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Frames and Finite-Rank Integral Representations of Positive Operator-Valued Measures

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Abstract Discrete and continuous frames can be considered as positive operator-valued measures (POVMs) that have integral representations using rank-one operators. However, not every POVM has an integral representation. One goal of this paper is to examine the POVMs that have finite-rank integral representations. More precisely, we present a necessary and sufficient condition under which a positive operator-valued measure $F : \Omega \to B(H)$ has an integral representation of the form

$$F(E) = \sum_{k=1}^{m} \int_E G_k(\omega) \otimes G_k(\omega)\, d\mu(\omega)$$

for some weakly measurable maps G_k $(1 \le k \le m)$ from a measurable space Ω to a Hilbert space $\mathcal{H}$ and some positive measure μ on Ω. Similar characterizations are also obtained for projection-valued measures. As special consequences of our characterization we settle negatively a problem of Ehler and Okoudjou about probability frame representations of probability POVMs, and prove that an integral representable probability POVM can be dilated to a integral representable projection-valued measure if and only if the corresponding measure is purely atomic.

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1 Introduction

It is well known that positive operator-valued measures (POVM) play important roles in the study of quantum information theory, and the most natural ones are the POVMs that are induced by a system of states (rank-one operators) in classical mechanics theory, and by other types of quantum states in quantum physics. This naturally leads to the study on the integral representations of POVMs with respect to systems of (quantum) states and (probability) distribution measures (cf. [3, 9]).

In recent years the theory of frames has been introduced in many areas of applications, and it also has natural connections with POVMs and even with non-commutative quantum measure theory (cf. [5–8, 12]). For example, a Parseval (discrete or continuous) frame naturally induces a probability positive operator-valued measure, and the dilations of a Parseval frame corresponds to an orthogonal projection-valued measure that is the Naimark dilation of the positive operator-valued measure. The converse is not true meaning that not every POVM arises in this way. In other words, a POVM may not be represented by a "frame-like" system. The main purpose of this paper is to investigate the conditions under which a POVM can be represented by a (discrete or continuous) "frame-like" system and its various generalizations. In particular, we obtain characterizations for all the POVMs that can be represented by a system of rank-k operators in both finite and infinite dimensional Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space and let $(\Omega, \mathcal{B})$ be a measure space, i.e. $\mathcal{B}$ is a σ-algebra on Ω. Let $B(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. An *operator-valued measure* on Ω (abbr. OVM) is a mapping $F : \mathcal{B} \to B(\mathcal{H})$ such that

- $F(\emptyset) = 0$.
- $F(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} F(E_n)$ in the strong operator topology, for every countable collection of $\{E_n\}_{n\ge1}$ of elements of $\mathcal{B}$ satisfying $E_n \cap E_m = \emptyset$ if $n, m \ge 1$ and $m \ne n$.

It is called a *positive operator-valued measure* on Ω (abbr. POVM) if, in addition, $F(E) \ge 0$ (as an operator) for any $E \in \mathcal{B}$. A *probability POVM* is a POVM satisfying the condition $F(\Omega) = I$, where I denotes the identity operator acting on $\mathcal{H}$.

Definition 1.1 A mapping $A : \Omega \to B(\mathcal{H})$ is *weakly measurable* if for every $\omega \in \Omega$, $A(\omega) \in B(\mathcal{H})$ and the mapping:

$$\omega \mapsto \langle A(\omega)x, y\rangle, \quad x, y \in \mathcal{H},$$

is measurable.

We say that a positive operator-valued measure F on Ω is *integral representable* if there exist a weakly measurable mapping $A : \Omega \to B(\mathcal{H})$ with $A(\omega) \ge 0$ for each $\omega \in \Omega$, a constant $B > 0$ and a positive measure μ on $(\Omega, \mathcal{B})$ such that

$$\int_\Omega \langle A(\omega)x, x\rangle \, d\mu(\omega) \le B\|x\|^2, \quad x \in \mathcal{H},$$

and

$$\langle F(E)x, y\rangle = \int_E \langle A(\omega)x, y\rangle \, d\mu(\omega), \quad x, y \in \mathcal{H}, \tag{1.1}$$

for every $E \in \mathcal{B}$.

If a POVM F can be represented by a pair (A, μ) as above, then we write $dF = A\, d\mu$ for short. Such a representation is clearly non-unique as we can always multiply $A(\omega)$

by a function $h(\omega) > 0$ and divide the measure μ by that same function to obtain a different representation of the same POVM. We say that a representation $(A(\omega), \mu)$ for a given POVM is *normalized* if $\|A(\omega)\| = 1$ for μ-a.e. $\omega \in \Omega$. If we have the property that $\mu(\{\omega \in \Omega, A(\omega) = 0\}) = 0$, we can replace the given representation of the POVM by a normalized one by using the function $h(\omega) = \|A(\omega)\|^{-1}$.

The following known result (cf. [10]) tells us that every POVM is integrable representable and we provide a short proof for self completeness.

Theorem 1.1 *Suppose that $F : \mathcal{B} \to B(\mathcal{H})$ is an POVM on Ω. Then, there exists a bounded positive measure μ defined on $\mathcal{B}$ and a continuous positive sesquilinear form $\sigma : \mathcal{H} \times \mathcal{H} \to L^1(\mu)$ such that*

$$\langle F(E)x, y\rangle = \int_E \sigma(x, y)(\omega)\, d\mu(\omega), \quad x, y \in \mathcal{H}. \tag{1.2}$$

Proof Let $\{x_k\}_{k\geq 1}$ be a dense sequence in $\mathcal{H}$. Since F is a POVM, for each $k \geq 1$, the mapping $\mathcal{B} \to \mathbb{R}$:

$$E \mapsto \langle F(E)x_k, x_k\rangle, \quad E \in \mathcal{B},$$

defines a bounded positive measure μ_k on $\mathcal{B}$ with

$$\mu_k(\Omega) = \langle F(\Omega)x_k, x_k\rangle \leq \|F(\Omega)\| \|x_k\|^2 < \infty.$$

Define

$$\mu = \sum_{k=1}^{\infty} \frac{1}{2^k \|x_k\|^2} \mu_k.$$

Clearly $\mu \geq 0$ and $\mu(\Omega) \leq \|F(\Omega)\|$, so μ is a bounded measure. If $x, y \in \mathcal{H}$, consider the (complex) measure $\nu_{x,y}$ defined by

$$\nu_{x,y}(E) = \langle F(E)x, y\rangle, \quad E \in \mathcal{B}.$$

If E is an element of $\mathcal{B}$ such that $\mu(E) = 0$, we have then $\langle F(E)x_k, x_k\rangle = 0$ for all $k \geq 1$. If $\{x_{k(n)}\}_{n\geq 1}$ is a subsequence converging to x in $\mathcal{H}$, we have

$$0 = \langle F(E)x_{k(n)}, x_{k(n)}\rangle = \langle F(E)x_{k(n)}, x\rangle + \langle F(E)x_{k(n)}, x_{k(n)} - x\rangle.$$

