

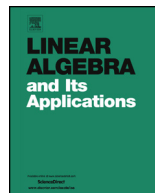


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Phase-retrievable operator-valued frames and representations of quantum channels[☆]

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ABSTRACT

We examine some connections among phase-retrievable (not necessarily self-adjoint) operator-valued frames, projective group representation frames and representations of quantum channels. We first present some characterizations of phase-retrievable frames for general operator systems acting on both finite and infinite dimensional Hilbert spaces, which generalize the known results for vector-valued frames, fusion frames and frames of Hermitian matrices. For an irreducible projective unitary representation of a finite group, the image system is automatically phase-retrievable and, moreover, it is a pointwisely tight operator-valued frames. We generalize this notion to more general operator-valued frames, and prove that pointwisely tight operator-valued frames are exactly the ones that are right equivalent to operator-valued tight frames. For an operator system that represent a quantum channel, we show that phase-retrievability of the system is independent of the choices of the representations of the quantum channel.

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1. Introduction

In some applications we often encounter the problem of recovering a signal x , up to a unimodular scalar, in a Hilbert space H from a system of measurements, which is the same as recovering the rank-one operator $x \otimes x$ from the given measurements. More precisely, let H be a Hilbert (signal) space over $\mathbb{F}$ ($= \mathbb{C}$ or $\mathbb{R}$) and let $F_j : H \rightarrow \mathbb{F}$ ($j \in \mathbb{J}$) be a system of measurement functions that satisfy the condition $F_j(\lambda x) = \lambda F_j(x)$ for all $x \in H$ and every unimodular scalar $\lambda \in \mathbb{F}$. We say that $\{F_j\}_{j \in \mathbb{J}}$ does *phase-retrieval* (or, is a *phase-retrievable system*) if the measurements $\{F_j(x) : j \in \mathbb{J}\}$ uniquely determines $x \otimes x$ for every $x \in H$, where the notation $x \otimes y$ denotes the rank-one operator defined by $(x \otimes y)(z) = \langle z, y \rangle x$ for every $z \in H$. In the case that $H = \mathbb{R}^d$ (or $\mathbb{C}^d$), $x \otimes y = xy^*$, where y^* is the transpose (or conjugate transpose) of y .

Depending on choices of the measurement functions, we will get into different scenarios of the phase-retrieval problems, some of which have been extensively investigated in recent literature (cf. [1–23], [25], [27–29], [34,36–43] and the references therein). For example, if measurement functions are given by $F_j(x) = |\langle x, x_j \rangle|^2$, then this leads to the recently well-studied problems for vector valued phase-retrievable frames $\{x_j\}_{j \in \mathbb{J}}$. Vector-valued phase-retrievable frames can also be viewed as a special case of a phase-retrievable problem from positive (or more generally self-adjoint) operator induced quadratic measurements. This paper examines several aspects of general phase-retrievable operator-valued frames. The elementary (both in terms of statements and proofs) characterizations of such frames will be presented in connection with several applications to, for example, group representation frames, point-wisely tight frames, almost point-wise phase-retrievable frames and representations of quantum channels etc.

Notations: H will be a separable Hilbert space and $B(H)$ denote the space of bounded linear operators on H . We also use $M_d(\mathbb{C})$ or $M_d(\mathbb{R})$ for $B(H)$ in the case that $H = \mathbb{C}^d$ or $\mathbb{R}^d$. While $\langle \cdot, \cdot \rangle$ usually denotes the inner product for a Hilbert space H , it is also used for the Hilbert-Schmidt inner product defined by $\langle S, T \rangle = \text{Tr}(ST^*)$.

2. Phase-retrieval operator-valued frames

This section will be devoted to presenting some characterizations for phase-retrievable operator-valued frames that generalize several well-known characterizations for vector valued phase-retrievable frames, fusion frames, or, more generally, frames of Hermitian matrices (cf. [4,17,22,25,40,42,43]). The characterizations are for arbitrary operator families and for both finite and infinite dimensional Hilbert spaces. Moreover, the operator family does not have to countable (i.e., the index set $\mathbb{J}$ can be uncountable, see Remark 2.2). We also discuss a few applications including the minimal length of phase-retrievable operator-valued frames, the phase-retrievability of the “complement” systems, and phase-retrieval by norms.

Phase-retrievable has natural connections with quantum information theory. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space H over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). An

operator family in $\mathcal{A}$ is a (finite or infinite) sequence of operators $\{T_j\}_{j \in \mathbb{J}}$ in $\mathcal{A}$. The set of all the normal positive linear functionals f is the (quantum) state space of $\mathcal{A}$ such that $f(I) = 1$, which will be denoted by $\mathcal{S}(\mathcal{A})$. A pure state is an extreme point of the state space. In what follows we will mostly consider the case when $\mathcal{A} = B(H)$. In this case the quantum state space of $B(H)$ is exactly the space of all positive trace-class operators of trace one.

Definition 2.1. Let $\{T_j\}_{j \in \mathbb{J}}$ be an operator system in $\mathcal{A}$. We say that $\{T_j\}_{j \in \mathbb{J}}$ is

- (i) an *operator-valued frame* for H if $\sum_{j \in \mathbb{J}} T_j^* T_j$ is bounded invertible;
- (ii) *phase-retrievable* from quadratic measurement (QM-phase-retrievable for short) if $\{\langle x, T_j x \rangle\}_{j \in \mathbb{J}}$ uniquely determines $x \otimes x$ for every $x \in H$;
- (iii) *state-retrievable* if $\{\rho(T_j)\}_{j \in \mathbb{J}}$ uniquely determines ρ for every $\rho \in \mathcal{S}(\mathcal{A})$.

Recall that a (vector-valued) frame for a Hilbert space H is a sequence $\{x_j\}_{j \in \mathbb{J}}$ such that

$$AI \leq \sum_{j \in \mathbb{J}} x_j \otimes x_j \leq BI$$

for some constants $A, B > 0$. Therefore if we let $T_j = x_j \otimes x_j$ then $\{T_j\}_{j \in \mathbb{J}}$ is a QM-phase-retrievable operator-valued frame if and only if $\{x_j\}_{j \in \mathbb{J}}$ is a phase-retrievable frame. The following is an easy consequence from the definition.

Proposition 2.1. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space H and $\{T_j\}_{j \in \mathbb{J}}$ be an operator family in $\mathcal{A}$ such that $\sum_{j \in \mathbb{J}} T_j^* T_j$ is bounded.

- (i) If $\{T_j\}_{j \in \mathbb{J}}$ is state-retrievable, then it is QM-phase-retrievable.
- (ii) If $\dim H < \infty$, then the QM-phase-retrievability of $\{T_j\}_{j \in \mathbb{J}}$ implies that it is an operator-valued frame.

2.1. Characterizations

There have been several well-known necessary and sufficient conditions for phase retrievable frames (cf. [4–8]). One of them is characterized in terms of the “complement property”: A frame $\{x_j\}_{j \in \mathbb{J}}$ is said to have the *complement property* if for every $\Omega \subseteq \mathbb{J}$ we have either $\overline{\text{span}}\{x_j\}_{j \in \Omega} = H$ or $\overline{\text{span}}\{x_j\}_{j \in \Omega^c} = H$. The complement property is necessary for a frame to be phase-retrievable. The condition is also sufficient for real Hilbert spaces, but not sufficient for complex Hilbert spaces.

It is easy to prove that the complement property is equivalent to the condition that

$$\overline{\text{span}}\{\langle x, x_j \rangle x_j : j \in \mathbb{J}\} = H$$

for every nonzero vector $x \in H$. The following slightly more general statement (for non-self-adjoint operators $T_j = x_j \otimes y_j$) remains to be true.

