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# Large deviations and continuity estimates for the derivative of a random model of $\log|\zeta|$ on the critical line

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**Abstract**

In this paper, we study the random field

$$X(h) \circeq \sum_{p\leq T} \frac{\mathrm{Re}(U_p\, p^{-ih})}{p^{1/2}}, \quad h\in[0,1],$$

where $(U_p,\, p \text{ primes})$ is an i.i.d. sequence of uniform random variables on the unit circle in $\mathbb{C}$. Harper (2013) showed that $(X(h),\, h\in(0,1))$ is a good model for the large values of $(\log|\zeta(\frac{1}{2}+i(T+h))|,\, h\in[0,1])$ when $T$ is large, if we assume the Riemann hypothesis. The asymptotics of the maximum were found in Arguin *et al.* (2017) up to the second order, but the tightness of the recentered maximum is still an open problem. As a first step, we provide large deviation estimates and continuity estimates for the field's derivative $X'(h)$. The main result shows that, with probability arbitrarily close to 1,

$$\max_{h\in[0,1]} X(h) - \max_{h\in\mathcal{S}} X(h) = O(1),$$

where $\mathcal{S}$ a discrete set containing $O(\log T\sqrt{\log\log T})$ points.

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## 1. Introduction

In Fyodorov *et al.* (2012) and Fyodorov & Keating (2014), it was conjectured that if $\tau$ is sampled uniformly in $[T,2T]$ for some large $T$, then the law of the maximum of $(\log|\zeta(\frac{1}{2}+i(\tau+h))|, h\in[0,1])$, where $\zeta$ denotes the Riemann zeta function, should be asymptotic to $\log\log T - \frac{3}{4}\log\log\log T + \mathcal{M}_T$ where $(\mathcal{M}_T, T\geq 2)$ is a sequence of random variables converging in distribution. At present, the first order of the maximum is proved conditionally on the Riemann hypothesis in Najnudel (2018) and unconditionally in Arguin *et al.* (2018).

In order to study this hard problem originally, a randomized version of the Riemann zeta function was introduced in Harper (2013), see (2.1). The first order of the maximum was proved in Harper (2013), the second order of the maximum was proved in Arguin *et al.* (2017), and a related study of the Gibbs measure can be found in Arguin & Tai (2018) and Ouimet (2018). The tightness of the recentered maximum is still open.

As a first step, our main result (Theorem 3.3) shows that the tightness of the "continuous" maximum $\max_{h\in[0,1]} X(h)$ (once recentered) can be reduced to the tightness of a "discrete" maximum $\max_{h\in\mathcal{S}} X(h)$ (once recentered) where $\mathcal{S}$ is a discrete set containing $O(\log T\sqrt{\log\log T})$ points. In order to prove Theorem 3.3, we will need continuity estimates and large deviation estimates for the field's derivative $X'(h)$, which can be found in Proposition 3.1 and Proposition 3.2, respectively.

The paper is organised as follows. In Section 2, we introduce the model $X(h)$. In Section 3, the main result is stated and proven. Proposition 3.1 and Proposition 3.2 are stated in Section 3 and proven in Section 4.

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## 2. The model

Let $(U_p,\, p \text{ primes})$ be an i.i.d. sequence of uniform random variables on the unit circle in $\mathbb{C}$. The random field of interest is

$$X(h) \mathrel{\hat=} \sum_{p\le T} W_p(h) \mathrel{\hat=} \sum_{p\le T} \frac{\mathrm{Re}(U_p\, p^{-ih})}{p^{1/2}}, \quad h\in[0,1]. \tag{2.1}$$