Since,

$$\langle F(E)x_{k(n)}, x\rangle \to \langle F(E)x, x\rangle, \quad n \to \infty,$$

and

$$\left|\langle F(E)x_{k(n)}, x_{k(n)} - x\rangle\right| \leq \|F(E)\| \, \|x_{k(n)}\| \, \|x_{k(n)} - x\| \to 0, \quad n \to \infty,$$

it follows that $\langle F(E)x, x\rangle = 0$ and, similarly, $\langle F(E)y, y\rangle = 0$. Therefore,

$$\left|\nu_{x,y}(E)\right| = \left|\langle F(E)x, y\rangle\right| \leq \left|\langle F(E)x, x\rangle\right|^{1/2} \left|\langle F(E)y, y\rangle\right|^{1/2} = 0.$$

This shows that $\nu_{x,y} \ll \mu$. By the Radon-Nikodym theorem, there exists thus a unique function in $L^1(\mu)$, denoted by $\sigma(x, y)$, such that

$$\nu_{x,y}(E) = \langle F(E)x, y\rangle = \int_E \sigma(x, y)(\omega)\, d\mu(\omega), \quad x, y \in \mathcal{H}.$$

It is routine to check that σ is indeed a positive sesquilinear mapping from $\mathcal{H}\times\mathcal{H}\to L^1(\mu)$. This completes the proof. $\square$

Motivated by the recent development on frame theory and its related topics we are interested in characterizing the POVMs that can be represented by finite-rank measurable mapping $A(\omega)$ (i.e. $A(\omega)$ is at most rank-k). One simple way to construct a POVM on Ω is to start with a mapping $G:\Omega\to\mathcal{H}$ which is weakly measurable, i.e. for every $x\in\mathcal{H}$, the mapping

$$\Omega\to\mathbb{C}:\omega\mapsto\langle x, G(\omega)\rangle \tag{1.3}$$

is measurable and a positive measure μ on Ω. If there exists a constant $B\ge 0$ such that

$$\int_\Omega \left|\langle x, G(\omega)\rangle\right|^2 d\mu(\omega)\le B\|x\|^2, \quad x\in\mathcal{H}, \tag{1.4}$$

we can associate with the pair (G,μ) a POVM by defining

$$\langle F(E)x, y\rangle=\int_E \langle x, G(\omega)\rangle\langle G(\omega), y\rangle d\mu(\omega), \quad x, y\in\mathcal{H},\ E\in\mathcal{B}. \tag{1.5}$$

We can thus think of a POVM defined by (1.5) as being defined by an integral on the set E of rank one operators $G(\omega)\otimes G(\omega)$:

$$F(E)=\int_E G(\omega)\otimes G(\omega)d\mu(\omega), \quad E\in\mathcal{B}. \tag{1.6}$$

Recall that a weakly measurable map $G:\Omega\to\mathcal{H}$ is called a frame for H if there exists positive constants C_1, C_2 such that

$$C_1\|x\|^2\le\int_\Omega \left|\langle x, G(\omega)\rangle\right|^2 d\mu(\omega)\le C_2\|x\|^2, \quad \forall x\in\mathcal{H}.$$

It is called a *Parseval frame* if $C_1=C_2=1$. In the case that Ω is discrete, say $\Omega=\mathbb{N}$, and μ is the counting measure, we recover the classical definition for a discrete collection $\{x_i\}$ to form a frame for H: it should satisfy

$$C_1\|x\|^2\le\sum_{i=1}^{\infty}\left|\langle x, x_i\rangle\right|^2\le C_2\|x\|^2, \quad \forall x\in\mathcal{H}$$

and it will form a Parseval frame if $\sum_{i=1}^{\infty}|\langle x, x_i\rangle|^2=\|x\|^2$ $(\forall x\in\mathcal{H})$.

Clearly every frame defines a POVM and every Parseval frame defines a probability POVM through (1.5) or (1.6), and such a POVM has a rank-one integral representation. More generally, a POVM has rank-k integral representation if it has the form

$$F(E)=\sum_{i=1}^{k}\int_E G_i(\omega)\otimes G_i(\omega)d\mu(\omega), \quad E\in\mathcal{B},$$

where $G_i:\Omega\to\mathcal{H}$ $(i=1,\dots,k)$ are weakly measurable maps and μ is a positive measure. In Sect. 2 we will present a characterization for this type of POVMs.

2 Finite-Rank Integral Representations

We first justify that (1.5) indeed defines a POVM.

Lemma 2.1 *Suppose that $G, H : \Omega \to \mathcal{H}$ are weakly measurable maps. Then, the real-valued functions*

$$f(\omega) = \|G(\omega)\| \quad \text{and} \quad g(\omega) = \langle G(\omega), H(\omega)\rangle, \quad \omega \in \Omega,$$

are measurable on Ω.

Proof If $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, we have, by Parseval's identity, that

$$g(\omega) = \langle G(\omega), H(\omega)\rangle = \sum_{i=1}^{\infty} \langle G(\omega), e_i\rangle\langle e_i, H(\omega)\rangle, \quad \omega \in \Omega,$$

showing that the function g is a pointwise limit of measurable functions and is thus measurable. Letting $H = G$, we obtain that f^2 is measurable. Since $f \geq 0$, f itself is measurable. □

Lemma 2.2 *If $G : \Omega \to \mathcal{H}$ is a weakly measurable mapping, let g_j, $j \in \mathbb{N}$, denote the measurable functions defined on Ω by*

$$g_j(\omega) = \langle e_j, G(\omega)\rangle, \quad \omega \in \Omega, \tag{2.1}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Then, if μ is a positive measure on Ω such that (1.4) holds, the functions g_j are all square-integrable with respect to μ on Ω and $\|g_j\|_{2,\mu} \leq B$. Furthermore, if we have $\|G(\omega)\| > 0$ for all $\omega \in \Omega$, then μ must be σ-finite.

