi(h^{-1}g^{-1}) \\ &= \alpha(gh, (gh)^{-1})^{-1}\alpha(h^{-1}, g^{-1})^{-1}{}^t(\pi(h^{-1})\pi(g^{-1})) \\ &= \alpha(gh, (gh)^{-1})^{-1}\alpha(h^{-1}, g^{-1})^{-1}\alpha(g, g^{-1})\alpha(h, h^{-1})\pi^*(g)\pi^*(h) \\ &= \alpha(g, h)\pi^*(g)\pi^*(h).\end{aligned} \tag{3.1}$$

Therefore, the multiplier attached to π^* is α^{-1}.

For any $x \in V$, denote by $x^* \in V^*$ the linear functional defined by $u \mapsto \langle u, x\rangle$. For any $u,\ v \in V$, we have a matrix coefficient $c_{u,v} : G \to \mathbb{C}$ defined by

$$c_{u,v}(h) = v^*(\pi(h)u) = \langle \pi(h)u, v\rangle.$$

Denote by $C_\pi \subset L^2(G)$ the spanning space of matrix coefficients of V, i.e.

$$C_\pi = \mathrm{Span}\{c_{u,v} \mid u \in V,\ v \in V\} \subset L^2(G).$$

Fix a basis $\{u_i \mid 1 \le i \le d\}$ of V, then $\{c_{u_i,u_j} \mid 1 \le i, j \le d\}$ is a spanning set of C_π. Define

$$C_{u,v} = \mathrm{Span}\{c_{\pi(g)u,\pi(g)v} \mid g \in G\} \subset L^2(G).$$

Certainly, every element $c_{\pi(g)u,\pi(g)v}$ is a matrix coefficient of V. Hence $\dim C_{u,v} \le (\dim V)^2$. If the equality holds, we say that (u, v) is a *maximal spanning pair* for V. If (x, x) is a maximal spanning pair for V, we say that $x \in V$ has the *maximal spanning property* and call x a *maximal spanning vector* for V.

A maximal spanning vector x is automatically *phase-retrievable* in the sense that the set of numbers $\{|\langle v, \pi(g)x\rangle| : g \in G\}$ uniquely (up to a unimodular scalar) determines v. We refer to [1–3,16] for more information on the relation between maximal spanning vectors and the phase retrieval problems. In this section, we focus on [16, Conjecture].

Conjecture 3.1. *If $\pi : G \to \mathbf{U}(V)$ is an irreducible projective representation with multiplier α, then there exists a maximal spanning vector for V.*

In [16], the authors verified this conjecture for abelian groups and certain metacyclic groups. In the following, we explain the difficulty of this conjecture and the reason why the abelian group case is *easy*.

Write gu for $\pi(g)u$, then

$$c_{gu,gv}(h) = \langle h(gu), gv\rangle = \alpha(g, g^{-1})^{-1}\alpha(g^{-1}, h)\alpha(g^{-1}h, g)c_{u,v}(g^{-1}hg). \tag{3.2}$$

From equation (3.2), there are two types of difficulties to find a maximal spanning pair (u, v).

A: The complication coming from the multiplier.
B: The complication coming from the conjugation in the group.

These two types are not independent, as simple computation shows that Type **A** involves conjugation. For G abelian, Type **B** difficulty disappears and we may also control Type **A** difficulty. In Section 3.2, we explain the strategy in detail and give another proof of [16, Theorem 1.7].

From (G, α)-frame point of view, if we consider the representation $\pi \otimes \pi^* : G \to \mathbf{U}(V \otimes V^*)$, it is easy to see that (u, v) is a maximal spanning pair for V if and only if $G(u \otimes v^*)$ is a frame for $V \otimes V^*$. From Theorem 2.12 and its corollary, to characterize maximal pairs (u, v), we need to know the decomposition of $V \otimes V^*$ as a direct sum of irreducible representations. In Section 3.4, we give another proof of the conjecture for abelian groups, as a special case of a more general result. On the other hand, $V \otimes V^*$ always admits tight G-frames.

