



An equivariant tensor product on Mackey functors

Michael A. Hill^a, Kristen Mazur^{b,*}

^a University of California Los Angeles, Los Angeles, CA 90095, United States of America

^b Elon University, Elon, NC 27244, United States of America

ARTICLE INFO

Article history:

Received 1 August 2016

Received in revised form 12 March 2019

Available online 16 April 2019

Communicated by C.A. Weibel

MSC:

19A22; 55P91; 18G99; 20J05

Keywords:

Mackey functors

Tambara functors

ABSTRACT

For all subgroups H of a cyclic p -group G we define norm functors that build a G -Mackey functor from an H -Mackey functor. We give an explicit construction of these functors in terms of generators and relations based solely on the intrinsic, algebraic properties of Mackey functors and Tambara functors. We use these norm functors to define a monoidal structure on the category of Mackey functors where Tambara functors are the commutative ring objects.

© 2019 Elsevier B.V. All rights reserved.

1. Introduction

For a finite group G , a commutative G -ring spectrum has norm maps that are multiplicative versions of the transfer maps. These maps are not seen in ordinary G -spectra. Moreover, we see the algebraic shadows of these norm maps in the zeroth stable homotopy groups of commutative G -ring spectra: π_0 of a G -spectrum is a Mackey functor (see for example [8]), but if X is a commutative G -ring spectrum, then $\pi_0(X)$ is a Tambara functor [2]. Thus, it is a Mackey functor with a ring structure (i.e. a Green functor) and an extra class of norm maps that are the multiplicative analogues of the transfer maps.

In this paper we show that the relationship between Mackey functors and Tambara functors mirrors the relationship between G -spectra and commutative G -ring spectra. We define an equivariant symmetric monoidal structure on the category of Mackey functors under which Tambara functors are the commutative monoid objects. The category of Mackey functors is symmetric monoidal, but the commutative ring objects under the symmetric monoidal product are not Tambara functors. Notably, they do not have norm maps.

Hill and Hopkins have developed an appropriate notion of equivariant symmetric monoidal, calling it G -symmetric monoidal [3]. They then call the commutative monoid objects under a G -symmetric monoidal

* Corresponding author.

E-mail address: kmazur@elon.edu (K. Mazur).

structure the G -commutative monoids [3]. We provide formal definitions of these concepts in Section 5. Hill and Hopkins [3], Ullman [12], and Hoyer [4] have independently defined G -symmetric monoidal structures on the category $\mathcal{Mack}_G$ of G -Mackey functors. In this paper, for G a cyclic p -group we define a very explicit G -symmetric monoidal structure on $\mathcal{Mack}_G$. The key to this structure is new norm functors $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ for all subgroups H of G .

Main Theorem. *Let G be a cyclic p -group. For all subgroups H of G there is an explicit construction of a norm functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ that has the following properties.*

- a. N_H^G is isomorphic to the composition of functors $N_K^G N_H^K$ whenever $H < K < G$.
- b. N_H^G is strong symmetric monoidal.

The word “construction” is deliberate. Given an H -Mackey functor $\underline{M}$ we build a G -Mackey functor $N_H^G \underline{M}$ based only on the intrinsic properties of Mackey functors and Tambara functors. We then define the functor N_H^G via the map $\underline{M} \mapsto N_H^G \underline{M}$ and use the collection of these norm functors $\{N_H^G \text{ for all } H \leq G\}$ to prove the following theorem.

Theorem 1.1. *Let G be a cyclic p -group. There is a G -symmetric monoidal structure on $\mathcal{Mack}_G$ so that a G -Mackey functor is a G -commutative monoid if and only if it has the structure of a G -Tambara functor.*

Because we restrict the Main Theorem and Theorem 1.1 to cyclic p -groups we are able to explicitly describe the G -symmetric monoidal structure defined on $\mathcal{Mack}_G$. Thus, we can work with this construction to build intuition into how the structure operates.

In this regard our definition of the norm functors in the Main Theorem is analogous to the explicit definition of the tensor product. Even though defining the tensor product via the universal property is more elegant and at times easier to work with, the explicit definition helps us understand how the tensor product combines two vector spaces. Similarly, the explicit construction of norm functors in this paper allows us to understand what the G -symmetric monoidal structure actually does to the objects of $\mathcal{Mack}_G$. Moreover, while we feel strongly that we can extend our results at least to all finite abelian groups, many aspects would become so much more complicated that the results would no longer develop the reader’s intuition for Mackey and Tambara functors.

Even before the emergence of Tambara functors in equivariant stable homotopy theory, there was interest in developing a structure on $\mathcal{Mack}_G$ that supported Tambara functors as ring objects. During an open problem session at the 1996 Seattle Conference on Cohomology, Representations and Actions of Finite Groups T. Yoshida posed the problem of defining a tensor induction for Mackey functors that preserves tensor products of Mackey functors and satisfies Tambara’s axioms for multiplicative transfer [1]. The G -symmetric monoidal structure that we create is such a tensor induction.

We organize this paper into the following sections. In Section 2 we provide two definitions of Tambara functors. The first is Tambara’s original definition. The second is an axiomatic and constructive definition. We base the construction of the norm functors in the Main Theorem on the second definition. In Section 3 we build the norm functors, and in Section 4 we prove that these functors satisfy the Main Theorem. Finally, we prove Theorem 1.1 in Section 5.

2. Tambara functors

Let $\mathcal{Set}$ be the category of sets, and let $\mathcal{Set}_G^{Fin}$ be the category of finite G -sets. Given morphisms $f: X \rightarrow Y$ and $p: A \rightarrow X$ in $\mathcal{Set}_G^{Fin}$ we define the *dependent product* $\prod_f A$ by

$$\prod_f A = \left\{ (y, \sigma) \left| \begin{array}{l} y \in Y, \\ \sigma: f^{-1}(y) \rightarrow A \text{ is a map of sets,} \\ p \circ \sigma(x) = x \text{ for all } x \in f^{-1}(y) \end{array} \right. \right\}.$$

The group G acts on $\prod_f A$ by $\gamma(y, \sigma) = (\gamma y, \gamma \sigma)$ where $(\gamma \sigma)(x) = \gamma \sigma(\gamma^{-1}x)$.

Definition 2.1. [11] For all morphisms $f: X \rightarrow Y$ and $p: A \rightarrow X$ in $\mathcal{S}et_G^{Fin}$ the *canonical exponential diagram* generated by f and p is the commutative diagram below, where $h(y, \sigma) = y$, the map e is the evaluation map $e(x, (y, \sigma)) = \sigma(x)$ and f' is the pullback of f by h .

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{e} & X \times_Y \prod_f A \\ f \downarrow & & & & \downarrow f' \\ Y & \xleftarrow{h} & & & \prod_f A \end{array}$$

We call a diagram in $\mathcal{S}et_G^{Fin}$ that is isomorphic to a canonical exponential diagram an *exponential diagram*.

Definition 2.2. [11] A G -Tambara functor $\underline{S}$ is a triple $(S^*, S_*, S_\star)$ consisting of two covariant functors

$$S_*: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et$$

$$S_\star: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et$$

and one contravariant functor

$$S^*: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et$$

such that the following properties hold.

1. All functors have the same object function $X \mapsto \underline{S}(X)$, and each $\underline{S}(X)$ is a commutative ring.
2. For all morphisms $f: X \rightarrow Y$ in $\mathcal{S}et_G^{Fin}$ the map $S_*(f)$ is a homomorphism of additive monoids, $S_\star(f)$ is a homomorphism of multiplicative monoids and $S^*(f)$ is a ring homomorphism.
3. The pair (S^*, S_*) is a G -Mackey functor and $(S^*, S_\star)$ is a semi- G -Mackey functor.
4. (Distributive Law) If

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{e} & X' \\ f \downarrow & & & & \downarrow f' \\ Y & \xleftarrow{h} & & & Y' \end{array}$$

is an exponential diagram then the induced diagram below commutes.

$$\begin{array}{ccccc} \underline{S}(X) & \xleftarrow{p_*} & \underline{S}(A) & \xrightarrow{e^*} & \underline{S}(X') \\ f_* \downarrow & & & & \downarrow f'_* \\ \underline{S}(Y) & \xleftarrow{h_*} & & & \underline{S}(Y') \end{array}$$

Given a morphism $f: X \rightarrow Y$ in $\mathcal{S}et_G^{Fin}$, we call $S_*(f)$ a *transfer* map, $S^*(f)$ a *restriction* map and $S_*(f)$ a *norm* map. In particular, if $f: G/H \rightarrow G/K$ is a morphism between orbits, we denote the norm map $S_*(f)$ by N_H^K , the transfer map $S_*(f)$ by tr_H^K , and the restriction map $S^*(f)$ by res_H^K . Further, given an element $g \in G$ there is also a morphism $c_g: G/H \rightarrow G/gHg^{-1}$. Thus, we have an induced map $S_*(c_g)$ (which is equal to $S_*(c_g)$ and $S^*(c_{g^{-1}})$) that we will denote by c_g .

The norm maps are not additive, but there is an explicit formula for the norm of a sum in a Tambara functor. By Property 3 of Definition 2.2 Tambara functors convert disjoint unions of G -sets to direct products of commutative rings. Since every finite G -set can be written as a disjoint union of orbits we only need this formula for the norm maps N_H^K for all subgroups $H < K \leq G$. Moreover, since the norm map N_H^K in a G -Tambara functor $\underline{S}$ must agree with N_H^K in the K -Tambara functor that results from applying the forgetful functor $i_K^*: \mathcal{T}amb_G \rightarrow \mathcal{T}amb_K$ to $\underline{S}$, it suffices to only state the formula for the norm N_H^G of a sum for any subgroup H of G .

We will only use two cases of exponential diagrams, and for these it is helpful to use the following construction. Let H be a subgroup of G and let X be a finite H -set. Then the G -set $Map_H(G, X)$ is the set of all H -equivariant maps $q: G \rightarrow X$ with G -action $(g \cdot q)(g') = q(g'g)$ for all g' and g in G .

Proposition 2.3. *Let $f: G/H \rightarrow G/G$ be the crush map, and let $p: A \rightarrow G/H$ be a morphism in $\mathcal{S}et_G^{Fin}$. Then the dependent product $\prod_f A$ is canonically isomorphic to $Map_H(G, p^{-1}(eH))$.*

Proof. The category of finite G -sets over G/H is equivalent to $\mathcal{S}et_H^{Fin}$, and the pullback functor from the category of finite G -sets over G/G to the category of finite G -sets over G/H is canonically isomorphic to the restriction functor $\mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et_H^{Fin}$. Moreover, the functor $(A \xrightarrow{p} G/H) \mapsto (\prod_f A \rightarrow G/G)$ is right adjoint to the pullback functor [11]. And, the functor $X \mapsto Map_H(G, X)$ is right adjoint to the restriction functor. It follows that these two right adjoints are canonically isomorphic. Therefore, for all $p: A \rightarrow G/H$, the dependent product $\prod_f A$ is canonically isomorphic to $Map_H(G, p^{-1}(eH))$. $\square$

Theorem 2.4 (Tambara reciprocity for sums). *Let G be a finite group with subgroup H , and let $\underline{S}$ be a G -Tambara functor.*

1. *If $*_1 \amalg *_2$ is the disjoint union of two single point G -sets then the following diagram of rings commutes.*

$$\begin{array}{ccc} \underline{S}(G/H) & \xleftarrow{\nabla_*} \underline{S}(G/H) \oplus \underline{S}(G/H) & \xrightarrow{e_*} \underline{S}(G/H \times Map_H(G, *_1 \amalg *_2)) \\ N_H^G \downarrow & & \downarrow \pi_* \\ \underline{S}(G/G) & \xleftarrow{h_*} & \underline{S}(Map_H(G, *_1 \amalg *_2)) \end{array}$$

2. *For all a and b in $\underline{S}(G/H)$*

$$N_H^G(a + b) = N_H^G(a) + N_H^G(b) + T(-)$$

where $T(-)$ is a polynomial in transfers from proper subgroups, norms, and restrictions that is universally determined by G in the sense that it depends only on G and not on $\underline{S}$, a , or b . Moreover, $T(-)$ lands in the ideal generated by the transfers.

Proof. To prove part 1, given the fold map $\nabla: G/H \amalg G/H \rightarrow G/H$ and the crush map $f: G/H \rightarrow G/G$, by Proposition 2.3, the dependent product $\prod_f (G/H \amalg G/H)$ is canonically isomorphic to

$$Map_H(G, *_1 \amalg *_2) \cong Map(G/H, *_1 \amalg *_2).$$

Therefore, the diagram below is an exponential diagram, and so the diagram in part 1 of the above theorem commutes by the Distributive Law of Definition 2.2.

$$\begin{array}{ccccc}
 G/H & \xleftarrow{\nabla} & G/H \amalg G/H & \xleftarrow{e} & G/H \times \operatorname{Map}(G/H, *_1 \amalg *_2) \\
 f \downarrow & & & & \downarrow \pi \\
 G/G & \xleftarrow{h} & & & \operatorname{Map}(G/H, *_1 \amalg *_2)
 \end{array}$$

Next, given a and b in $\underline{S}(G/H)$, by part 1, $N_H^G \nabla_*(a, b) = h_* \pi_* e^*(a, b)$. Since $N_H^G \nabla_*(a, b) = N_H^G(a + b)$, we can develop the formula given in part 2 by determining $h_* \pi_* e^*(a, b)$.

We begin by examining the decomposition of $\operatorname{Map}(G/H, *_1 \amalg *_2)$ and $G/H \times \operatorname{Map}(G/H, *_1 \amalg *_2)$ into disjoint unions of G -orbits. We can rewrite this as

$$\operatorname{Map}(G/H, *_1 \amalg *_2) \cong G/G \amalg G/G \amalg \coprod_i Z_i$$

where each Z_i is of the form G/H_i for some proper subgroup H_i . This union is difficult to determine, however it suffices that it is universally determined by the group G , and none of the orbits Z_i are isomorphic to G/G . Further, $G/H \times \operatorname{Map}_H(G, *_1 \amalg *_2)$ is isomorphic to $G/H \amalg G/H \amalg \coprod_i (G/H \times Z_i)$, and so the composition $h_* \pi_* e^*$ is given below.

$$\begin{array}{c}
 \underline{S}(G/H) \oplus \underline{S}(G/H) \\
 \downarrow e^* \\
 \underline{S}(G/H) \oplus \underline{S}(G/H) \oplus \bigoplus_i \underline{S}(G/H \times Z_i) \\
 \downarrow \pi_* \\
 \underline{S}(G/G) \oplus \underline{S}(G/G) \oplus \bigoplus_i \underline{S}(Z_i) \\
 \downarrow h_* \\
 \underline{S}(G/G)
 \end{array}$$

The map e^* sends (a, b) to $(a, b, \mathbf{x})$ where $\mathbf{x}$ is an element in $\bigoplus_i \underline{S}(G/H \times Z_i)$ that is universally determined by G .

Next, we determine π_* . First, $\underline{S}(G/H \times Z_i)$ is isomorphic to $\coprod G/J_i$ where G/J_i is the largest orbit that contains both G/H and Z_i . Thus, let $f_i: \underline{S}(G/H \times Z_i) \rightarrow \underline{S}(Z_i)$ be the multiplication map composed with the appropriate norm map, and let $F: \bigoplus_i \underline{S}(G/H \times Z_i) \rightarrow \bigoplus_i \underline{S}(Z_i)$ be $\bigoplus_i f_i$. Then $\pi_* = N_H^G \oplus N_H^G \oplus F$, and hence $\pi_* e^*(a, b) = N_H^G(a) \oplus N_H^G(b) \oplus F(\mathbf{x})$.

Finally, h_* is the identity map on the first two summands and the appropriate transfer maps on the remaining summands. Therefore, $h_* \pi_* e^*(a, b)$ is

$$N_H^G(a) + N_H^G(b) + T(-)$$

where $T(-)$ is as given in the above theorem. $\square$

We explicitly describe the Tambara reciprocity relations for sums in a C_4 -Tambara functor in Example 2.13. In fact, when G is a finite *abelian* group we can determine more details regarding the polynomial

$T(-)$. Thus, if G is finite abelian, we can state an even more explicit formula for the norm of a sum in a G -Tambara functor.

Theorem 2.5 (Tambara reciprocity for sums when G is finite abelian). *Let G be a finite abelian group and let $\underline{S}$ be a G -Tambara functor. If $H < G$, then for all a and b in $\underline{S}(G/H)$*

$$N_H^G(a + b) = N_H^G(a) + N_H^G(b) + \sum_{H < K < G} \text{tr}_K^G \left(\sum_{k=1}^{i_K} N_H^K((\underline{ab})_k^K) \right) + \text{tr}_H^G(g_H(a, b))$$

where $g_H(a, b)$ is a polynomial in some of the $W_G(H)$ -conjugates of a and b , and each $(\underline{ab})_k^K$ is a monomial in some of the $W_G(K)$ -conjugates of a and b . These polynomials are universally determined by the group G in the sense that they depend only on G and not on $\underline{S}$. Each integer i_K is also universally determined by G .