Proof If $j \geq 1$, we have

$$\int_{\Omega} |g_j(\omega)|^2 \, d\mu(\omega) = \int_{\Omega} |\langle e_j, G(\omega)\rangle|^2 \, d\mu(\omega) \leq B.$$

If $\|G(\omega)\|^2 = \sum_{i=1}^{\infty} |g_i(\omega)|^2 > 0$ for all $\omega \in \Omega$, we have then

$$\Omega = \bigcup_{i \in \mathbb{N}} \{\omega, \ |g_i(\omega)|^2 > 0\} = \bigcup_{i \in \mathbb{N}} \bigcup_{k \geq 1} \{\omega, \ |g_i(\omega)|^2 > 1/k\}.$$

Since

$$\mu(\{|g_i|^2 > 1/k\}) \leq k \int_{\{|g_i|^2 > 1/k\}} |g_i(\omega)|^2 \, d\mu(\omega) \leq k \int_{\Omega} |g_i(\omega)|^2 \, d\mu(\omega) \leq kB < \infty,$$

it follows that μ is σ-finite. □

It is clear that, if POVM is defined by formula (1.5), we can assume that $\|G(\omega)\| > 0$ for $\omega \in \Omega$ since (1.5) is unchanged if Ω is replaced by the measurable subset

$$\Omega^0 = \Omega \cap \{\omega \in \Omega, \ \|G(\omega)\| > 0\}$$

and the σ-algebra $\mathcal{B}$ is replaced by $\mathcal{B}_0 = \{B \cap \Omega^0 : B \in \mathcal{B}\}$.

Proposition 2.3 *Given a weakly-measurable mapping $G : \Omega \to \mathcal{H}$ and a positive measure μ on Ω satisfying* (1.4), *the mapping $F : \mathcal{B} \to B(\mathcal{H})$ defined by* (1.5) *is a POVM.*

Proof Using the Cauchy-Schwarz inequality together with (1.4), we obtain that

$$\left|\langle F(E)x, y\rangle\right| \le B\|x\|\|y\|, \quad x, y \in H, \; E \in \mathcal{B},$$

showing that $F(E)$ is a bounded operator in $B(\mathcal{H})$ with norm bounded above by B. Each operator $F(E)$ is a positive since

$$\langle F(E)x, x\rangle = \int_\Omega \left|\langle x, G(\omega)\rangle\right|^2 d\mu(\omega) \ge 0.$$

Now let $\{E_n\}_{n\ge 1}$ be a countable collection of elements of $\mathcal{B}$ satisfying $E_n \cap E_m = \emptyset$ if $n, m \ge 1$ and $m \ne n$. Let $H = \bigcup_{n=1}^\infty E_n$ and $H_N = \bigcup_{n=1}^N E_n$ for $N \ge 1$. The sequence of positive (and thus Hermitian) operators $F(H_N)$ is increasing and bounded. Hence it is convergent in the strong operator topology (see e.g. [4]). Since, for any $x, y \in \mathcal{H}$, we have

$$\langle F(H_N)x, y\rangle = \int_{H_N} \langle x, g(\omega)\rangle\langle g(\omega), y\rangle \to \int_H \langle x, g(\omega)\rangle\langle g(\omega), y\rangle\, d\mu(\omega) = \langle F(H)x, y\rangle,$$

as $N \to \infty$, using the Lebesgue dominated convergence theorem, it follows that $F(H_N) \to F(H)$ strongly as $N \to \infty$. We have thus

$$F\left(\bigcup_{n=1}^\infty E_n\right)x = F(H)x = \lim_{N\to\infty} F(H_N)x = \lim_{N\to\infty} \sum_{n=1}^N F(E_n)x = \sum_{n=1}^\infty F(E_n)x, \quad x \in \mathcal{H},$$

showing that $F(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty F(E_n)$ in the strong operator topology, as claimed. □

Now we consider the question as to whether or not a given POVM on Ω can be represented by an integral of operators of rank at most $m < \infty$ with respect to a positive measure μ on $(\Omega, \mathcal{B})$. Since a finite sum of POVMs is clearly a POVM again, given m weakly-measurable functions $G_k : \Omega \to \mathcal{H}$ and a positive measure μ on Ω such that

$$\sum_{k=1}^m \int_\Omega \left|\langle x, G_k(\omega)\rangle\right|^2 d\mu(\omega) \le B\|x\|^2, \quad x \in \mathcal{H}, \tag{2.2}$$

for some positive constant B, we can consider thus the POVM defined by

$$F(E) = \sum_{k=1}^m \int_E G_k(\omega) \otimes G_k(\omega) d\mu(\omega), \quad E \in \mathcal{B}, \tag{2.3}$$

or, equivalently by

$$\langle F(E)x, y\rangle = \sum_{k=1}^m \int_E \langle x, G_k(\omega)\rangle\langle G_k(\omega), y\rangle\, d\mu(\omega), \quad E \in \mathcal{B}, \; x, y \in \mathcal{H}.$$

A POVM defined is this fashion is thus the sum of k POVMs, each admitting a rank at most 1 integral representation. As before we fix an orthonormal basis $\{e_i\}_{i=1}^\infty$ for $\mathcal{H}$ and we defined the functions

$$g_{k,i}(\omega) = \langle e_i, G_k(\omega)\rangle, \quad \omega \in \Omega, \; 1 \le k \le m, \; i \ge 1. \tag{2.4}$$

Given a POVM F on Ω and $i \in \mathbb{N}$, let us define the set $E_i \in \mathcal{B}$

$$E_i = \left\{ \omega \in \Omega, \sum_{k=1}^{m} \left| g_{k,i}(\omega) \right|^2 \neq 0 \right\}. \tag{2.5}$$

Lemma 2.4 *Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Let $G_k : \Omega \to \mathcal{H}$ be weakly-measurable for $1 \le k \le m$ and let μ be a positive measure μ on Ω. Suppose that $F : \mathcal{B} \to B(\mathcal{H})$ is a POVM such that*

$$\left\langle F(E)e_i, e_j \right\rangle = \int_E \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)}\, d\mu(\omega), \quad i, j \in \mathbb{N},\ E \in \mathcal{B},$$

where $g_{k,i}$ is defined in (2.4). Then, (2.3) holds.

Proof Let $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} a_i e_i$ and let $x_n = \sum_{i=1}^{n} a_i e_i$. If $\omega \in \Omega$, we have thus

$$\left\langle x, G_k(\omega) \right\rangle = \sum_{i=1}^{\infty} a_i g_{k,i}(\omega) := s_k(\omega), \qquad \left\langle x_n, G_k(\omega) \right\rangle = \sum_{i=1}^{n} a_i g_{k,i}(\omega) := s_{k,n}(\omega).$$

Hence, if $n, n' \ge 1$,

$$\left\langle F(E)x_n, x_{n'} \right\rangle = \int_E \sum_{k=1}^{m} s_{k,n}(\omega)\overline{s_{k,n'}(\omega)}\, d\mu(\omega), \quad E \in \mathcal{B}.$$

