s-wave contacts of quantum gases in quasi-one-dimensional and quasi-two-dimensional traps

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In quasi-one- or quasi-two-dimensional traps with strong transverse confinements, quantum gases behave like strictly one- or two-dimensional systems at large length scales. However, at short distance, the two-body scattering intrinsically has three-dimensional characteristics such that an exact description of any universal thermodynamic relation requires three-dimensional contacts since the range of interaction (a few nm) is orders of magnitude smaller than the harmonic-oscillator length of the transverse confinement ($\sim 10^2$ nm for a 100 kHz trap). A fundamental question arises as to whether one- or two-dimensional contacts, which were originally defined for strictly one or two dimensions, are capable of describing quantum gases in quasi-one- or quasi-two-dimensional traps. Here, we point out an exact relation between the three- and low-dimensional contacts in these highly anisotropic traps. Such relation allows us to directly connect physical quantities at different length scales and to characterize the quasi-one- or quasi-two-dimensional traps using universal thermodynamic relations that were derived for strict one or two dimensions.

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I. INTRODUCTION

A striking property of dilute quantum gases is that only a few physical quantities, the so-called contacts, fully govern a complex quantum many-body system. Contacts connect distinct physical observables through universal thermodynamic relations and provide physicists a unique and powerful tool to bridge few-body and many-body physics. In the past decade, the study of contacts and universal thermodynamic relations has become a fundamentally important topic in quantum gases [1–20] and has attracted significant interest from nuclear physicists and other communities [21–23]. Whereas the original work on contact focused on the s-wave one [1–3], recent studies have generalized such concept to high partial-wave contacts [24–29]. It has also been realized that to have a complete description of the universal thermodynamic relations, contacts should be defined as a matrix [30,31].

Similar to other physical quantities and phenomena, contacts and universal thermodynamic relations exhibit distinct behaviors in different dimensions [6–9]. The three-dimensional (3D) s-wave contact, C_{3D}, is proportional to $\frac{\partial E}{\partial(-1/a_{3D})}$ at the ground state, where E is the total energy and a_{3D} is the 3D scattering length. In contrast, contacts in one dimension (1D) and two dimensions (2D), C_{1D} and C_{2D}, are proportional to $\frac{\partial E}{\partial a_{1D}}$ and $\frac{\partial E}{\partial \ln(a_{2D})}$, where a_{1D} and a_{2D} are the scattering lengths in 1D and 2D, respectively. Other universal thermodynamic relations also have qualitative differences in different dimensions. Universal relations have also been derived in arbitrary, either integer or noninteger, dimensions [9].

So far, studies of contacts at low dimensions have been mainly focusing on the theoretical investigation of strictly 1D and 2D systems, where the transverse degree of freedom is absent. Contacts and universal relations in realistic low-dimensional systems have not been established. A crucial question remains unanswered as to whether universal relations theoretically derived for strictly 1D and 2D systems apply to realistic experiments on quasi-1D and quasi-2D traps in laboratories. It is well known that the origin of universal relations is the asymptotic behaviors of the many-body wave function in the limit where the distance between any two particles approaches zero. In strictly 1D (2D) systems, the asymptotic form of the two-body wave functions behaves like $|z|$ ($\ln\rho$) when $z \to 0$ ($\rho \to 0$), where z (ρ) is the relative coordinate of two particles. Such asymptotic behaviors lay the foundation for all universal relations in strictly 1D and 2D systems. However, these asymptotic forms do not apply to quasi-1D or quasi-2D traps when the separation between two particles approaches zero. In laboratories, a 1D or 2D system is created by applying a tight confinement, for instance, a strong harmonic trap of a harmonic-oscillator length d and frequency ω, along one or two spatial directions, as shown in Fig. 1. Such systems are often referred to as quasi-1D or quasi-2D traps. When the distance between two particles is much smaller than d, the two-body interaction inevitably has 3D characteristics, as the confining potential can barely affect the two-body wave function in such regime. The asymptotic form of the two-body wave function behaves like $1/r$, where r is the relative coordinate of two particles, similar to a strictly 3D system, and C_{3D} is required to describe universal thermodynamic relations in quasi-1D and quasi-2D traps, no matter how strong the transverse confinement is. Thus, fundamental questions arise regarding how to define C_{1D} and C_{2D} in quasi-1D and quasi-2D traps and whether they control universal relations in such highly anisotropic 3D traps.