Proposition 3.2. *If $\pi : G \to \mathbf{U}(V)$ is irreducible, then there always exists $w \in V \otimes V^*$, such that Gw is a tight G-frame for $V \otimes V^*$.*

Proof. First we show that there exists w such that Gw is a G-frame for $V \otimes V^*$. In order to prove this, by Corollary 2.12, it suffices to show that every irreducible $W_i \in \mathrm{Rep}_G$ appears at most $(\dim W_i)$-times in $V \otimes V^*$. Since

$$\mathrm{Hom}_{\mathrm{Rep}_G}(W_i, V \otimes V^*) = \mathrm{Hom}_{\mathrm{Rep}_G^\alpha}(W_i \otimes V, V),$$

and V is irreducible, the claim follows immediately. Then write $w = w_1 + \cdots + w_s$ according to the decomposition of $V \otimes V^*$ and adjust the length of the components w_i, we may make Gw tight. $\square$

Remark 3.3. The same argument shows that $V \otimes V$ admits tight (G, α^2)-frames. Continuing with the Gabor frame example, assume that $\pi : G \to \mathbf{U}(V)$ is from Remark 2.7 with $G = H \times \hat{H}$. Assume further that $2 \nmid |G|$. Then α^2 represents an element with maximal order in $H^2(G, \mathbb{C}^\times)$. Every irreducible element of $\mathrm{Rep}_G^{\alpha^2}$ has the same dimension, which equals $\dim V = |H|$. Then one sees that $V \otimes V$ is isomorphic to the α^2-regular representation of G and it admits a central tight (G, α^2)-frame. By Theorem 2.20, every tight (G, α^2)-frame vector for $V \otimes V$ is central.

If π in Conjecture 3.1 is reducible, then the statement is false. For example, we have the following results.

Proposition 3.4. *Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional projective representation of G with multiplier α. Then the following claims hold.*

1. *If G is abelian, α is trivial, and $\dim V \geq 2$, then V admits no maximal spanning vector.*
2. *If $\dim V \geq 2$ and $x \in V$ is a central (G, α)-frame vector for V, then x is not a maximal spanning vector for V.*
3. *Let $r := \pi \oplus \pi : G \to \mathbf{U}(V \oplus V)$. Then $(r, V \oplus V)$ admits no maximal spanning vector.*

Proof. The first claim follows from equation (3.2), as in this case $c_{gu,gv} = c_{u,v}$ for any $g \in G$ and $u, v \in V$. The second claim follows from Theorem 2.20. For the third claim, let $x = u \oplus v \in V \oplus V$ be a nonzero vector. Then

$$\begin{aligned} c_{r(g)x,r(g)x}(h) &= \langle r(h)r(g)x, r(g)x\rangle \\ &= \langle \pi(h)\pi(g)u, \pi(g)u\rangle + \langle \pi(h)\pi(g)v, \pi(g)v\rangle. \end{aligned}$$

Hence $C_{x,x} \subset C_\pi$ and $\dim C_{x,x} \leq (\dim V)^2 < (\dim(V \oplus V))^2$. Therefore $(r, V \oplus V)$ admits no maximal spanning vector. $\square$

Since a maximal spanning vector for a representation automatically yields a maximal spanning vector for any of its subrepresentations, we immediately have the following consequence of Proposition 3.4.

Corollary 3.5. *Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional projective representation of G with multiplier α. If π has a subrepresentation that is equivalent to a subrepresentation of the α-regular representation of G corresponding to a central projection P of rank at least 2, then π does not admit any maximal spanning vector.*

Remark 3.6. Starting with the first claim in Proposition 3.4, a natural question to ask is that, with the same assumption there, does V admit phase retrievable vector? We give the following $\mathbb{R}$-vector space, for which the answer is yes.