Since $F(\Omega)$ is a bounded operator, $\langle F(\Omega)x_n, x_n \rangle$ converges to $\langle F(\Omega)x, x \rangle$ as $n \to \infty$. Therefore,

$$\begin{aligned}\int_\Omega \sum_{k=1}^{m} \left| s_{k,n}(\omega) - s_{k,n'}(\omega) \right|^2 d\mu(\omega) &= \int_\Omega \sum_{k=1}^{m} \left| \left\langle x_n - x_{n'}, G_k(\omega) \right\rangle \right|^2 d\mu(\omega) \\ &= \left\langle F(\Omega)(x_n - x_{n'}), x_n - x_{n'} \right\rangle \le \left\| F(\Omega) \right\| \|x_n - x_{n'}\|^2.\end{aligned}$$

It follows that for any k with $1 \le k \le m$, the sequence $\{s_{k,n}\}_{n \ge 1}$ is Cauchy in $L^2(\Omega, \mu)$. Since $s_{k,n} \to s_k$ pointwise on Ω, the function s_k is square-integrable with respect to μ on Ω and $s_{k,n} \to s_k$ in $L^2(\Omega, \mu)$ as $n \to \infty$. Hence,

$$\begin{aligned}\left\langle F(E)x, x \right\rangle &= \lim_{n\to\infty} \left\langle F(E)x_n, x_n \right\rangle = \lim_{n\to\infty} \int_E \sum_{k=1}^{m} \left| s_{k,n}(\omega) \right|^2 d\mu(\omega) \\ &= \int_E \sum_{k=1}^{m} \left| s_k(\omega) \right|^2 d\mu(\omega), \quad E \in \mathcal{B}.\end{aligned}$$

If $y = \sum_{i=1}^{\infty} b_i e_i \in \mathcal{H}$, let $y_n = \sum_{i=1}^{n} b_i e_i$. Define also,

$$t_k(\omega) := \left\langle y, G_k(\omega) \right\rangle = \sum_{i=1}^{\infty} b_i g_{k,i}(\omega) \quad \text{and}$$

$$t_{k,n}(\omega) := \left\langle y_n, G_k(\omega) \right\rangle = \sum_{i=1}^{n} b_i g_{k,i}(\omega), \quad \omega \in \Omega,$$

for $n \geq 1$. We have then,

$$\langle F(E)x, y\rangle = \lim_{n\to\infty}\langle F(E)x_n, y_n\rangle = \lim_{n\to\infty}\int_E \sum_{k=1}^m s_{k,n}(\omega)\overline{t_{k,n}(\omega)}\, d\mu(\omega)$$
$$= \int_E \sum_{k=1}^m s_k(\omega)\overline{t_k(\omega)}\, d\mu(\omega), \quad E \in \mathcal{B},$$

as claimed. □

Given weakly-measurable mapping $G_k : \Omega \to \mathcal{H}$ ($k = 1, \dots, m$), our next goal will be to give a characterization of the POVM of the form (2.3) for some positive measure μ on Ω. Before doing so, we need to introduce some notation. Let $\{e_i\}_{i=1}^\infty$ be a fixed orthonormal basis for $\mathcal{H}$ and F be a POVM on Ω. We consider the complex measures $\nu_{i,j}$, $i, j \geq 1$ defined by

$$\nu_{i,j}(E) = \langle F(E)e_i, e_j\rangle, \quad E \in \mathcal{B}. \tag{2.6}$$

Note that $\nu_{i,i}$ is a positive measure for $i \geq 1$ and that

$$\nu_{i,j}(E) = 0 \quad \text{if either } \nu_{i,i}(E) = 0 \quad \text{or} \quad \nu_{j,j}(E) = 0.$$

If $i \in \mathbb{N}$, let us define the set $E_i \in \mathcal{B}$

$$E_i = \left\{\omega \in \Omega, \sum_{k=1}^m \left|\langle e_i, G_k(\omega)\rangle\right|^2 > 0\right\}. \tag{2.7}$$

If we assume that $\sum_{k=1}^m \|G_k\|^2 > 0$ on Ω, it follows that

$$\Omega = \bigcup_{i=1}^\infty E_i.$$

Theorem 2.5 *Let $(\Omega, \mathcal{B})$ be a measurable space and let $F : \mathcal{B} \to B(\mathcal{H})$ be a POVM on Ω. If $\{e_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, let $\nu_{i,j}$ be the bounded measures on Ω defined by (2.6) for $i, j \in \mathbb{N}$. Given weakly-measurable mappings $G_k : \Omega \to \mathcal{H}$, $1 \leq k \leq m$, let $g_{k,j}$, $1 \leq k \leq m$, $j \in \mathbb{N}$, be the measurable functions on Ω defined by (2.4). Then, the following are equivalent*

(a) *There exists a positive σ-finite measure μ on Ω satisfying (2.2), such that F is defined as in (2.3).*
(b) (i) *For every $i \geq 1$, we have $\nu_{i,i}(\Omega \setminus E_i) = 0$ and the positive measure*

$$d\rho_i := \frac{\chi_{E_i}}{\sum_{k=1}^m |g_{k,i}|^2}\, d\nu_{i,i}$$

is σ-finite.

(ii) *If $i, j \geq 1$ and $i \neq j$, we have $\rho_i = \rho_j$ on the set $E_i \cap E_j$. Furthermore,*

$$d\nu_{i,j} = \left(\sum_{k=1}^m g_{k,i}\overline{g_{k,j}}\right) d\rho_i = \left(\sum_{k=1}^m g_{k,i}\overline{g_{k,j}}\right) d\rho_j.$$

Proof Suppose first that (a) holds. If $i \in \mathbb{N}$,

$$\nu_{i,i}(E) = \langle F(E)e_i, e_i\rangle = \int_E \sum_{k=1}^{m} \left|\langle e_i, G_k(\omega)\rangle\right|^2 d\mu(\omega) = \int_E \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\mu(\omega), \quad E \in \mathcal{B}.$$

Using the definition of E_i, we have

$$\nu_{i,i}(\Omega \setminus E_i) = \int_{\{\sum_{k=1}^{m} |g_{k,i}|^2 = 0\}} \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\mu(\omega) = 0.$$

Furthermore, since μ is σ-finite and

$$d\rho_i = \frac{\chi_{E_i}}{\sum_{k=1}^{m} |g_{k,i}|^2}\, d\nu_{i,i} = \chi_{E_i}\, d\mu,$$

so is ρ_i. In addition, it is immediate that, if $i \neq j$,

$$\rho_i = \rho_j = \mu \quad \text{on} \quad E_i \cap E_j.$$

If $i, j \in \mathbb{N}$ and $i \neq j$, we have

$$\begin{aligned}
\nu_{i,j}(E) = \langle F(E)e_i, e_j\rangle &= \int_E \sum_{k=1}^{m} \langle e_i, G_k(\omega)\rangle \langle G_k(\omega), e_j\rangle\, d\mu(\omega) \\
&= \int_E \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)}\, d\mu(\omega) = \int_{E\cap E_i} \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)}\, d\mu(\omega) \\
&= \int_{E\cap E_j} \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)}\, d\mu(\omega), \quad E \in \mathcal{B},
\end{aligned}$$

which yields

$$d\nu_{i,j} = \left(\sum_{k=1}^{m} g_{k,i}\overline{g_{k,j}}\right) d\mu = \left(\sum_{k=1}^{m} g_{k,i}\overline{g_{k,j}}\right) d\rho_i = \left(\sum_{k=1}^{m} g_{k,i}\overline{g_{k,j}}\right) d\rho_j.$$