The main results of this paper are summarized as follows. (i) In quasi-1D (quasi-2D) traps, C_{1D} (C_{2D}) needs to be defined from the momentum distribution $n_\sigma(\mathbf{k})$ in the regime, $k_F \ll k \ll d^{-1}$, where k_F is the Fermi momentum, $k = |\mathbf{k}|$, and $\sigma =$

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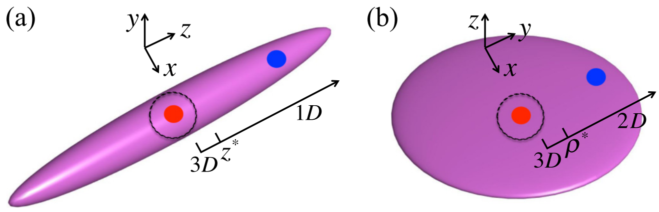

FIG. 1. (a) A quasi-1D trap. Atom cloud (purple cloud) with a strong harmonic confinement in the x-y plane. Red and blue spheres represent a spin-up and spin-down atom, respectively. When their separation is much larger (smaller) than $z^* \sim d$, two-body scatterings have 1D (3D) features, and C_{1D} (C_{3D}) controls all physical quantities in the corresponding large (small) length and small (large) momentum scales. (b) A quasi-2D trap with a strong harmonic confinement along the z direction. C_{2D} (C_{3D}) controls the system in a scale $\rho \gg \rho^* \sim d$ ($\rho \ll \rho^*$).

$\uparrow, \downarrow$ is the spin index. To be explicit, we define $\mathbf{k} = (\mathbf{k}_\perp, k_z)$ and obtain

$$n_\sigma^{1D}(k_z) \equiv \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} n_\sigma(\mathbf{k}) \xrightarrow{k_F \ll k_z \ll d^{-1}} \frac{C_{1D}}{k_z^4}, \tag{1}$$

$$n_\sigma^{2D}(\mathbf{k}_\perp) \equiv \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} n_\sigma(\mathbf{k}) \xrightarrow{k_F \ll k_\perp \ll d^{-1}} \frac{C_{2D}}{k_\perp^4}. \tag{2}$$

In the regime $k \gg d^{-1}$, C_{3D} determines $n_\sigma(\mathbf{k})$ in the large momentum tail,

$$n_\sigma(\mathbf{k}) \xrightarrow{k \gg d^{-1}} \frac{C_{3D}}{k^4}. \tag{3}$$

(ii) We establish an exact relation between C_{1D} (C_{2D}) and C_{3D} in quasi-1D (quasi-2D) traps, which is

$$C_{3D} = \pi d^2 C_{1D}, \tag{4}$$

$$C_{3D} = \sqrt{\pi d^2} C_{2D}. \tag{5}$$

Equations (4) and (5) provide us with a means to explore universal thermodynamic relations using two equivalent schemes, i.e., either through C_{3D} that controls any physical systems, including highly anisotropic traps, or using C_{1D} (C_{2D}), which governs $n_\sigma(\mathbf{k})$ in the intermediate momentum regime. These two equations also enable an alternative means to explore the fundamentally important problem on dimension crossover in ultracold atoms and related fields [32–36]. (iii) Using Eqs. (4) and (5), we obtain a rigorous proof that the adiabatic relation derived for a strictly 1D (2D) system is exact in quasi-1D (quasi-2D) traps.