(A sum over the variable $p$ always denotes a sum over primes.) This is a good model for the large values of $(\log|\zeta(\frac{1}{2}+i(\tau+h))|, h\in[0,1])$ for the following reason. Proposition 1 in Harper (2013) proves that, assuming the Riemann hypothesis, and for $T$ large enough, there exists a set $B\subseteq[T,T+1]$, of Lebesgue measure at least 0.99, such that

$$\log|\zeta(\frac{1}{2}+it)| = \mathrm{Re}\left(\sum_{p\le T}\frac{1}{p^{1/2+it}}\frac{\log(T/p)}{\log T}\right) + O(1), \quad t\in B. \tag{2.2}$$

If we ignore the smoothing term $\log(T/p)/\log T$ and note that the process $(p^{-i\tau}, p \text{ primes})$, where $\tau$ is sampled uniformly in $[T,2T]$, converges, as $T\to\infty$ (in the sense of convergence of its finite-dimensional distributions), to a sequence of independent random variables distributed uniformly on the unit circle (by computing the moments), then the model (2.1) follows. For more information, see Section 1.1 in Arguin *et al.* (2017).

More generally, for $-1\le r\le k$, denote the increments of the field by

$$X_{r,k}(h) \mathrel{\hat=} \sum_{2^r<\log p\le 2^k} \frac{\mathrm{Re}(U_p\, p^{-ih})}{p^{1/2}}, \quad h\in[0,1]. \tag{2.3}$$

Differentiation of (2.3) yields

$$X'_{r,k}(h) = \sum_{2^r<\log p\le 2^k} W'_p(h) = \sum_{2^r<\log p\le 2^k} \frac{\mathrm{Im}(U_p\, p^{-ih})\log p}{p^{1/2}}. \tag{2.4}$$

## 3. Main result

Throughout the paper, we will write $c$, $\widetilde{c}$, $c'$, and $c''$, for generic positive constants whose value may change at different occurrences. Here are the main side results of this paper.

**Proposition 3.1** (Continuity estimates)**.** *Let $C>0$. For any $-1\le r\le k$, $0\le x\le C(2^{2k}-2^{2r})$, $2\le a\le 2^{6k}-x$ and $h\in\mathbb{R}$,*

$$\mathbb{P}\left(\max_{h':|h'-h|\le 2^{-3k-1}} X'_{r,k}(h')\ge x+a, X'_{r,k}(h)\le x\right)\le c\exp\left(-2\frac{x^2}{2^{2k}-2^{2r}}-\widetilde{c}\,a^{3/2}\right), \tag{3.1}$$

*where the constants $c$ and $\widetilde{c}$ only depend on $C$.*

**Proposition 3.2** (Large deviation estimates)**.** *Let $C>0$. For any $-1\le r\le k$, $0\le x\le C(2^{2k}-2^{2r})$ and $h\in\mathbb{R}$,*

$$\mathbb{P}\left(\max_{h':|h'-h|\le 2^{-3k-1}} X'_{r,k}(h')\ge x\right)\le c\exp\left(-2\frac{x^2}{2^{2k}-2^{2r}}\right), \tag{3.2}$$

*where the constant $c$ only depends on $C$.*

From the last proposition, we obtain the following theorem.

**Theorem 3.3** (Main result)**.** *Let $-1\le r\le k$. For all $L>0$, let $\mathcal{S}_{r,k,L}$ be a set of equidistant points in $[0,1]$ such that $|\mathcal{S}_{r,k,L}| = \lceil L\sqrt{2^{2k}-2^{2r}}\sqrt{k\log 2}\rceil$ and $|h'-h|\ge|\mathcal{S}_{r,k,L}|^{-1}$ for different $h,h'\in\mathcal{S}_{r,k,L}$. Then, for any $K>0$, there exists $L\mathrel{\hat=}L(K)>0$ large enough that*

$$\mathbb{P}\left(\left|\max_{h\in[0,1]} X_{r,k}(h) - \max_{h\in\mathcal{S}_{r,k,L}} X_{r,k}(h)\right| > K\right) < e^{-\frac{k}{4}(1-e^{-K})^2L^2}. \tag{3.3}$$