Proposition 2.2. *Let $T_j = x_j \otimes y_j$ with $x_j, y_j \in H$ ($j \in \mathbb{J}$). Then the following are equivalent:*

- (i) $\overline{\text{span}}\{T_j x : j \in \mathbb{J}\} = H$ for every nonzero vector $x \in H$;
- (ii) For any $\Omega \subseteq \mathbb{J}$ we have either $\overline{\text{span}}\{x_j\}_{j \in \Omega} = H$ or $\overline{\text{span}}\{y_j\}_{j \in \Omega^c} = H$.

Condition (i) (and its variations) of Proposition 2.2 can be stated for any operator-valued frames and in fact it has been used in the characterization of phase-retrievable frames of self-adjoint operators (cf. [25,40]). Here we generalize some of those to arbitrary operator-valued frames.

Lemma 2.3. *An operator-valued frame $\{T_j\}_{j \in \mathbb{J}}$ is not QM-phase-retrievable if and only if there exist nonzero vectors x, y such that $x \notin i\mathbb{R}y$ and*

$$\langle x, T_j y \rangle + \overline{\langle x, T_j^* y \rangle} = 0$$

for $j \in \mathbb{J}$.

(Note that in the real Hilbert space case, the condition $x \notin i\mathbb{R}y$ is automatically satisfied.)

Proof. For $x, y \in H$, write $u = x + y$ and $v = x - y$. Then we have that $\langle u, T_j u \rangle = \langle v, T_j v \rangle$ if and only if $\langle x, T_j y \rangle + \overline{\langle x, T_j^* y \rangle} = 0$.

Now first assume that $\{T_j\}_{j \in \mathbb{J}}$ is not QM-phase-retrievable. Then there exists u, v such that $u \otimes u \neq v \otimes v$ but $\langle u, T_j u \rangle = \langle v, T_j v \rangle$ for every j . Let $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then $u = x + y$ and $v = x - y$, and $\langle x, T_j y \rangle + \overline{\langle x, T_j^* y \rangle} = 0$. Clearly x and y are nonzero. If $x = iay$ for some $a \in \mathbb{R}$ in the complex Hilbert space case, then we have $u = (1 + ia)y$ and $v = (1 - ia)y$, which implies that $u \otimes u = v \otimes v$. Thus we also have $x \notin i\mathbb{R}y$.

Conversely, assume that $\langle x, T_j y \rangle + \overline{\langle x, T_j^* y \rangle} = 0$ for some nonzero vectors x and y with $x \notin i\mathbb{R}y$. Then $\langle u, T_j u \rangle = \langle v, T_j v \rangle$ with $u = x + y$ and $v = x - y$. If $u \otimes u = v \otimes v$, then u and v are linearly dependent and $\|u\| = \|v\|$, which implies that x and y are linearly dependent. Write $x = cy$ with $c \neq 0$. Thus we have $|c + 1| = |c - 1|$ since $\|u\| = \|v\|$. If H is a real Hilbert space, then c is real and hence $c = 0$, which leads to a contradiction. If H is a complex Hilbert space, then $c \in \mathbb{C}$. Thus $|c + 1| = |c - 1|$ holds only if c is purely imaginary, i.e. $c = ia$ for some $a \in \mathbb{R}$, which contradicts with the assumption that $x \notin i\mathbb{R}y$. Thus we obtain that $u \otimes u \neq v \otimes v$ and so $\{T_j\}_{j \in \mathbb{J}}$ is not QM-phase-retrievable. $\square$

Proposition 2.4. *Let $\{T_j\}_{j \in \mathbb{J}}$ be an operator-valued frame for a **real** Hilbert space H . Then the following are equivalent:*

- (i) $\{T_j\}_{j \in \mathbb{J}}$ is QM-phase-retrievable.
- (ii) $\overline{\text{span}}\{(T_j + T_j^*)x\}_{j \in \mathbb{J}} = H$ for every nonzero vector $x \in H$.
In the finite-dimensional and $|\mathbb{J}| < \infty$ case, this condition is also equivalent to:
- (iii) $\sum_{j \in \mathbb{J}} (T_j + T_j^*)(x \otimes x)(T_j + T_j^*)$ is invertible for every nonzero vector $x \in H$.

Proof. Clearly (ii) and (iii) are equivalent in the finite-dimensional case. Write $S_j = T_j + T_j^*$. Then $\langle x, S_j y \rangle + \langle x, S_j^* y \rangle = 2(\langle x, T_j y \rangle + \langle x, T_j^* y \rangle)$. So $\langle x, S_j y \rangle + \langle x, S_j^* y \rangle = 0$ if and only if $\langle x, T_j y \rangle + \langle x, T_j^* y \rangle = 0$. Thus, by Lemma 2.3, we have that $\{T_j\}$ is QM-phase-retrievable if and only if $\{T_j + T_j^*\}$ is QM-phase-retrievable. Therefore we can assume that $T_j^* = T_j$. By Lemma 2.3 we have that $\{T_j\}$ is not QM-phase-retrievable if and only if there exist nonzero vectors x, y such that $\langle y, T_j x \rangle = 0$. Thus we establish the equivalence of (i) and (ii). $\square$

Remark 2.1.

- (i) If T_j is a self-adjoint matrix for every j , then Proposition 2.4 recovers Theorem 2.1 in [40]. So this can be considered as an extension since here T_j is not required to be self-adjoint and H does not have to be finite-dimensional.
- (ii) If we do not assume that T_j is self-adjoint, then the condition that $\text{span}\{T_j x\} = H$ for every nonzero vector $x \in H$ is neither necessary nor sufficient for the QM-phase-retrievability of $\{T_j\}$. For example, in $\mathbb{R}^2$, let $T_1 = e_1 \otimes e_1$, $T_2 = e_2 \otimes e_2$ and $T_3 = e_1 \otimes e_2$. Then it is easy to verify that $\text{span}\{(T_j + T_j^*)x\}_{j=1}^3 = \mathbb{R}^2$ for every nonzero vector $x \in \mathbb{R}^2$. Thus, by Proposition 2.4, $\{T_j\}_{j=1}^3$ is QM-phase-retrievable. However, $\{T_j e_1\}_{j=1}^3$ does not span $\mathbb{R}^2$.

Conversely, if we let $T_1 = e_1 \otimes e_1$, $T_2 = e_2 \otimes e_2$ and $T_3 = e_1 \otimes e_2 - e_2 \otimes e_1$. Then $\sum_{j=1}^3 T_j(x \otimes x)T_j^*$ is invertible for every $x \neq 0$. However, by using the fact that $T_3 + T_3^* = 0$, we get that $\text{span}\{(T_j + T_j^*)x\}_{j=1}^3 \neq \mathbb{R}^2$ for $x = e_1$, which shows by Proposition 2.4 that $\{T_j\}$ is not QM-phase-retrievable.

For complex Hilbert spaces, without losing the generality we will work on the concrete space $H = \ell_2^d(\mathbb{C})$, where $\ell_2^d(\mathbb{C}) = \mathbb{C}^d$ when d is finite and $\ell_2^d(\mathbb{C}) = \ell_2(\mathbb{C})$ is the space of square-summable sequences when $d = \infty$. Similarly $\ell_2^d(\mathbb{R})$ represents the real Hilbert space. For every vector $x \in \ell_2^d(\mathbb{C})$, we write $x = \text{Re}(x) + i\text{Im}(x)$ with $\text{Re}(x), \text{Im}(x) \in \ell_2^d(\mathbb{R})$. A closed subspace of a Hilbert space W is said to have co-dimension one if $\dim W^\perp = 1$.

Proposition 2.5. *An operator-valued frame for $\ell_2^d(\mathbb{C})$ is QM-phase-retrievable if and only if for every nonzero vector $x \in \ell_2^d(\mathbb{C})$ the subspace*

$$W_x := \overline{\text{span}}\{Re((T_j + T_j^*)x) \oplus Im((T_j + T_j^*)x), Im((T_j^* - T_j)x) \oplus Re((T_j - T_j^*)x) : j \in \mathbb{J}\}$$

has co-dimension one in $\ell_2^d(\mathbb{R}) \oplus \ell_2^d(\mathbb{R})$.