It is worth pointing out that formulas similar to Eqs. (4) and (5) were derived in [9] by assuming the validity of adiabatic relations in quasi-low-dimensional traps. As we have explained in detail, adiabatic relations derived for strictly 1D (2D) systems cannot be taken for granted in quasi-1D (quasi-2D) traps, and even the definition of C_{1D} and C_{2D} in these traps is questionable. Thus, the full asymptotic forms of the many-body wave functions in all length scales in quasi-1D (quasi-2D) traps need to be taken as the starting point. This allows us to obtain Eqs. (1)–(5), provide a precise definition of C_{1D} (C_{2D}) in quasi-1D (quasi-2D) traps, reveal their relations with C_{3D}, and access the full structure of the large momentum tail, which includes two plateaus in $n_\sigma(\mathbf{k})k^4$, unlike strictly 1D and 2D systems with only one plateau. Eventually, adiabatic relations in quasi-1D (quasi-2D) traps are proved rigorously as the consequence, instead of the prerequisite, of Eqs. (4) and (5).

II. CONTACTS AND UNIVERSAL RELATIONS IN QUASI-1D TRAPS

We focus on quantum gases with zero-range interactions such that only s-wave scatterings and s-wave contacts are relevant. We first consider two-component fermion gases with total numbers $N_\uparrow$ and $N_\downarrow$ in each component in a quasi-1D trap. The Hamiltonian is written as

$$H = -\sum_i \frac{\hbar^2\nabla_i^2}{2M} + \sum_i V(\rho_i) + g\sum_{i=1}^{N_\uparrow}\sum_{j=N_\uparrow+1}^{N_\uparrow+N_\downarrow} \delta(\mathbf{r}_{ij})\frac{\partial(r_{ij}\cdot)}{\partial r_{ij}}, \tag{6}$$

where M is the mass of each atom, $\mathbf{r}_i = (\boldsymbol{\rho}_i, z_i)$ is the spatial coordinate of the ith atom, $\rho_i = |\boldsymbol{\rho}_i|$, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $r_{ij} = |\mathbf{r}_{ij}|$, and $V(\rho_i) = \frac{1}{2}M\omega^2\rho_i^2$ is a harmonic trapping potential for the ith atom in the x-y plane. Atoms are free along the z direction. $g = 4\pi\hbar^2 a_{3D}/M$ is the strength of the Huang-Yang pseudopotential. $V(\rho_i)$ is sufficiently strong such that $d = \sqrt{2\hbar/(M\omega)} \ll k_F^{-1}$ is satisfied. This is equivalent to saying that the chemical potential μ is much smaller than $2\hbar\omega$, i.e., the energy separation between the ground and the first vibration level of the harmonic trap. When the distance between a spin-up and spin-down atom, which is denoted by $r = |\mathbf{r}|$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, is much smaller than k_F^{-1}, the wave function of a many-body eigenstate has a universal asymptotic form,

$$\Psi \xrightarrow{r \ll k_F^{-1}} \int d\epsilon_q \phi(\mathbf{r};\epsilon_q) G\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}, \mathbf{r}_{i\neq 1,2}; \sigma_i; E-\epsilon_q\right), \tag{7}$$

where $\phi(\mathbf{r};\epsilon_q)$ is the wave function of the relative motion of two atoms, $\epsilon_q = \hbar\omega + \hbar^2q^2/M$ is the colliding energy, q is the corresponding momentum, and E is the total energy of the system. σ_i is the spin index of the ith atom. Whereas Eq. (7) is valid for any 3D system, it is useful to make use of the explicit form of $\phi(\mathbf{r};\epsilon_q)$ in quasi-1D traps,

$$\phi(\mathbf{r};\epsilon_q) = \Phi_{00}(\boldsymbol{\rho})[\cos(qz) + f(q)e^{iq|z|}] - f(q)\sum_{n>0}\frac{iq}{q_n}\Phi_{n0}(\boldsymbol{\rho})e^{-q_n|z|}, \tag{8}$$