**Remark 3.4.** *When $r=-1$ and $2^k=\log T$, $X_{r,k}(h)$ is just the original model $X(h)$. In that case, (3.3) shows that, with probability as close to 1 as we want, there exists a discrete set $\mathcal{S}\subseteq[0,1]$ such that*

$$\max_{h\in[0,1]} X(h) - \max_{h\in\mathcal{S}} X(h) = O(1), \tag{3.4}$$

*where $|\mathcal{S}| = O(\log T\sqrt{\log\log T})$.*

We prove Theorem 3.3 right away and we will prove Proposition 3.1 and Proposition 3.2 in Section 4.

*Proof of Theorem 3.3.* For $M > 0$, define the event

$$E = \left\{ \max_{h \in [0,1]} |X'_{r,k}(h)| \geq M \sqrt{2^{2k} - 2^{2r}} \sqrt{k \log 2} \right\}. \tag{3.5}$$

Let $\mathcal{H}_k \doteq 2^{-3k}\mathbb{Z}$ and note that $|\mathcal{H}_k \cap [0,1]| = 2^{3k} + 1$. By a union bound, the symmetry of $X'_{r,k}(h)$'s distribution, and Proposition 3.2, we obtain

$$\mathbb{P}(E) \leq \sum_{h \in \mathcal{H}_k \cap [0,1]} 2 \cdot \mathbb{P}\left( \max_{h' : |h'-h| \leq 2^{-3k-1}} X'_{r,k}(h') \geq M \sqrt{2^{2k} - 2^{2r}} \sqrt{k \log 2} \right) \leq (2^{3k} + 1) \cdot c\, 2^{-2kM^2}. \tag{3.6}$$

For every realisation $\omega$ of the field $\{X_{r,k}(h)\}_{h \in [0,1]}$, let $h^\star(\omega)$ be a point where the maximum is attained. When $\omega \in E^c$, the mean value theorem yields that, for any $h(\omega) \in \mathcal{S}_{r,k,L}$ such that $|h^\star(\omega) - h(\omega)| \leq 2/|\mathcal{S}_{r,k,L}|$, we have

$$e^{X_{r,k}(h^\star(\omega))} - e^{X_{r,k}(h(\omega))} = X'_{r,k}(\xi(\omega)) e^{X_{r,k}(\xi(\omega))} (h^\star(\omega) - h(\omega)) \leq \frac{2M}{L} e^{X_{r,k}(h^\star(\omega))}, \tag{3.7}$$

for some $\xi(\omega)$ lying between $h(\omega)$ and $h^\star(\omega)$. By taking $L \doteq L(K) \doteq 2M/(1 - e^{-K})$, we deduce $e^{X_{r,k}(h(\omega))} \geq e^{-K} e^{X_{r,k}(h^\star(\omega))}$. This reasoning shows that, on the event $E^c$,

$$\max_{h \in \mathcal{S}_{r,k,L}} X_{r,k}(h) \geq \max_{h \in [0,1]} X_{r,k}(h) - K. \tag{3.8}$$

The conclusion follows from (3.8) and (3.6) with $M = \frac{1}{2}(1 - e^{-K})L$. □

## 4. Proof of Proposition 3.1 and Proposition 3.2

We start by controlling the tail probabilities for a single point of the field's derivative.

**Lemma 4.1.** *Let $C > 0$. For any $-1 \leq r \leq k$, $0 \leq x \leq C(2^{2k} - 2^{2r})$ and $h \in \mathbb{R}$,*

$$\mathbb{P}\big(X'_{r,k}(h) \geq x\big) \leq c \exp\left( -2 \frac{x^2}{2^{2k} - 2^{2r}} \right), \tag{4.1}$$