Conversely, let us assume that (b) holds and let $D_i \in \mathcal{B}$, $i \geq 1$, be defined by $D_1 = E_1$ and

$$D_{i+1} = E_{i+1} \Big\backslash \left(\bigcup_{j=1}^{i} E_j\right), \quad i \geq 1.$$

The collection $\{D_i\}_{i\geq 1}$ is thus pairwise disjoint; we have also $D_i \subset E_i$ for $i \geq 1$ and

$$\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} E_i.$$

Define the positive measure

$$d\mu = \sum_{r=1}^{\infty} \chi_{D_r}\, d\rho_r.$$

Since each measure ρ_i is σ-finite and the sets D_r, $r \geq 1$, are pairwise disjoint, μ is also σ-finite. Using the condition (i) in (b), we have $d\nu_{i,i} = \sum_{k=1}^{m} |g_{k,i}|^2 \, d\rho_i$

$$\begin{aligned}
\langle F(E)e_i, e_i\rangle = \nu_{i,i}(E) &= \int_E \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\rho_i(\omega) \\
&= \sum_{r=1}^{\infty} \int_{E\cap D_r \cap E_i} \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\rho_i(\omega) \\
&= \sum_{r=1}^{\infty} \int_{E\cap D_r \cap E_i} \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\rho_r(\omega) \\
&= \sum_{r=1}^{\infty} \int_{E\cap D_r} \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\rho_r(\omega) \\
&= \int_E \sum_{k=1}^{m} \left|g_{k,i}(\omega)\right|^2 d\mu(\omega), \qquad E \in \mathcal{B}.
\end{aligned}$$

Furthermore, using the condition (ii) in (b), we have also if $i, j \geq 1$ and $i \neq j$,

$$\begin{aligned}
\langle F(E)e_i, e_j\rangle = \nu_{i,j}(E) &= \int_E \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)} \, d\rho_i(\omega) \\
&= \sum_{r=1}^{\infty} \int_{E\cap D_r \cap E_i} \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)} \, d\rho_i(\omega) \\
&= \sum_{r=1}^{\infty} \int_{E\cap D_r \cap E_i} \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)} \, d\rho_r(\omega) \\
&= \sum_{r=1}^{\infty} \int_{E\cap D_r} \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)} \, d\rho_r(\omega) \\
&= \int_E \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)} \, d\mu(\omega), \qquad E \in \mathcal{B}.
\end{aligned}$$

Hence, we deduce that

$$\langle F(E)e_i, e_j\rangle = \int_E \sum_{k=1}^{m} g_{k,i}(\omega)\overline{g_{k,j}(\omega)} \, d\mu(\omega), \qquad i, j \in \mathbb{N},\ E \in \mathcal{B}.$$

Our claim follows then from Lemma 2.4. □

3 Integral Representations for Projection Valued Measures

The first goal of this section is to show that if a PVOM admits an integral representation by some operator-valued function then the values of that function must be positive operators almost everywhere with respect to the measure used in the integral representation.

Recall that a PVOM F on Ω is called a *projection-valued measure* if each operator $F(E)$, $E \in \mathcal{B}$, is an orthogonal projection on $\mathcal{H}$ and $F(\Omega) = I$, the identity operator on $\mathcal{H}$. We will also prove in this section that, if a projection-valued measure has an integral representation as in (1.1), then μ has to be purely atomic.

Proposition 3.1 *Let $(\Omega, \mathcal{B})$ be a measure space and let $A : \Omega \to B(\mathcal{H})$ be a weakly measurable mapping satisfying*

$$\int_{\Omega} \left|\langle A(\omega)x, y\rangle\right| d\mu(\omega) \leq B\|x\|\|y\|, \quad x, y \in \mathcal{H}, \tag{3.1}$$

for some constant $B > 0$. Then, there exists a unique OVM $F : \mathcal{B} \to B(\mathcal{H})$ on Ω such that

$$\langle F(E)x, y\rangle = \int_E \langle A(\omega)x, y\rangle d\mu(\omega). \tag{3.2}$$

Proof Since, if $E \in \mathcal{B}$, the mapping $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$:

$$(x, y) \mapsto \int_E \langle A(\omega)x, y\rangle d\mu(\omega), \quad x, y \in \mathcal{H},$$

defines a bounded bilinear form on $\mathcal{H}$, we can easily show, using standard arguments, that there exists a unique bounded operator on $\mathcal{H}$, denoted by $F(E)$, satisfying (3.2), which shows in particular the uniqueness of the mapping F. To show that F defines an OVM, consider a sequence $\{E_n\}_{n\geq 1}$ of pairwise disjoints elements of $\mathcal{B}$ and define the sets $G = \bigcup_{n\geq 1} E_n$ and $G_N = \bigcup_{1\leq n\leq N} E_n$. If $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{n=1}^{N}\langle F(E_n)x, y\rangle &= \sum_{n=1}^{N}\int_{E_n}\langle A(\omega)x, y\rangle d\mu(\omega) = \int_{G_N}\langle A(\omega)x, y\rangle d\mu(\omega)\\ &\to \int_G \langle A(\omega)x, y\rangle d\mu(\omega), \quad \text{as} \quad N \to \infty,\end{aligned}$$

using the Lebesgue dominated convergence theorem. It follows that $\sum_{n=1}^{\infty} F(E_n)x = F(G)x$ where the series converges weakly. Since we have also that

$$\begin{aligned}\sum_{n=1}^{N}\left|\langle F(E_n)x, y\rangle\right| &\leq \sum_{n=1}^{N}\int_{E_n}\left|\langle A(\omega)x, y\rangle\right| d\mu(\omega) = \int_{G_N}\left|\langle A(\omega)x, y\rangle\right| d\mu(\omega)\\ &\to \int_G \left|\langle A(\omega)x, y\rangle\right| d\mu(\omega), \quad \text{as } N \to \infty,\end{aligned}$$

the series $\sum_{n=1}^{\infty} F(E_n)x$ converges weakly unconditionally. By the Orlicz-Pettis theorem (see [1]), this implies that the series converges strongly. Hence, $\sum_{n=1}^{\infty} F(E_n)$ converges to $F(G)$ in the strong operator topology. □