where $\Phi_{nm}(\boldsymbol{\rho})$ is the eigenstate of the harmonic trap with eigenenergy $E_\perp^{nm} = \hbar\omega(2n + |m| + 1)$ in the x-y plane, n is the quantum number for the radial part of the wave function, and m is the angular momentum quantum number. $f(q) = i/[\cot\eta_{1D}(q) - i]$ is the scattering amplitude and $\eta_{1D}(q)$ is the phase shift in 1D. The first line in Eq. (8) is the contribution from the ground state of the harmonic trap, the second line is the contribution from excited states, and $q_n = \sqrt{(E_\perp^{n0} - \epsilon_q)M/\hbar^2}$. For s-wave scatterings, only wave functions with $m = 0$ are relevant. Since $\hbar^2q^2/M$ is typically of the order of $\mu \ll 2\hbar\omega$, q_n is positive for all $n > 0$. Thus, the second line in Eq. (8) decays exponentially in the quasi-1D regime where the energy of the incoming wave in the scattering problem is smaller than the gap between the ground and the first-excited vibration levels. When $|z| \gg z^* \equiv 1/q_1$, Eq. (8) reduces to a wave function in strict 1D. It is also easy to see that $z^* \sim d \ll k_F^{-1}$. Correspondingly, based on the definition $n_\sigma(\mathbf{k}) = \sum_{i=1+N_\uparrow\delta_{\downarrow,\sigma}}^{N_\uparrow+N_\downarrow\delta_{\downarrow,\sigma}} \int \prod_{j\neq i} d^3\mathbf{r}_j |\int d^3\mathbf{r}_i \Psi e^{-i\mathbf{k}\cdot\mathbf{r}_i}|^2$, where $\delta_{i,j}$ is the Kronecker delta, we obtain the momentum distribution of the many-body system in the regime $k_F \ll k \ll d^{-1}$,

$$n_\sigma(\mathbf{k}) \xrightarrow{k_F \ll k \ll d^{-1}} |\Phi_{00}(\mathbf{k}_\perp)|^2 \frac{C_{1\mathrm{D}}}{k_z^4}, \quad \sigma = \uparrow, \downarrow, \tag{9}$$

where $\mathbf{k} = (\mathbf{k}_\perp, k_z)$, $\Phi_{00}(\mathbf{k}_\perp) = \int d^2\boldsymbol{\rho}\Phi_{00}(\boldsymbol{\rho})e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}}$,

$$C_{1\mathrm{D}} = 4N_\uparrow N_\downarrow \int d^3\mathbf{R}_{12} \left| \int d\epsilon_q q f(q) G(\mathbf{R}_{12}; E-\epsilon_q) \right|^2, \tag{10}$$

and $\mathbf{R}_{12}$ is a shorthand notation for a set of coordinates $\{(\mathbf{r}_1+\mathbf{r}_2)/2, \mathbf{r}_{i\neq 1,2}; \sigma_i\}$, $d^3\mathbf{R}_{12} = \prod_{i\neq 1,2} d^3\mathbf{r}_i d^3(\mathbf{r}_1+\mathbf{r}_2)/2$. Though this power-law tail comes from the singular behavior of the relative wave function of a pair of particles when they approach each other, it does show up in the momentum distribution when k is much larger than k_F and other momentum scales, such as the center-of-mass momentum of a pair of particles and the inverse of the scattering length. Thus, for simplicity, we have just specified that $k \gg k_F$, as the center-of-mass momentum of a pair of particles is, in general, much smaller than k_F, and so is the inverse of the scattering length in the strongly interacting regime. In this regime, $n_\sigma(\mathbf{k})$ is a broad distribution along the k_x and k_y directions, as expected for a quasi-1D system. For $k_F \ll k_z \ll d^{-1}$, the expression in Eq. (9) could be extend to $k_\perp \to \infty$. Integrating over $\mathbf{k}_\perp$, we obtain Eq. (1).