Let $G = \mathbb{Z}/3\mathbb{Z}$ and $V = \mathbb{R}^2$. Let $\rho : G \to \mathbf{U}(V)$ be the $\mathbb{R}$-representation of G sending $1 \in G$ to the rotation by $2\pi/3$. In this case (π, V) is irreducible. Let $v_0 = (0, 1) \in \mathbb{R}^2$. Then $Gv_0 = \{v_0, v_1, v_2\}$ is the Mercedes–Benz frame. Consider the map

$$\begin{aligned} A : V &\to \mathbb{R}^3 \\ v &\mapsto (|\langle v, v_0\rangle|, |\langle v, v_1\rangle|, |\langle v, v_2\rangle|). \end{aligned}$$

If (a, b) and (c, d) have the same image under A, then $(a, b) = \pm(c, d)$. This tells us that Gv is a phase retrievable frame for V.

On the other hand, any phase-retrievable frame for $\mathbb{C}^2$ needs at least four vectors (cf. [1]) and so every phase-retrievable frame vector for a representation on $\mathbb{C}^2$ is a maximal spanning vector. This naturally leads to the following question.

Question 3.7. *Let G be a finite group and V a finite dimensional $\mathbb{C}$-space with dimension ≥ 2. Let $\pi : G \to \mathbf{U}(V)$ be a finite dimensional projective representation of G with multiplier α. Is it possible that (π, V) admits a phase retrievable frame vector which is also central? In particular, can this happen if G is abelian and π is a linear representation?*

3.2. The abelian case

In this section, G is a finite abelian group. We study projective representations of G in detail. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be a unitary multiplier. Let (π, V, α) be an irreducible projective representation of G with multiplier α. Let $(\overline{\pi}, \overline{V}, \overline{\alpha} = \alpha^{-1})$ be the projective representation defined by $\overline{V} = V$ and $\overline{\pi}(g) = \overline{\pi(g)}$, where the last $^-$ means complex conjugation. Then the tensor product $\pi \otimes \overline{\pi}$ is a projective representation of G with multiplier $\alpha\overline{\alpha} = 1$, i.e., it is a linear representation. The character of $\pi \otimes \overline{\pi}$ is $\chi_\pi \chi_{\overline{\pi}}$, where χ_π and $\chi_{\overline{\pi}}$ are the characters of π and $\overline{\pi}$ respectively. Let $\rho : G \to \mathbb{C}^\times$ be a one-dimensional linear representation that appears in $\pi \otimes \overline{\pi}$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\rho, \pi \otimes \overline{\pi}) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\overline{\chi_\pi(g)\chi_{\overline{\pi}}(g)} = \dim_{\mathbb{C}} \operatorname{Hom}(\pi \otimes \rho, \pi) \leq 1. \tag{3.3}$$

Therefore, the dimension must be one, i.e., every one-dimensional representation has multiplicity at most one in $\pi \otimes \overline{\pi}$. Define

$$\mathcal{H}(V) := \{\rho \in \widehat{G} := \operatorname{Hom}(G, \mathbb{C}^\times) \mid \dim_{\mathbb{C}} \operatorname{Hom}(\rho, \pi \otimes \overline{\pi}) = 1\}.$$

Lemma 3.8. *With the notation as above, $\mathcal{H}(V)$ is a subgroup of $\widehat{G}$. Moreover, $\overline{\mathcal{H}(V)} := \{\overline{\rho} \mid \rho \in \mathcal{H}(V)\}$ coincides with $\mathcal{H}(V)$.*

Proof. By equation (3.3), a one-dimensional linear representation ρ is an element of $\mathcal{H}(V)$ if and only if $\pi \otimes \rho$ is isomorphic to π as projective representations. From this observation, it is easy to see that $\mathcal{H}(V)$ is a subgroup of $\widehat{G}$.