Lemma 3.2 *Let $(\Omega, \mathcal{B})$ be a measure space and let $F : \mathcal{B} \to B(\mathcal{H})$ be an OVM on Ω defined by (3.2) where A is a weakly measurable mapping $A : \Omega \to B(H)$ satisfying (3.1). Then, $F = 0$ if and only if $A(\omega) = 0$ for μ-a.e. $\omega \in \Omega$.*

Proof Clearly $F=0$ if $A(\omega)=0$ for μ-a.e. $\omega\in\Omega$. In the other direction, suppose that $F=0$ and consider an orthonormal basis $\{e_i\}_{i\geq 1}$ for $\mathcal{H}$. We have

$$\{\omega\in\Omega,\ A(\omega)\neq 0\}=\bigcup_{i,j\geq 1}\{\omega\in\Omega,\ \langle A(\omega)e_i,e_j\rangle\neq 0\}.$$

Thus, if $\mu(\{A(\omega)\neq 0\})>0$, there would exists $i,j\geq 1$ such that

$$\mu\big(\{\omega\in\Omega,\langle A(\omega)e_i,e_j\rangle\neq 0\}\big)>0.$$

Hence, one of the sets

$$\begin{aligned}&\{\omega\in\Omega,\operatorname{Re}(\langle A(\omega)e_i,e_j\rangle)>0\}, &&\{\omega\in\Omega,\operatorname{Re}(\langle A(\omega)e_i,e_j\rangle)<0\},\\ &\{\omega\in\Omega,\operatorname{Im}(\langle A(\omega)e_i,e_j\rangle)>0\}, &&\{\omega\in\Omega,\operatorname{Im}(\langle A(\omega)e_i,e_j\rangle)<0\},\end{aligned}$$

has positive μ-measure. If $E=\{\omega\in\Omega,\operatorname{Re}(\langle A(\omega)e_i,e_j\rangle)>0\}$ and $\mu(E)>0$, we have

$$0=\operatorname{Re}(\langle F(E)e_i,e_j\rangle)=\operatorname{Re}\bigg(\int_E\langle A(\omega)e_i,e_j\rangle\,d\mu(\omega)\bigg)=\int_E\langle\operatorname{Re}(A(\omega)e_i,e_j\rangle)\,d\mu(\omega),$$

which leads to a contradiction. The other three cases are similar. Hence, it follows that the set $\{\omega\in\Omega,A(\omega)\neq 0\}$ has μ-measure zero as claimed. □

If $F:\mathcal{B}\to B(\mathcal{H})$ is an OVM on Ω, we denote by F^* the OVM defined by

$$F^*(E)=F(E)^*,\quad E\in\mathcal{B}.$$

Lemma 3.3 *Let $(\Omega,\mathcal{B})$ be a measure space and let $F:\mathcal{B}\to B(\mathcal{H})$ be an OVM on Ω defined by (3.2) where A is a weakly measurable mapping $A:\Omega\to B(H)$ satisfying (3.1). Then,*

(a) *$F=F^*$ if and only if $A(\omega)=A^*(\omega)$ for μ-a.e. $\omega\in\Omega$.*
(b) *$F\geq 0$ (i.e. F is a POVM if and only if $A(\omega)\geq 0$ for μ-a.e. $\omega\in\Omega$.*

Proof Since for any $x,y\in\mathcal{H}$,

$$\langle F(E)^*x,y\rangle=\langle x,F(E)y\rangle=\int_E\langle x,A(\omega)y\rangle\,d\mu(\omega)=\int_E\langle A(\omega)^*x,y\rangle\,d\mu(\omega),$$

it follows that $dF^*=A^*\,d\mu$ and $d(F-F^*)=(A-A^*)\,d\mu$. Hence, the statement (a) follows immediately from Lemma 3.2. Statement (b) can be proved using a similar argument as the one used in the proof of Lemma 3.2. □

Lemma 3.4 *Let $(\Omega,\mathcal{B})$ be a measure space and let $F:\mathcal{B}\to B(\mathcal{H})$ be an POVM on Ω defined by (1.1) where A is a weakly measurable mapping $A:\Omega\to B(H)$. Then, if $\mu(\{\omega\in\Omega:\ A(\omega)=0\})=0$, μ must be σ-finite.*

Proof Let $X\subset\mathcal{H}$ be a countable dense set and note that, since $A(\omega)\geq 0$ for $\omega\in\Omega$, we have

$$\Omega=\bigcup_{x\in X}\{\omega\in\Omega:\langle A(\omega)x,x\rangle>0\}\cup H,$$

where $H \in \mathcal{B}$ and $\mu(H) = 0$. Furthermore, if $x \in X$,

$$\{\omega \in \Omega : \langle A(\omega)x, x\rangle > 0\} = \bigcup_{k\geq 1}\{\omega \in \Omega : \langle A(\omega)x, x\rangle > 1/k\}$$

and

$$\begin{aligned}\mu(\{\omega \in \Omega : \langle A(\omega)x, x\rangle > 1/k\}) &\leq k \int_{\{\omega\in\Omega:\,\langle A(\omega)x,x\rangle>1/k\}} \langle A(\omega)x, x\rangle \, d\mu(\omega)\\ &\leq k \int_{\Omega} \langle A(\omega)x, x\rangle \, d\mu(\omega) \leq kB\|x\|^2 < \infty,\end{aligned}$$

from which the claim immediately follows. □

Recall that a set $E \in \mathcal{B}$ is an atom for the measure μ defined on $\mathcal{B}$ if $\mu(E) > 0$ and whenever $F \in \mathcal{B}$ and $F \subset E$, either $\mu(F) = \mu(E)$ or $\mu(F) = 0$. A σ-finite measure μ is atomic if there is a countable partition of X made of atoms or sets of μ-measure zero. In this case, μ is called finitely atomic if this partition is finite.

Theorem 3.5 *Suppose that F is a projection-valued measure and that there exists a weakly measurable mapping $A : \Omega \to B(H)$ with $A(\omega) \geq 0$ for $\omega \in \Omega$ and a positive measure μ on Ω) such that $F = A\,d\mu$. If $\mu(\{\omega \in \Omega :\, A(\omega) = 0\}) = 0$ and the representation (A, μ) is normalized, then μ must be atomic. Furthermore, there exists a partition of Ω into a finite or countable collection of measurable subsets Ω_n, $n \in \mathcal{N}$, and a corresponding decomposition of the Hilbert $\mathcal{H}$ into an orthogonal direct sum of closed subspaces $\mathcal{H}_n$, $n \in \mathcal{N}$, with the following properties*:

(a) $\mu(\Omega_n) = 1$, $n \in \mathcal{N}$.
(b) $A(\omega) = \sum_{n\in\mathcal{N}} P_{\mathcal{H}_n} \chi_{\Omega_n}(\omega)$, *for a.e.* $\omega \in \Omega$, *where* $P_{\mathcal{H}_n}$ *denotes the orthogonal projection onto the subspace* $\mathcal{H}_n$, $n \in \mathcal{N}$.