We now consider $r \ll d$, where we have

$$\Psi \xrightarrow{r \ll d} \left(\frac{1}{r} - \frac{1}{a_{3\mathrm{D}}}\right) \int d\epsilon_q G_{3\mathrm{D}}(\mathbf{R}_{12}; E-\epsilon_q). \tag{11}$$

Correspondingly, $n_\sigma(\mathbf{k})$ has a large momentum tail. It is given by Eq. (3) and

$$C_{3\mathrm{D}} = (4\pi)^2 N_\uparrow N_\downarrow \int d^3\mathbf{R}_{12} \left| \int d\epsilon_q G_{3\mathrm{D}}(\mathbf{R}_{12}; E-\epsilon_q) \right|^2. \tag{12}$$

Indeed, Eq. (8) becomes $\frac{-iqf(q)}{2}\frac{d}{\sqrt{\pi}}(\frac{1}{|z|} - \frac{1}{a_{3\mathrm{D}}})$ when $|z| \ll d$ for $\rho = 0$, and [32]

$$a_{1\mathrm{D}} = -\frac{d^2}{2a_{3\mathrm{D}}}\left(1 - 1.4603\frac{a_{3\mathrm{D}}}{d}\right) \tag{13}$$

where $\cot\eta(q)/q = a_{1\mathrm{D}}$ and $G_{3\mathrm{D}}(\mathbf{R}_{12}; E-\epsilon_q) = \frac{-iqf(q)}{2}\frac{d}{\sqrt{\pi}}G(\mathbf{R}_{12}; E-\epsilon_q)$. Compare Eq. (10) and Eq. (12), we immediately see that Eq. (4) holds.

It is interesting to note that Eq. (4) has a simple geometric interpretation. Though the quasi-1D trap is highly nonuniform in the transverse directions, it can be viewed as a cylinder with a uniform distribution of contact density on the cross section of radius d. Since the total contact in 3D is the contact density multiplied by the total volume, one can view $C_{1\mathrm{D}}$ as the linear contact density. Thus, $C_{3\mathrm{D}}$ is simply $C_{1\mathrm{D}}$ multiplied by the cross-sectional area πd^2. Equation (4) also allows one to establish an exact relation between $n_\sigma(\mathbf{k})$ in different

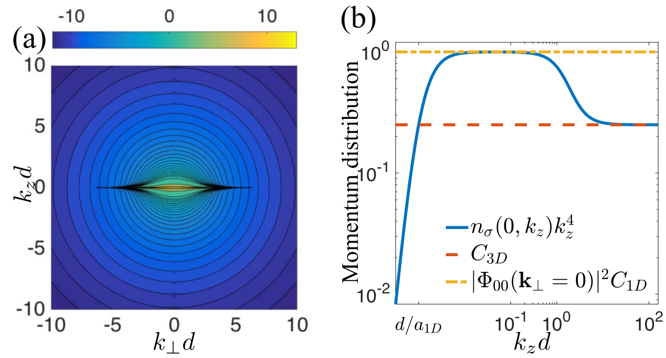

FIG. 2. (a) A contour plot of the exact momentum distribution $\ln[n_\sigma(\mathbf{k})]$ of a two-body system, with $n_\sigma(\mathbf{k})$ in units of $d^4|\Phi_{00}(\mathbf{k}_\perp = 0)|^2 C_{1\mathrm{D}}$. The total number of vibration levels considered is $N = 300$, and $a_{1\mathrm{D}} = 1000d$. (b) Scaled momentum $n_\sigma(0, k_z)k_z^4$. It is determined by $C_{1\mathrm{D}}$ and $C_{3\mathrm{D}}$ in the regime $a_{1\mathrm{D}}^{-1} \ll k_z \ll d^{-1}$ and $k_z \gg d^{-1}$, respectively.

momentum scales. From Eq. (1) and Eq. (3), we obtain

$$n_\sigma(\mathbf{k})k^4\big|_{k\gg d^{-1}} = (\pi d^2) n_\sigma^{1\mathrm{D}}(k_z)k_z^4\big|_{k_F \ll k_z \ll d^{-1}}, \tag{14}$$

a result originated from the exact relation between $C_{3\mathrm{D}}$ and $C_{1\mathrm{D}}$.