Moreover, if $\rho \in \mathcal{H}(V)$, then $V \otimes \overline{\rho} \cong (V \otimes \rho) \otimes \overline{\rho} \cong V \otimes (\rho \otimes \overline{\rho}) \cong V$. Thus $\overline{\rho} \in \mathcal{H}(V)$ and the last claim follows. $\square$

Let $H(V) \subset G$ be $\mathcal{H}(V)^{\perp} := \{g \in G \mid \rho(g) = 1 \text{ for any } \rho \in \mathcal{H}(V)\}$. Then

$$[G : H(V)] = (\dim_{\mathbb{C}} V)^2. \tag{3.4}$$

By [6, Propositions 2.3, 2.14], the number of irreducible projective representations in Rep_G^{α} is $|H(V)|$. Let (π', V', α) be another irreducible projective representation of G with multiplier α. Considering the tensor product $\overline{V} \otimes V'$ as a projective representation of G with trivial multiplier, by the same argument as above, it is easy to see that $V' \cong V \otimes \rho$ for some $\rho \in \widehat{G}$. Therefore,

$$V \otimes \overline{V} \cong V' \otimes \overline{V}'.$$

Thus $H(V) = H(V')$. This group is independent of the choice of V. From now on, we denote it by H_α. The product $(V \otimes \overline{V})|_{H_\alpha}$ decomposes as $(\dim_{\mathbb{C}} V)^2$-copies of the trivial linear representation of H_α.

Lemma 3.9. *With the notation as above, $\alpha|_{H_\alpha \times H_\alpha}$ is a coboundary.*

Proof. Suppose that $\alpha|_{H_\alpha \times H_\alpha}$ is not a coboundary, then the irreducible projective representations of H_α with multiplier $\alpha|_{H_\alpha \times H_\alpha}$ have dimension at least two. Let $W \subset V|_{H_\alpha}$ be such an irreducible object. Then $W \otimes \overline{W} \subset (V \otimes \overline{V})|_{H_\alpha}$ is a sub-representation. By the same argument as before, $W \otimes \overline{W} \cong \oplus_{j \in J} \chi_j$, where χ_j are different one-dimensional linear representations of H_α. This contradicts to the fact that $(V \otimes \overline{V})|_{H_\alpha}$ is a direct sum of trivial representations of H_α. □

Consider the map $f : G \times G \to \mathbb{C}^{\times}$ defined by

$$f(g, h) = \frac{\alpha(g, h)}{\alpha(h, g)}$$

for any $g, h \in G$. It is easy to check that f is a bi-homomorphism and thus induces a homomorphism $\lambda : G \to \widehat{G}$ given by $g \mapsto \lambda_g := \frac{\alpha(g, \cdot)}{\alpha(\cdot, g)}$.

Lemma 3.10. *For any $h \in H_\alpha$ and $g \in G$, $\alpha(h, g) = \alpha(g, h)$. Moreover,* $\operatorname{Ker} \lambda = H_\alpha$.

Proof. A subgroup X of G is called *α-symmetric* if $\alpha(x, y) = \alpha(y, x)$ for any $x, y \in X$. Let K be a maximal α-symmetric subgroup of G such that $H_\alpha \subset K$. Such K exists since H_α is α-symmetric by Lemma 3.9.

By [6, Proposition 2.14], $|K| \cdot \dim_{\mathbb{C}} V = |G|$ and $V \cong \alpha \operatorname{Ind}_K^G \chi$. Here χ is a one-dimensional projective representation of K with multiplier $\alpha|_{K \times K}$ and $\alpha \operatorname{Ind}$ is the induction of projective representations with respect to α (see for example [6, Section 2.2]). Moreover, by [6, Corollary 2.11],

$$V|_K \cong \oplus_{s \in G/K} \chi^s,$$

where $\chi^s(k) = \frac{\alpha(s^{-1},k)}{\alpha(k,s^{-1})}\chi(k)$ for $k \in K$. Therefore,

$$(V \otimes \overline{V})|_{H_\alpha} = \oplus_{s,t \in G/K} \chi^s|_{H_\alpha} \otimes \overline{\chi^t}|_{H_\alpha}.$$

By the definition of H_α, $\chi^s|_{H_\alpha} \otimes \overline{\chi^t}|_{H_\alpha} = 1$ for any $s, t \in G/K$. Let t be the unity element, then one obtains that $\alpha(s^{-1}, h) = \alpha(h, s^{-1})$ for any $h \in H_\alpha$ and $s \in G/K$. The first claim follows and $H_\alpha \subset \operatorname{Ker}(\lambda)$.