Proof. Given a and b in $\underline{S}(G/H)$, by Theorem 2.4, $N_H^G(a + b) = N_H^G(a) + N_H^G(b) + T(-)$, but since G is finite abelian we can further determine $T(-)$ by developing a more concrete formula for the composition $h_*\pi_*e^*(a, b)$.

The entire analysis hinges on the structure of the dependent product, which as we showed in Proposition 2.3, is the G -set $\text{Map}(G/H, *_1 \amalg *_2)$. In order to make later formulae more transparent, we write the set $*_1 \amalg *_2$ as $\{a, b\}$ in all that follows.

Because G is an abelian group, the stabilizer of any element $f \in \text{Map}(G/H, \{a, b\})$ contains H . If $H < K$, then composition with the quotient map

$$G/H \rightarrow G/K$$

induces an isomorphism

$$\text{Map}(G/H, \{a, b\})^K \cong \text{Map}(G/K, \{a, b\}).$$

Thus, since i_K is the number of orbits of functions with stabilizer exactly K , the above isomorphism allows us to identify i_K as the following cardinality:

$$i_K = \left| \text{Map}(G/K, \{a, b\}) - \bigcup_{K \subsetneq K'} \text{Map}(G/K', \{a, b\}) \right| / [G : K].$$

This gives us our orbit decomposition

$$\text{Map}(G/H, \{a, b\}) \cong G/G \amalg G/G \amalg \coprod_{H < K < G} \coprod_{i=1}^{i_K} G/K.$$

Therefore, the composition $h_*\pi_*e^*$ is given below where $S = \underline{S}(G/H)$.

$$\begin{array}{c}
 S \oplus S \\
 \downarrow e^* \\
 S \oplus S \oplus \bigoplus_{H < K < G} \left[\bigoplus_{k=1}^{i_K} \left(\bigoplus_{|G/K|} S \right) \right] \oplus \bigoplus_{j=1}^{i_H} \left(\bigoplus_{|G/H|} S \right) \\
 \downarrow \pi_* \\
 \underline{S}(G/G) \oplus \underline{S}(G/G) \oplus \bigoplus_{H < K < G} \left(\bigoplus_{k=1}^{i_K} \underline{S}(G/K) \right) \oplus \bigoplus_{j=1}^{i_H} S \\
 \downarrow h_* \\
 \underline{S}(G/G)
 \end{array}$$

The map e^* sends (a, b) to a sequence of a 's and b 's, the order of which is determined as follows. The codomain of e^* is grouped by the orbits of $\text{Map}_H(G, \{a, b\})$, and each grouping is indexed by the elements of G/G , G/H , or G/K for $H < K < G$. For example, each $\bigoplus_{|G/H|} S$ corresponds to an orbit isomorphic to G/H , and thus we can write $\bigoplus_{|G/H|} S$ as

$$S_e \oplus S_{g_1} \oplus \cdots \oplus S_{g_{|G/H|-1}}$$

where the indexing elements $e, g_1, \dots, g_{|G/H|-1}$ are representatives for the elements of G/H . To define e^* , we choose a representative map $f: G \rightarrow \{a, b\}$ from each orbit. (The choice of representative does not matter.) We then determine the image under f of each of the indexing elements. For every indexing element g_i , if $f(g_i) = a$, then $e^*(a, b) = a$ on the corresponding summand, and if $f(g_i) = b$, then $e^*(a, b) = b$ on the summand. For example, the first summand corresponds to the constant function that sends all elements of G to a . Hence, $e^*(a, b) = a$ on that summand. Similarly, $e^*(a, b) = b$ on the second summand.

Next, π_* is a combination of norm maps and maps that send a direct sum to a product over a Weyl action. More specifically, let the map $\pi_H: \bigoplus_{|G/H|} \underline{S}(G/H) \rightarrow \underline{S}(G/H)$ be

$$(s_e, s_{g_1}, s_{g_2}, \dots, s_{g_{|G/H|-1}}) \mapsto s_e g_1 s_{g_1} g_2 s_{g_2} \cdots g_{|G/H|-1} s_{g_{|G/H|-1}}.$$

The map $\pi_K: \bigoplus_{|G/K|} \underline{S}(G/H) \rightarrow \underline{S}(G/H)$ is analogous. Then π_* is

$$N_H^G \oplus N_H^G \oplus \bigoplus_{H < K < G} \bigoplus_{k=1}^{i_K} [N_H^K \circ \pi_K]_k \oplus \bigoplus_{j=1}^{i_H} (\pi_H)_j,$$

where N_H^G and N_H^K are the norm maps.

Thus, if each $(\underline{ab})_j^K$ is as defined in the theorem and $(\underline{ab})_j^H$ is defined analogously, then $\pi_*e^*(a, b)$ equals

$$N_H^G(a) \oplus N_H^G(b) \oplus \bigoplus_{H < K < G} \left(\bigoplus_{k=1}^{i_K} N_H^K((\underline{ab})_k^K) \right) \oplus \bigoplus_{j=1}^{i_H} (\underline{ab})_j^H.$$

Finally, $h_*\pi_*e^*(a, b)$ is

$$N_H^G(a) + N_H^G(b) + \sum_{H < K < G} \text{tr}_K^G \left(\sum_{k=1}^{i_K} N_H^K((\underline{ab})_k^K) \right) + \text{tr}_H^G(g_H(a, b))$$

where $g_H(a, b) = \sum_{j=1}^{i_H} (\underline{ab})_j^H$ and thus is as defined in the above theorem. $\square$

We use the following corollary frequently in the upcoming sections.

Corollary 2.6 (Tambara reciprocity for sums when G is a cyclic p -group). Let $G = C_{p^n}$ be a cyclic p -group, let $H = C_{p^k}$ be a subgroup of G , and let $\underline{S}$ be a G -Tambara functor. Then for all a and b in $\underline{S}(G/H)$,

$$N_H^G(a + b) = N_H^G(a) + N_H^G(b) + \sum_{H < C_{p^j} < G} \text{tr}_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N_H^{C_{p^j}}((\underline{ab})_f) \right) + \text{tr}_H^G(g_H(a, b)),$$

where

$$\mathcal{I}_j = (\text{Map}(G/C_{p^j}, \{a, b\}) - \text{Map}(G/C_{p^{j+1}}, \{a, b\})),$$

where

$$(\underline{ab})_f = \prod_{gC_{p^j} \in G/C_{p^j}} gf(gC_{p^j}),$$

and where

$$g_H(a, b) = \sum_{f \in \mathcal{I}_k/G} (\underline{ab})_f = \sum_{f \in \mathcal{I}_k/G} \left(\prod_{gH \in G/H} gf(gH) \right).$$

We also have a formula for the norm of a transfer in a Tambara functor. As we did for the norm of a sum, we first provide a general formula that holds for all finite groups. We then give a more detailed formula for the norm of a transfer in a G -Tambara functor when G is finite abelian.

Theorem 2.7 (Tambara reciprocity for transfers). Let G be a finite group with subgroups $H' < H$, and let $\underline{S}$ be a G -Tambara functor.

1. The following diagram of rings commutes.

$$\begin{array}{ccccc} \underline{S}(G/H) & \xleftarrow{\text{tr}_{H'}^H} & \underline{S}(G/H') & \xrightarrow{e^*} & \underline{S}(G/H \times \text{Map}_H(G, H/H')) \\ N_H^G \downarrow & & & & \downarrow \pi_* \\ \underline{S}(G/G) & \xleftarrow{h_*} & & & \underline{S}(\text{Map}_H(G, H/H')) \end{array}$$

2. For all x in $\underline{S}(G/H')$,

$$N_H^G \text{tr}_{H'}^H(x) = T(-)$$

where $T(-)$ is an element in the subgroup of $\underline{S}(G/G)$ generated by all transfer terms. Moreover, it is universally determined by G .

Proof. By Proposition 2.3, the dependent product of the composition $G/H' \xrightarrow{p} G/H \xrightarrow{f} G/G$ is canonically isomorphic to $\text{Map}_H(G, H/H')$. It follows that the diagram below is an exponential diagram.

$$\begin{array}{ccccc}
G/H & \xleftarrow{p} & G/H' & \xleftarrow{e} & G/H \times \text{Map}_H(G, H/H') \\
f \downarrow & & & & \downarrow \pi \\
G/G & \xleftarrow{h} & & & \text{Map}_H(G, H/H')
\end{array}$$

Next we prove part 2 of the theorem. The G -set $\text{Map}_H(G, H/H')$ has no G -fixed points. Thus, there are no copies of G/G in its decomposition into G -orbits. This means that in the commutative diagram of rings given in part 1, the map h_* is a sum of various transfer maps. Hence, for all x in $\underline{S}(G/H')$, $h_*\pi_*e^*(x)$ is some element in the subgroup of $\underline{S}(G/G)$ generated by the transfer terms. $\square$

Theorem 2.8 (Tambara reciprocity for transfers when G is finite abelian). *Let G be a finite abelian group, and let $\underline{S}$ be a G -Tambara functor. For all subgroups $H' < H < G$ and all x in $\underline{S}(G/H')$*

$$N_H^G \text{tr}_{H'}^H(x) = \text{tr}_{H'}^G(f(x)) + \sum_{K \in \Omega} \left(\sum_{j=1}^{q_K} \text{tr}_K^G N_{H'}^K(\underline{x}_j) \right)$$

where Ω is the set of all subgroups K of G such that $H' = H \cap K$, $f(x)$ is a polynomial in some of the $W_G(H')$ -conjugates of x , and each $\underline{x}_j$ is a monomial in some of the $W_G(H')$ -conjugates of x . Moreover, $f(x)$, $\underline{x}_j$, and the integers q_K are universally determined by the group G .

Proof. Since G is abelian we can create a more complete picture of the composition $h_*\pi_*e^*(x)$ given in Theorem 2.7. In particular, we are now able to better describe the map π_* .

First, as a G -set, $\text{Map}_H(G, H/H')$ is isomorphic to

$$\coprod_{i=1}^r (G/H')_i \amalg \coprod_{K \in \Omega} \left(\coprod_{j=1}^{q_K} (G/K)_j \right).$$

We do not need to know the exact values for r and each q_K , since it suffices that they are universally determined by G . So, $G/H \times \text{Map}_H(G, H/H')$ is isomorphic to

$$\coprod_{i=1}^r \left(\coprod_{|G/H|} G/H' \right)_i \amalg \coprod_{K \in \Omega} \left[\coprod_{j=1}^{q_K} \left(\coprod_m G/H' \right)_j \right]$$

where $m = \frac{|G/K||G/H|}{|G/H'|}$.

Thus, $h_*\pi_*e^*$ becomes the composition below where $S = \underline{S}(G/H')$.

$$\begin{array}{c}
S \\
\downarrow e^* \\
\oplus_{i=1}^r \left(\oplus_{|G/H|} S \right) \oplus \oplus_{K \in \Omega} \left[\oplus_{j=1}^{q_K} \left(\oplus_m S \right) \right] \\
\downarrow \pi_* \\
\oplus_{i=1}^r S \oplus \oplus_{K \in \Omega} \left[\oplus_{j=1}^{q_K} \underline{S}(G/K) \right] \\
\downarrow h_* \\
\underline{S}(G/G)
\end{array}$$

To describe the map π_* let

$$\pi_i: \bigoplus_{|G/H|} S \rightarrow S$$

be the map that sends $(s_e, s_{g_1}, \dots, s_{g_{|G/H|-1}})$ to a specific product

$$g_{t_e} s_e g_{t_1} s_{g_1} \cdots g_{t_{|G/H|-1}} s_{g_{|G/H|-1}}$$

where $g_{t_i} s_{g_i}$ is a $W_G(H')$ -conjugate of s_{g_i} . While it is difficult to give further detail on this product of Weyl conjugates, it suffices that it is universally determined by the group G . Define $\pi_j: \bigoplus_m S \rightarrow S$ analogously. Then $\pi_* = \bigoplus_{i=1}^r \pi_i \oplus \bigoplus_{j=1}^{q_K} (N_{H'}^K \circ \pi_j)$.

The map e^* is the diagonal map and h_* is a composition of addition and transfer maps. Therefore,

$$N_H^G \text{tr}_{H'}^H(x) = h_* \pi_* e^*(x) = \text{tr}_{H'}^G(f(x)) + \sum_{K \in \Omega} \left(\sum_{j=1}^{q_K} \text{tr}_K^G N_{H'}^K(\underline{x}_j) \right)$$

where $f(x)$ is a polynomial in some of the $W_G(H')$ -conjugates of x , and each $\underline{x}_j$ is a monomial in some of the $W_G(H')$ -conjugates of x . $\square$

When G is a cyclic p -group, since all subgroups are nested, $H' = H \cap K$ if and only if $K = H'$. Thus, in this case the Tambara reciprocity formula for transfers simplifies nicely.

Corollary 2.9 (Tambara reciprocity for transfers when G is a cyclic p -group). *Let $G = C_{p^n}$ be a cyclic p -group with chosen generator γ , and let $H' < H$ be subgroups of G . Then for all x in $\underline{S}(G/H')$,*

$$N_H^G \text{tr}_{H'}^H(x) = \text{tr}_H^G(f(x))$$

where $f(x)$ is a polynomial in some of the Weyl conjugates of x . More specifically,

$$f(x) = \sum_{s=1}^r \prod_{i=0}^{|G/H|-1} \gamma^i \gamma^{m_{i,s}} x$$

where each $\gamma^{m_{i,s}} x$ is a $W_H(H')$ -conjugate of x and the integers r and $m_{i,s}$ are universally determined by G .

Fact 2.10. The following fact regarding Tambara reciprocity will allow us to create Tambara reciprocity-like relations in the Mackey functor $N_H^G \underline{M}$ that we define in the next section. Since we only use this fact when G is a cyclic p -group we restrict our attention to this case. It appears however that this fact holds for all finite abelian groups.

When G is a cyclic p -group the monomials of $g_H(a, b)$ and $f(x)$ and the monomials $(ab)_f$ given in Corollaries 2.6 and 2.9 do not contain repeated factors. If γ generates G , then for every γ^m the elements $\gamma^m a$ and $\gamma^m b$ appear at most once in any monomial in the formula for the norm of a sum, and it is impossible for both $\gamma^m a$ and $\gamma^m b$ to appear in the same monomial. Similarly, the element $\gamma^m x$ appears at most once in any monomial in the formula for the norm of a transfer. These facts follow from the formulas for the maps π_* in the proofs of Theorems 2.5 and 2.8.

Let G be a finite group and let $\mathcal{O}_G$ be the orbit category of G . Motivated by an analogous definition of Mackey functors from [14], we also define G -Tambara functors as follows.

Definition 2.11. Let G be a finite group. A G -Tambara functor $\underline{S}$ consists of a collection of commutative rings

$$\{\underline{S}(G/H) : G/H \in \mathcal{O}_G\}$$

along with the following maps for all orbits G/H and G/K such that $H < K$ and all g in G :

- the *restriction* map $\text{res}_H^K : \underline{S}(G/K) \rightarrow \underline{S}(G/H)$,
- the *transfer* map $\text{tr}_H^K : \underline{S}(G/H) \rightarrow \underline{S}(G/K)$,
- the *norm* map $N_H^K : \underline{S}(G/H) \rightarrow \underline{S}(G/K)$, and
- the *conjugation* map $c_g : \underline{S}(G/H) \rightarrow \underline{S}(G/gHg^{-1})$.

These rings and maps satisfy the following conditions.

1. All restriction maps are ring homomorphisms, all transfer maps are homomorphisms of additive monoids, and all norm maps are homomorphisms of multiplicative monoids.
2. (Transitivity) For all $H' < H < K$ and all g and h in G

$$\begin{aligned} \text{res}_{H'}^K &= \text{res}_{H'}^H \text{res}_H^K \\ \text{tr}_{H'}^K &= \text{tr}_H^K \text{tr}_{H'}^H \\ N_{H'}^K &= N_H^K N_{H'}^H \\ c_g c_h &= c_{gh}. \end{aligned}$$

3. (Frobenius Reciprocity) If $H < K$, then for all x in $\underline{S}(G/H)$ and y in $\underline{S}(G/K)$

$$y \text{tr}_H^K(x) = \text{tr}_H^K(\text{res}_H^K(y)x).$$

4. If $H < K$ and $g \in G$, then

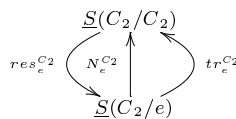
$$\begin{aligned} c_g \text{res}_H^K &= \text{res}_{gHg^{-1}}^{gKg^{-1}} c_g \\ c_g \text{tr}_H^K &= \text{tr}_{gHg^{-1}}^{gKg^{-1}} c_g \\ c_g N_H^K &= N_{gHg^{-1}}^{gKg^{-1}} c_g. \end{aligned}$$

5. For subgroups H' and H of K , let $[H \backslash K / H']$ denote a set of representatives in G for the double cosets $H \backslash K / H'$. Then

$$\begin{aligned} \text{res}_H^K \text{tr}_{H'}^K &= \sum_{g \in [H \backslash K / H']} \text{tr}_{H \cap gH'g^{-1}}^H c_g \text{res}_{gHg^{-1} \cap H'}^{H'} \\ \text{res}_H^K N_{H'}^K &= \prod_{g \in [H \backslash K / H']} N_{H \cap gH'g^{-1}}^H c_g \text{res}_{gHg^{-1} \cap H'}^{H'}. \end{aligned}$$

6. (Tambara Reciprocity) $\underline{S}$ satisfies Theorems 2.4 and 2.7.