Proof. A simple calculation shows that

$$\begin{aligned} \langle y, T_j x \rangle + \overline{\langle y, T_j^* x \rangle} &= \langle Re(y) \oplus Im(y), Re((T_j + T_j^*)x) \oplus Im((T_j + T_j^*)x) \rangle \\ &\quad + i \langle Re(y) \oplus Im(y), Im((T_j^* - T_j)x) \oplus Re((T_j - T_j^*)x) \rangle. \end{aligned}$$

Thus $\langle y, T_j x \rangle + \overline{\langle y, T_j^* x \rangle} = 0$ for all j if and only if $Re(y) \oplus Im(y)$ is orthogonal to W_x .

If $y = iax$ for some nonzero vector x and nonzero scalar $a \in \mathbb{R}$, then $\langle y, T_j x \rangle + \overline{\langle y, T_j^* x \rangle} = 0$. Note that $Re(y) \oplus Im(y) = a((-Im(x)) \oplus Re(x))$. Thus $\text{span}\{(-Im(x)) \oplus Re(x)\}$ is a one-dimensional subspace of the $W_x^\perp$. Thus $\dim W_x^\perp \geq 1$.

On the other hand, we also have $\dim W_x^\perp > 1$ if and only if there exist vectors $u, v \in \ell_2^d(\mathbb{R})$ such that $u \oplus v \notin \text{span}\{(-Im(x)) \oplus Re(x)\}$ and $u \oplus v \in W_x^\perp$, which, in turn, is equivalent to the condition that there exists $y = u + iv \notin i\mathbb{R}x$ such that $\langle y, T_j x \rangle + \overline{\langle y, T_j^* x \rangle} = 0$ for every $j \in \mathbb{J}$. Therefore we get that for a nonzero vector x , $\dim W_x^\perp = 1$ if and only if there exists no nonzero vector y such that $\langle y, T_j x \rangle + \overline{\langle y, T_j^* x \rangle} = 0$ for every $j \in \mathbb{J}$. Hence, Lemma 2.3 implies that $\{T_j\}_{j \in \mathbb{J}}$ is QM-phase-retrievable if and only if the following W_x has co-dimension one for every nonzero vector $x \in \ell_2^d(\mathbb{C})$. $\square$

Remark 2.2. Although we have assumed that $\{T_j\}_{j \in \mathbb{J}}$ is a finite or countable set, from their proofs it is clear that all the above results remain to be true even when the index set $\mathbb{J}$ is not countable (in this case we drop off the requirement that $\{T_j\}_{j \in \mathbb{J}}$ is an operator-valued frame).

In the case that H is finite-dimensional and each T_j is self-adjoint we get the following consequence that was due to Wang and Xu [40] (also see the work of P. Casazza and his collaborators for the case where each T_j is an orthogonal projection).

Corollary 2.6. Let $H = \mathbb{C}^d$ and assume that each $T_j \in B(H)$ is self-adjoint. Then $\{T_j\}$ is QM-phase-retrievable if and only if

$$\dim \text{span}\{Re(T_j x) \oplus Im(T_j x) : j \in \mathbb{J}\} = 2d - 1$$

holds for every nonzero $x \in \mathbb{C}^d$. In particular $|\mathbb{J}| \geq 2d - 1$ if $\{T_j\}_{j \in \mathbb{J}}$ does QM-phase-retrieval.

Remark 2.3. Note that for a finite sequence $\{z_j\}_{j=1}^k$, $\text{span}\{z_j\}$ is m -dimensional if and only if $\text{rank}(\sum_{j=1}^k z_j \otimes z_j) = m$. Therefore for self-adjoint matrices $T_j \in M_d(\mathbb{C})$, we have that $\{T_j\}_{j=1}^k$ is QM-phase-retrievable if and only if

$$\text{rank} \begin{bmatrix} \sum_{j=1}^k \text{Re}(T_j x) \otimes \text{Re}(T_j x) & \sum_{j=1}^k \text{Re}(T_j x) \otimes \text{Im}(T_j x) \\ \sum_{j=1}^k \text{Im}(T_j x) \otimes \text{Re}(T_j x) & \sum_{j=1}^k \text{Im}(T_j x) \otimes \text{Im}(T_j x) \end{bmatrix} = 2d - 1$$

for every nonzero vector $x \in \mathbb{C}^d$.

2.2. Applications

The following corollary is a simple consequence of the characterizations of QM-phase-retrievable operator-valued frames.

Corollary 2.7. *Let $\{T_j\}_{j \in \mathbb{J}}$ be an operator-valued frame on a Hilbert space H . Then QM-phase-retrievability of $\{T_j + T_j^*\}$ implies the QM-phase-retrievability of $\{T_j\}$. The converse is also true for real Hilbert case but false for the complex case.*

Proof. The real Hilbert space case follows from the proof of Proposition 2.4, and the complex Hilbert space case follows from Proposition 2.5. For a counterexample of the converse, let $H = \mathbb{C}^2$, $T_1 = i(e_1 \otimes e_1)$, $T_2 = i(e_2 \otimes e_2)$, $T_3 = e_1 \otimes e_2$ and $T_4 = e_2 \otimes e_1$. Then $\text{span}\{T_j\}_{j=1}^4 = M_{2 \times 2}(\mathbb{C})$. Thus $\{T_j\}_{j=1}^4$ is QM-phase-retrievable. However $\{T_j + T_j^*\}_{j=1}^4 = \{0, 0, T_3 + T_4, T_3 + T_4\}$ is clearly not QM-phase-retrievable. $\square$

One of the interesting and difficult problems in phase-retrieval and in quantum information theory is to determine the minimal number of measurements required to performing phase-retrieval (see [35,42,43] and the references therein). In the d -dimensional real Hilbert space H case, the minimal number N such that there exists a QM-phase-retrievable frame of the form $\{x_j \otimes x_j\}_{j=1}^N$ is $2d - 1$. However, the minimal number N could be smaller than $2d - 1$ for arbitrary operator-valued QM-phase-retrievable frames. For example, Z. Xu constructed an example in [42] of a QM-phase-retrievable frame of six symmetric matrices for $\mathbb{R}^4$.

For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $m(\mathbb{F}^d)$ (respectively $m_s(\mathbb{F}^d)$) be the smallest number N such that there exists a QM-phase-retrievable operator-valued frames of length N (respectively, QM-phase-retrievable self-adjoint operator-valued frames of length N). Then the above corollary shows that $m(\mathbb{R}^d) = m_s(\mathbb{R}^d)$. It would be interesting to know whether we also have $m(\mathbb{C}^d) = m_s(\mathbb{C}^d)$.

Recall that a family of orthogonal projections $\{P_j\}$ does *phase retrieval* if $\{\|P_j x\|\}$ determines x uniquely up to a unimodular scalar [2,17,18,22,27]. Since $\|P_j x\|^2 = \langle x, P_j x \rangle$, we have that this is the same as saying that $\{P_j\}$ is a QM-phase-retrievable projection-valued frame. P. Casazza etc proved that if $\sum_{j \in \mathbb{J}} P_j = I$, then $\{P_j^\perp\}$ is QM-phase-retrievable if and only if $\{P_j\}$ is QM-phase-retrievable. The following generalizes the to arbitrary operator-valued frame.