Moreover, from the above discussion, one sees that $(V \otimes \overline{V})|_{\operatorname{Ker}(\lambda)}$ is a direct sum of trivial representations of $\operatorname{Ker}(\lambda)$. Since H_α is maximal with respect to this property, $\operatorname{Ker}(\lambda) \subset H_\alpha$. The second claim follows. □

Return to maximal spanning vectors. In equation (3.2), since G is abelian,

$$\alpha(g, g^{-1})^{-1}\alpha(g^{-1}, h)\alpha(g^{-1}h, g) = \frac{\alpha(g^{-1}, h)}{\alpha(h, g^{-1})}.$$

Therefore, $c_{gu,gv} = \lambda_{g^{-1}} c_{u,v}$. We have the following result.

Proposition 3.11. *Let G be an abelian group and $\pi : G \to \mathbf{U}(V)$ be an irreducible projective representation of G. Then (u, v) is a maximal pair for V if and only if $c_{u,v}(g) \neq 0$ for all $g \in G$.*

Proof. Suppose that (u, v) is maximal and $c_{u,v}(h) = 0$ for an $h \in G$. Then $c_{gu,gv}(h) = 0$ for all $g \in G$. This contradicts to the fact that $\pi(h) \in \mathbf{U}(V)$ and some matrix entries of $\pi(h)$ must be nonzero.

Conversely, assume $c_{u,v}(g) \neq 0$ for all $g \in G$. By Lemma 3.10, $|\operatorname{Im}\lambda| = d^2$. Let $\{\lambda_i \mid 1 \le i \le d^2\} \subset \hat{G}$ be the image of λ. Suppose $a_i \in \mathbb{C}$ and

$$\sum_i a_i(\lambda_i c_{u,v}) = 0.$$

Then for any $g \in G$, we have $\sum_i a_i\lambda_i(g) = 0$ since $c_{u,v}(g) \neq 0$. Therefore $a_i = 0$ since the characters are linearly independent. Hence $\{\lambda_i c_{u,v} \mid 1 \le i \le d^2\}$ are linearly independent and $\dim C_{u,v} = d^2$. The proposition follows. □

Combining Proposition 3.11 and [16, Lemma 2.2], we obtain [16, Theorem 1.7].

3.3. Non-abelian examples

Consider a projective irreducible representation $\pi : G \to \mathbf{U}(V)$ with multiplier α. Let $G_1 \subset G$ be a subgroup. By restriction, we may consider $\pi_1 := \pi|_{G_1} : G_1 \to \mathbf{U}(V)$ as a projective representation of G_1 with multiplier $\alpha|_{G_1 \times G_1}$. If $v \in V$ is a maximal spanning vector for $\pi_1 : G_1 \to \mathbf{U}(V)$, then v is a maximal spanning vector for $\pi : G \to \mathbf{U}(V)$. We could use this simple idea to verify the conjecture for certain representations of solvable groups. See for example [16, Example 3.3] for metacyclic groups of the form $C_m \rtimes C_p$ with p a prime.

Let $G = A_4$, the alternating group for four points. Since $A_4 \cong (C_2 \times C_2) \rtimes C_3$, it is not metacyclic. In this case $H^2(G, \mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be a cocycle that represents the nontrivial element in $H^2(G, \mathbb{C}^\times)$.