To verify the above results, we evaluate exactly $n_\sigma(\mathbf{k})$ of a two-body system using Eqs. (7) and (8). Its scaling behaviors also describe those of $n_\sigma(\mathbf{k})$ in a generic many-body system in the regime $k \ll k_F$. By taking into account a large enough number of excited states, we obtain numerically $n_\sigma(\mathbf{k})$, as shown in Fig. 2(a). Indeed, in the regime $k_F \ll k \ll d^{-1}$, $n_\sigma(\mathbf{k})$ decays slowly with increasing k_x and k_y. As mentioned above, the width of the wave function $\phi_{00}(\mathbf{k}_\perp)$ is given by the inverse of the harmonic-oscillator length. Thus, for a strong confinement, $n_\sigma(\mathbf{k})$ exhibits a 1D feature in such momentum scale. In contrast, in the regime $k \gg d^{-1}$, $n_\sigma(\mathbf{k})$ becomes isotropic, which is a 3D characteristic as expected. Figure 2(b) shows the scaled momentum distribution $n_\sigma(\mathbf{k})k^4$, which clearly demonstrates how $n_\sigma(0, k_z)$ gradually changes from $|\Phi_{00}(\mathbf{k}_\perp = 0)|^2 C_{1\mathrm{D}}/k_z^4$ to $C_{3\mathrm{D}}/k_z^4$.

Besides $n_\sigma(\mathbf{k})$, Eq. (4) allows us to connect other universal thermodynamic relations in 1D and 3D. Here, we focus on the adiabatic relations. In strictly 1D systems, where the transverse degrees of freedom are absent, the adiabatic relation is written as [8]

$$\frac{dE}{da_{1\mathrm{D}}} = \frac{\hbar^2 C_{1\mathrm{D}}}{2M}. \tag{15}$$

In quasi-1D systems, as mentioned above, $C_{1\mathrm{D}}$ controls physical quantities in a large length scale $z \gg d$ or, equivalently, in the momentum scale $k \ll d^{-1}$. A complete description of the system needs the introduction of $C_{3\mathrm{D}}$ to capture physics in the length scale $z < d$ or momentum scale $k > d^{-1}$. A natural question is then whether Eq. (15) is still valid.

Interestingly, a simple calculation shows that Eq. (15) holds for the quasi-1D system. The reason is that Eq. (4) provides an exact relation between $C_{1\mathrm{D}}$ and $C_{3\mathrm{D}}$, the latter of which governs any 3D system, including a quasi-1D trap that is highly anisotropic. Thus the 3D adiabatic relation [2],

$$\frac{dE}{d(-1/a_{3\mathrm{D}})} = \frac{\hbar^2 C_{3\mathrm{D}}}{4\pi M}, \tag{16}$$

is always valid in a quasi-1D trap. It is also known that $a_{3\mathrm{D}}$ and $a_{1\mathrm{D}}$ are related by Eq. (13). Substitute this expression and Eq. (4) to Eq. (16), and Eq. (15) is obtained. This immediately tells us that the adiabatic relation derived for strictly 1D systems applies to quasi-1D traps. In practice, Eqs. (1) and (15) are also particularly useful, as experimentalists do not need to extract $C_{3\mathrm{D}}$ from $n_\sigma(\mathbf{k})$ in the very large momentum regime $k \gg d^{-1}$, which may become too small to detect. Instead, a measurement of $n_\sigma(\mathbf{k})$ in the intermediate regime $k_F \ll k \ll d^{-1}$, which has a much larger amplitude, is sufficient to obtain $C_{1\mathrm{D}}$ that could also fully govern the quasi-1D trap.