Proposition 2.8. *Let $T_j \in B(H)$ and $T = \sum_{j \in \mathbb{J}} T_j$, where the sum is convergent in the sense of weak-operator-topology if $|\mathbb{J}| = \infty$. Then $\{T_j\}$ is QM-phase-retrievable if and only if $\{T - T_j\}$ is QM-phase-retrievable.*

Proof. Assume that $\{T_j\}$ is QM-phase-retrievable but $\{T - T_j\}$ is not QM-phase-retrievable. Then by Lemma 2.3 we get that there exist nonzero vectors x, y such that $x \notin i\mathbb{R}y$ and

$$\langle x, (T - T_j)y \rangle + \overline{\langle x, (T^* - T_j^*)y \rangle} = 0$$

for each j .

Suppose that $\mathbb{J} = \{1, \dots, N\}$ is a finite set. Summing up over j and using the assumption $T = \sum_{j=1}^N T_j$ we get that

$$N(\langle x, Ty \rangle + \overline{\langle x, T^*y \rangle}) - (\langle x, Ty \rangle + \overline{\langle x, T^*y \rangle}) = 0.$$

Thus $\langle x, Ty \rangle + \overline{\langle x, T^*y \rangle} = 0$ since $N > 1$, and therefore we get that $\langle x, T_j y \rangle + \overline{\langle x, T_j^* y \rangle} = 0$ for each j , which leads to the contradiction that $\{T_j\}$ is not QM-phase-retrievable. Suppose that $\mathbb{J} = \{1, 2, \dots\}$ is infinite. Then we get

$$\langle x, Ty \rangle + \overline{\langle x, T^*y \rangle} = \frac{1}{N}(\langle x, \sum_{j=1}^N T_j y \rangle + \overline{\langle x, \sum_{j=1}^N T_j^* y \rangle})$$

for every N . This implies by letting $N \rightarrow \infty$ that $\langle x, Ty \rangle + \overline{\langle x, T^*y \rangle} = 0$, which also leads to the contradiction that $\{T_j\}$ is not QM-phase-retrievable. Therefore we have that $\{T - T_j\}$ is QM-phase-retrievable.

Conversely, assume that $\{T - T_j\}$ is QM-phase-retrievable. Suppose that $\{T_j\}$ is not QM-phase-retrievable. Again by Lemma 2.3 there exist nonzero vectors x, y such that $x \notin i\mathbb{R}y$ and

$$\langle x, T_j y \rangle + \overline{\langle x, T_j^* y \rangle} = 0$$

for each j . Summing up over j we get $\langle x, Ty \rangle + \overline{\langle x, T^*y \rangle} = 0$. Thus we obtain that

$$\langle x, (T - T_j)y \rangle + \overline{\langle x, (T^* - T_j^*)y \rangle} = 0$$

which implies that $\{T - T_j\}$ is not QM-phase-retrievable. Therefore $\{T_j\}$ must be QM-phase-retrievable. $\square$

Example 2.1. Let $\{x_j\}$ and $\{y_j\}$ be a dual frame pair, i.e., $\sum_{j \in \mathbb{J}} x_j \otimes y_j = I$. Then $\{x_j \otimes y_j\}$ is QM-phase-retrievable if and only if $\{I - x_j \otimes y_j\}$ is QM-phase-retrievable.

For projection-valued frames, norm-retrieval is the same as QM-phase-retrieval since $\|Px\|^2 = \langle x, Px \rangle$ for any orthogonal projection P and any $x \in H$. There are many other situations (cf. the forward imaging model studied in [16], and quantum measurements discussed in [30,35]) where norm measurement is also a natural choice. For example if

the measurement operators T_j are operators from H to a space W_j , where W_j is not necessarily a subset of H , then clearly the quadratic measurement does not apply. In this case the natural measurement seems to be $\|T_j x\|$.

Definition 2.2. Let H and W_j be Hilbert spaces and $T_j \in B(H, W_j)$ for $j \in \mathbb{J}$. We say that $\{T_j\}$ does phase retrieval from norm measurements (NM-phase-retrieval for short) if $\{\|T_j x\|\}_{j \in \mathbb{J}}$ uniquely determines $x \otimes x$.

It follows from $\|T_j x\|^2 = \langle x, T_j^* T_j x \rangle$ that $\{T_j\}$ does NM-phase-retrieval if and only if $\{T_j^* T_j\}$ does QM-phase-retrieval.

Corollary 2.9. Let H and W_j be Hilbert spaces and $T_j \in B(H, W_j)$ for $j \in \mathbb{J}$.

- (i) If H, W_j are real Hilbert spaces, then $\{T_j\}$ does NM-phase-retrieval if and only if $\overline{\text{span}}\{T_j^* T_j x\} = H$ for every nonzero vector $x \in H$.
- (ii) If $H = \ell_2^d(\mathbb{C})$, then $\{T_j\}$ does NM-phase-retrieval if and only if for every nonzero vector $x \in H$,

$$\overline{\text{span}}\{\text{Re}T_j^* T_j x \oplus \text{Im}T_j^* T_j x : j \in \mathbb{J}\}$$

has co-dimension one in $\ell_2^d(\mathbb{R}) \oplus \ell_2^d(\mathbb{R})$.

3. Almost everywhere point-wise and point-wise tight phase-retrievable operator-valued frames

For an operator-valued frames $\{T_j\}_{j \in \mathbb{J}}$, we establish in this section some connections among the phase-retrievability of the operator-valued frames, the (almost everywhere) point-wise phase-retrievability, and point-wise tight frame property for $\{T_j\}$. We will assume that $H = \mathbb{R}^d$ or $\mathbb{C}^d$ is finite-dimensional and $\mathbb{J} = \{1, \dots, N\}$ is finite.

For the real Hilbert space $\mathbb{R}^d$ case, it is easy to prove that $\{T_j x\}_{j=1}^N$ is phase-retrievable for some x if and only if $\{T_j x\}_{j=1}^N$ is phase-retrievable for any generic vector x . Moreover, if $T_j = x_j \otimes x_j$, then $\{T_j\}_{j=1}^N$ is QM-phase-retrievable if and only if and if $\{T_j x\}_{j=1}^N$ is phase-retrievable for some $x \in H$. However, this is no longer true in general as demonstrated by the following example.

Example 3.1. Again we use the example of Z. Xu [42] which is an QM-phase-retrievable operator-valued frame with six Hermitian operators $\{T_j\}_{j=1}^6$ for $\mathbb{R}^4$. Clearly $\{T_j \xi\}_{j=1}^6$ is not phase retrievable for any $\xi \in \mathbb{R}^4$ since it requires at least 7 vectors for a vector-valued frame to be phase-retrievable for $\mathbb{R}^4$.

Conversely, let $H = \mathbb{R}^2$ and $\{e_1, e_2\}$ be its standard orthonormal basis. Define $T_1 = e_1 \otimes e_1 + e_2 \otimes e_2$, $T_2 = e_1 \otimes e_1 + 2e_2 \otimes e_2$ and $T_3 = e_1 \otimes e_1 + 3e_2 \otimes e_2$. Then $\{T_1, T_2, T_3\}$ is an operator-valued frame for $\mathbb{R}^2$. For $x = e_1$ we have $\text{span}\{T_1 x, T_2 x, T_3 x\} = \mathbb{R}e_1 \neq \mathbb{R}^2$. Thus, by Proposition 2.4, $\{T_1, T_2, T_3\}$ is not QM-phase-retrievable. However, for

$x = e_1 + e_2$ we have $\{T_1x, T_2x, T_3x\} = \{e_1 + e_2, e_1 + 2e_2, e_1 + 3e_2\}$, which is clearly is phase-retrievable for $\mathbb{R}^2$.