Proposition 3.12. *Conjecture 3.1 holds for the pair (A_4, α).*

Proof. We sketch the proof here. Since α is nontrivial, every simple object in $\mathrm{Rep}^\alpha_{A_4}$ has dimension at least 2. Since the dimension divides 12 and the sum of their squares is 12, there are 3 simple objects in $\mathrm{Rep}^\alpha_{A_4}$, all of which have dimension two. Let V_4 be the abelian normal subgroup of A_4. It is the Klein four group and isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. It is easy to check that $\alpha|_{V_4\times V_4}$ represents the nontrivial element in $H^2(V_4, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $\pi|_{V_4}$ is irreducible as an element in $\mathrm{Rep}^\alpha_{V_4}$. As V_4 is abelian and we have verified the conjecture for abelian groups, the conjecture holds for (A_4, α) too. $\square$

3.4. On the $V \otimes V^*$-side

If we approach the problem from the $V \otimes V^*$-side, then by Theorem 2.11 and its corollary, to find a maximal spanning vector, we need to understand the decomposition of the tensor product $V\otimes V^*$ into a direct sum of irreducible objects. Note that $V\otimes V^*$ is a projective representation with trivial multiplier, i.e. it is a linear representation. Assume that $V \otimes V^*$ is *unramified* in the sense that every irreducible object in Rep_G appears at most once in $V\otimes V^*$. Write $V\otimes V^* = V_1\oplus\cdots\oplus V_k$ with V_i irreducible and non-isomorphic. Let η_i be the character of V_i. We have canonical projections $V \otimes V^* \to V_i$, given by

$$P_i = \sum_{t\in G} \eta_i(t^{-1})\pi \otimes \pi^*(t).$$

Then we see that (u, v) is a maximal pair for V if and only if $P_i(u\otimes v^*) \neq 0$ for $1 \leq i \leq k$ (cf. Corollary 2.12). This gives us a general characterization for maximal spanning vectors in the unramified case.

Remark 3.13. If G is abelian, then $V \otimes V^* = V \otimes \overline{V}$ and it is unramified as explained in Section 3.2. In this case, the η_i's form the set $H(V)$ in Section 3.2, which is the same as the set $\mathrm{Im}(\lambda)$ (cf. Lemma 3.10). One then may verify that the conditions

$$P_i(u \otimes v^*) = \sum_{t\in G} \eta_i(t^{-1})\pi \otimes \pi^*(t)(u \otimes v^*) \neq 0 \quad (1 \leq i \leq k)$$

are equivalent to the conditions $\langle\pi(g)u, v\rangle \neq 0$ for all $g \in G$. This gives us another proof of Proposition 3.11.

Indeed, suppose that $\langle \pi(g)u, v\rangle \neq 0$ for all $g \in G$ and $P_i(u \otimes v^*) = 0$ for some $1 \leq i \leq k$. Then for any $h \in G$, we have

$$\begin{aligned} 0 = P_i(u \otimes v^*)(h) &= \sum_{t\in G} \eta_i(t^{-1})\pi \otimes \pi^*(t)(u \otimes v^*)(h) \\ &= \sum_{t\in G} \eta_i(t^{-1})\langle \pi(h)\pi(t)u, \pi(t)v\rangle \\ &= \sum_{t\in G} \eta_i(t^{-1})\alpha(t, t^{-1})^{-1}\alpha(t^{-1}, h)\alpha(t^{-1}h, t)\langle \pi(h)u, v\rangle \end{aligned}$$

We may choose $h \in G$ so that $\eta_i(t^{-1})\alpha(t, t^{-1})^{-1}\alpha(t^{-1}, h)\alpha(t^{-1}h, t) = 1$ is the trivial character on $t \in G$ and obtain a contradiction.

Remark 3.14. Unfortunately, $V \otimes V^*$ could be *ramified*. For example, let $G = A_4$ from Section 3.3 and consider the irreducible 3-dimensional representation of A_4 on $V = \mathbb{C}^3$. In this case, V^* is a 3-dimensional irreducible representation, hence $V \cong V^*$. From the character table in [18, Chap. 5, Section 5.7], $V \otimes V^* \cong V^{\oplus 2} \oplus \chi_0 \oplus \chi_1 \oplus \chi_2$ and it is ramified.