Whereas we focus on the adiabatic relation here, discussions can be directly generalized to other universal thermodynamic relations. Equation (4) shows that any universal thermodynamic relations established by $C_{3\mathrm{D}}$ can be rewritten in terms of $C_{1\mathrm{D}}$ that governs the behaviors of the quasi-1D systems in the large length scale $z \gg d$. Thus, universal thermodynamic relations in 3D can be directly transformed to those in 1D.

III. CONTACTS AND UNIVERSAL RELATIONS IN QUASI-2D TRAPS

We now turn to a quasi-2D trap. The Hamiltonian is written as

$$H = -\sum_i \frac{\hbar^2\nabla_i^2}{2M} + \sum_i V(z_i) + g\sum_{i=1}^{N_\uparrow}\sum_{j=N_\uparrow+1}^{N_\uparrow+N_\downarrow} \delta(\mathbf{r}_{ij})\frac{\partial(r_{ij}\cdot)}{\partial r_{ij}}, \tag{17}$$

where $V(z_i) = \frac{1}{2}M\omega^2 z_i^2$ is a harmonic trapping potential for the ith atom along the z direction. The system is free in the x-y plane. The discussions are essentially parallel to those in quasi-1D traps. Starting from Eq. (7) and the two-body wave function in a quasi-2D trap for s-wave scattering,

$$\phi(\mathbf{r};\epsilon_q) = \frac{\pi}{2}\cot\eta_{2\mathrm{D}}(q)[J_0(q\rho) - \tan\eta_{2\mathrm{D}}(q)N_0(q\rho)]\Phi_0(z) + \frac{i\pi}{2}\sum_{n>0}(-1)^n\sqrt{\frac{(2n-1)!!}{(2n)!!}}\Phi_{2n}(z)H_0^{(1)}(iq_n\rho), \tag{18}$$

it is straightforward to derive Eq. (5), the tails of the momentum distribution and the adiabatic relation. In Eq. (18), $\eta_{2\mathrm{D}}(q)$ is the 2D phase shift, J_0 (N_0) is the Bessel function of the first (second) kind, $H_0^{(1)}$ is the Hankel function of the first kind, $\Phi_n(z)$ is the eigenfunction of the harmonic oscillator along the z axis with eigenenergy $E_z^n = \hbar\omega(n+1/2)$, $\epsilon_q = \hbar\omega/2 + \hbar^2q^2/M$, and $q_n = \sqrt{(E_z^{2n} - \epsilon_q)M/\hbar^2}$. When $\rho > \rho^* \equiv 1/q_1$ ($\rho < \rho^*$), the wave function in Eq. (18) is 2D-like (3D-like).

Figure 3 shows the numerical results for the momentum distribution of a two-body system. Again, its scaling behaviors capture those of a generic many-body system in the regime, $k \ll k_F$. When $k_F \ll k_\perp \ll d^{-1}$, we obtain the 2D analogy of Eq. (9),

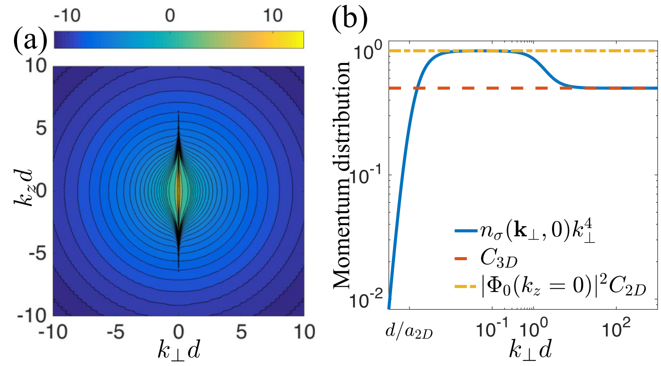

FIG. 3. (a) A contour plot of the exact momentum distribution $\ln[n_\sigma(\mathbf{k})]$ of a two-body system, with $n_\sigma(\mathbf{k})$ in units of $d^4|\Phi_0(k_z=0)|^2C_{2\mathrm{D}}$. The total number of vibration levels considered is $N = 300$, and $a_{2\mathrm{D}} = 1000d$. (b) Scaled momentum $n_\sigma(\mathbf{k}_\perp, 0)k_\perp^4$. It is determined by $C_{2\mathrm{D}}$ and $C_{3\mathrm{D}}$ in the regime $a_{2\mathrm{D}}^{-1} \ll k_\perp \ll d^{-1}$ and $k_\perp \gg d^{-1}$, respectively.