If an operator-valued frame $\{T_j\}_{j \in \mathbb{J}}$ has the property that $\{T_jx\}_{j \in \mathbb{J}}$ is phase-retrievable for some $x \in H$ (and hence for almost all $x \in H$), then we say that $\{T_j\}_{j \in \mathbb{J}}$ is *almost everywhere point-wise phase-retrievable*. The above example naturally leads to the question of identifying a large class of systems which are QM-phase-retrievable if and only if they are almost everywhere point-wise phase-retrievable. We will prove in Theorem 3.3 that, like QM-phase-retrievable operator-valued frames, every generic operator-valued frame of length N in $\mathbb{R}^d$ is almost everywhere point-wise phase-retrievable when $N \geq 2d - 1$. For a concrete class of such examples, it was proved in [37] that if π is an irreducible unitary representation of a finite abelian group G , then $\{\pi(g)\}_{g \in G}$ is almost everywhere point-wise phase-retrievable. It remains open whether it is still true for non-abelian groups [37,23].

The following was recently proved by Y. Wang and Z. Xu.

Theorem 3.1 ([40], Theorem 4.1). *Let $N \geq 2d - 1$. Then a generic operator-valued frame $A = (A_1, \dots, A_N)$ of Hermitian matrices is QM-phase-retrievable for $\mathbb{R}^d$.*

By Corollary 2.7 and the above theorem we immediately get

Corollary 3.2. *Let $N \geq 2d - 1$. Then a generic operator-valued frame $A = (A_1, \dots, A_N)$ is QM-phase-retrievable for $\mathbb{R}^d$.*

Example 3.1 tells that the set of almost everywhere point-wise phase-retrievable operator-valued frames is different from the set of QM-phase-retrievable operator-valued frames. Therefore the following is not a generalization of Theorem 3.3 or Theorem 4.1 in [40], but should be considered as a supplement to these results.

Theorem 3.3. *Assume that $N \geq 2d - 1$. Let $\mathcal{P}$ the set of all n -tuples $(A_1, \dots, A_N)$, where $A_j \in M_d(\mathbb{R})$, such that $\{A_jx\}_{j=1}^N$ is phase-retrievable for some $x \in \mathbb{R}^d$. Then $\mathcal{P}$ is open dense in the direct sum space $M_d(\mathbb{R}) \oplus \dots \oplus M_d(\mathbb{R})$ (N -copies).*

Proof. Write $A = (A_1, \dots, A_N)$. Let $\{x_j\}_{j=1}^N$ be a phase-retrievable frame for $\mathbb{R}^d$ such that $x_j \neq 0$ for each j . Set $A_j = x_j \otimes x_j$, and pick $x \in \mathbb{R}^d$ such that $\langle x, x_j \rangle \neq 0$ for every j . Then clearly $\{A_jx\}$ is phase-retrievable and hence $\mathcal{P}$ is nonempty.

Now let $A = (A_1, \dots, A_N) \in \mathcal{P}$ and $x \in \mathbb{R}^d$ be such that $\{A_jx\}$ is phase-retrievable. We clearly can assume that $\|x\| = 1$. Since the set of all the phase-retrievable vector-valued frames of length N is open in $\mathbb{R}^d \oplus \dots \oplus \mathbb{R}^d$, there exists $\delta > 0$ such that $\{y_j\}_{j=1}^N$ is phase-retrievable whenever $\sum_{j=1}^N \|A_jx - y_j\|^2 < \delta$. This implies that if $\sum_{j=1}^N \|A_j - B_j\|^2 < \delta$, then $\{B_jx\}_{j=1}^N$ is phase-retrievable and consequently $B = (B_1, \dots, B_N) \in \mathcal{P}$. Thus $\mathcal{P}$ is open.

For density, let $B = (B_1, \dots, B_N) \in M_d(\mathbb{R}) \oplus \dots \oplus M_d(\mathbb{R})$ be an arbitrary element and let $A = (A_1, \dots, A_N) \in \mathcal{P}$ be a fixed element with $\{A_j x\}$ being phase-retrievable for some $x \in \mathbb{R}^d$. Consider $C(t) = tA + (1-t)B$. We show that $C(t)$ is in $\mathcal{P}$ for all but finitely many numbers of t 's, which will imply that B is a limit point of $\mathcal{P}$. Since $\{A_j x\}$ is phase-retrievable, we have that either $\text{span}\{A_j x : j \in \Lambda\} = \mathbb{R}^d$ or $\text{span}\{A_j x : j \in \Lambda^c\} = \mathbb{R}^d$ for every $\Lambda \subseteq \{1, \dots, N\}$. Thus we can associate every Λ with a set $\Phi(\Lambda)$ of cardinality d such that it is either a subset of Λ or a subset of Λ^c and $\det[A_j x]_{j \in \Phi(\Lambda)} \neq 0$. Define

$$f_\Lambda(t) = \det[(tA_j + (1-t)B_j)x]_{j \in \Phi(\Lambda)}.$$

Then these are nonzero polynomials since $f_\Lambda(1) \neq 0$ for every Λ . By the complement property for phase-retrievable frames, we clearly have that $\{tA_j + (1-t)B_j\}x_{j=1}^N$ is phase retrievable if $f_\Lambda(t) \neq 0$ for every Λ . Since the union of the zero sets of f_Λ is finite, we conclude that $C(t)$ is in $\mathcal{P}$ for all but finitely many numbers of t 's. $\square$

We have seen from section 2 that the characterizations of QM-phase-retrievable frames are much simpler for frames of self-adjoint operators. However this might be too restrictive since there are many useful and interesting examples (e.g. frames of unitary operators) do not fall into this category. In what follows, we will call an operator family $\mathcal{S}$ a *self-adjoint family* if $T \in \mathcal{S}$ implies $T^* \in \mathcal{S}$.

Lemma 3.4. *Let $\{T_j\}_{j=1}^N$ be a self-adjoint family. If $\{T_j\}_{j=1}^N$ is QM-phase-retrievable, then for every nonzero vector $x \in H$ we have that $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ is invertible.*

Proof. We only need to prove for the complex Hilbert space case. Assume that $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ is not invertible for some $x \neq 0$. Then there exists $y \neq 0$ such that $\langle y, T_j x \rangle = 0$ for every j . If $y = iax$ for some $0 \neq a \in \mathbb{R}$, then we get $\langle x, T_j^* x \rangle = 0$, which implies that $\{T_j\}$ is not QM-phase-retrievable. So we have that $y \notin i\mathbb{R}x$. Since $\mathcal{S}$ is self-adjoint we obtain that $\langle y, T_j^* x \rangle = 0$ for every j . Thus $\langle y, T_j x \rangle + \overline{\langle y, T_j^* x \rangle} = 0$, which implies by Lemma 2.3 that $\{T_j\}$ is not QM-phase-retrievable. This contradiction shows that $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ must be invertible for every $x \neq 0$. $\square$

Remark 3.1. The converse of the above lemma is not true. For the complex case, let $\{x_j\}_{j=1}^N$ be a frame for $\mathbb{C}^d$ such that it has the complement property but not phase-retrievable (existence of such a frame is guaranteed for complex Hilbert spaces). Let $T_j = x_j \otimes x_j$. Clearly $\{T_j\}_{j=1}^N$ is a self-adjoint family, $\sum_{j=1}^N T_j(x \otimes x)T_j^*$ is invertible for every nonzero vector x and $\{T_j\}$ is not QM-phase-retrievable. However, this phenomenon can not happen for some well-structured operator-valued frames. Here we examine the example of projective unitary group representation frames. For a counterexample for real space case, we use the modified example of Remark 2.1: Conversely, if we let $T_1 = e_1 \otimes e_1$, $T_2 = e_2 \otimes e_2$, $T_3 = e_1 \otimes e_2 - e_2 \otimes e_1$ and $T_4 = T_3^*$. Then $\sum_{j=1}^4 T_j(x \otimes x)T_j^*$ is invertible for every $x \neq 0$. However, by using the fact that $T_3 + T_3^* = 0 = T_4 + T_4^*$, we get

that $\text{span}\{(T_j + T_j^*)x\}_{j=1}^4 \neq \mathbb{R}^2$ for $x = e_1$, which shows by Proposition 2.4 that the self-adjoint family $\{T_j\}_{j=1}^4$ is not QM-phase-retrievable.