*where the constant $c$ only depends on $C$.*

*Proof.* Using Chernoff's inequality, the independence of the $U_p$'s and translation invariance, we have that, for all $\lambda \geq 0$,

$$\mathbb{P}\big(X'_{r,k}(h) \geq x\big) \leq e^{-\lambda x}\, \mathbb{E}\big[e^{\lambda X'_{r,k}(h)}\big] = e^{-\lambda x} \prod_{2^r < \log p \leq 2^k} \mathbb{E}\big[e^{\lambda W'_p(0)}\big]. \tag{4.2}$$

Note that

$$\mathbb{E}\big[e^{\lambda W'_p(0)}\big] = \frac{1}{2\pi} \int_0^{2\pi} \exp\left( \frac{\lambda \log p}{p^{1/2}} \sin(\theta) \right) d\theta = I_0\left( \frac{\lambda \log p}{p^{1/2}} \right), \tag{4.3}$$

(Abramowitz & Stegun, 1964, 9.6.16, p.376), where $I_0$ denotes the *modified Bessel function of the first kind.* The function $I_0$ has the following series representation : $I_0(u) = 1 + \frac{u^2}{4} + \frac{u^4}{64} + O(u^6),\ u \in \mathbb{R}$. In turn,

$$\log(I_0(u)) = \frac{u^2}{4} - \frac{u^4}{64} + O(u^6), \quad u \in (-1,1), \tag{4.4}$$

because $\log(1+y) = y - \frac{y^2}{2} + O(y^3)$ for $y \in (-1,1)$, and $|I_0(u) - 1| < 1$ for $u \in (-1,1)$. Choose $\lambda = 4x/(2^{2k} - 2^{2r})$. By applying (4.4) in (4.3), the right-hand side of (4.2) is bounded from above by

$$c\, e^{-\lambda x} \exp\left( \sum_{2^r < \log p \leq 2^k} \frac{\lambda^2 (\log p)^2}{4p} + \widetilde{c} \sum_{2^r < \log p \leq 2^k} \frac{\lambda^6 (\log p)^6}{p^3} \right). \tag{4.5}$$

For the finite number of primes $p$ for which we cannot apply (4.4) in (4.3) (note that $\lambda \log p < p^{1/2}$ holds for $p$ large enough since $\lambda \leq 4C$ by the assumption on $x$), the correction terms needed for (4.5) to hold are absorbed in the constant $c$ in front of the first exponential in (4.5). The second sum in the big exponential is bounded by a constant independent from $r$ and $k$ since $\lambda \leq 4C$ and $\sum_p (\log p)^6 p^{-3} < \infty$. By applying Lemma Appendix A.1 with $m = 2$, $\log P = 2^r$ and $\log Q = 2^k$, the first sum in the big exponential is bounded by $2x^2/(2^{2k} - 2^{2r})$ up to an additive constant that only depends on $C$. The conclusion of the lemma follows. □

In the next lemma, we complement Lemma 4.1 by proving a large deviation estimate for $X'_{r,k}(0)$ and the difference $X'_{r,k}(h_2) - X'_{r,k}(h_1)$ jointly, where $|h_2 - h_1| \leq 2^{-3k}$.

**Lemma 4.2.** *Let $C > 0$. For any $-1 \leq r \leq k$, $0 \leq x \leq C(2^{2k} - 2^{2r})$, $0 \leq y \leq 2^{6k}$, and any distinct $h_1, h_2 \in \mathbb{R}$ such that $-2^{-3k-1} \leq h_1, h_2 \leq 2^{-3k-1}$,*

$$\mathbb{P}\big(X'_{r,k}(0) \geq x, X'_{r,k}(h_2) - X'_{r,k}(h_1) \geq y\big) \leq c \exp\left(-2\frac{x^2}{2^{2k} - 2^{2r}} - \frac{\widetilde{c}\, y^{3/2}}{|h_2 - h_1|\, 2^{3k}}\right), \tag{4.6}$$