Write $V \otimes V^* = V_1^{\oplus r_1} \oplus \cdots \oplus V_k^{\oplus r_k}$ with V_i irreducible and non-isomorphic. Let η_i be the character of V_i. Then P_i is a projection from $V \otimes V^*$ to $V_i^{\oplus r_i}$. The non-vanishing property is a necessary but not sufficient condition for being maximal.

Definition 3.15. Let X and Y be two finite dimensional Hilbert spaces with $\dim X \geq \dim Y$. Let $v \in X \otimes Y$ be a nonzero vector. We call v *optimal* if $\min\{s \in \mathbb{Z} \mid v = x_1 \otimes y_1 + \cdots + x_s \otimes y_s\} = \dim Y$.

For any nonzero $v \in X \otimes Y$, let s be the minimal number such that we may write $v = x'_1 \otimes y'_1 + \cdots + x'_s \otimes y'_s$ with $x'_i \in X$ and $y'_i \in Y$ $(1 \leq i \leq s)$. In this case, $x'_1, \ldots, x'_s$ are linearly independent in X and $y'_1, \ldots, y'_s$ are linearly independent in Y. Let $e_1, \ldots, e_d$ be a basis of Y such that at least one y'_i has no zero coordinate under this basis. Write

$$(y'_1, \ldots, y'_s) = (e_1, \ldots, e_d) \begin{pmatrix} a_{11} & \cdots & a_{s1} \\ \cdots & \cdots & \cdots \\ a_{1d} & \cdots & a_{sd} \end{pmatrix}.$$

Let $A = (a_{ij})_{1\leq i\leq s, 1\leq j\leq d}$. Then

$$\begin{aligned} v &= x'_1 \otimes y'_1 + \cdots + x'_s \otimes y'_s \\ &= x'_1 \otimes (a_{11}e_1 + \cdots + a_{1d}e_d) + \cdots + x'_s \otimes (a_{s1}e_1 + \cdots + a_{sd}e_d) \\ &= (a_{11}x_1 + \cdots + a_{s1}x_s) \otimes e_1 + \cdots (a_{1d}x_1 + \cdots + a_{sd}x_s) \otimes e_d \\ &= w_1 \otimes e_1 + \cdots + w_d \otimes e_d. \end{aligned}$$

First, $w_i \neq 0$ for $1 \leq i \leq d$ since each column of A has at least one nonzero element and x_i's are linearly independent. Moreover, assume that v is optimal (i.e. $s = \dim Y$), then w_i $(1 \leq i \leq d)$ are linearly independent. Applying Gram–Schmidt process to w_i, we may write $v = x_1 \otimes y_1 + \cdots + x_d \otimes y_d$, such that $d = \dim Y$, $y_1, \ldots, y_d$ form a basis of Y, $x_1, \ldots, x_d$ are pairwise orthogonal. Combining with Corollary 2.12, we obtain the following result.

Proposition 3.16. *Let $\pi : G \to \mathbf{U}(V)$ be an irreducible projective representation of G. Assume that $V \otimes V^* = V_1^{\oplus r_1} \oplus \cdots \oplus V_k^{\oplus r_k}$ with V_i irreducible and non-isomorphic. Let η_i be the character of V_i. Let P_i be the projection from $V \otimes V^*$ to $V_i^{\oplus r_i}$ defined by*

$$P_i = \sum_{t \in G} \eta_i(t^{-1}) \pi \otimes \pi^*(t).$$

Then (u, v) is a maximal pair for V if and only if $P_i(u \otimes v^*) \in V_i^{\oplus r_i} = V_i \otimes \mathbb{C}^{r_i}$ is optimal for any $1 \leq i \leq k$. Here G acts on $\mathbb{C}^{r_i}$ trivially.

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