Now we examine a special type of structure frames that have been systematically investigated in the literature (cf. [23,26,32,31,33,37]). Recall that a *projective unitary representation* π for a finite group G is a mapping $g \mapsto \pi(g)$ from G into the group $U(H)$ of all the unitary operators on a finite dimensional Hilbert space H such that $\pi(g)\pi(h) = \mu(g, h)\pi(gh)$ for all $g, h \in G$, where $\mu(g, h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. In this case we also say that π is a μ -projective unitary representation. It is clear from the definition that we have

- (i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$.
- (ii) $\mu(g, e) = \mu(e, g) = 1$ for all $g \in G$, where e denotes the group unit of G .

Any function $\mu : G \times G \mapsto \mathbb{T}$ satisfying (i)–(ii) above will be called a *multiplier* or *2-cocycle* of G . It follows from (i) and (ii) that we also have

- (iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ holds for all $g \in G$.

A projective unitary representation π of G on H is *irreducible* if $\text{span}\{\pi(g) : g \in G\} = B(H)$.

Proposition 3.5. *Let π be a μ -projective unitary representation of G on a complex Hilbert space H . Then the following are equivalent:*

- (i) $\{\pi(g)\}_{g \in G}$ is QM-phase-retrievable.
- (ii) $\sum_{g \in G} \pi(g)(x \otimes x)\pi(g)^*$ is invertible for every $x \neq 0$.
- (iii) π is irreducible.
- (iv) There exists scalar $\lambda > 0$ such that $\sum_{g \in G} \pi(g)(x \otimes y)\pi(g)^* = \lambda \langle x, y \rangle I$ for all $x, y \in H$.
- (v) $\{\pi(g)\}_{g \in G}$ is tight frame for $B(H)$.
- (vi) $\{\pi(g)\}_{g \in G}$ is a frame for $B(H)$.

Proof. Clearly, we have the equivalence of (ii) and (iii) and the implications (v) $\Rightarrow$ (vi) $\Rightarrow$ (i). The implication (i) $\Rightarrow$ (ii) follows from Lemma 3.4.

For (iii) $\Rightarrow$ (v), it is a routine exercise to check that $\sum_{g \in G} \pi(g)(x \otimes y)\pi(g)^*$ commutes with $\pi(h)$ for every $h \in G$. Since π is irreducible, there exists scalar $b(x, y)$ such that $\sum_{g \in G} \pi(g)(x \otimes y)\pi(g)^* = b(x, y)I$. Clearly, $b(x, y)$ is bilinear and thus there is an operator T such that $b(x, y) = \langle Tx, y \rangle$ for all $x, y \in H$. Now for every $h \in G$ and $x, y \in H$, we have

$$\langle T\pi(h)x, y \rangle = \sum_{g \in G} \pi(g)(\pi(h)x \otimes y)\pi(g)^*$$

$$\begin{aligned}
&= \sum_{g \in G} \pi(gh)(x \otimes \pi(h)^*y)\pi(gh)^* \\
&= \sum_{g' \in G} \pi(g')(x \otimes \pi(h)^*y)\pi(g')^* \\
&= b(x, \pi(h)^*y) = \langle Tx, \pi(h)^*, y \rangle = \langle \pi(h)Tx, y \rangle.
\end{aligned}$$

This implies that T commutes with $\pi(h)$ for all $h \in G$ and therefore $T = \lambda I$ for some scalar λ . Since $\lambda \|x\|^2 I = b(x, x)I = \sum_{g \in G} \pi(g)(x \otimes x)\pi(g)^*$ is a positive operator for every $x \neq 0$ we obtain that $\lambda > 0$.

For (iv) $\Rightarrow$ (v), it is enough to show that $\sum_{g \in G} \langle x \otimes y, \pi(g) \rangle \pi(g) = \lambda(x \otimes y)$ for every $x, y \in H$. Let

$$A = \sum_{g \in G} \langle x \otimes y, \pi(g) \rangle \pi(g).$$

Then for every $z \in H$, we have

$$\begin{aligned}
Az &= \sum_{g \in G} \langle x \otimes y, \pi(g) \rangle \pi(g)z \\
&= \sum_{g \in G} \text{tr}((x \otimes y)\pi(g)^*)\pi(g)z \\
&= \sum_{g \in G} \text{tr}(x \otimes \pi(g)y)\pi(g)z \\
&= \sum_{g \in G} \langle x, \pi(g)y \rangle \pi(g)z \\
&= \sum_{g \in G} (\pi(g)z \otimes \pi(g)y)(x) \\
&= \lambda \langle z, y \rangle x \\
&= \lambda(x \otimes y)z.
\end{aligned}$$

Thus $A = \lambda(x \otimes y)$. Since every operator T is the sum of rank-one operators we get

$$\sum_{g \in G} \langle T, \pi(g) \rangle \pi(g) = \lambda T$$

hold every $T \in B(H)$. Thus $\{\pi(g)\}_{g \in G}$ is a λ -tight frame for $B(H)$. $\square$

We would like to see how much of the previous proposition can be generalized to more general operator-valued frames. For this purpose we introduce:

Definition 3.1. We say an operator family $\{T_j\}_{j \in \mathbb{J}}$ is point-wisely tight if for every $x \neq 0$, $\{T_j x\}_{j \in \mathbb{J}}$ is a tight frame for H , i.e., $\sum_{j \in \mathbb{J}} T_j x \otimes T_j x = \lambda_x I$ for some $\lambda_x > 0$.

Theorem 3.6. Let $\{T_j\}_{j \in \mathbb{J}}$ be an operator family for a complex Hilbert space H . Then the following are equivalent:

- (i) There exists a positive invertible operator B such that $\{T_j B\}_{j \in \mathbb{J}}$ is a Parseval frame for $B(H)$.
- (ii) $\{T_j\}_{j \in \mathbb{J}}$ is point-wisely tight.
- (iii) For every $x, y \in H$, there exists $\lambda_{x,y}$ such that

$$\sum_{j \in \mathbb{J}} T_j(x \otimes y) T_j^* = \lambda_{x,y} I,$$

and $\lambda_{x,x} > 0$ when $x \neq 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose that there exists a positive invertible operator B such that $\{T_j B\}$ is a Parseval frame for $B(H)$. Then for any $T \in B(H)$ we have

$$\sum_{j \in \mathbb{J}} \langle T, T_j B \rangle T_j B = T.$$

Note that

$$\langle T B^{-1}, T_j B \rangle = \text{Tr}(T B^{-1} (T_j B)^*) = \text{Tr}(T T_j^*) = \langle T, T_j \rangle.$$

Thus, by replacing T with $T B^{-1}$, we get $\sum_{j \in \mathbb{J}} \langle T, T_j \rangle T_j B = T B^{-1}$ and hence we have $\sum_{j \in \mathbb{J}} \langle T, T_j \rangle T_j = T (B^{-1})^2$.

Let $S = (B^{-1})^2$. Now for a fixed $x \neq 0$, define

$$A = \sum_{j \in \mathbb{J}} T_j x \otimes T_j x.$$

Then for any $z, y \in H$ we have that

$$\begin{aligned} \langle Az, y \rangle &= \sum_{j \in \mathbb{J}} \langle z, T_j x \rangle \cdot \langle T_j x, y \rangle = \sum_{j \in \mathbb{J}} \langle z \otimes x, T_j \rangle \langle T_j, y \otimes x \rangle \\ &= \left\langle \sum_{j \in \mathbb{J}} \langle z \otimes x, T_j \rangle T_j, y \otimes x \right\rangle \\ &= \langle (z \otimes x) S, y \otimes x \rangle = \langle x, Sx \rangle \langle z, y \rangle. \end{aligned}$$

This implies that $Az = \langle x, Sx \rangle z$ for any $z \in H$, and hence $A = \langle x, Sx \rangle I$. Therefore $\{T_j\}_{j \in \mathbb{J}}$ is point-wisely tight.