*where the constants $c$ and $\widetilde{c}$ only depend on $C$.*

*Proof.* Assume that $y \geq \widetilde{C}|h_2 - h_1|2^{3k}$ for a large constant $\widetilde{C} \geq 1$ because otherwise (4.6) follows from (4.1). Since $|h_2 - h_1|2^{3k} \leq 1$, note that this assumption also implies $y^{1/2} \geq \widetilde{C}^{1/2}|h_2 - h_1|2^{3k}$. For all $\lambda_1, \lambda_2 \geq 0$, the left-hand side of (4.6) is bounded from above (using Chernoff's inequality) by

$$\mathbb{E}\big[\exp(\lambda_1 X'_{r,k}(0) + \lambda_2(X'_{r,k}(h_2) - X'_{r,k}(h_1)))\big] \exp(-\lambda_1 x - \lambda_2 y). \tag{4.7}$$

We will show that if $0 \leq \lambda_1 \leq 4C$ and $0 \leq \lambda_2 \leq |h_2 - h_1|^{-1}$, then

$$\begin{aligned} &\mathbb{E}\big[\exp(\lambda_1 X'_{r,k}(0) + \lambda_2(X'_{r,k}(h_2) - X'_{r,k}(h_1)))\big] \\ &\quad \leq c \exp\left(\frac{\lambda_1^2}{8}(2^{2k} - 2^{2r}) + c\,\lambda_2|h_2 - h_1|\, 2^{3k} + c^2\lambda_2^2|h_2 - h_1|^2\, 2^{4k}\right). \end{aligned} \tag{4.8}$$

The result (4.6) follows by choosing $\lambda_1 = 4x/(2^{2k} - 2^{2r})$, $\lambda_2 = y^{1/2}|h_2 - h_1|^{-1}\, 2^{-3k}$ and $\widetilde{C}$ large enough (with respect to $c$) in (4.7) and (4.8). The assumptions on $x$, $y$, $h_1$ and $h_2$ ensure that $0 \leq \lambda_1 \leq 4C$ and $0 \leq \lambda_2 \leq |h_2 - h_1|^{-1}$. We now prove (4.8). For $2^r < \log p \leq 2^k$, the quantity

$$\mathbb{E}\big[\exp(\lambda_1 W'_p(0) + \lambda_2(W'_p(h_2) - W'_p(h_1)))\big] \tag{4.9}$$

(recall $W'_p(h)$ from (2.4)) can be written as

$$\frac{1}{2\pi}\int_0^{2\pi} \exp\left(\frac{\log p}{p^{1/2}}\Big\{\lambda_1 \sin\theta + \lambda_2(\sin(\theta - h_2\log p) - \sin(\theta - h_1 \log p))\Big\}\right) d\theta. \tag{4.10}$$

Since $\sin(\theta - \eta) = \sin(\theta)\cos(\eta) - \cos(\theta)\sin(\eta)$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(a\cos\theta + b\sin\theta) d\theta = I_0(\sqrt{a^2 + b^2}), \tag{4.11}$$

(Abramowitz & Stegun, 1964, 9.6.16, p.376), then (4.9) is equal to

$$I_0\left(\sqrt{\frac{(\log p)^2}{p}\left\{\begin{array}{l} \big(\lambda_1 + \lambda_2(\cos(h_2\log p) - \cos(h_1 \log p))\big)^2 \\ +\big(\lambda_2(\sin(h_1 \log p) - \sin(h_2 \log p))\big)^2 \end{array}\right\}}\right). \tag{4.12}$$

From (4.4), note that

$$\log(I_0(\sqrt{u})) = \frac{u}{4} - \frac{u^2}{64} + O(u^3), \quad u \in (-1, 1). \tag{4.13}$$

Also, note that

$$\begin{aligned} \sin(h_1\log p) - \sin(h_2 \log p) &= O(|h_2 - h_1|\log p), \\ \cos(h_2 \log p) - \cos(h_1\log p) &= O(|h_2 - h_1|\log p). \end{aligned} \tag{4.14}$$