Proof By replacing $A(\omega)$ by $A(\omega)/\|A(\omega)\|$ for $\omega \in \Omega$ and $d\mu$ by $\|A(\omega)\|\,d\mu$, we can assume that $\|A(\omega)\| = 1$ for all $\omega \in \Omega$. Let $E \subset \mathcal{B}$ with $\mu(E) > 0$. Let $X \subset \mathcal{H}$ be a countable dense set. Writing X as the collection $X = \{x_k, k \geq 1\}$, note that

$$\sum_{k\geq 1} 2^{-k} \frac{\langle A(\omega)x_k, x_k\rangle}{\|x_k\|^2} > 0, \quad \text{for a.e. } \omega \in \Omega.$$

In particular, there must exist $x \in X$ such that

$$\langle F(E)x, x\rangle = \int_E \langle A(\omega)x, x\rangle \, d\mu(\omega) > 0,$$

which shows that $F(E)$ is a non-zero projection. Let $M \neq \{0\}$ be the range of the projection $F(E)$ and let $x_0 \in M$ with $\|x_0\|^2 = 1$. We have then,

$$\begin{aligned}1 = \|x_0\|^2 = \langle F(E)x_0, x_0\rangle &= \int_E \langle A(\omega)x_0, x_0\rangle \, d\mu(\omega)\\ &\leq \int_E \langle x_0, x_0\rangle \, d\mu(\omega) = \mu(E)\|x_0\|^2 = \mu(E),\end{aligned}$$

which shows that $\mu(E) \geq 1$ if $\mu(E) \neq 0$. Consider now an at most countable measurable partition $\{E_k\}$ of the set Ω with $\mu(E_k) < \infty$. If $\mu(E_k) \leq N$, an integer, there exist at most N disjoint measurable subsets of E_k with positive μ-measure. Choose such a collection with a maximum number of elements and call that collection $\{E_{k,j}\}$. Then, these $E_{k,j}$ must be atoms by construction which shows that μ must be atomic. It is clear that the collection of all such sets $E_{k,j}$ so constructed forms an at most countable partition of Ω which we rename Ω_n, $n \in \mathcal{N}$. Since, for every $x, y \in \mathcal{H}$, the function $\omega \mapsto \langle A(\omega)x, y\rangle$ is constant on Ω_n, for each $n \in \mathcal{N}$, we have thus $A(\omega) = A_n$ μ-a.e. on Ω_n, for each $n \in \mathcal{N}$. In particular,

$$\big\langle F(\Omega_n)x, y\big\rangle = \int_{\Omega_n} \big\langle A(\omega)x, y\big\rangle \, d\mu(\omega) = \langle A_n x, y\rangle \mu(\Omega_n), \quad n \in \mathcal{N},$$

which shows that

$$F(\Omega_n) = A_n \mu(\Omega_n), \quad n \in \mathcal{N}.$$

Since $F(\Omega_n)$ is a non-zero projection, we have $\|F(\Omega_n)\| = 1$. Furthermore, the representation (A, μ) being normalized, if follows that $\mu(\Omega_n) = 1$ and thus that $A_n = F(\Omega_n)$ for each $n \in \mathcal{N}$. Let $\mathcal{H}_n$ be the range of the projection $F(\Omega_n)$ for $n \in \mathcal{N}$. If $x \in \mathcal{H}_n$, we have

$$\|x\|^2 = \big\langle F(\Omega_n)x, x\big\rangle = \int_{\Omega_n} \big\langle A(\omega)x, x\big\rangle \, d\mu(\omega) = \int_{\Omega} \big\langle A(\omega)x, x\big\rangle \, d\mu(\omega)$$

which shows that

$$\int_{\Omega \setminus \Omega_n} \big\langle A(\omega)x, x\big\rangle \, d\mu(\omega) = 0.$$

In particular, if $m, n \in \mathcal{N}$ with $m \neq n$, $x \in \mathcal{H}_n$ and $y \in \mathcal{H}_m$, we have

$$\begin{aligned} |\langle x, y\rangle| &= \big|\big\langle F(\Omega_n)x, y\big\rangle\big| = \left|\int_{\Omega_n} \big\langle A(\omega)x, y\big\rangle \, d\mu(\omega)\right| \\ &\leq \left(\int_{\Omega_n} \big|\big\langle A(\omega)x, x\big\rangle\big|^2 \, d\mu(\omega)\right)^{1/2} \left(\int_{\Omega_n} \big|\big\langle A(\omega)y, y\big\rangle\big|^2 \, d\mu(\omega)\right)^{1/2} = 0, \end{aligned}$$

since $\Omega_m \cap \Omega_n = \emptyset$. This shows that $\mathcal{H}_n \perp \mathcal{H}_n$ if $n, m \in \mathcal{N}$ and $n \neq m$. Since

$$I = F(\Omega) = F\left(\bigcup_{n \in \mathcal{N}} \Omega_n\right) = \sum_{n \in \mathcal{N}} F(\Omega_n) = \sum_{n \in \mathcal{N}} P_{\mathcal{H}_n},$$

it follows that

$$\mathcal{H} = \bigoplus_{n \in \mathcal{N}} \mathcal{H}_n,$$

as an orthogonal direct sum and since, $A(\omega) = P_{\mathcal{H}_n}$ a.e. on Ω_n, the assertion in (b) follows, which concludes the proof. □

Corollary 3.6 *If, in addition to the assumptions of Theorem 3.5, we have also that*

$$\int_{\Omega} \big\|A(\omega)\big\| \, d\mu < \infty,$$

then the measure μ must be finitely atomic

Proof We have

$$\int_{\Omega} \|A(\omega)\| \, d\mu = \sum_{n \in \mathcal{N}} \int_{\Omega_n} \|A(\omega)\| \, d\mu = \sum_{n \in \mathcal{N}} \|P_{\mathcal{H}_n}\| \mu(\Omega_n) = \sum_{n \in \mathcal{N}} 1 = \operatorname{card}(\mathcal{N}) < \infty. \qquad \square$$

Note that there are plenty of projection-valued measures for which the measure μ is not discrete. For example, we can take $\mathcal{H} = L^2([0, 1])$ and define

$$F(E)f = \chi_E f, \qquad f \in L^2\big([0, 1]\big)$$

for any Borel subset E of $[0, 1]$, where χ_E denotes the characteristic function of the set E. We have then

$$\big\langle F(E)f, g\big\rangle = \int_E f(x)\overline{g(x)}\, dx, \qquad f, g \in L^2\big([0, 1]\big).$$

Clearly, $F(E)$ is the orthogonal projection onto the subspace of $L^2([0, 1])$ consisting of those functions in $L^2([0, 1])$ vanishing a.e. (with respect to the Lebesgue measure) outside of E and $F([0, 1]) = I$. The measure used in the integral representation for this example is the Lebesgue measure on $[0, 1]$ which is certainly not discrete.