(ii) $\Rightarrow$ (iii): Assume that $\sum_{j \in \mathbb{J}} T_j x \otimes T_j x = \lambda_x I$ for each x with $\lambda_x > 0$ when $x \neq 0$. Now fix any $x, y \in H$. We have

$$\begin{aligned}
\lambda_{x+y}I &= \sum_{j \in \mathbb{J}} T_j(x+y) \otimes T_j(x+y) \\
&= (\lambda_x + \lambda_y)I + \sum_{j \in \mathbb{J}} T_j y \otimes T_j x + \sum_{j \in \mathbb{J}} T_j x \otimes T_j y,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{x+iy}I &= \sum_{j \in \mathbb{J}} T_j(x+iy) \otimes T_j(x+iy) \\
&= (\lambda_x + \lambda_y)I + i \sum_{j \in \mathbb{J}} T_j y \otimes T_j x - i \sum_{j \in \mathbb{J}} T_j x \otimes T_j y.
\end{aligned}$$

This implies that

$$\sum_{j \in \mathbb{J}} T_j x \otimes T_j y = \frac{1}{2} [\lambda_{x+y} + i\lambda_{x+iy} - (1+i)(\lambda_x + \lambda_y)]I.$$

Thus we get (iii) by setting $\lambda_{x,y} = \frac{1}{2} [\lambda_{x+y} + i\lambda_{x+iy} - (1+i)(\lambda_x + \lambda_y)]$.

(iii) $\Rightarrow$ (i): Assume that

$$\sum_{j \in \mathbb{J}} T_j(x \otimes y) T_j^* = \lambda_{x,y} I,$$

and $\lambda_{x,x} > 0$ when $x \neq 0$. Then clearly $\lambda_{x,y}$ defines a positive bilinear form on H , and hence there exists a positive invertible operator S such that $\lambda_{x,y} = \langle Sx, y \rangle$ for all $x, y \in H$. Let $B = S^{-1/2}$. A simple calculation shows that (iii) implies

$$\sum_{j \in \mathbb{J}} \langle u \otimes v, T_j B \rangle T_j B = u \otimes v$$

for all $u, v \in H$. Since $\text{span}\{u \otimes v : u, v \in H\} = B(H)$, we get that $\{T_j B\}_{j \in \mathbb{J}}$ is a Parseval frame for $B(H)$. $\square$

From the proof of Theorem 3.6, we have that $\lambda_{x,y} = \langle B^{-1/2}x, y \rangle$ when $\{T_j B\}_{j \in \mathbb{J}}$ is a Parseval frame for $B(H)$. Thus we obtain the following consequence:

Corollary 3.7. *Let $\{T_j\}_{j \in \mathbb{J}}$ be an operator family for a complex Hilbert space H . Then the following are equivalent:*

- (i) $\{T_j\}_{j \in \mathbb{J}}$ is a tight frame for $B(H)$.
- (ii) There exists $\lambda > 0$ such that $\sum_{j \in \mathbb{J}} T_j(x \otimes x) T_j^* = \lambda \langle x, x \rangle I$.
- (iii) There exists $\lambda > 0$ such that $\sum_{j \in \mathbb{J}} T_j(x \otimes y) T_j^* = \lambda \langle x, y \rangle I$.

Using the fact that every operator is a linear combination of rank-one operators, we also have:

Corollary 3.8. *An operator family $\{T_j\}_{j \in \mathbb{J}}$ is a point-wisely tight operator system for a complex Hilbert space H if and only if there exists a positive operator B such that*

$$\sum_{j \in \mathbb{J}} T_j A T_j^* = \text{tr}(AB)I$$

for every $A \in B(H)$.

Remark 3.2. In the real Hilbert space case, $\lambda_{x,y}$ in (iii) is positive symmetric. So it is obvious from the proof of Theorem 3.6 that we still have the equivalence between (i) and (iii).

4. Phase-retrievability of representations of quantum channels

In quantum information and tomography theory, special type of operator families are used for quantum channels and measurements of unknown quantum states (cf. [30,32,35] and the references therein). While quantum measurements are described by positive operator-valued measures, quantum channels are described as completely positive (CP) trace-preserving maps between spaces of operators. In this section we discuss the QM-phase-retrievable operator-valued frames and representations of quantum channels. We prove that phase-retrievability is independent of the operator families that are used to represent the quantum channel.

Let A be a C^* -algebra and $M_n(\mathcal{A})$ be the C^* -algebra consisting of all the $n \times n$ matrices with entries from $\mathcal{A}$. A linear map Φ from a C^* -algebra $\mathcal{A}$ to a C^* -algebra $\mathcal{B}$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. For a linear map Φ from a C^* -algebra $\mathcal{A}$ to a C^* -algebra $\mathcal{B}$, we define for each n a linear map $\Phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ by

$$\Phi_n([A_{ij}]) = [\Phi(A_{ij})], \quad \text{where } [A_{ij}] \in M_n(\mathcal{A}).$$

A linear map Φ from a C^* -algebra $\mathcal{A}$ to a C^* -algebra $\mathcal{B}$ is called *completely bounded* if $\sup_{n \in \mathbb{N}} \|\Phi_n\| < \infty$, and *completely positive* (CP map for short) if Φ_n is positive for every $n \in \mathbb{N}$.

Now we consider a linear map Φ from the C^* -algebra $M_d(\mathbb{C})$ to the C^* -algebra $M_k(\mathbb{C})$, where $M_d(\mathbb{C})$ denote the matrix algebra of $d \times d$ matrices over $\mathbb{C}$. Let $E_{ij} = e_i \otimes e_j$ ($i, j = 1, \dots, d$) be the standard matrix units of $M_d(\mathbb{C})$ and $P_\Phi = [\Phi(E_{ij})] \in M_d(M_k(\mathbb{C}))$.

The characterization of quantum channels from $M_d(\mathbb{C})$ to $M_k(\mathbb{C})$ are described by the following well-known Choi-Kraus representation theorem [24].

Theorem 4.1 (Choi's first theorem). *The following are equivalent:*

- (i) Φ is completely positive.
- (ii) $\Phi(A) = \sum_{j=1}^m T_j A T_j^*$ for some matrices $T_j \in M_{k \times d}(\mathbb{C})$ for $1 \leq j \leq m$.
- (iii) P_Φ is positive.

Suppose that $\Phi : M_d(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ is a CP map. Then, by Theorem 4.1

$$\Phi(A) = \sum_{j=1}^m T_j A T_j^*$$

for some matrices $T_j \in M_{k \times d}(\mathbb{C})$ for $1 \leq j \leq m$, here we will call $\{T_j\}$ a *representing operator system* for Φ . The *Choi rank* of Φ (denoted by $cr(\Phi)$) is defined to be the smallest r such that $\Phi(A) = \sum_{j=1}^r S_j A S_j^*$ for some $S_j \in M_{k \times d}(\mathbb{C})$.

Lemma 4.2 ([30]). *Let $\Phi : M_d(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ be a CP map. Then $cr(\Phi) = rank(P_\Phi)$.*

In quantum information theory, Choi's rank of a quantum channel is an important piece of information that allows us minimizing the number of quantum measurements. While the above lemma tells us that it is equal to the rank of the matrix P_Φ , it is also natural to ask what more can be said about Choi's rank if a quantum channel can be represented by a special type of operator systems. In this section we consider the ones that are represented by QM-phase-retrievable frames. We will first prove in the following result that the Choi's rank for a quantum channel acting on a $\mathbb{C}^d$ is exactly d^2 if it can be represented by a point-wisely tight operator-valued frames (recall from Theorem 3.6 that such a frame is "right" similar to a QM-phase-retrievable frames). Then we will prove in Proposition 4.5 that if a quantum channel is represented by a QM-phase-retrievable operator-valued frame, then all of its representing operator systems are QM-phase-retrievable.