If we put (4.9), (4.12), (4.13) and (4.14) together, we get, for $p$ large enough,

$$\begin{aligned} \log(4.9) &\leq \frac{(\log p)^2}{4p}\left\{\big(\lambda_1 + c\,\lambda_2|h_2 - h_1|\log p\big)^2 + \big(c\,\lambda_2|h_2 - h_1|\log p)\big)^2\right\} + \frac{\widetilde{c}}{p^2} \\ &\leq \frac{\lambda_1^2}{4}\frac{(\log p)^2}{p} + c\,\lambda_2|h_2 - h_1|\frac{(\log p)^3}{p} + c^2\lambda_2^2|h_2 - h_1|^2\frac{(\log p)^4}{p} + \frac{\widetilde{c}}{p^2}. \end{aligned} \tag{4.15}$$

To obtain the last inequality, we used the fact that $\lambda_1 \leq 4C$. After summing (4.15) over $2^r < \log p \leq 2^k$ and using Lemma Appendix A.1, we deduce

$$
\begin{aligned}
&\log \mathbb{E}\big[\exp(\lambda_1 X'_{r,k}(0) + \lambda_2(X'_{r,k}(h_2) - X'_{r,k}(h_1)))\big] \\
&\quad \leq \widetilde{c} + \frac{\lambda_1^2}{8}(2^{2k} - 2^{2r}) + c\,\lambda_2 |h_2 - h_1|\, 2^{3k} + c^2 \lambda_2^2 |h_2 - h_1|^2\, 2^{4k},
\end{aligned}
\tag{4.16}
$$

where the constants $c$ and $\widetilde{c}$ only depend on $C$. This is exactly (4.8). $\square$

We are now ready to prove Proposition 3.1. For $k \in \mathbb{N}_0$, recall that $\mathcal{H}_k \mathrel{\hat{=}} 2^{-3k}\mathbb{Z}$, so that $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \ldots \subseteq \mathcal{H}_k \subseteq \ldots \subseteq \mathbb{R}$ is a nested sequence of sets of equidistant points and $|\mathcal{H}_k \cap [0,1)| = 2^{3k}$.

*Proof of Proposition 3.1.* Without loss of generality, we may assume $h = 0$. We can also round $x$ up to the nearest larger integer and decrease $a$ so that we may assume that $x \in \mathbb{N}_0$ and $a \geq 1$. To see why this is possible, define the new values of $x$ and $a$ by $\widetilde{x} \mathrel{\hat{=}} \lceil x \rceil$ and $\widetilde{a} \mathrel{\hat{=}} a - \widetilde{x} + x$, respectively. Since $x + a = \widetilde{x} + \widetilde{a}$ and $x \leq \widetilde{x}$, and assuming that we can show (3.1) with $\widetilde{x}$ and $\widetilde{a}$, we would have

$$
\begin{aligned}
\mathbb{P}\bigg(\max_{h' : |h' - h| \leq 2^{-3k-1}} X'_{r,k}(h') \geq x + a, X'_{r,k}(h) \leq x\bigg) &\leq \mathbb{P}\bigg(\max_{h' : |h' - h| \leq 2^{-3k-1}} X'_{r,k}(h') \geq \widetilde{x} + \widetilde{a}, X'_{r,k}(h) \leq \widetilde{x}\bigg) \\
&\leq c\, \exp\left(-2\frac{\widetilde{x}^2}{2^{2k} - 2^{2r}} - \widetilde{c}\, \widetilde{a}^{3/2}\right) \\
&\leq c' \exp\left(-2\frac{x^2}{2^{2k} - 2^{2r}} - c'' a^{3/2}\right),
\end{aligned}
\tag{4.17}
$$

where the constants $c'$ and $c''$ only depend on $C$.