If G is a finite group, and $\underline{S}$ is a G -Tambara functor as defined in Definition 2.2, showing that $\underline{S}$ satisfies Definition 2.11 is straightforward.

Fig. 1. $\underline{S}$ is a C_2 -Tambara functor.

Remark 2.12 (*Weyl actions*). Every $\underline{S}(G/H)$ is equipped with actions of the Weyl groups $W_K(H)$ for all subgroups K such that $H < K \leq G$. Here we state a few properties of these actions that we subsequently use to define the Mackey functor $N_H^G \underline{M}$. By Property 4 of Definition 2.11, for all g in $W_K(H)$, all x in $\underline{S}(G/H)$, and all y in $\underline{S}(G/K)$, $\text{res}_H^K(x) = \text{res}_H^K(x)$, $\text{tr}_H^K(gx) = \text{tr}_H^K(x)$ and $N_H^K(gx) = N_H^K(x)$. Moreover, when G is abelian, by Property 5,

$$\text{res}_H^K \text{tr}_H^K(x) = \sum_{g \in W_K(H)} gx \quad \text{and} \quad \text{res}_H^K N_H^K(x) = \prod_{g \in W_K(H)} gx.$$

We collect the properties of Definition 2.11 into a lattice-like diagram. For example, a C_2 -Tambara functor is pictured in Fig. 1. We do not draw the Weyl action.

Example 2.13. We can find explicit Tambara reciprocity formulas. For example, let $\underline{S}$ be a C_4 -Tambara functor, let a and b be elements in $\underline{S}(C_4/e)$, and let γ generate C_4 . Then

$$N_e^{C_4}(a+b) = N_e^{C_4}(a) + N_e^{C_4}(b) + \text{tr}_{C_2}^{C_4} \left(N_e^{C_2}((\underline{ab})_1^{C_2}) \right) + \text{tr}_e^{C_4}(g_e(a,b))$$

where $(\underline{ab})_1^{C_2} = a\gamma b$ and

$$g_e(a,b) = a\gamma a\gamma^2 a\gamma^3 b + b\gamma b\gamma^2 b\gamma^3 a + a\gamma b\gamma^2 b\gamma^3 a.$$

Further, for x in $\underline{S}(C_4/e)$, $N_{C_2}^{C_4} \text{tr}_e^{C_2}(x) = \text{tr}_e^{C_4}(f(x))$ where $f(x) = x\gamma x$.

3. Constructing the norm functors

From here on, let G be a cyclic group of order p^n with chosen generator γ . In this section, for a subgroup $H = C_{p^k}$ of G and H -Mackey functor $\underline{M}$, we build a G -Mackey functor $N_H^G \underline{M}$ that we will use to define the norm functor $N_H^G: \text{Mack}_H \rightarrow \text{Mack}_G$. Our Mackey functor norm will be built inductively, using the linear order of the subgroups of a cyclic p -group. This allows for a clean, conceptual description of the norm.

Our definition of $N_H^G \underline{M}$ is motivated by the explicit description of the symmetric monoidal product in the category of G -Mackey functors. This product is called the *box product* $\square$. The category theoretic definition of the box product can be found in [5] or [13], and while this definition is central to the theory of Mackey functors, it is difficult to actually compute the box product of two Mackey functors.

Lewis gives an explicit construction of the box product for C_p -Mackey functors, and this description immediately generalizes to other cyclic p -groups. Despite its constraint to C_p -Mackey functors, this definition allows us to visualize the box product of two Mackey functors. In a similar vein, even though we only define $N_H^G \underline{M}$ when G is a cyclic p -group, our construction provides insight into the norm functors that is otherwise difficult to see.

Definition 3.1. [[9], [6]] Given C_p -Mackey functors $\underline{M}$ and $\underline{L}$ we define their box product $\underline{M} \square \underline{L}$ by the diagram in Fig. 2.

The transfer map is the quotient map onto the second summand. The restriction map is induced from $\text{res}_e^{C_p} \otimes \text{res}_e^{C_p}$ on the first summand and by the trace of the Weyl action on the second summand. The Weyl

$$\begin{array}{c}
 \overbrace{[\underline{M}(C_p/C_p) \otimes \underline{L}(C_p/C_p) \oplus (\underline{M}(C_p/e) \otimes \underline{L}(C_p/e))/w_{C_p(e)}]}^{Im(tr_e^{C_p})} \\
 \downarrow \quad \quad \quad \uparrow \\
 \text{res}_e^{C_p} \quad \quad \quad tr_e^{C_p} \\
 \underline{M}(C_p/e) \otimes \underline{L}(C_p/e)
 \end{array}$$

Fig. 2. The box product of C_p -Mackey functors.

action on $\underline{M}(C_p/e) \otimes \underline{L}(C_p/e)$ is the diagonal action. Finally, FR is the *Frobenius reciprocity* submodule and is generated by all elements of the form

$$m' \otimes tr_e^{C_p}(l) - tr_e^{C_p}(res_e^{C_p}(m') \otimes l)$$

and

$$tr_e^{C_p}(m) \otimes l' - tr_e^{C_p}(m \otimes res_e^{C_p}(l'))$$

for all m' in $\underline{M}(C_p/C_p)$, l' in $\underline{L}(C_p/C_p)$, m in $\underline{M}(C_p/e)$, and l in $\underline{L}(C_p/e)$.

If G is a cyclic p -group C_{p^n} we can extend the above definition to the box product of G -Mackey functors as follows. Because all subgroups of G are nested, for any G -Mackey functor and any subgroups $H < C_{p^j} \leq G$, $tr_H^{C_{p^j}} = tr_{C_{p^{j-1}}}^{C_{p^j}} tr_H^{C_{p^{j-1}}}$. Therefore, if $\underline{M}$ and $\underline{L}$ are G -Mackey functors, we can inductively build $\underline{M} \square \underline{L}$ so that

$$(\underline{M} \square \underline{L})(G/C_{p^j}) = [\underline{M}(G/C_{p^j}) \otimes \underline{L}(G/C_{p^j}) \oplus \underbrace{(\underline{M} \square \underline{L})(G/C_{p^{j-1}})/w_{C_{p^j}(C_{p^{j-1}})}}_{Im(tr_{j-1}^j)}] / FR.$$

The restriction and transfer maps are identical to those given in Definition 3.1.

The above definition naturally extends to an m -fold box product for any positive integer m . It also gives us another way to view the universal property of the box product. This is equivalent to the usual description of the universal property; we find this a helpful reformulation. The proof is immediate from Lewis' description of the box product.

Proposition 3.2. [5,9] Let $\underline{M}$, $\underline{L}$, and $\underline{N}$ be G -Mackey functors. Maps $\underline{M} \square \underline{L} \rightarrow \underline{N}$ are in natural bijective correspondence with the following data: A collection of Weyl equivariant maps

$$f_j: \underline{M}(G/C_{p^j}) \otimes \underline{L}(G/C_{p^j}) \rightarrow \underline{N}(G/C_{p^j})$$

for all $0 \leq j \leq n$ that satisfy the following compatibility conditions.

1. For all j , we have

$$res_{j-1}^j \circ f_j = f_{j-1} \circ (res_{j-1}^j \otimes res_{j-1}^j),$$

2. for all j , we have

$$f_j \circ (tr_{j-1}^j \otimes Id) = tr_{j-1}^j \circ f_{j-1} \circ (Id \otimes res_{j-1}^j),$$

3. and for all j , we have

$$f_j \circ (Id \otimes tr_{j-1}^j) = tr_{j-1}^j \circ f_{j-1} \circ (res_{j-1}^j \otimes Id).$$

Alternatively, we can define $\underline{M} \square \underline{L}$ as a quotient of Mackey functors. If we let $\text{TENS} \underline{M} \underline{L}$ be the Mackey functor with

$$(\text{TENS} \underline{M} \underline{L})(G/K) = [\underline{M}(G/K) \otimes \underline{L}(G/K)] \oplus \text{Im}(tr)$$

for all subgroups K of G , then the Frobenius reciprocity submodules form a subMackey functor $\text{FR} \underline{M} \underline{L}$ of $\text{TENS} \underline{M} \underline{L}$. We can then define $\underline{M} \square \underline{L}$ by

$$\underline{M} \square \underline{L} := \text{TENS} \underline{M} \underline{L} / \text{FR} \underline{M} \underline{L}.$$

We play a similar game to define $N_H^G \underline{M}$. We start by building a “free” G -Mackey functor from an H -Mackey functor $\underline{M}$. Since we want $N_H^G \underline{M}$ to look like a Tambara functor we will quotient the “free” Mackey functor by a subMackey functor (analogous to $\text{FR} \underline{M} \underline{L}$) that creates Tambara reciprocity-like relations.

3.1. The “free” G -Mackey functor

Definition 3.3 (The “free” G -Mackey functor). Let $H = C_{p^k}$ be a subgroup of G and let $\underline{M}$ be an H -Mackey functor. Define the G -Mackey functor $\text{FREE}^G \underline{M}$ as follows.

For all subgroups H' of H define

$$(\text{FREE}^G \underline{M})(G/H') := \underline{M}^{\square |G/H|}(H/H'),$$

and if $H'' < H' \leq H$ then $res_{H''}^{H'}$ and $tr_{H''}^{H'}$ are the box product restriction and transfer maps given in Definition 3.1. The Weyl action is via tensor induction.

Now let $K = C_{p^j}$ be a subgroup such that $H < K \leq G$. Define $(\text{FREE}^G \underline{M})(G/K)$ inductively by

$$(\text{FREE}^G \underline{M})(G/K) := \mathbb{Z}\{\text{Map}(G/K, \underline{M}(H/H))\} \oplus \text{Im}(tr_{j-1}^j),$$

where

$$\text{Im}(tr_{j-1}^j) = ((\text{FREE}^G \underline{M})(G/C_{p^{j-1}})) / w_K(C_{p^{j-1}}).$$

The transfer map tr_{j-1}^j is the canonical quotient map onto that summand. The Weyl action is via the action on $\text{Map}(G/K, \underline{M}(H/H))$ by precomposition and the induced action on the other summand.

The restriction on $\text{Im}(tr_{j-1}^j)$ is defined by the universal formula

$$res_{j-1}^j tr_{j-1}^j(x) = \sum_{g \in W_{C_{p^j}}(C_{p^{j-1}})} gx$$

in a Mackey functor. On the “free” summand, when $j > k + 1$ we take

$$res_{j-1}^j : \text{Map}(G/K, \underline{M}(H/H)) \rightarrow \text{Map}(G/C_{p^{j-1}}, \underline{M}(H/H))$$

to be the map induced by the quotient map

$$G/C_{p^{j-1}} \rightarrow G/K.$$

Finally, the restriction map res_k^{k+1} on the free summand is given by the composite

$$\mathbb{Z}\{\text{Map}(G/C_{p^{k+1}}, \underline{M}(H/H))\} \rightarrow \mathbb{Z}\{\text{Map}(G/H, \underline{M}(H/H))\} \rightarrow \underline{M}(H/H)^{\otimes |G/H|} \rightarrow \underline{M}^{\square |G/H|}(H/H),$$

where the latter maps are the canonical maps.

Remark 3.4. There is a map $N: \underline{M}(H/H) \rightarrow (\text{FREE}^G \underline{M})(G/G)$ defined by the composition

$$\underline{M}(H/H) \rightarrow \mathbb{Z}\{\underline{M}(H/H)\} \rightarrow (\text{FREE}^G \underline{M})(G/G)$$

that sends an element in $\underline{M}(H/H)$ to the corresponding generator in the free summand in $(\text{FREE}^G \underline{M})(G/G)$. This map N will be the universal norm map, and in Section 5, we use it to define the internal norm maps of a Tambara functor. The restriction map on this summand (and the other factors) is defined to mirror the relation in a Tambara functor

$$\text{res}_{j-1}^j N_{j-1}^j(x) = \prod_{g \in W_{C_{p^j}}(C_{p^{j-1}})} gx.$$

Hence, with this idea in mind we will use the notation $N(-)$ to denote generators of the free summand.

In order to make some computations more explicit, we need to introduce some notation. Since whenever $H' < H$, $(\text{FREE}^G \underline{M})(G/H') = \underline{M}^{\square |G/H|}(H/H')$, we index $(\text{FREE}^G \underline{M})(G/H')$ by elements of G/H . Moreover,

$$\underline{M}^{\square |G/H|}(H/H') = \left(\underline{M}(H/H')^{\otimes |G/H|} \oplus \text{Im}(\text{tr}) \right) /_{FR},$$

and we denote a simple tensor in the $\underline{M}(H/H')^{\otimes |G/H|}$ summand of this module by

$$\mathbf{m}^{\otimes |G/H|} = m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma|G/H|-1}.$$

Similarly, $\text{Map}(G/K, \underline{M}(H/H))$ in the free summand of $(\text{FREE}^G \underline{M})(G/K)$ is the product over G/K and can be indexed by elements of G/K . We denote an element in $\text{Map}(G/K, \underline{M}(H/H))$ by

$$\mathbf{m}^{\times |G/K|} = m_e \times m_\gamma \times \cdots \times m_{\gamma|G/K|-1}.$$

Let $N(\mathbf{m}^{\times |G/K|})$ denote the corresponding generator of the free summand $\mathbb{Z}\{\text{Map}(G/K, \underline{M}(H/H))\}$ in $(\text{FREE}^G \underline{M})(G/K)$.

This gives us a universal property for $\text{FREE}^G \underline{M}$ that follows immediately from the universal property of the free abelian group functor.

Proposition 3.5. A map $\text{FREE}^G \underline{M} \rightarrow \underline{M}'$ is given by two pieces of data:

1. A map of Mackey functors $\underline{M}^{\square |G/H|} \rightarrow i_H^* \underline{M}'$, where the source is the analogue of tensor induction in Mackey functors, and for both sides, the Weyl action by $W_H(H')$ comes equipped with an extension to an action of $W_G(H')$,
2. for $K = C_{p^j}$ such that $H < K \leq G$, G/K -equivariant set maps

$$f_j: \text{Map}(G/K, \underline{M}(H/H)) \rightarrow \underline{M}'(G/K)$$

that satisfy the following properties.

(a) For each $k < j - 1 < n$, we have

$$\text{res}_{j-1}^j \circ f_j = f_{j-1} \circ \text{res}_{j-1}^j$$

(b) when $j - 1 = k$ we have

$$\begin{array}{ccc} \text{Map}(G/C_{p^{k+1}}, \underline{M}(H/H)) & \xrightarrow{f_{k+1}} & \underline{M}'(G/C_{p^{k+1}}) \\ \downarrow & & \downarrow \\ \underline{M}^{\square|G/H|}(H/H) & \longrightarrow & \underline{M}'(H/H). \end{array}$$

3.2. The Tambara reciprocity subMackey functor $\text{TR}^G \underline{M}$

As discussed in Remark 3.4, in Section 5 we will use $N_H^G \underline{M}$ to define the internal norm maps of a Tambara functor, so $N_H^G \underline{M}$ must reflect the Tambara reciprocity property of Definition 2.11. Hence, we will define $N_H^G \underline{M}$ by taking the quotient of $\text{FREE}^G \underline{M}$ with a subMackey functor to produce relations that mimic the Tambara reciprocity relations of a Tambara functor. Thus, we need to quotient $\text{FREE}^G \underline{M}$ by a subMackey functor in order to identify the analogues of “the norm of a sum” and “the norm of a transfer” in the free summand with appropriate elements in the transfer summand. We will call this subMackey functor the *Tambara reciprocity subMackey functor* $\text{TR}^G \underline{M}$.

We will define $\text{TR}^G \underline{M}$ inductively. We begin with a lemma identifying $\text{FREE}^G \underline{M}$.