4 Applications

We first discuss a simple application of Theorem 2.5 to a conjecture [2] of Ehler and Okoudjou about representations of positive operator-valued measures by means of tight probabilistic frames in $\mathbb{R}^d$.

Let $\mathcal{H} = \Omega = \mathbb{R}^n$. A Parseval probability frame is a probability measure μ on $\mathbb{R}^n$ that satisfies the condition that

$$\|x\|^2 = \int_{\mathbb{R}^n} \big|\langle x, \omega\rangle\big|^2 d\mu(\omega), \qquad \forall \omega \in \mathbb{R}^n.$$

In other world, $G(\omega) = \omega$ is a Parseval frame for $\mathbb{R}^d$ with respect to the Borel measurable space $\Omega = \mathbb{R}^d$. As discussed in Sect. 1, every Parseval probability frame induces a probability POVM:

$$F(E) = \int_E \omega \otimes \omega \, d\mu(\omega) \tag{4.1}$$

Ehler and Okoudjou asked whether the converse is not true (Problem 2 in [2]). The characterization of rank-one ($m = 1$) integral representation for POVM in Theorem 2.5 clearly shows that the converse is not true in general.

Secondly we examine the dilation aspect of the integral representable POVM. Possibly the first well-known dilation result for operator-valued measures is due to Naimark (see [6]).

Theorem 4.1 *Let F be a probability POVM acting on Ω and a Hilbert space $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a projection-valued measure $\tilde{F}$ acting on Ω and $\mathcal{K}$ such that*

$$F(E) = P\tilde{F}(E)P, \qquad \forall E \in \mathcal{B},$$

where P is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$.

The projection-valued measure $\tilde{F}$ is called a *dilation* of F. Clearly if a dilation of F is integral (resp. rank-m integral) representable with respect to a positive measure on Ω, then so is F with respect to the same measure μ. An natural question is whether the converse is true. We answer this question in Theorem 4.3.

Lemma 4.2 (Cf. [5]) *If $\{u_j\}$ is a Parseval frame for a separable Hilbert space $\mathcal{H}$, then there exist a Hilbert space $\mathcal{K}$ and an orthonormal basis $\{v_j\}$ for $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $u_j = Pv_j$, where P is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$.*

Theorem 4.3 *Let F be a probability POVM acting on Ω and a Hilbert space $\mathcal{H}$ such that*

$$\langle F(E)x, y\rangle = \int_E \langle A(\omega)x, y\rangle \, d\mu(\omega), \quad \forall E \in \mathcal{B},$$

where $A(\omega)$ is compact for almost all $\omega \in \Omega$ and $\mu(\{A(\omega) = 0\}) = 0$. Then F can be dilated to an integral representable projection-valued measure $\tilde{F}$ if and only if μ is purely atomic. Moreover, if $A(\omega)$ has at most rank m almost everywhere, then $\tilde{F}$ has an at most rank-m integral representation.

Proof The necessary part follows from Theorem 3.5. For the sufficient part, let assume that μ is purely atomic and

$$\langle F(E)x, y\rangle = \int_E \langle A(\omega)x, y\rangle \, d\mu(\omega), \quad \forall E \in \mathcal{B}.$$

Without losing the generality, we can assume that $\Omega = \mathbb{N}$ and $\mu(\{i\}) = a_i > 0$. Thus we have

$$\langle F(E)x, y\rangle = \sum_{i\in E} a_i \langle A(i)x, y\rangle, \quad \forall E \subset \mathbb{N}.$$

By Lemma 3.3, we get that $A(i)$ is positive and compact. Thus we can write

$$A(i) = \sum_{k=1}^{r_i} g_{i,k} \otimes g_{i,k},$$

where r_i is the rank of $A(i)$. This implies that

$$\|x\|^2 = \langle F(\mathbb{N})x, x\rangle = \sum_{i\in\mathbb{N}} a_i \langle A(i)x, x\rangle = \sum_{i\in\mathbb{N}} \sum_{k=1}^{r_i} a_i \left|\langle x, g_{i,k}\rangle\right|^2$$

for every $x \in \mathcal{H}$. Let $u_{i,k} = \sqrt{a_i}\, g_{i,k}$. Then we have the Parseval identity:

$$\|x\|^2 = \sum_{i\in\mathbb{N}} \sum_{k=1}^{r_i} \left|\langle x, u_{i,k}\rangle\right|^2,$$

and hence $\{u_{i,k}\}$ is a Parseval frame for $\mathcal{H}$. By Lemma 4.2, there exist a Hilbert space $\mathcal{K}$ and an orthonormal basis $\{v_{i,k}\}$ for $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $u_{i,k} = Pv_{i,k}$, where P is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Let $\tilde{g}_{i,k} = \frac{1}{\sqrt{a_i}} v_{i,k}$, and

$$\tilde{A}(i) = \sum_{k=1}^{r_i} \tilde{g}_{i,k} \otimes \tilde{g}_{i,k}.$$

Then for any $E \subset \mathbb{N}$, we get

$$\tilde{F}(E) := \int_E \tilde{A}(i) d\mu(i) = \sum_{i \in E} a_i \tilde{g}_{i,k} \otimes \tilde{g}_{i,k} = \sum_{i \in E} v_{i,k} \otimes v_{i,k},$$

which is an orthogonal projection. Thus $\tilde{F}$ is a projection-valued probability measure which has the integral representation by $\tilde{A}$ and μ, and $\tilde{F}(E) = P\tilde{F}(E)P$ holds for every $E \subset \Omega$. From the proof it is clear that $\mathrm{rank}(A(i)) = \mathrm{rank}(\tilde{A}(i))$, and thus we also obtain the "moreover" part of the theorem. □

Remark This work was motivated to answer a problem posed by M. Ehler and K. Okoudjou [2] and it was completed when J.-P. Gabardo was visiting the University of Central Florida in 2015. During the finalization of the manuscript, the authors noticed a recent paper [11] of M. Maslouhi and S. Loukili who also answered the same problem with the help a special form of rank one integral representation (Theorem 2.1 of [11]) for probability POVM on $\mathbb{R}^d$. The two papers clearly have some overlaps on this regard but also have significant differences in terms of the scope of the interests.

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