$$n_\sigma(\mathbf{k}) \xrightarrow{k_F \ll k \ll d^{-1}} |\Phi_0(k_z)|^2\frac{C_{2\mathrm{D}}}{k_\perp^4}, \tag{19}$$

which shows that $n_\sigma(\mathbf{k})$ decays slowly in the k_z direction, a characteristic quasi-2D feature. Integrating over k_z, we obtain Eq. (2) and

$$C_{2\mathrm{D}} = (2\pi)^2 N_\uparrow N_\downarrow \int d^3\mathbf{R}_{12}\left|\int d\epsilon_q G(\mathbf{R}_{12}, E-\epsilon_q)\right|^2. \tag{20}$$

By considering the asymptotic behavior of $\phi(\mathbf{r};\epsilon_q)$ at $\rho \ll d$ and $z = 0$, one can also obtain that

$$\phi(\boldsymbol{\rho}, 0; \epsilon_q) \xrightarrow{\rho \ll d} \frac{\sqrt{d\sqrt{\pi}}}{2}\left(\frac{1}{\rho} - \frac{1}{a_{3\mathrm{D}}}\right), \tag{21}$$

which is consistent with Eq. (11), and [36]

$$a_{2\mathrm{D}} = \sqrt{\frac{2\pi}{\tau}}d\exp\left(-\frac{\sqrt{\pi}}{2}\frac{d}{a_{3\mathrm{D}}} - \gamma\right), \tag{22}$$

where $\tau = 0.915\cdots$ and γ is the Euler's constant, $\cot\eta_{2\mathrm{D}} = \frac{2}{\pi}\ln(qa_{2\mathrm{D}}e^\gamma/2)$, and $G_{3\mathrm{D}}(\mathbf{R}_{12}; E-\epsilon_q) = \sqrt{d\sqrt{\pi}}/4G(\mathbf{R}_{12}; E-\epsilon_q)$. Thus, when $r \ll d$ or, equivalently, $k \gg d^{-1}$, the system is 3D-like, as shown in Fig. 3. $n_\sigma(\mathbf{k})$ becomes isotropic and is governed by $C_{3\mathrm{D}}$. Compare Eq. (12) with Eq. (20), it is clear that Eq. (5) holds. We can also see that

$$n_\sigma(\mathbf{k})k^4|_{k\gg d^{-1}} = \sqrt{\pi}d^2 n_\sigma^{2\mathrm{D}}(\mathbf{k}_\perp)k_\perp^4|_{k_F \ll k_\perp \ll d^{-1}}. \tag{23}$$

Similar to the discussions in quasi-1D cases, we find out that the adiabatic relation,

$$\frac{dE}{d\ln a_{2\mathrm{D}}} = \frac{\hbar^2 C_{2\mathrm{D}}}{2\pi M}, \tag{24}$$

which was originally derived for strictly 2D systems [6], still holds for quasi-2D traps. By taking Eq. (22) and Eq. (5) into Eq. (24), it recovers the 3D adiabatic relation in Eq. (16).

IV. CONCLUSION

In conclusion, we have shown an exact relation between C_{3D} and C_{1D} (C_{2D}) in quasi-1D (quasi-2D) traps, which correlates not only physical quantities at different length or momentum scales, but also universal relations in different dimensions. We hope that our work will provide physicists with an alternative angle to explore the dimension crossover, and inspire more studies of the central role of contacts in many-body quantum phenomena of quantum gases and related systems.

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