Lemma 3.6. For any $K = C_{p^j}$ such that $H < K \leq G$ we have a natural isomorphism

$$i_K^* \text{FREE}^G \underline{M} \cong (\text{FREE}^K \underline{M})^{\square|G/K|}.$$

Proof. We will use induction on the order of the group. First, by definition of $\text{FREE}^G \underline{M}$, for $H' \leq H$, $(i_K^* \text{FREE}^G \underline{M})(K/H')$ is isomorphic to $(\text{FREE}^K \underline{M})^{\square|G/K|}(K/H')$. Then by induction it remains to show that $(i_K^* \text{FREE}^G \underline{M})(K/K)$ is isomorphic to $(\text{FREE}^K \underline{M})^{\square|G/K|}(K/K)$. First, we have

$$(i_K^* \text{FREE}^G \underline{M})(K/K) = \mathbb{Z}\{\text{Map}(G/K, \underline{M}(H/H))\} \oplus \text{Im}(tr_{j-1}^j),$$

and

$$(\text{FREE}^K)^{\square|G/K|}(K/K) = \left(\left(\mathbb{Z}\{\text{Map}(K/K, \underline{M}(H/H))\} \oplus \text{Im}(tr_{j-1}^j) \right)^{\otimes|G/K|} \oplus \text{Im}^{\square}(tr_{j-1}^j) \right) /_{FR}.$$

Then by Frobenius reciprocity we identify all transfer terms of

$$\left(\mathbb{Z}\{\text{Map}(K/K, \underline{M}(H/H))\} \oplus \text{Im}(tr_{j-1}^j) \right)^{\otimes|G/K|}$$

with elements of $\text{Im}^{\square}(tr_{j-1}^j)$. Thus, $(\text{FREE}^K)^{\square|G/K|}(K/K)$ is isomorphic to

$$\mathbb{Z}\{\text{Map}(K/K, \underline{M}(H/H))\}^{\otimes|G/K|} \oplus \text{Im}^{\square}(tr_{j-1}^j).$$

Then $\mathbb{Z}\{\text{Map}(K/K, \underline{M}(H/H))\}^{\otimes|G/K|}$ is isomorphic to $\mathbb{Z}\{\text{Map}(G/K, \underline{M}(H/H))\}$, and by induction $\text{Im}^{\square}(tr_{j-1}^j)$ is isomorphic to the $\text{Im}(tr_{j-1}^j)$ summand of $(i_K^* \text{FREE}^G \underline{M})(K/K)$. Therefore, $(\text{FREE}^K)^{\square|G/K|}(K/K)$ is isomorphic to $(i_K^* \text{FREE}^G \underline{M})(K/K)$. $\square$

Our definition of $\mathrm{tr}^G \underline{M}$ uses a similar inductive description: we build $\mathrm{tr}^G \underline{M}$ so that for $H < K \leq G$ we have an isomorphism

$$i_K^* \mathrm{FREE}^G \underline{M} / \mathrm{tr}^G \underline{M} \cong (\mathrm{FREE}^K \underline{M} / \mathrm{tr}^K \underline{M})^{\square^{|G/K|}}.$$

We start by defining the elements that generate the submodule $(\mathrm{tr}^G \underline{M})(G/G)$ of $(\mathrm{FREE}^G \underline{M})(G/G)$. These generators are of two types, one of which will create relations like Tambara reciprocity for sums in $N_H^G \underline{M}$ and the other will create relations like Tambara reciprocity for transfers.

To develop the first type of generators, recall by Corollary 2.6 that when G is a cyclic p -group, the Tambara reciprocity formula for sums is given by

$$N_H^G(a+b) = N_H^G(a) + N_H^G(b) + \sum_{H < C_{p^j} < G} \mathrm{tr}_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N_H^{C_{p^j}}((\underline{ab})_f) \right) + \mathrm{tr}_H^G(g_H(a, b)).$$

Thus, we will develop elements in $(\mathrm{FREE}^G \underline{M})(G/G)$ that are analogous to these formulas. We have grouped the various terms so that they match the summands which show up in $\mathrm{FREE}^G \underline{M}$. Thus, for any a and b in $\underline{M}(H/H)$ we need to start with the element

$$N(a+b) - N(a) - N(b)$$

in the free summand of $(\mathrm{FREE}^G \underline{M})(G/G)$ and subtract elements in the transfer summand that are analogous to each $\mathrm{tr}_{C_{p^j}}^G(N_H^{C_{p^j}}((\underline{ab})_f))$ (which arise from the transfers of the other free summands) and to $\mathrm{tr}_H^G(g_H(a, b))$ (which comes from the box powers of $\underline{M}$).

We will first construct an element analogous to $(\underline{ab})_f$.

Definition 3.7. Let $f \in \mathcal{I}_j/G$, and let $K = C_{p^j}$. The map $\{a, b\} \rightarrow \underline{M}(H/H)$ then defines an element

$$(\underline{ab})_f = \prod_{gK \in G/K} f(gK) = f(eK) \times \cdots \times f(\gamma^{|G/K|-1}K) \in \mathrm{Map}(G/K, \underline{M}(H/H)).$$

Let $N((\underline{ab})_f)$ be the corresponding element in the free summand of $(\mathrm{FREE}^G \underline{M})(G/K)$.

By Fact 2.10 we can define an element in $(\mathrm{FREE}^G \underline{M})(G/H)$ that is analogous to $g_H(a, b)$.

Definition 3.8. Define $g_H(\underline{a}, \underline{b})$ in $\underline{M}(H/H)^{\otimes |G/H|}$ by

$$g_H(\underline{a}, \underline{b}) = \sum_{f \in \mathcal{I}_k/G} \left(\bigotimes_{gH \in G/H} f(gH) \right) = \sum_{f \in \mathcal{I}_k/G} \left(f(eH) \otimes \cdots \otimes f(\gamma^{|G/H|-1}H) \right).$$

Thus, one type of generator of $(\mathrm{tr}^G \underline{M})(G/G)$ is of the form

$$N(a+b) - N(a) - N(b) - \sum_{H < C_{p^j} < G} \mathrm{tr}_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N((\underline{ab})_f) \right) - \mathrm{tr}_H^G(g_H(\underline{a}, \underline{b})).$$

Hence, when we quotient by $\mathrm{tr}^G \underline{M}$ to define $N_H^G \underline{M}$ we have the following relation in $(N_H^G \underline{M})(G/G)$.

$$N(a+b) = N(a) + N(b) + \sum_{H < C_{p^j} < G} tr_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N((\mathbf{a}\mathbf{b})_f) \right) + tr_H^G(g_H(\mathbf{a}, \mathbf{b}))$$

Similarly, we want to mimic the Tambara reciprocity formula for transfers given in Corollary 2.9;

$$N_H^G tr_{H'}^H(x) = tr_{H'}^G(f(x))$$

where

$$f(x) = \sum_{s=1}^r \prod_{i=0}^{|G/H|-1} \gamma^i \gamma^{m_{i,s}} x.$$

Definition 3.9. Define $f(\mathbf{x})$ in $\underline{M}(H/H')^{\otimes |G/H|}$ by

$$f(\mathbf{x}) = \sum_{s=1}^r \gamma^{m_{0,s}} x \otimes \gamma^{m_{1,s}} x \otimes \cdots \otimes \gamma^{m_{|G/H|-1,s}} x.$$

So, $tr_{H'}^G(f(\mathbf{x}))$ is in the transfer summand of $(\text{FREE}^G \underline{M})(G/G)$.

Hence, the other type of generator of $(\text{TR}^G \underline{M})(G/G)$ is of the form

$$N(tr_{H'}^H(x)) - tr_{H'}^G(f(\mathbf{x})),$$

so that in $(N_H^G \underline{M})(G/G)$ we have the relation

$$N(tr_{H'}^H(x)) = tr_{H'}^G(f(\mathbf{x})).$$

We now define the Tambara reciprocity sub-Mackey functor $\text{TR}^G \underline{M}$. For now it is an abuse of terminology to call $\text{TR}^G \underline{M}$ a Mackey functor. However, we prove that it is in fact a Mackey functor in Theorem 3.12.

Definition 3.10 (*The Tambara reciprocity sub-Mackey functor, $\text{TR}^G \underline{M}$*). We define $\text{TR}^G \underline{M}$ as follows.

For $H' \leq H$ define $(\text{TR}^G \underline{M})(G/H') = 0$.

If $H < K < G$ then we inductively define $(\text{TR}^G \underline{M})(G/K)$ by

$$(\text{TR}^G \underline{M})(G/K) = \sum_{i=1}^{|G/K|} \left((\text{FREE}^K \underline{M})^{\square i-1} \square \text{TR}^K \underline{M} \square (\text{FREE}^K \underline{M})^{\square |G/K|-i} \right) (K/K)$$

If $K = G$, then let $\text{TR}^G \underline{M}(G/G)$ be the subgroup generated by

1. the “addition relations”:

$$N(a+b) - N(a) - N(b) - \sum_{H < C_{p^j} < G} tr_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N((\mathbf{a}\mathbf{b})_f) \right) - tr_H^G(g_H(\mathbf{a}, \mathbf{b}))$$

for all $a, b \in \underline{M}(H/H)$,

2. the “transfer relations”:

$$N(tr_{H'}^H(x)) - tr_{H'}^G(f(\mathbf{x}))$$

for all $x \in \underline{M}(H/H')$, and

3. the transfer from smaller subgroups:

$$Im\left(tr_{G'}^G((\text{TR}^G \underline{M})(G/G'))\right),$$

where G' is the maximal subgroup of G .

Remark 3.11. The values of $\text{TR}^G \underline{M}(G/K)$ with K a proper subgroup are so that we have a copy of the Tambara reciprocity relations for each box factors in $i_K^* \text{FREE}^G \underline{M}$.

We now show that $\text{TR}^G \underline{M}$ is a Mackey functor.

Theorem 3.12. *The collection of submodules $\text{TR}^G \underline{M}$ given in Definition 3.10 forms a subMackey functor of $\text{FREE}^G \underline{M}$.*

Proof. We will use induction on the order of the group. The base case holds since for $H' \leq H$, $(\text{TR}^G \underline{M})(G/H')$ is zero. Assume for induction that for all K such that $H < K < G$, $\text{TR}^K \underline{M}$ is a Mackey functor. Since each $\text{TR}^K \underline{M}$ is a Mackey functor for $H < K < G$, it follows that $\text{TR}^G \underline{M}$ is well-defined. Further, by definition, $\text{TR}^G \underline{M}$ is closed under the transfer maps and under the Weyl action, since this action is just the permutation action. Thus, it remains to show that $\text{TR}^G \underline{M}$ is closed under the restriction maps. In particular, let G' be the maximal subgroup of G . We will show that $\text{res}_{G'}^G((\text{TR}^G \underline{M})(G/G))$ is contained in $(\text{TR}^G \underline{M})(G/G')$.

From the formula

$$\text{res}_{G'}^G tr_{G'}^G(x) = \sum_{gG' \in G/G'} gx,$$

we know that the restriction of the elements in the image of the transfer lands in the desired submodule. We need only check the “addition” generators and the “transfer” generators. These are very similar, combinatorial arguments, the main difficulty of which is in good, consistent notation. We spell the argument out carefully for the addition generators; the transfer case is similar enough that including it is unenlightening.

We now begin considering the “addition relations”

$$N(a+b) - N(a) - N(b) - \sum_{H < C_{p^j} < G} tr_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N((\mathbf{a}\mathbf{b})_f) \right) - tr_H^G(g_H(\mathbf{a}, \mathbf{b})).$$

By definition,

$$\begin{aligned} N(a) + N(b) + \sum_{H < C_{p^j} < G} tr_{C_{p^j}}^G \left(\sum_{f \in \mathcal{I}_j/G} N((\mathbf{a}\mathbf{b})_f) \right) + tr_H^G(g_H(\mathbf{a}, \mathbf{b})) \\ = \sum_{f \in \text{Map}(G/H, \{a, b\})/G} tr_{K_f}^G \left(\bigotimes_{i=0}^{|G/K_f|-1} f(\gamma^i K_f) \right) \end{aligned}$$

where each K_f is the stabilizer of f . Thus, we need to show that

$$\text{res}_{G'}^G(N(a+b)) = \text{res}_{G'}^G \left(\sum_{f \in \text{Map}(G/H, \{a, b\})/G} tr_{K_f}^G \left(\bigotimes_{i=0}^{|G/K_f|-1} f(\gamma^i K_f) \right) \right)$$

$$= \sum_{f \in \text{Map}(G/H, \{a, b\})/_{G'}} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G/K_f|-1} f(\gamma^i K_f) \right).$$

By definition of the restriction in $\text{FREE}^G \underline{M}$, we have

$$\text{res}_{G'}^G(N(a+b)) = (N(a+b))^{\otimes |G/G'|},$$

and by our induction hypothesis, in $(\text{TR}^G \underline{M})(G/G')$ we have

$$\begin{aligned} N(a+b) &= N(a) + N(b) + \sum_{H < C_{p^j} < G'} \text{tr}_{C_{p^j}}^{G'} \left(\sum_{f \in \mathcal{I}_j/G'} N((\mathbf{a}\mathbf{b})_f) \right) + \text{tr}_H^{G'}(g_H(\mathbf{a}, \mathbf{b})) \\ &= \sum_{f \in \text{Map}(G'/H, \{a, b\})/_{G'}} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G'/K_f|-1} f(\gamma^{ip} K_f) \right). \end{aligned}$$

Combining these, we have

$$\text{res}_{G'}^G(N(a+b)) = \left(\sum_{f \in \text{Map}(G'/H, \{a, b\})/_{G'}} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G'/K_f|-1} f(\gamma^{ip} K_f) \right) \right)^{\otimes |G/G'|}.$$

Therefore, we need to show the following equality.

$$\begin{aligned} \sum_{f \in \text{Map}(G/H, \{a, b\})/_{G'}} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G/K_f|-1} f(\gamma^i K_f) \right) \\ = \left(\sum_{f \in \text{Map}(G'/H, \{a, b\})/_{G'}} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G'/K_f|-1} f(\gamma^{ip} K_f) \right) \right)^{\otimes |G/G'|} \end{aligned} \quad (3.1)$$

We pause here to draw the reader's attention to the apparent asymmetry in the exponents of γ on the two sides of the equation. This is because since γ is a chosen generator of G , the element γ^p is then a chosen generator for G' . The left-hand side is the restriction of a formula from G , and hence uses γ ; the right-hand side is the corresponding formula for G' and hence uses its generator γ^p .

Here the description in terms of the box product is most useful. The tensor product on the right-hand side of Equation 3.1 distributes over the sum via the usual formula:

$$\begin{aligned} \left(\sum_{f \in \text{Map}(G'/H, \{a, b\})/_{G'}} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G'/K_f|-1} f(\gamma^{ip} K_f) \right) \right)^{\otimes |G/G'|} = \\ \sum_{\vec{f} \in (\text{Map}(G'/H, \{a, b\})/_{G'})^{\times p}} \bigotimes_{j=0}^{p-1} \left(\text{tr}_{K_{f_j}}^{G'} \left(\bigotimes_{i=0}^{|G'/K_{f_j}|-1} f_j(\gamma^{ip} K_{f_j}) \right) \right), \end{aligned} \quad (3.2)$$

where f_i is the i th factor of $\vec{f}$.

Frobenius reciprocity allows us to rewrite the right-hand side of Equation 3.2. For each $\vec{f} \in (\text{Map}(G'/H, \{a, b\})/_{G'})^{\times p}$, let K_f be the intersection of the stabilizers K_{f_i} . Since the subgroups are nested

and since we have only p terms, $K_f = K_{f_j}$ for some j , and without loss of generality, we may assume that $K_f = K_{f_0}$ (otherwise, we simply make the obvious bookkeeping change). Frobenius reciprocity shows that we have an equality

$$\bigotimes_{j=0}^{p-1} \left(\text{tr}_{K_{f_j}}^{G'} \left(\bigotimes_{i=0}^{|G'/K_{f_j}|-1} f_j(\gamma^{pi} K_{f_j}) \right) \right) = \text{tr}_{K_f}^{G'} \left(\left(\bigotimes_{i=0}^{|G'/K_{f_0}|-1} f_0(\gamma^{pi} K_{f_0}) \right) \otimes \bigotimes_{j=1}^{p-1} \left(\sum_{i=0}^{[G':K_{f_j}]-1} \gamma^{pi} \text{res}_{K_f}^{K_{f_j}} \left(\bigotimes_{k=0}^{|G'/K_{f_j}|-1} f_j(\gamma^{pk} K_{f_j}) \right) \right) \right). \quad (3.3)$$

By construction,

$$\text{res}_{K_f}^{K_j} \left(\bigotimes_{k=0}^{|G'/K_{f_j}|-1} f_j(\gamma^{pk} K_{f_j}) \right) = \bigotimes_{k=0}^{|G'/K_f|-1} f_j(\gamma^{pk} K_f),$$

and the action of γ^{pi} simply rotates the tensor factors as tensor induction.