It remains to show (3.1) when $x \in \mathbb{N}_0$ and $a \geq 1$. We choose to adapt the chaining argument found in (Arguin *et al.*, 2017, Proposition 2.5). Define the events

$$
B_x \mathrel{\hat{=}} \{X'_{r,k}(0) \leq 0\} \quad \text{and} \quad B_q \mathrel{\hat{=}} \{X'_{r,k}(0) \in [x - q - 1, x - q]\}, \quad q \in \{0, 1, \ldots, x - 1\}. \tag{4.18}
$$

Note that the left-hand side of (3.1) is at most

$$
\sum_{q=0}^{x} \mathbb{P}\left(B_q \cap \left\{\max_{h' \in A}\{X'_{r,k}(h') - X'_{r,k}(0)\} \geq a + q\right\}\right), \tag{4.19}
$$

where $A = [-2^{-3k-1}, 2^{-3k-1}]$. Let $(h_i, i \in \mathbb{N}_0)$ be a sequence such that $h_0 = 0$, $h_i \in \mathcal{H}_{k+i} \cap A$, $\lim_{i\to\infty} h_i = h'$ and $|h_{i+1} - h_i| \in \{0, \frac{1}{8}2^{-3(k+i)}, \frac{2}{8}2^{-3(k+i)}, \frac{3}{8}2^{-3(k+i)}, \frac{4}{8}2^{-3(k+i)}\}$ for all $i$. Because the map $h \mapsto X'_{r,k}(h)$ is almost-surely continuous,

$$
X'_{r,k}(h') - X'_{r,k}(0) = \sum_{i=0}^{\infty}(X'_{r,k}(h_{i+1}) - X'_{r,k}(h_i)). \tag{4.20}
$$

Since $\sum_{i=0}^{\infty} \frac{1}{2(i+1)^2} \leq 1$, we have the inclusion of events,

$$
\{X'_{r,k}(h') - X'_{r,k}(0) \geq a + q\} \subseteq \bigcup_{i=0}^{\infty}\left\{X'_{r,k}(h_{i+1}) - X'_{r,k}(h_i) \geq \frac{a+q}{2(i+1)^2}\right\}. \tag{4.21}
$$

This implies that $\{\max_{h' \in A} X'_{r,k}(h') - X'_{r,k}(0) \geq a + q\}$ is included in

$$
\bigcup_{i=0}^{\infty} \bigcup_{\substack{h_1 \in \mathcal{H}_{k+i} \cap A \\ |h_2 - h_1| = \frac{j}{8}2^{-3(k+i)} \\ \text{for some } j \in \{1,2,3,4\}}} \left\{X'_{r,k}(h_2) - X'_{r,k}(h_1) \geq \frac{a+q}{2(i+1)^2}\right\}, \tag{4.22}
$$

where we have ignored the case $h_1 = h_2$ since the event $\{X'_{r,k}(h_2) - X'_{r,k}(h_1) \geq \frac{a+q}{2(i+1)^2}\}$ is the empty set. Because $|\mathcal{H}_{k+i} \cap A| \leq c\, 2^{3i}$, the $q$-th summand in (4.19) is at most,

$$
\sum_{i=0}^{\infty} c\, 2^{3i} \sup_{\substack{h_1 \in \mathcal{H}_{k+i} \cap A \\ |h_2 - h_1| = \frac{j}{8}2^{-3(k+i)} \\ \text{for some } j \in \{1,2,3,4\}}} \mathbb{P}\left(B_q \cap \left\{X'_{r,k}(h_2) - X'_{r,k}(h_1) \geq \frac{a+q}{2(i+1)^2}\right\}\right). \tag{4.23}
$$