Corollary 4.3. *Let $H = \mathbb{C}^d$. Suppose that $\{T_j\}_{j=1}^m$ is a point-wisely tight system and $\Phi(A) = \sum_{j=1}^m T_j A T_j^*$. Then $cr(\Phi) = d^2$.*

Proof. By Theorem 3.6, there exists a positive invertible matrix B such that $\{T_j B\}_{j=1}^m$ is a Parseval frame for $M_d(\mathbb{C})$ and

$$\Phi(x \otimes y) = \sum_{j=1}^m T_j (x \otimes y) T_j^* = \langle x, (B^{-1})^2 y \rangle I$$

for every $x, y \in H$. This implies that

$$\Phi(E_{ij}) = \sum_{j=1}^m T_j (e_i \otimes e_j) T_j^* = \langle e_i, (B^{-1})^2 e_j \rangle I$$

Write $(B^{-1})^2 = [c_{ij}]_{d \times d}$. Then clearly $c_{ij} = \langle e_i, (B^{-1})^2 e_j \rangle$. Thus we get that

$$P_{\Phi} = [\Phi(E_{ij})] = [\langle x, (B^{-1})^2 y \rangle I] = [c_{ij} I] = (B^{-1})^2 \otimes I,$$

where in the last equality we write the matrix $[c_{ij} I] \in M_d(M_d(\mathbb{C}))$ as the tensor product of the matrices $(B^{-1})^2$ and I , acting on the tensor product Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$. Hence P_{Φ} is invertible and therefor, by Lemma 4.2, $cr(\Phi) = rank(P_{\Phi}) = d^2$. $\square$

This can also be deduced from the following Proposition 4.7 which is based on Choi's second theorem [30].

Theorem 4.4 (Choi's second theorem). *Suppose that $\Phi : M_d(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ is completely positive and $cr(\Phi) = r$. If $\Phi(A) = \sum_{j=1}^r S_j A S_j^* = \sum_{j=1}^m T_j A T_j^*$ for some matrices $S_j, T_j \in M_{k \times d}(\mathbb{C})$, then there exists matrix $U = (u_{ij}) \in M_{m,r}(\mathbb{C})$ such that $U^* U = I_r$ and $S_i = \sum_{j=1}^r u_{ij} T_j$ for $i = 1, \dots, m$ and $span\{T_1, \dots, T_m\} = span\{S_1, \dots, S_r\}$.*

The following shows that the QM-phase-retrievability of the representation system for CP maps is independent of the representation.

Proposition 4.5. *Let $\{T_j\}_{j=1}^n$ and $\{S_j\}_{j=1}^m$ be two representations of a completely positive map $\Phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$ (i.e. $\Phi(A) = \sum_{j=1}^n T_j A T_j^* = \sum_{j=1}^m S_j A S_j^*$ for every A in $M_d(\mathbb{C})$). Then $\{T_j\}_{j=1}^n$ is QM-phase-retrievable if and only if $\{S_j\}_{j=1}^m$ is QM-phase-retrievable.*

Proof. First note, by definition, that if two operator systems have the same linear span, then one does QM-phase-retrieval if and only if the other one does QM-phase-retrieval.

Let $r = cr(\Phi)$. Then there exists $L_j \in M_d$ ($j = 1, \dots, r$) such that $\Phi(A) = \sum_{j=1}^r L_j A L_j^*$ for every $A \in M_d$. By Choi's second theorem we get both $span\{L_1, \dots, L_r\} = span\{S_1, \dots, S_m\}$ and $span\{L_1, \dots, L_r\} = span\{T_1, \dots, T_n\}$. Thus we have

$$span\{S_1, \dots, S_m\} = span\{T_1, \dots, T_n\},$$

which implies that $\{T_j\}_{j=1}^n$ is QM-phase-retrievable if and only if $\{S_j\}_{j=1}^m$ is QM-phase-retrievable. $\square$

In the complex Hilbert space spaces case, two operator systems $\{T_j\}_{j=1}^n$ and $\{S_j\}_{j=1}^m$ induce the same completely positive map if and only if $\sum_{j=1}^n T_j x \otimes T_j x = \sum_{j=1}^m S_j x \otimes S_j x$ for every $x \in \mathbb{C}^d$. If we set $T_j = u_j \otimes u_j$ and $S_i = v_i \otimes v_i$ for $j = 1, \dots, n$ and $i = 1, \dots, m$, then we have $\sum_{j=1}^n T_j x \otimes T_j x = \sum_{j=1}^m |\langle x, u_j \rangle|^2 u_j \otimes u_j$ and $\sum_{i=1}^m S_j x \otimes S_j x = \sum_{i=1}^m |\langle x, v_i \rangle|^2 v_i \otimes v_i$. Thus from Proposition 4.5 we obtain the following consequence:

Corollary 4.6. *Let $u_j, v_i \in \mathbb{C}^d$ be such that*

$$\sum_{j=1}^m |\langle x, u_j \rangle|^2 u_j \otimes u_j = \sum_{i=1}^m |\langle x, v_i \rangle|^2 v_i \otimes v_i$$

holds for every $x \in \mathbb{C}^d$. Then $\{u_j\}_{j=1}^n$ is phase-retrievable if and only if $\{v_i\}_{i=1}^m$ is phase-retrievable.

By Choi's second theorem, we know that $cr(\Phi) \geq \dim \text{span}\{T_1, \dots, T_m\}$ if $\Phi(A) = \sum_{j=1}^m T_j A T_j^*$. In fact they are equal due to the following simple argument.

Proposition 4.7. *Let $\Phi(A) = \sum_{j=1}^m T_j A T_j^*$. Then $cr(\Phi) = \dim \text{span}\{T_1, \dots, T_m\}$.*

Proof. Let $r = \dim \text{span}\{T_1, \dots, T_m\}$. We already know by Choi's second theorem that $cr(\Phi) \geq r$. So it is enough to show there exist $\{L_1, \dots, L_r\}$ such that $\Phi(A) = \sum_{j=1}^r L_j A L_j^*$. Without losing the generality, we can assume that $\{T_1, \dots, T_r\}$ are linearly independent. Let $S_i = T_i$ for $i = 1, \dots, r$ and write $T_j = \sum_{i=1}^r c_{ij} S_i$ for $i = 1, \dots, m$. The $C = [c_{ij}]$ is an $m \times r$ matrix of rank r . Let $C = U(C^*C)^{1/2}$ be the polar decomposition of C . Then $U = [u_{ij}]$ is an $m \times r$ full rank partial isometry such that $U^*U = I_r$. Define $L_i = \sum_{j=1}^r \alpha_{ij} S_j$ for $i = 1, \dots, r$, where $(C^*C)^{1/2} = [\alpha_{ij}]$. Then we get that $T_i = \sum_{j=1}^r u_{ij} L_j$ for $i = 1, \dots, m$, and so

$$\begin{aligned} \Phi(A) &= \sum_{i=1}^m T_i A T_i^* \\ &= \sum_{i=1}^m \left(\sum_{j=1}^r u_{ij} L_j \right) A \left(\sum_{j=1}^r \bar{u}_{ij} L_j^* \right) \\ &= \sum_{i=1}^m \left(\sum_{j,j'=1}^r u_{ij} \bar{u}_{ij'} L_j A L_{j'}^* \right) \\ &= \sum_{j,j'=1}^r \left(\sum_{i=1}^m u_{ij} \bar{u}_{ij'} \right) L_j A L_{j'}^* \\ &= \sum_{j=1}^r L_j A L_j^*, \end{aligned}$$

where in the last equality we use the fact that $U^*U = I_r$. This completes the proof. $\square$

Declaration of Competing Interest

No competing interest.

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