We can further simplify the right-hand side of Equation 3.3 by noticing that there is a coordinate-wise action of $G'^{\times p}$ on $\text{Map}(G'/H, \{a, b\})^{\times p}$. We see that in the i th factor of our formula, we are summing over the orbit G'/K_{f_i} . There is an apparent asymmetry here too, because the first factor appears in isolation. However, if we remember that the transfer factors through the Weyl action then we notice that the sum can be taken over the set

$$\mathcal{G}_f = \left(\prod_{i=0}^{p-1} G'/K_{f_i} \right) / G'.$$

If $\vec{g} \in \mathcal{G}_f$, then let g_i be the i th coordinate of the representative where the first coordinate is 1. We can then rewrite the right-hand side of Equation 3.3 as

$$\sum_{\vec{g} \in \mathcal{G}_f} \text{tr}_{K_f}^{G'} \bigotimes_{i=0}^{p-1} \left(g_i \bigotimes_{j=0}^{|G'/K_f|-1} f_i(\gamma^{pj} K_f) \right). \quad (3.4)$$

This gives us a refined version of what we need to show, so now we need to show

$$\sum_{f \in \text{Map}(G/H, \{a, b\})/G'} \text{tr}_{K_f}^{G'} \left(\bigotimes_{i=0}^{|G/K_f|-1} f(\gamma^i K_f) \right) = \sum_{\vec{f} \in (\text{Map}(G'/H, \{a, b\})/G')^{\times p}} \sum_{\vec{g} \in \mathcal{G}_f} \text{tr}_{K_f}^{G'} \bigotimes_{i=0}^{p-1} \left(g_i \bigotimes_{j=0}^{|G'/K_f|-1} f_i(\gamma^{pj} K_f) \right). \quad (3.5)$$

The argument now is combinatorial. It is helpful to think about the indexing set

$$\text{Map}(G/H, \{a, b\}),$$

both as a G -set and as a G' -set. First, the cosets G/G' partition G/H into p copies of G'/H , indexed as G'/H , $\gamma G'/H$, $\dots$, $\gamma^{p-1} G'/H$. This partitioning is not G -equivariant, but since G is abelian, it is G' -equivariant, and this gives us a G' -equivariant isomorphism

$$G/H \cong \{0, \dots, p-1\} \times G'/H.$$

Using this isomorphism, we have G' -equivariant isomorphisms

$$\text{Map}(G/H, \{a, b\}) \cong \text{Map}(\{0, \dots, p-1\} \times G'/H, \{a, b\}) \cong (\text{Map}(G'/H, \{a, b\}))^{\times p}.$$

Thus, given a function $f: G/H \rightarrow \{a, b\}$, these isomorphisms allows us to write f as

$$f \leftrightarrow \prod_{i=0}^{p-1} f_i = (f_0, \dots, f_{p-1}) =: \vec{f},$$

where f_i is the restriction of f to the coset $\gamma^i G'/H$.

Although the isomorphisms used are not G -equivariant, we can still describe the action of γ on this product:

$$\gamma(f_0, \dots, f_{p-1}) = (f_1, \dots, f_{p-1}, \gamma^p f_0).$$

We can use this to determine the stabilizer of f . We know that the only G -fixed functions are the constant ones, and here, that means that each f_i is constant and they all agree. The G' -fixed functions come from those sequences $\vec{f}$, where each f_i is a constant function. Finally, for a non-constant function, the stabilizer of $f = (f_0, \dots, f_{p-1})$ is just

$$K_f = \text{Stab}(f) = \bigcap_{i=0}^{p-1} K_{f_i},$$

where K_{f_i} is the stabilizer of f_i .

The stabilizer of $\vec{f}$ in $(G')^{\times p}$ is

$$K_{\vec{f}} = K_{f_0} \times \dots \times K_{f_{p-1}}.$$

This means that using the coordinate-wise action of G' , each (non-constant) p -tuple $\vec{f}$ contributes

$$|(G')^{\times p} / K_{\vec{f}}| = \prod_{i=0}^{p-1} [G' : K_{f_i}]$$

distinct functions to $\text{Map}(G/H, \{a, b\})$. Putting this together, we see that the two sides of Equation 3.5 are giving the two ways to express this sum over $\text{Map}(G/H, \{a, b\})$, completing the proof. $\square$

Definition 3.13. Let M be an H -Mackey functor. Define $N_H^G \underline{M}$ by

$$N_H^G \underline{M} := \text{FREE}^G \underline{M} / \text{TR}^G \underline{M}.$$

Lemma 3.14. For $H < K < G$, $i_K^* N_H^G \underline{M}$ is isomorphic to $(N_H^K \underline{M})^{\square[G/K]}$.

Proof. This result follows directly from Lemma 3.6 and from the fact that the box powers of a quotient is the quotient of box powers. $\square$

4. Proof of the Main Theorem

In this section we verify that the construction $N_H^G \underline{M}$ satisfies the Main Theorem. Recall that a morphism $\phi: \underline{M} \rightarrow \underline{L}$ of Mackey functors consists of a collection of homomorphisms $\{\phi_H: \underline{M}(G/H) \rightarrow \underline{L}(G/H) : H \leq G\}$ that commute with the appropriate restriction and transfer maps.

Theorem 4.1. *For all subgroups H of G , the map $N_H^G: \text{Mack}_H \rightarrow \text{Mack}_G$ given by $\underline{M} \mapsto N_H^G \underline{M}$ is a functor.*

Proof. Given a morphism $\phi: \underline{M} \rightarrow \underline{L}$ in Mack_H we define the associated morphism $N_H^G(\phi): N_H^G \underline{M} \rightarrow N_H^G \underline{L}$ in Mack_G as follows.

For all subgroups H' in H define $N_{H'}^G(\phi)_{H'}$ to be $\phi_{H'}^{\square|G/H|}$. If $K = C_{p^j}$ is a subgroup such that $H < K \leq G$, then we inductively define $N_H^G(\phi)_K$ so that it is compatible with the appropriate restriction and transfer maps. More specifically, for all $tr(x)$ in the $Im(tr_{j-1}^j)$ summand of $(N_H^G \underline{M})(G/K)$ we define $N_H^G(\phi)_K(tr(x))$ to be

$$tr_{j-1}^j(N_H^G(\phi)_{C_{p^{j-1}}}(x)).$$

If $N(\mathbf{m}^{\times|G/K|})$ is a generator in the image of the free summand in $(N_H^G \underline{M})(G/K)$, define $N_H^G(\phi)_K(N(\mathbf{m}^{\times|G/K|}))$ to be

$$N(\phi_H(m_e) \times \phi_H(m_\gamma) \times \cdots \times \phi_H(m_{\gamma|G/K|-1})).$$

Further, $N_H^G(\phi)$ maps $\text{TR}^G \underline{M}$ to $\text{TR}^G \underline{L}$ since each $(\mathbf{ab})_f$, $g_H(\mathbf{a}, \mathbf{b})$ and $f(\mathbf{x})$ are universally determined by the group G . Then, by definition, the maps $\{N_H^G(\phi)_K : K \leq G\}$ form a natural transformation of G -Mackey functors, and the assignment $\phi \mapsto N_H^G(\phi)$ is functorial. $\square$

4.1. N_H^G is isomorphic to the composition $N_K^G N_H^K$

We next prove that the norm functors satisfy Property (a) of the Main Theorem. Thus, we will prove that $N_H^G: \text{Mack}_H \rightarrow \text{Mack}_G$ (given by $\underline{M} \mapsto N_H^G \underline{M}$) is isomorphic to the composition of functors $N_K^G N_H^K$ whenever $H < K < G$. By induction it suffices to show that for G' maximal in G , $N_{G'}^G N_H^{G'} \underline{M}$ is isomorphic to $N_H^G \underline{M}$.

Theorem 4.2. *Let H be a subgroup of G and let G' be the maximal subgroup of G . Then $N_{G'}^G N_H^{G'} \underline{M}$ is isomorphic to $N_H^G \underline{M}$.*

Proof. First, we will show that $(N_{G'}^G N_H^{G'} \underline{M})(G/K)$ is isomorphic to $(N_H^G \underline{M})(G/K)$ for all $K \leq G'$. By Lemma 3.14 $(N_H^G \underline{M})(G/K)$ is isomorphic to $(N_H^K \underline{M})^{\square|G/K|}(K/K)$. Further, by definition,

$$(N_{G'}^G N_H^{G'} \underline{M})(G/K) = (N_H^{G'} \underline{M})^{\square|G/G'|}(G'/K),$$

and by Lemma 3.14 $(N_H^{G'} \underline{M})(G'/K)$ is isomorphic to $(N_H^K \underline{M})^{\square|G'/K|}$. Hence, $(N_{G'}^G N_H^{G'} \underline{M})(G/K)$ is isomorphic to $\left((N_H^K \underline{M})^{\square|G'/K|}\right)^{\square|G/G'|}(K/K)$, which is isomorphic to $(N_H^K \underline{M})^{\square|G/K|}(K/K)$.

It remains to show that $(N_{G'}^G N_H^{G'} \underline{M})(G/G)$ is isomorphic to $(N_H^G \underline{M})(G/G)$. By definition,

$$\begin{aligned} (N_H^G \underline{M})(G/G) &= (\text{FREE}^G \underline{M})(G/G)/(\text{TR}^G \underline{M})(G/G) \\ &= (\mathbb{Z}\{\underline{M}(H/H)\} \oplus (N_H^G \underline{M})(G/G')/_{W_G(G')}) / (\text{TR}^G \underline{M})(G/G). \end{aligned}$$

Then the module $(N_G^G, N_H^{G'} \underline{M})(G/G)$ is

$$\left(\mathbb{Z}\{(N_H^{G'} \underline{M})(G'/G')\} \oplus (N_G^G, N_H^{G'} \underline{M})(G/G') /_{W_G(G')} \right) /_{(\mathrm{TR}^G \underline{M})(G/G)},$$

which equals

$$\left(\mathbb{Z}\{\mathbb{Z}\{\underline{M}(H/H)\} \oplus \mathrm{Im}(\mathrm{tr})\} /_{(\mathrm{TR}^{G'} \underline{M})(G'/G')} \oplus \mathrm{Im}(\mathrm{tr}_{G'}^G) \right) /_{(\mathrm{TR}^G \underline{M})(G/G)}.$$

Quotienting by $(\mathrm{TR}^G \underline{M})(G/G)$ identifies all elements in the transfer summand of $(N_H^{G'} \underline{M})(G'/G')$ with elements in $\mathrm{Im}(\mathrm{tr}_{G'}^G)$. Thus,

$$(N_G^G, N_H^{G'} \underline{M})(G/G) \cong (\mathbb{Z}\{\mathbb{Z}\{\underline{M}(H/H)\} \oplus \mathrm{Im}(\mathrm{tr}_{G'}^G)\} /_{(\mathrm{TR}^G \underline{M})(G/G)}).$$

Finally, since the Tambara reciprocity submodule identifies sums of generators in the free summand with elements in the transfer summand, it follows that $(N_G^G, N_H^{G'} \underline{M})(G/G)$ is isomorphic to $(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \mathrm{Im}(\mathrm{tr}_{G'}^G)) /_{(\mathrm{TR}^G \underline{M})(G/G)}$. Since $\mathrm{Im}(\mathrm{tr}_{G'}^G)$ is isomorphic to $(N_H^G \underline{M})(G/G') /_{W_G(G')}$, it follows that

$$(N_G^G, N_H^{G'} \underline{M})(G/G) \cong (N_H^G \underline{M})(G/G). \quad \square$$

Corollary 4.3. *For all subgroups H and K of G such that $H < K < G$, the norm functor $N_H^G : \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ is isomorphic to the composition of functors $N_K^G N_H^K$.*

4.2. The norm functors are strong symmetric monoidal

We now show that for all subgroups H of G , $N_H^G : \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ is strong symmetric monoidal, and thus the norm functors satisfy Property (b) of the Main Theorem. By Corollary 4.3, it suffices to let H be maximal in G . In this case $N_H^G \underline{M}$ simplifies nicely. Indeed, if H' is a subgroup of H , then $(N_H^G \underline{M})(G/H') = \underline{M}^{\square|G/H'|}(H/H')$. The only remaining module is $(N_H^G \underline{M})(G/G)$.

Fact 4.4. The module $(N_H^G \underline{M})(G/G)$ is $(\mathrm{FREE}^G \underline{M})(G/G) /_{(\mathrm{TR}^G \underline{M})(G/G)}$, which is

$$\left(\mathbb{Z}\{\underline{M}(H/H)\} \oplus \underline{M}^{\square|G/H|}(H/H) /_{W_G(H)} \right) /_{\mathrm{TR}},$$

where TR is the “Tambara reciprocity” submodule generated by elements of the forms

$$N(a+b) - N(a) - N(b) - \mathrm{tr}_H^G(g_H(\mathbf{a}, \mathbf{b}))$$

and

$$N(\mathrm{tr}_{H'}^H(x)) - \mathrm{tr}_{H'}^G(f(\mathbf{x}))$$

for all a and b in $\underline{M}(H/H)$, x in $\underline{M}(H/H')$ and $H' < H$.

To show that N_H^G is strong symmetric monoidal, given H -Mackey functors $\underline{M}$ and $\underline{L}$ we will build an isomorphism

$$\Psi : N_H^G \underline{M} \square N_H^G \underline{L} \rightarrow N_H^G(\underline{M} \square \underline{L})$$

by defining a collection of isomorphisms

$$\{\Psi_K: (N_H^G \underline{M} \square N_H^G \underline{L})(G/K) \rightarrow N_H^G(\underline{M} \square \underline{L})(G/K) \text{ for all } K \leq G\}.$$

For $H' \leq H$ the isomorphism $\Psi_{H'}$ will be induced from properties of the box product. So, the work lies in defining Ψ_G . Before we do so, we explicitly describe $(N_H^G \underline{M} \square N_H^G \underline{L})(G/G)$ and $N_H^G(\underline{M} \square \underline{L})(G/G)$.

Lemma 4.5. *Let H be maximal in G and let $\underline{M}$ and $\underline{L}$ be H -Mackey functors. The module $(N_H^G \underline{M} \square N_H^G \underline{L})(G/G)$ is isomorphic to*

$$(\mathbb{Z}\{\underline{M}(H/H) \times \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G)) / \widetilde{FR},$$

where $\text{Im}(\text{tr}_H^G)$ is $(\underline{M}^{\square|G/H|} \square \underline{L}^{\square|G/H|})(H/H)/_{W_G(H)}$. The submodule $\widetilde{FR}$ is generated by

$$\begin{aligned} & N((a+b) \times l) - N(a \times l) - N(b \times l) - \text{tr}_H^G \left(g_H(\mathbf{a}, \mathbf{b}) \otimes \bigotimes_{|G/H|} l \right), \\ & N(m \times (y+z)) - N(m \times y) - N(m \times z) - \text{tr}_H^G \left(\left(\bigotimes_{|G/H|} m \right) \otimes g_H(\mathbf{y}, \mathbf{z}) \right), \\ & N(\text{tr}_{H'}^H(d) \times l) - \text{tr}_{H'}^G \left(f(\mathbf{d}) \otimes \bigotimes_{|G/H|} \text{res}_{H'}^H(l) \right), \end{aligned}$$

and

$$N(m \times \text{tr}_{H'}^H(x)) - \text{tr}_{H'}^G \left(\bigotimes_{|G/H|} \text{res}_{H'}^H(m) \otimes f(\mathbf{x}) \right)$$

for all a, b , and m in $\underline{M}(H/H)$, d in $\underline{M}(H/H')$, y, z , and l in $\underline{L}(H/H)$, x in $\underline{L}(H/H')$ and subgroups H' of H .

Proof. First,

$$(N_H^G \underline{M} \square N_H^G \underline{L})(G/G) = (N_H^G \underline{M}(G/G) \otimes N_H^G \underline{L}(G/G) \oplus \text{Im}(\text{tr}_H^G)) / FR.$$

The submodule $\widetilde{FR}$ stems from combining the relations defined by the FR submodule with the relations from each TR submodule. For example, by the TR submodule of $N_H^G \underline{M}(G/G)$ we have

$$N(a+b) \otimes N(l) - N(a) \otimes N(l) - N(b) \otimes N(l) - \text{tr}_H^G(g_H(\mathbf{a}, \mathbf{b})) \otimes N(l).$$

But, now we use the FR submodule and the fact that $\text{res}_H^G(N(l)) = \bigotimes_{|G/H|} l$ to identify $\text{tr}_H^G(g_H(\mathbf{a}, \mathbf{b})) \otimes N(l)$ with $\text{tr}_H^G \left(g_H(\mathbf{a}, \mathbf{b}) \otimes \bigotimes_{|G/H|} l \right)$. Finally, we use the fact that for any modules R and S , $\mathbb{Z}\{R\} \otimes \mathbb{Z}\{S\}$ is isomorphic to $\mathbb{Z}\{R \times S\}$ to arrive at the module given in the lemma. $\square$

We introduce some notation before describing $N_H^G(\underline{M} \square \underline{L})$. First, let $g_H(\mathbf{a} \otimes \mathbf{y}, \mathbf{b} \otimes \mathbf{z})$ be a polynomial similar to $g_H(\mathbf{a}, \mathbf{b})$. Specifically, let

$$g_H(\mathbf{a} \otimes \mathbf{y}, \mathbf{b} \otimes \mathbf{z}) = \sum_{f \in \mathcal{I}_k/G} \left(\bigotimes_{gH \in G/H} f(gH) \right)$$

where now $\mathcal{I}_k = (\text{Map}(G/H, \{a \otimes y, b \otimes z\}) - \text{Map}(G/G, \{a \otimes y, b \otimes z\}))$.