Note that $a + q \le a + x \le 2^{6k}$ by assumption. Lemma 4.2 can thus be applied to get that (4.23) is at most

$$c\sum_{i=0}^{\infty} 2^{3i} \exp\left(-2\frac{(x-q-1)^2}{2^{2k}-2^{2r}} - \widetilde{c}\, 2^{3i}\frac{(a+q)^{3/2}}{(i+1)^3}\right) \le c' e^{-2\frac{(x-q-1)^2}{2^{2k}-2^{2r}} - \widetilde{c}(a+q)^{3/2}}. \tag{4.24}$$

Since $e^{-\widetilde{c}(a+q)^{3/2}} \le e^{-\widetilde{c}a^{3/2} - \widetilde{c}q^{3/2}}$, (4.19) is at most

$$\begin{aligned} c'\, e^{-\widetilde{c}a^{3/2}} \sum_{q=0}^{x} e^{-2\frac{(x-q-1)^2}{2^{2k}-2^{2r}} - \widetilde{c}q^{3/2}} &\le c'\, e^{-\frac{2x^2}{2^{2k}-2^{2r}} - \widetilde{c}a^{3/2}} \sum_{q=0}^{x} e^{4C(q+1) - \widetilde{c}q^{3/2}} \\ &\le c'' e^{-\frac{2x^2}{2^{2k}-2^{2r}} - \widetilde{c}a^{3/2}}, \end{aligned} \tag{4.25}$$

where we used the assumption $x \le C(2^{2k} - 2^{2r})$ to obtain the first inequality in (4.25). This proves (3.1). $\square$

*Proof of Proposition 3.2.* The left-hand side of (3.2) is at most

$$\mathbb{P}\big(X'_{r,k}(h) \ge x-2\big) + \mathbb{P}\begin{pmatrix} \max_{h':|h'-h|\le 2^{-3k-1}} X'_{r,k}(h') \ge (x-2)+2, \\ X'_{r,k}(h) \le x-2 \end{pmatrix} \tag{4.26}$$

The conclusion follows from Lemma 4.1 and Proposition 3.1 with $x-2$ in place of $x$ and $a = 2$. $\square$

## Appendix A. Technical lemma

**Lemma Appendix A.1.** *Let $m \ge 1$ and $1 \le P < Q$, then*

$$\left| \sum_{P<p\le Q} \frac{(\log p)^m}{p} - \left(\frac{(\log Q)^m}{m} - \frac{(\log P)^m}{m}\right) \right| \le D, \tag{A.1}$$

*where $D > 0$ is a constant that only depends on $m$.*

*Proof.* Without loss of generality, assume that $P \ge 2$. We use a standard form of the prime number theorem (Montgomery & Vaughan, 2007, Theorem 6.9) which states that

$$\#\{p \text{ prime} : p \le x\} = \int_2^x \frac{1}{\log u} du + R(x), \tag{A.2}$$

where $R(x) = O(xe^{-c\sqrt{\log x}})$, uniformly for $x \ge 2$. Using (A.2) and integration by parts, we have

$$\begin{aligned} \sum_{P<p\le Q} \frac{(\log p)^m}{p} &= \int_P^Q \frac{(\log u)^{m-1}}{u} du + \int_P^Q \frac{(\log u)^m}{u} dR(u) \\ &= \frac{(\log Q)^m}{m} - \frac{(\log P)^m}{m} + \frac{(\log Q)^m}{Q} R(Q) - \frac{(\log P)^m}{P} R(P) \\ &\quad - \int_P^Q \frac{(m-\log u)(\log u)^{m-1}}{u^2} R(u) du. \end{aligned} \tag{A.3}$$

By making the change of variable $z = c\sqrt{\log u}$ on the right-hand side of (A.3), note that

$$\left| \int_P^Q \frac{(m-\log u)(\log u)^{m-1}}{u^2} R(u) du \right| \le \widetilde{D} \int_0^\infty z^{2m+1} e^{-z} dz = \widetilde{D}\,\Gamma(2m+2), \tag{A.4}$$

where $\widetilde{D} > 0$ is a constant that only depends on $m$. This ends the proof. $\square$


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