Further, we define $f(\mathbf{d} \otimes \text{res}_{H'}^H(\mathbf{y}))$ similarly to $f(\mathbf{x})$. So,

$$f(\mathbf{d} \otimes \text{res}_{H'}^H(\mathbf{y})) = \sum_{s=1}^r \bigotimes_{i=0}^{|G/H|-1} (\gamma^{m_{i,s}} \mathbf{d} \otimes \text{res}_{H'}^H(\mathbf{y})).$$

We define $f(\text{res}_{H'}^H(\mathbf{a}) \otimes \mathbf{x})$ analogously.

Lemma 4.6. *Let H be maximal in G and let $\underline{M}$ and $\underline{L}$ be H -Mackey functors. The module $N_H^G(\underline{M} \square \underline{L})(G/G)$ is isomorphic to*

$$(\mathbb{Z}\{\underline{M}(H/H) \otimes \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G)) / \widetilde{TR},$$

where $\text{Im}(\text{tr}_H^G)$ is $(\underline{M} \square \underline{L})^{\square|G/H|}(H/H)/_{W_G(H)}$, and $\widetilde{TR}$ is generated by the following elements for all $a \otimes y$ and $b \otimes z$ in $\underline{M}(H/H) \otimes \underline{L}(H/H)$, d in $\underline{M}(H/H')$, x in $\underline{L}(H/H')$, and subgroups H' of H :

$$\begin{aligned} N(a \otimes y + b \otimes z) - N(a \otimes y) - N(b \otimes z) - \text{tr}_H^G(g_H(\mathbf{a} \otimes \mathbf{y}, \mathbf{b} \otimes \mathbf{z})), \\ N(\text{tr}_{H'}^H(d) \otimes y) - \text{tr}_{H'}^G(f(\mathbf{d} \otimes \text{res}_{H'}^H(\mathbf{y}))), \\ N(a \otimes \text{tr}_{H'}^H(x)) - \text{tr}_{H'}^G(f(\text{res}_{H'}^H(\mathbf{a}) \otimes \mathbf{x})). \end{aligned}$$

Proof. The module

$$N_H^G(\underline{M} \square \underline{L})(G/G) = (\mathbb{Z}\{(\underline{M} \square \underline{L})(H/H)\} \oplus \text{Im}(\text{tr}_H^G)) / TR,$$

and by Definition 3.1, $(\underline{M} \square \underline{L})(H/H) = (\underline{M}(H/H) \otimes \underline{L}(H/H) \oplus \text{Im}(\text{tr})) /_{FR}$. Thus, since we quotient by the TR submodule in $N_H^G(\underline{M} \square \underline{L})(G/G)$, we can identify all elements in $\text{Im}(\text{tr})$ with the appropriate elements in $\text{Im}(\text{tr}_H^G)$. If we then combine the Frobenius reciprocity submodule of $(\underline{M} \square \underline{L})(H/H)$ with Tambara reciprocity the resulting module is isomorphic to

$$(\mathbb{Z}\{\underline{M}(H/H) \otimes \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G)) / \widetilde{TR}. \quad \square$$

Theorem 4.7. *For all subgroups H of G , the norm functor $N_H^G: \text{Mack}_H \rightarrow \text{Mack}_G$ is a strong symmetric monoidal functor.*

Proof. By Corollary 4.3 it suffices to let H be maximal in G . We will build an isomorphism $\Psi: N_H^G \underline{M} \square N_H^G \underline{L} \rightarrow N_H^G(\underline{M} \square \underline{L})$ by defining a collection of isomorphisms

$$\{\Psi_K: (N_H^G \underline{M} \square N_H^G \underline{L})(G/K) \rightarrow N_H^G(\underline{M} \square \underline{L})(G/K) \text{ for all } K \leq G\}.$$

First, let H' be a subgroup of H . By definition, $(N_H^G \underline{M} \square N_H^G \underline{L})(G/H') = (\underline{M}^{\square|G/H|} \square \underline{L}^{\square|G/H|})(H/H')$ and $N_H^G(\underline{M} \square \underline{L})(G/H') = (\underline{M} \square \underline{L})^{\square|G/H|}(H/H')$. Since $\square$ is symmetric monoidal we define $\Psi_{H'}$ to be the natural isomorphism

$$(\underline{M}^{\square|G/H|} \square \underline{L}^{\square|G/H|})(H/H') \rightarrow (\underline{M} \square \underline{L})^{\square|G/H|}(H/H').$$

It remains to define the isomorphism Ψ_G . By Lemmas 4.5 and 4.6,

$$(N_H^G \underline{M} \square N_H^G \underline{L})(G/G) \cong (\mathbb{Z}\{\underline{M}(H/H) \times \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G)) / \widetilde{FR},$$

and

$$N_H^G(\underline{M} \square \underline{L})(G/G) \cong (\mathbb{Z}\{\underline{M}(H/H) \otimes \underline{L}(H/H)\} \oplus \text{Im}(\text{tr}_H^G)) / \widetilde{TR}.$$

Thus, we define Ψ_G to be the direct sum of the following two maps. First, let $\psi_G: \text{Im}(\text{tr}_H^G) \rightarrow \text{Im}(\text{tr}_H^G)$ be the isomorphism induced from Ψ_H . Then define

$$\psi'_G: \mathbb{Z}\{\underline{M}(H/H) \times \underline{L}(H/H)\} \rightarrow \mathbb{Z}\{\underline{M}(H/H) \otimes \underline{L}(H/H)\}$$

by $\psi'_G(N(a \times y)) = N(a \otimes y)$, and so we define Ψ_G to be $\psi'_G \oplus \psi_G$. Because of the way ψ_G reindexes the elements of $\text{Im}(\text{tr}_H^G)$, the map Ψ_G sends all elements in $\widetilde{FR}$ to elements in $\widetilde{TR}$. Further, by construction, $\Psi_G \text{tr}_H^G = \text{tr}_H^G \Psi_H$ and $\text{res}_H^G \Psi_G = \Psi_H \text{res}_H^G$. Finally, the relations defined by $\widetilde{FR}$ are analogous to those that define a tensor product from a Cartesian product. It follows that Ψ_G is an isomorphism. $\square$

We end this section with a proof of the Main Theorem.

Proof of the Main Theorem. For all subgroups H of G and H -Mackey functors $\underline{M}$, let $N_H^G \underline{M}$ be the G -Mackey functor defined in Definition 3.13. Then define the norm functors $N_H^G: \text{Mack}_H \rightarrow \text{Mack}_G$ by $\underline{M} \mapsto N_H^G \underline{M}$. These maps satisfy all properties given in the Main Theorem by Theorem 4.1, Corollary 4.3, and Theorem 4.7. $\square$

5. Proof of Theorem 1.1

Finally, given a cyclic p -group G , we use the norm functors $N_H^G: \text{Mack}_H \rightarrow \text{Mack}_G$ to define a G -symmetric monoidal structure on the category of G -Mackey functors. We will then show that G -Tambara functors are the G -commutative monoids under this structure, thus proving Theorem 1.1.

We begin with Hill and Hopkins' definition of a G -symmetric monoidal structure [3].

Definition 5.1. Let $\mathcal{S}et_G^{\text{Fin}, \text{Iso}}$ be the category whose objects are finite G -sets and whose morphisms are isomorphisms of G -sets. Further, let $(\mathcal{C}, \boxtimes, e)$ be a symmetric monoidal category. A G -symmetric monoidal structure on $\mathcal{C}$ consists of a functor

$$(-) \otimes (-): \mathcal{S}et_G^{\text{Fin}, \text{Iso}} \times \mathcal{C} \rightarrow \mathcal{C}$$

that satisfies the following properties.

1. $(X \amalg Y) \otimes C$ is naturally isomorphic to $(X \otimes C) \boxtimes (Y \otimes C)$ and $X \otimes (C \boxtimes D)$ is naturally isomorphic to $(X \otimes C) \boxtimes (X \otimes D)$.
2. When restricted to $\mathcal{S}et_G^{\text{Fin}, \text{Iso}} \times \mathcal{C}$ this functor is the canonical exponentiation map given by $X \otimes C = {}_C \boxtimes^{X|}$.
3. $X \otimes (Y \otimes C)$ is naturally isomorphic to $(X \times Y) \otimes C$.

Theorem 5.2. Let $i_H^*: \text{Mack}_G \rightarrow \text{Mack}_H$ be the forgetful functor. For a cyclic p -group G the functor $(-) \otimes (-): \mathcal{S}et_G^{\text{Fin}, \text{Iso}} \times \text{Mack}_G \rightarrow \text{Mack}_G$ defined by

- $\emptyset \otimes \underline{M} := \underline{A}$, where $\underline{A}$ is the Burnside Mackey functor,
- $G/H \otimes \underline{M} := N_H^G i_H^* \underline{M}$ for all orbits G/H of G , and
- $(X \amalg Y) \otimes \underline{M} := (X \otimes \underline{M}) \square (Y \otimes \underline{M})$ for all X and Y in $\mathcal{S}et_G^{\text{Fin}, \text{Iso}}$

is a G -symmetric monoidal structure on Mack_G .

Proof. Let $\underline{M}$ be a G -Mackey functor. The above functor $(-) \otimes (-): \mathcal{S}et_G^{Fin, Iso} \times \mathcal{M}ack_G \rightarrow \mathcal{M}ack_G$ satisfies Property 1 of Definition 5.1 because the norm functors N_H^G are strong symmetric monoidal for all subgroups H of G . Further, if X is a finite set, then we can regard it as a disjoint union of $|X|$ -many copies of the G -orbit G/G . Thus, $(-) \otimes (-)$ satisfies Property 2 of Definition 5.1 since

$$X \otimes \underline{M} = (G/G \otimes \underline{M})^{\square|X|} = \underline{M}^{\square|X|}.$$

Finally, to show that Property 3 of Definition 5.1 holds it suffices to show that $(G/K \times G/H) \otimes \underline{M} \cong G/K \otimes (G/H \otimes \underline{M})$ for all orbits G/H and G/K of G . We first assume that H is a subgroup of K , so $G/K \times G/H$ is isomorphic to $\Pi_{|G/K|} G/H$. Then

$$(G/K \times G/H) \otimes \underline{M} \cong (G/H \otimes \underline{M})^{\square|G/K|} \cong N_H^G i_H^* (\underline{M}^{\square|G/K|}).$$

On the other hand, $G/K \otimes (G/H \otimes \underline{M}) = N_K^G i_K^* N_H^G i_H^* \underline{M}$, and using Lemma 3.14, $i_K^* N_H^G i_H^* \underline{M}$ is isomorphic to $(N_H^K i_H^* \underline{M})^{\square|G/K|}$. Then via Theorem 4.7, $(N_H^K i_H^* \underline{M})^{\square|G/K|}$ is isomorphic to $N_H^K i_H^* (\underline{M}^{\square|G/K|})$, and therefore,

$$G/K \otimes (G/H \otimes \underline{M}) \cong N_K^G N_H^K i_H^* (\underline{M}^{\square|G/K|}) \cong N_H^G i_H^* (\underline{M}^{\square|G/H|}).$$

Next, if $K \leq H$ then $(G/K \times G/H) \otimes \underline{M} \cong N_K^G i_K^* (\underline{M}^{\square|G/H|})$. Moreover,

$$G/K \otimes (G/H \otimes \underline{M}) = N_K^G i_K^* N_H^G i_H^* \underline{M}.$$

But, $i_K^* N_H^G i_H^* \underline{M} = i_K^* \underline{M}^{\square|G/H|}$, and so $G/K \otimes (G/H \otimes \underline{M}) = N_K^G i_K^* (\underline{M}^{\square|G/H|})$ as well. $\square$

To define the commutative ring objects under a G -symmetric monoidal structure let $\mathcal{C}$ be a symmetric monoidal category with a G -symmetric monoidal structure $(-) \otimes (-)$. Every object C in $\mathcal{C}$ defines a functor

$$(-) \otimes C: \mathcal{S}et_G^{Fin, Iso} \rightarrow \mathcal{C}.$$

Definition 5.3. [3] A G -commutative monoid is an object C in $\mathcal{C}$ together with an extension of $(-) \otimes C$ as given below.

$$\begin{array}{ccc} \mathcal{S}et_G^{Fin, Iso} & \xrightarrow{(-) \otimes C} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathcal{S}et_G^{Fin} & & \end{array}$$

We will finish proving Theorem 1.1 by showing that if we endow $\mathcal{M}ack_G$ with the G -symmetric monoidal structure defined in Theorem 5.2, then a Mackey functor $\underline{M}$ is a Tambara functor if and only if it is a G -commutative monoid. We start by proving the forward implication in Proposition 5.4 and leave the reverse implication to Proposition 5.7.

Proposition 5.4. Let $\underline{M}$ be a G -Mackey functor and endow $\mathcal{M}ack_G$ with the G -symmetric monoidal structure of Theorem 5.2. If $\underline{M}$ has the structure of a Tambara functor, then $\underline{M}$ is a G -commutative monoid.

To prove Proposition 5.4 we need to show that if $\underline{S}$ is a Tambara functor, then a map $X \rightarrow Y$ of G -sets induces a map $X \otimes \underline{S} \rightarrow Y \otimes \underline{S}$ of Mackey functors. Thus, we will extend the norm functors

$N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ to functors $\mathcal{N}_H^G: \mathcal{Tamb}_H \rightarrow \mathcal{Tamb}_G$ on Tambara functors and show that $\mathcal{N}_H^G$ is left adjoint to the forgetful functor $i_H^*: \mathcal{Tamb}_G \rightarrow \mathcal{Tamb}_H$. Then given a Tambara functor $\underline{S}$ we will use the counit of the above adjunction along with properties of finite G -sets and the fact that the box product is the coproduct in $\mathcal{Tamb}_G$ ([10]) to induce a map $X \otimes \underline{S} \rightarrow Y \otimes \underline{S}$.

Lemma 5.5. *For all subgroups H of G the functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$ extends to a functor $\mathcal{N}_H^G: \mathcal{Tamb}_H \rightarrow \mathcal{Tamb}_G$.*

Proof. We need to show that for all subgroups H of G , if $\underline{S}$ is an H -Tambara functor then $N_H^G \underline{S}$ is a G -Tambara functor. However, by Corollary 4.3 it suffices to let H be the maximal subgroup in G . Since N_H^G is strong symmetric monoidal it naturally extends to a functor $\mathcal{Green}_H \rightarrow \mathcal{Green}_G$ where $\mathcal{Green}_G$ is the category of G -Green functors. Hence, it remains to define the internal norm maps $N_{K'}^K: (N_H^G \underline{S})(G/K') \rightarrow (N_H^G \underline{S})(G/K)$ for all subgroups $K' < K$ in G .

Since the box product is the coproduct in $\mathcal{Tamb}_H$ [10, Prop 9.1], if both H' and H'' are subgroups of H with $H'' < H'$, then we define $N_{H''}^{H'}$ to be the $|G/H|$ -fold box product of the norm map $N_{H''}^{H'}$ in $\underline{S}$. Lastly, we must define the norm $N_H^G: \underline{S}^{\square|G/H|}(H/H) \rightarrow (N_H^G \underline{S})(G/G)$. This norm is the composition of the multiplication map of $\underline{S}$ with the map $N: \underline{S}(H/H) \rightarrow (N_H^G \underline{S})(G/G)$ defined in Remark 3.4. Thus, letting $\mu: \underline{S}^{\square|G/H|} \rightarrow \underline{S}$ be the multiplication map of $\underline{S}$, we define N_H^G to be the composition

$$\underline{S}^{\square|G/H|}(H/H) \xrightarrow{\mu} \underline{S}(H/H) \xrightarrow{N} (N_H^G \underline{S})(G/G).$$

This composition satisfies Properties 1, 2, 4, and 5 of Definition 2.11 by construction of the functor $N_H^G: \mathcal{Mack}_H \rightarrow \mathcal{Mack}_G$. (Property 3 is Frobenius Reciprocity.) It remains to show that this norm map satisfies Tambara Reciprocity (Property 6 of Definition 2.11).

First, we will show that N_H^G satisfies Tambara Reciprocity for sums (Corollary 2.6). Let $\bigotimes a_j = a_0 \otimes \cdots \otimes a_{|G/H|-1}$ and $\bigotimes b_j = b_0 \otimes \cdots \otimes b_{|G/H|-1}$ be simple tensors in $\underline{S}^{\square|G/H|}(H/H)$. We need to show that

$$N_H^G(\bigotimes a_j + \bigotimes b_j) = N_H^G(\bigotimes a_j) + N_H^G(\bigotimes b_j) + tr_H^G(g_H(\bigotimes a_j, \bigotimes b_j)),$$

where

$$g_H(\bigotimes a_j, \bigotimes b_j) = \sum_{f \in \mathcal{I}_k/G} \left(\prod_{gH \in G/H} gf(gH) \right) = \sum_{f \in \mathcal{I}_k/G} \left(\prod_{i=0}^{|G/H|-1} \gamma^i f(\gamma^i H) \right),$$

and

$$\mathcal{I}_k = \left(Map\left(G/H, \left\{ \bigotimes a_j, \bigotimes b_j \right\}\right) - Map\left(G/G, \left\{ \bigotimes a_j, \bigotimes b_j \right\}\right) \right).$$

Now, in $\mathcal{N}_H^G \underline{S}$,

$$\begin{aligned} N_H^G(\bigotimes a_j + \bigotimes b_j) &= (N \circ \mu)(\bigotimes a_j + \bigotimes b_j) \\ &= N(a_0 a_1 \cdots a_{|G/H|-1} + b_0 b_1 \cdots b_{|G/H|-1}). \end{aligned}$$

Let $a_0 a_1 \cdots a_{|G/H|-1} = \prod a_j$ and $b_0 b_1 \cdots b_{|G/H|-1} = \prod b_j$. Since we quotient by the Tambara reciprocity submodule TR in $(\mathcal{N}_H^G \underline{S})(G/G)$, it follows that

$$N(\prod a_j + \prod b_j) = N(\prod a_j) + N(\prod b_j) + tr_H^G(g_H(\prod a_j, \prod b_j)),$$

and $g_H(\prod \mathbf{a}_j, \prod \mathbf{b}_j) = \sum_{f \in \mathcal{I}_k/G} \left(\bigotimes_{i=0}^{|G/H|-1} f(\gamma^i H) \right)$ where now

$$\mathcal{I}_k = \left(\text{Map}\left(G/H, \left\{ \prod \mathbf{a}_j, \prod \mathbf{b}_j \right\}\right) - \text{Map}\left(G/G, \left\{ \prod \mathbf{a}_j, \prod \mathbf{b}_j \right\}\right) \right).$$

Further, since in $(\mathcal{N}_H^G \underline{M})(G/G)$, $\text{tr}_H^G(g_H(\prod \mathbf{a}_j, \prod \mathbf{b}_j))$ lies in $\underline{M}^{\square|G/H|}(H/H)/_{W_G(H)}$, it follows that we can write each $\bigotimes_{i=0}^{|G/H|-1} f(\gamma^i H)$ as

$$f(H)f(\gamma H) \cdots f(\gamma^{|G/H|-1} H) \otimes 1^{\otimes |G/H|-1}.$$

Therefore, $\text{tr}_H^G(g_H(\prod \mathbf{a}_j, \prod \mathbf{b}_j)) = \text{tr}_H^G(g_H(\bigotimes \mathbf{a}_j, \bigotimes \mathbf{b}_j))$, and so the norm map satisfies Tambara reciprocity for sums.

If we employ a strategy analogous to the one above we can show that the norm N_H^G in $\mathcal{N}_H^G \underline{S}$ also satisfies Tambara reciprocity for transfers (Corollary 2.9). Specifically, because we quotient by the TR submodule in $(\mathcal{N}_H^G \underline{S})(G/G)$ and by the appropriate Weyl action in the image of each transfer map, it follows that

$$N_H^G \text{tr}_{H'}^H(x) = N(\text{tr}_{H'}^H(\mu(x))) = \text{tr}_{H'}^G(f(\mu(\mathbf{x}))) = \text{tr}_{H'}^G(f(x)).$$

Therefore, $\mathcal{N}_H^G \underline{S}$ is a Tambara functor. $\square$

Lemma 5.6. *The functor $\mathcal{N}_H^G: \mathcal{Tamb}_H \rightarrow \mathcal{Tamb}_G$ is left adjoint to the restriction functor $i_H^*: \mathcal{Tamb}_G \rightarrow \mathcal{Tamb}_H$.*

Proof. Since we can compose adjunctions in a natural fashion [7], by Corollary 4.3, it suffices to let H be maximal in G . Let $\underline{R}$ be in $\mathcal{Tamb}_G$ and $\underline{S}$ be in $\mathcal{Tamb}_H$. Further, let $\mathcal{Tamb}_H(\underline{S}, i_H^* \underline{R})$ be the set of morphisms from $\underline{S}$ to $i_H^* \underline{R}$ in $\mathcal{Tamb}_H$. We will show that $\mathcal{Tamb}_H(\underline{S}, i_H^* \underline{R})$ is in natural bijective correspondence with $\mathcal{Tamb}_G(\mathcal{N}_H^G \underline{S}, \underline{R})$ by showing that every morphism in $\mathcal{Tamb}_G(\mathcal{N}_H^G \underline{S}, \underline{R})$ determines and is determined by a morphism in $\mathcal{Tamb}_H(\underline{S}, i_H^* \underline{R})$.

A morphism Ω in $\mathcal{Tamb}_G(\mathcal{N}_H^G \underline{S}, \underline{R})$ consists of a collection of ring homomorphisms $\{\Omega_P: (\mathcal{N}_H^G \underline{S})(G/P) \rightarrow \underline{R}(G/P) \text{ for all } P \leq G\}$ that commute with the appropriate restriction, transfer, and norm maps. Further, every element in $(\mathcal{N}_H^G \underline{S})(G/G)$ is a sum consisting of elements in the image of the transfer map and sums of elements in the image of the norm map. (Indeed, every generator $N(s)$ in the image of the free summand $\mathbb{Z}\{\underline{S}(H/H)\}$ of $(\mathcal{N}_H^G \underline{S})(G/G)$ is the norm of the element $s \otimes 1^{\otimes |G/H|-1}$ in $(\mathcal{N}_H^G \underline{S})(G/H)$.) Thus, the ring homomorphism Ω_G is completely determined by Ω_H , and since H is maximal in G , the morphism Ω is completely determined by the collection of ring homomorphisms

$$\{\Omega_{H'}: \underline{S}^{\square|G/H|}(H/H') \rightarrow \underline{R}(G/H') \text{ for all } H' \leq H\}.$$

By Proposition 3.2, the above collection of maps determines and is determined by a collection of Weyl equivariant maps

$$\{\theta_{H'}: \underline{S}(H/H')^{\otimes |G/H|} \rightarrow \underline{R}(G/H') \text{ for all } H' \leq H\}$$

that satisfies the compatibility conditions given in Proposition 3.2. But we can write every $\mathbf{s}^{\otimes |G/H|}$ in $\underline{S}(H/H')^{\otimes |G/H|}$ as a product over the $W_G(H')$ -action. Thus, each $\theta_{H'}$ determines and is determined by a $W_G(H')$ -equivariant homomorphism $\Lambda_{H'}: \underline{S}(H/H') \rightarrow \underline{R}(G/H')$ because we can write $\theta_{H'}(\mathbf{s}^{\otimes |G/H|})$ as the following product.

$$\begin{aligned}
\theta_{H'}(\mathbf{s}^{\otimes |G/H|}) &= \theta_{H'} \left[\prod_{i=0}^{|G/H|-1} \gamma^i \left(s_{\gamma^i} \otimes 1^{\otimes |G/H|-1} \right) \right] \\
&= \prod_{i=0}^{|G/H|-1} \gamma^i \theta_{H'} \left(s_{\gamma^i} \otimes 1^{\otimes |G/H|-1} \right) \\
&= \prod_{i=0}^{|G/H|-1} \gamma^i \Lambda_{H'}(s_{\gamma^i}).
\end{aligned}$$

Lastly, we show that Ω is well-defined by showing that Ω_G sends the TR submodule of $(\mathcal{N}_H^G \underline{\mathcal{S}})(G/G)$ to zero. Consider the element $N(a+b) - N(a) - N(b) - tr_H^G(g_H(\mathbf{a}, \mathbf{b}))$ in TR . Then $\Omega_G(N(a+b) - N(a) - N(b) - tr_H^G(g_H(\mathbf{a}, \mathbf{b})))$ equals

$$\begin{aligned}
&N_H^G \Omega_H \left((a+b) \otimes 1^{\otimes |G/H|-1} \right) - \\
&N_H^G \Omega_H \left(a \otimes 1^{\otimes |G/H|-1} \right) - N_H^G \Omega_H \left(b \otimes 1^{\otimes |G/H|-1} \right) - tr_H^G \Omega_H(g_H(\mathbf{a}, \mathbf{b})),
\end{aligned}$$

and we must show that this element is zero in $\underline{R}(G/G)$. First,

$$N_H^G \Omega_H \left((a+b) \otimes 1^{\otimes |G/H|-1} \right) = N_H^G (\Lambda_H(a+b)) = N_H^G (\Lambda_H(a) + \Lambda_H(b)).$$

Then since $\underline{R}$ is a Tambara functor, by Tambara reciprocity for sums,

$$N_H^G (\Lambda_H(a) + \Lambda_H(b)) - N_H^G (\Lambda_H(a)) - N_H^G (\Lambda_H(b)) - tr_H^G(g_H(\Lambda_H(a), \Lambda_H(b))) = 0.$$

Thus, it remains to show that

$$tr_H^G(\Omega_H(g_H(\mathbf{a}, \mathbf{b}))) = tr_H^G(g_H(\Lambda_H(a), \Lambda_H(b))).$$

But, $g_H(\mathbf{a}, \mathbf{b}) = \sum_{f \in \mathcal{I}_k} \left(\bigotimes_{i=0}^{|G/H|-1} f(\gamma^i H) \right)$, and since $\mathcal{N}_H^G \underline{\mathcal{S}}$ is a Tambara functor (and thus $(\mathcal{N}_H^G \underline{\mathcal{S}})(G/H)$ has a multiplication), it follows that

$$\bigotimes_{i=0}^{|G/H|-1} f(\gamma^i H) = \prod_{i=0}^{|G/H|-1} \gamma^i \left(f(\gamma^i H) \otimes 1^{\otimes |G/H|-1} \right).$$

Thus,

$$\begin{aligned}
tr_H^G \Omega_H(g_H(\mathbf{a}, \mathbf{b})) &= tr_H^G \Omega_H \left(\sum_{f \in \mathcal{I}_k/G} \left(\prod_{i=0}^{|G/H|-1} \gamma^i \left(f(\gamma^i H) \otimes 1^{\otimes |G/H|-1} \right) \right) \right) \\
&= tr_H^G \left(\sum_{f \in \mathcal{I}_k/G} \left(\prod_{i=0}^{|G/H|-1} \gamma^i \Lambda_H(f(\gamma^i H)) \right) \right) \\
&= tr_H^G(g_H(\Lambda_H(a), \Lambda_H(b))).
\end{aligned}$$

Next we show that $\Omega_G(N(tr_{H'}^H(x)) - tr_{H'}^G(f(\mathbf{x}))) = 0$. First,

$$\Omega_G(N(tr_{H'}^H(x)) - tr_{H'}^G(f(\mathbf{x}))) = N_H^G \Omega_H \left(tr_{H'}^H(x) \otimes 1^{\otimes |G/H|-1} \right) - tr_{H'}^G \Omega_{H'}(f(\mathbf{x})),$$

and

$$N_H^G \Omega_H \left(\text{tr}_{H'}^H(x) \otimes 1^{\otimes |G/H|-1} \right) = N_H^G \Lambda_H(\text{tr}_{H'}^H(x)) = N_H^G \text{tr}_{H'}^H(\Lambda_{H'}(x)).$$

Since $\underline{R}$ is a Tambara functor it follows that

$$N_H^G \text{tr}_{H'}^H(\Lambda_{H'}(x)) - \text{tr}_{H'}^G(f(\Lambda_{H'}(x))) = 0,$$

and so we need to show that $\Omega_{H'}(f(x)) = f(\Lambda_{H'}(x))$.

As detailed in Corollary 2.9, $f(\Lambda_{H'}(x)) = \sum_{s=1}^r \left(\prod_{i=0}^{|G/H|-1} \gamma^i \gamma^{m_{i,s}} \Lambda_{H'}(x) \right)_s$. Further,

$$\begin{aligned} \Omega_{H'}(f(x)) &= \sum_{s=1}^r \theta_{H'}(\gamma^{m_{0,s}} x \otimes \gamma^{m_{1,s}} x \otimes \cdots \otimes \gamma^{m_{|G/H|-1,s}} x)_s \\ &= \sum_{s=1}^r \left(\prod_{i=0}^{|G/H|-1} \gamma^i \Lambda_{H'}(\gamma^{m_{i,s}} x) \right)_s. \end{aligned}$$

Since $\Lambda_{H'}$ is Weyl equivariant it follows that $\Omega_{H'}(f(x)) = f(\Lambda_{H'}(x))$. Therefore, since Ω_G maps all elements of the Tambara reciprocity submodule of $(\mathcal{N}_H^G \underline{S})(G/G)$ to zero, Ω is well defined. $\square$

We now prove Proposition 5.4.

Proof of Proposition 5.4. Let $\underline{S}$ be a G -Tambara functor. To show that $\underline{S}$ is a G -commutative monoid we first show that a map of orbits $G/H \rightarrow G/K$ induces a map $G/H \otimes \underline{S} \rightarrow G/K \otimes \underline{S}$. Consider the K -Tambara functor $i_K^* \underline{S}$. By Lemma 5.6 there is an adjunction between $\mathcal{N}_K^K$ and i_K^* , and hence a counit map $\mathcal{N}_H^K i_H^* i_K^* \underline{S} \rightarrow i_K^* \underline{S}$. To define $G/H \otimes \underline{S} \rightarrow G/K \otimes \underline{S}$ we apply $\mathcal{N}_K^G$ to the above counit map, which yields a map $\mathcal{N}_K^G \mathcal{N}_H^K i_H^* i_K^* \underline{S} \rightarrow \mathcal{N}_K^G i_K^* \underline{S}$. Since $\mathcal{N}_K^G \mathcal{N}_H^K$ is isomorphic to $\mathcal{N}_H^G$ and $i_H^* i_K^*$ is isomorphic to i_H^* it follows that the above map is a map $\mathcal{N}_H^G i_H^* \underline{S} \rightarrow \mathcal{N}_K^G i_K^* \underline{S}$.

If $X \rightarrow Y$ is a map of arbitrary G -sets we define the induced map $X \otimes \underline{S} \rightarrow Y \otimes \underline{S}$ as follows. First, we write X and Y as disjoint unions of orbits, so $X \cong \coprod_i G/H_i$ and $Y \cong \coprod_j G/K_j$. Thus, the map $X \rightarrow Y$ consists of a combination of fold maps, automorphisms, and canonical maps $G/H_i \rightarrow G/K_j$ where H_i is a subgroup of K_j . Further,

$$X \otimes \underline{S} \cong \left(\prod_i G/H_i \right) \otimes \underline{S} \cong \square_i (G/H_i \otimes \underline{S}) = \square_i \mathcal{N}_{H_i}^G i_{H_i}^* \underline{S},$$

and similarly, $Y \otimes \underline{S}$ is isomorphic to $\square_j \mathcal{N}_{K_j}^G i_{K_j}^* \underline{S}$.

Thus, because the box product is the coproduct in $\mathcal{Tamb}_G$, the induced map $X \otimes \underline{S} \rightarrow Y \otimes \underline{S}$ is a map

$$\square_i \mathcal{N}_{H_i}^G i_{H_i}^* \underline{S} \rightarrow \square_j \mathcal{N}_{K_j}^G i_{K_j}^* \underline{S}$$

that is a combination of multiplication maps, Weyl actions, and maps $\mathcal{N}_{H_i}^G i_{H_i}^* \underline{S} \rightarrow \mathcal{N}_{K_j}^G i_{K_j}^* \underline{S}$ induced from the counit map. $\square$

It remains to prove the following statement.

Proposition 5.7. *If $\mathcal{Mack}_G$ has the G -symmetric monoidal structure of Theorem 5.2 and $\underline{M}$ is a G -commutative monoid in $\mathcal{Mack}_G$, then $\underline{M}$ is a G -Tambara functor.*

We show that if a Mackey functor $\underline{M}$ is a G -commutative monoid then it is a commutative Green functor using the basic properties of the functor $(-)\otimes(-)$. Defining the internal norm maps N_H^K for all $H < K \leq G$ that make $\underline{M}$ into a Tambara functor takes more work. However, we will only define the norm map N_H^G since any norm map N_H^K in a G -Tambara functor $\underline{S}$ must agree with the analogous norm map in $i_K^*\underline{S}$.

To define the norm N_H^G we first recognize that the Mackey functor $i_H^*\underline{M}$ is an H -Mackey functor that maintains the Weyl action defined on $\underline{M}$. Hence, $N_H^G i_H^*\underline{M}$ should also remember this Weyl action, and in particular, for all subgroups H' of H , $(i_H^*\underline{M})(H/H')$ is isomorphic to $\underline{M}(G/H')$. Then since $\underline{M}$ is a G -commutative monoid the map of orbits $G/H \rightarrow G/G$ induces a map $N_H^G i_H^*\underline{M} \rightarrow \underline{M}$. Therefore, we define the norm map N_H^G to be the composition below where N is the map from Remark 3.4.

$$\underline{M}(G/H) \xrightarrow{N} (N_H^G i_H^*\underline{M})(G/G) \rightarrow \underline{M}(G/G)$$

In the proof of Proposition 5.7 we show that this composition satisfies all of the properties of a norm in a Tambara functor. But, in order to show that this composition specifically satisfies Property 6 of Definition 2.11 we first need to redefine $N_H^G i_H^*\underline{M}$ with an alternate Weyl action and restriction map and then show that this new Mackey functor is isomorphic to the original. Specifically, to show Property 6 of Definition 2.11 we use the fact that $i_H^*\underline{M}$ maintains the Weyl action on $\underline{M}$, and hence there is a Weyl action on $N_H^G i_H^*\underline{M}$ that combines this action with the original action. Moreover, since the restriction must map into the Weyl fixed points, using this alternate Weyl action requires us to tweak the restriction map as well.

Before stating this new Weyl action we provide further details regarding the original Weyl action and restriction maps of $N_H^G \underline{M}$.

Fact 5.8 (*Explicit description of the Weyl action on $N_H^G \underline{M}$*). The Weyl action on $N_H^G \underline{M}$ is as follows. When $H' \leq H$, $(N_H^G \underline{M})(G/H')$ is $\underline{M}^{\square|G/H|}(H/H')$, and thus, the generator γ of $W_G(H')$ acts on a simple tensor by

$$\gamma(\mathbf{m}^{\otimes|G/H|}) = \left(\gamma^{|G/H|} m_{\gamma|G/H|-1} \right) \otimes m_e \otimes m_\gamma \otimes \cdots \otimes m_{\gamma|G/H|-2},$$

where $\gamma^{|G/H|}$ generates $W_H(H')$.

For all subgroups K such that $H < K \leq G$, the generator γ of $W_G(K)$ acts on the generator $N(\mathbf{m}^{\times|G/K|})$ in $(N_H^G \underline{M})(G/K)$ by cyclically permuting the factors of $\mathbf{m}^{\times|G/K|}$ in the following way:

$$\begin{aligned} \gamma N(\mathbf{m}^{\times|G/K|}) &= N(\gamma(m_e \times m_\gamma \times \cdots \times m_{\gamma|G/K|-1})) \\ &= N(m_{\gamma|G/K|-1} \times m_e \times m_\gamma \times \cdots \times m_{\gamma|G/K|-2}). \end{aligned}$$

Fact 5.9 (*Explicit description of the restriction maps of $N_H^G \underline{M}$*). Recall that $H = C_{p^k}$, and let $K = C_{p^j}$ be a subgroup of G . If $j \leq k$ then res_{j-1}^j is the box product restriction of Definition 3.1. If $j > k$ then for $tr(x)$ in $Im(tr_{j-1}^j)$,

$$res_{j-1}^j(tr(x)) = \sum_{g \in W_K(C_{p^{j-1}})} gx,$$

and for a generator $N(\mathbf{m}^{\times|G/K|})$ in the image of the free summand in $(N_H^G \underline{M})(G/K)$,

$$res_{j-1}^j(N(\mathbf{m}^{\times|G/K|})) = \begin{cases} N\left(\prod_{i=0}^{p-1} \mathbf{m}^{\times|G/K|}\right) & \text{if } j-1 > k \\ \otimes_{i=0}^{p-1} \mathbf{m}^{\otimes|G/K|} & \text{if } j-1 = k \end{cases},$$

where $\prod_{i=0}^{p-1} (\mathbf{m}^{\times |G/K|})$ is the p -fold Cartesian product of $\mathbf{m}^{\times |G/K|}$ and $\bigotimes_{i=0}^{p-1} \mathbf{m}^{\otimes |G/K|}$ is the analogous tensor product.

Proposition 5.10. *Let $\underline{M}$ be a G -Mackey functor. There is a Weyl action on $N_H^G i_H^* \underline{M}$ that combines the Weyl action given in Fact 5.8 with the Weyl action defined on $\underline{M}$.*

Proof. We define the Weyl action on $N_H^G i_H^* \underline{M}$ as follows.

- For all subgroups H' of H , the generator γ of $W_G(H')$ acts on a simple tensor of $(N_H^G i_H^* \underline{M})(G/H')$ by cyclically permuting the factors of the tensor product *and* by acting on each factor. Thus,

$$\gamma \left(\mathbf{m}^{\otimes |G/H|} \right) = \gamma m_{\gamma|G/H|-1} \otimes \gamma m_e \otimes \gamma m_\gamma \otimes \cdots \otimes \gamma m_{\gamma|G/H|-2}.$$

- Let $r = |G/H| - |G/K| + 1$. For all subgroups K such that $H < K \leq G$ the generator γ of $W_G(K)$ acts on a generator $N(\mathbf{m}^{\times |G/K|})$ of the image of the free summand of $(N_H^G i_H^* \underline{M})(G/K)$ by

$$\gamma N \left(\mathbf{m}^{\times |G/K|} \right) = N \left(\gamma^r m_{\gamma|G/K|-1} \times \gamma m_e \times \gamma m_\gamma \times \cdots \times \gamma m_{\gamma|G/K|-2} \right).$$

Further, the restriction maps of $N_H^G i_H^* \underline{M}$ must remain compatible with the Weyl action. So, let $(g\mathbf{m})^{\otimes |G/H|}$ (or $(g\mathbf{m})^{\times |G/K|}$) denote g acting on each factor of the product:

$$(g\mathbf{m})^{\otimes |G/H|} = gm_e \otimes gm_\gamma \otimes \cdots \otimes gm_{\gamma|G/H|-1}.$$

If $K = C_{p^j}$ is a subgroup such that $H < K \leq G$, then

$$\text{res}_{j-1}^j(N(\mathbf{m}^{\times |G/K|})) = \begin{cases} N \left(\prod_{i=0}^{p-1} (\gamma^{i|G/K|} \mathbf{m})^{\times |G/K|} \right) & \text{if } j-1 > k \\ \bigotimes_{i=0}^{p-1} (\gamma^{i|G/K|} \mathbf{m})^{\otimes |G/K|} & \text{if } j-1 = k \end{cases}. \quad \square$$

Theorem 5.11. *Let $\underline{M}$ be a G -Mackey functor. The G -Mackey functor $N_H^G i_H^* \underline{M}$ with the Weyl action from Proposition 5.10 is isomorphic to $N_H^G i_H^* \underline{M}$ with the original Weyl action described in Fact 5.8.*

Proof. Let $U i_H^* \underline{M}$ denote the underlying H -Mackey functor of $i_H^* \underline{M}$. So, $U i_H^* \underline{M}$ does not remember the Weyl action from $\underline{M}$. Hence, we can let $N_H^G U i_H^* \underline{M}$ denote $N_H^G i_H^* \underline{M}$ with the Weyl action as defined in Definition 5.8. We will define an isomorphism $\chi: N_H^G U i_H^* \underline{M} \rightarrow N_H^G i_H^* \underline{M}$ by defining a collection of isomorphisms

$$\{\chi_P: (N_H^G U i_H^* \underline{M})(G/P) \rightarrow (N_H^G i_H^* \underline{M})(G/P) \text{ for all } P \leq G\}.$$

First, if $H' \leq H$, then define

$$\chi_{H'}: (U i_H^* \underline{M})^{\square |G/H|}(H/H') \rightarrow (i_H^* \underline{M})^{\square |G/H|}(G/H')$$

to be $1 \otimes \gamma \otimes \cdots \otimes \gamma^{|G/H|-1}$ on the tensor summand. On the image of the transfer map define $\chi_{H'}$ so that the appropriate diagram commutes. Similarly, for subgroups K such that $H < K \leq G$ let χ_K be $1 \times \gamma \times \cdots \times \gamma^{|G/K|-1}$ on the image of the free summand and on the image of the transfer require that the appropriate diagram commutes. $\square$

Finally, we complete the proof of Theorem 1.1 by proving Proposition 5.7.

Proof of Proposition 5.7. Since $\underline{M}$ is a G -commutative monoid, $(-) \otimes \underline{M}$ extends to a functor $\mathcal{S}et_G^{Fin} \rightarrow \mathcal{M}ack_G$. We will first show that $\underline{M}$ is a commutative G -Green functor by showing that $\underline{M}$ satisfies the categorical definition of a Green functor as given in [5] or [13]. We will then show that the codomain of $(-) \otimes \underline{M}$ is $\mathcal{G}reen_G$. We need the latter fact so that the internal norm maps that we will define to make $\underline{M}$ into a Tambara functor are multiplicative.

Let $*$ be the orbit G/G in $\mathcal{S}et_G^{Fin}$. The projection map $p: * \amalg * \rightarrow *$ induces a multiplication map $\underline{M} \square \underline{M} \rightarrow \underline{M}$ on $\underline{M}$, and the inclusion map $i: \emptyset \hookrightarrow *$ induces a unit map $\underline{A} \rightarrow \underline{M}$. Applying $(-) \otimes \underline{M}$ to the following three diagrams in $\mathcal{S}et_G^{Fin}$ results in the commutative diagrams in $\mathcal{M}ack_G$ needed to make $\underline{M}$ a G -Green functor.

$$\begin{array}{ccc}
 * \amalg * \amalg * & \xrightarrow{id \amalg p} & * \amalg * \\
 p \amalg id \downarrow & & \downarrow p \\
 * \amalg * & \xrightarrow{p} & *
 \end{array}
 \qquad
 \begin{array}{ccc}
 \emptyset \amalg * & \xrightarrow{i \amalg id} & * \amalg * \\
 \searrow = & & \swarrow = \\
 & * &
 \end{array}
 \qquad
 \begin{array}{ccc}
 * \amalg * & \xrightarrow{\tau} & * \amalg * \\
 p \searrow & & \swarrow p \\
 & * &
 \end{array}$$

To show that the codomain of $(-) \otimes \underline{M}$ is $\mathcal{G}reen_G$ we note that $G/H \otimes \underline{M}$ is a commutative Green functor for all orbits of G because $\underline{M}$ is a commutative Green functor and both functors N_H^G and i_H^* are strong symmetric monoidal. Then given a map $f: G/H \rightarrow G/K$ in $\mathcal{S}et_G^{Fin}$ we show that the induced map $G/H \otimes \underline{M} \rightarrow G/K \otimes \underline{M}$ is a morphism in $\mathcal{G}reen_G$ by applying $(-) \otimes \underline{M}$ to the diagrams below.

$$\begin{array}{ccc}
 G/H \amalg G/H & \xrightarrow{p} & G/H \\
 f \amalg f \downarrow & & \downarrow f \\
 G/K \amalg G/K & \xrightarrow{p} & G/K
 \end{array}
 \qquad
 \begin{array}{ccc}
 \emptyset & \xrightarrow{i} & G/H \\
 \searrow i & & \downarrow f \\
 & & G/K
 \end{array}$$

It remains to define norm maps $N_H^K: \underline{M}(G/H) \rightarrow \underline{M}(G/K)$ for all subgroups $H < K \leq G$. However, we need only construct the norm maps N_H^G since we can subsequently build every N_H^K by applying the process below to $i_K^* \underline{M}$. Let $\pi^*: N_H^G i_H^* \underline{M} \rightarrow \underline{M}$ be the map induced from $\pi: G/H \rightarrow G/G$. Since $N_H^G i_H^* \underline{M}$ has the Weyl action described in Proposition 5.10, $(i_H^* \underline{M})(H/H)$ is isomorphic to $\underline{M}(G/H)$. Then let the map $N: \underline{M}(G/H) \rightarrow (N_H^G i_H^* \underline{M})(G/G)$ be as given in Remark 3.4, and define the norm map N_H^G by the composition

$$\underline{M}(G/H) \xrightarrow{N} (N_H^G i_H^* \underline{M})(G/G) \xrightarrow{\pi_G^*} \underline{M}(G/G).$$

Since the functor $N_H^G: \mathcal{M}ack_H \rightarrow \mathcal{M}ack_G$ satisfies Property (a) of the Main Theorem, the above composition satisfies Property 3 of Definition 2.11. The composition satisfies Tambara reciprocity by the construction of the functor N_H^G .

Next we show that the norm map $\pi_G^* N$ factors through the Weyl action (i.e. that $\pi_G^* N$ satisfies Property 5 of Definition 2.11). The Weyl action on $G/H \otimes \underline{M}$ is induced from automorphisms of G/H , which are given by multiplication by γ^j for some γ^j in $W_G(H)$. Hence, the commutative diagram of G -sets on the left below induces the commutative diagram of Mackey functors on the right.

$$\begin{array}{ccc}
 G/H & \xrightarrow{\gamma^j \cdot} & G/H \\
 \searrow \pi & & \downarrow \pi \\
 & & G/G
 \end{array}
 \qquad
 \begin{array}{ccc}
 G/H \otimes \underline{M} & \xrightarrow{(\gamma^j \cdot)^*} & G/H \otimes \underline{M} \\
 \searrow \pi^* & & \downarrow \pi^* \\
 & & \underline{M}
 \end{array}$$

It follows that $\pi_G^*(N(\gamma^j x)) = \pi_G^*(N(x))$ for all x in $\underline{M}(G/H)$.

Finally, we show that $\text{res}_H^G \pi_G^* N(x) = \prod_{\gamma^j \in W_G(H)} \gamma^j x$ for all x in $\underline{M}(G/H)$. By Proposition 5.10 and properties of morphisms of Mackey functors we have

$$\text{res}_H^G \pi_G^* N(a) = \pi_H^* \text{res}_H^G N(a) = \pi_H^* \left(a \otimes \gamma a \otimes \cdots \otimes \gamma^{|G/H|-1} a \right).$$

Since the G -symmetric monoidal structure is compatible with the forgetful functor $i_H: \mathcal{S}et_G^{Fin} \rightarrow \mathcal{S}et_H^{Fin}$, it follows that π_H^* is induced from $i_H \pi$, which is the fold map $\coprod_{|G/H|} H/H \rightarrow H/H$. Therefore,

$$\pi_H^* \left(a \otimes \gamma a \otimes \cdots \otimes \gamma^{|G/H|-1} a \right) = a \gamma a \cdots \gamma^{|G/H|-1} a. \quad \square$$

Acknowledgements

The authors were supported by NSF Grant DMS-1207774. The authors thank Kate Ponto for her help and suggestions in organizing and writing this paper. They also thank the referee for their meticulous reading of this paper and their numerous valuable recommendations.

References

- [1] D. Benson, Problem session, in: A. Adem, J. Carlson, S. Priddy, P. Webb (Eds.), *Group Representations: Cohomology, Group Actions and Topology*, Seattle 1996, in: *Proceedings of Symposia in Pure Mathematics*, vol. 63, American Mathematical Society, 1998, pp. 519–532.
- [2] M. Brun, Witt vectors and equivariant ring spectra applied to cobordism, *Proc. Lond. Math. Soc.* 94 (2) (2007) 351–385.
- [3] M.A. Hill, M.J. Hopkins, Equivariant symmetric monoidal structures, arXiv:1610.03114, 2016.
- [4] R. Hoyer, Two Topics in Stable Homotopy Theory, Ph.D. thesis, University of Chicago, 2014.
- [5] L.G. Lewis Jr., The theory of Green functors, 1981, unpublished manuscript.
- [6] L.G. Lewis Jr., The $RO(G)$ -graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions, *Lect. Notes Math.* 1361 (1988) 53–122.
- [7] S. Mac Lane, *Categories for the Working Mathematician*, vol. 5, Springer-Verlag, 1998.
- [8] J.P. May, *Equivariant Homotopy and Cohomology Theory*, CBMS Regional Conference Series in Mathematics, vol. 91, 1996, Published for the Conference Board of the Mathematical Sciences, Washington, DC, with contributions by M. Cole, G. Comezana, S. Costenoble, A.D. Elmendorf, J.P.C. Greenlees, L.G. Lewis, Jr., R.J. Piacenza, G. Triantafillou, and S. Waner.
- [9] M. Shulman, *Equivariant Local Coefficients and the $RO(G)$ -Graded Cohomology of Classifying Spaces*, Ph.D. thesis, University of Chicago, 2010.
- [10] N. Strickland, Tambara functors, arXiv:1205.2516, 2012.
- [11] D. Tambara, On multiplicative transfer, *Commun. Algebra* 21 (4) (1993) 1393–1420.
- [12] J. Ullman, Symmetric powers and norms of Mackey functors, arXiv:1304.5648v2, 2013.
- [13] J. Ventura, Homological algebra for the representation Green functor for abelian groups, *Trans. Am. Math. Soc.* 357 (2005) 2253–2289.
- [14] P. Webb, A guide to Mackey functors, in: M. Hazewinkel (Ed.), *Handbook of Algebra*, vol. 2, Elsevier, 2000, pp. 805–836.